Ann. Inst. Henri Poincaré,

Vol. 5, n° 2, 1988, p. 141-178. *Analyse non linéaire*

On the singular limit in a phase field model of phase transitions

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ABSTRACT. — The limits of families of stable solutions for the equation $\varepsilon^2 \Delta u_\varepsilon - f(u_\varepsilon) + T(|x|) = 0$ over radially symmetric domains with no-flux boundary conditions are discribed. Particular emphasis is placed on the characterization of points of discontinuity of these limits (interfaces) and on the description of the graph of u_ε for small ε. Sufficient conditions for existence of interfaces in terms of the temperature function, T, are given. The analysis is more complete for families of global minimizers of the associated energy functional.

Key words : Singular perturbation, phase transition, interior layer, radial symmetry.

RÉSUMÉ. — On étudie les limites de familles de solutions stables de l'équation $\varepsilon^2 \Delta u_\varepsilon - f(u_\varepsilon) + T(|x|) - 0$ sur des domaines à symétrie radiale sans flux au bord. On s'attache particulièrement à caractériser les points de discontinuité de ces limites (interfaces) et à décrire le graphe de u_ε pour

Classification A.M.S. : 34 D 15, 35 B 25, 35 J 65.
(*) Partially supported by NSF DMS 8601 790.

petit ε. On donne des conditions suffisantes sur la température T pour qu'il existe des interfaces. On analyse de manière plus approfondie les familles minimisant l'énergie totale.

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0. INTRODUCTION

In this paper we consider the equation

$$\begin{gathered}\varepsilon^2 \Delta u - f(u) + \varepsilon \mathrm{T}(r) = 0, \qquad r = |x|, \quad x \in \Omega \\ \left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} = 0\end{gathered} \tag{1}$$

where Ω is a ball or an annulus in $\mathbb{R}^N$ and $\varepsilon > 0$ is a small parameter. The basic structure hypotheses are:

(i) $\mathrm{T} = \mathrm{T}(r)$, T' is continuous and changes sign finitely many times;

(ii) $f(u) = \mathrm{F}'(u)$ is of class C^1 and has precisely three zeroes, $f(-1) = f(1) = f(0) = 0$ with $f'(-1) > 0$, $f'(+1) > 0$, $f'(0) < 0$;

(iii) $\mathrm{F}(-1) = \mathrm{F}(1)$.

Note that Equation (1) reduces as $\varepsilon \to 0$, to the algebraic equation

$$-f(u) = 0 \tag{2}$$

Our study is concerned with the solutions of (2) that can be captured as limits of stable solutions of (1). In our investigation we exploit the fact that (1) is the Euler-Lagrange equation of the functional

$$\mathrm{J}_\varepsilon(u) = \frac{\varepsilon^2}{2} \int_\Omega |\nabla u|^2 \, dx + \int_\Omega \mathrm{F}(u) \, dx - \varepsilon \int_\Omega \mathrm{T}(x) u \, dx \tag{3}$$

over $\mathrm{W}^{1,2}(\Omega)$.

Equation (1) arises as a model in solidification theory, known as the *phase field model.* In that context u represents the phase of the material with say $u > 0$ representing liquid and $u < 0$ representing solid, T is the temperature, $\varepsilon>0$ is related to the correlation length or interaction distance between molecules, F is a double well potential and J_ε is the free energy. If $\varepsilon=0$, then any u_0 with $u_0(x)\in\{+1,-1\}$ is a global minimizer of J_0. We think of (1) in terms of a stability mechanism that selects appropriate solutions of (2). We refer the reader to Caginalp [C] for the physical background for the phase field model.

Part of this work is concerned with the characterization of the interior points of discontinuity of limits of global minimizers. We call these points *interfaces*. We give a detailed description of the asymptotic shape of the graphs of global minimizers, u_ε, as $\varepsilon\to 0$ and in particular we study their structure in appropriate neighborhoods of the interfaces where rapid variations of u_ε are to be expected (*interior layers*). Finally we give sufficient conditions so that interfaces exist.

Equation (1) has been studied rigorously by Caginalp and Fife [CF] in a general annular domain in two space dimensions and for a general function $T=T(x_1, x_2)$. The emphasis in that work is directed more in identifying conditions which guarantee the existence of a simple closed curve in Ω that can serve as an interface and in constructing families of solutions to (1) with interior layers at locations which converge to such a prescribed interface.

Caginalp and McLeod [CM] study all radial solutions of (1) in an annular domain for T identically constant, with inhomogeneous Dirichlet conditions and for a special choice of F. In that work the location of possible interfaces is characterized but no sufficient conditions for their existence are given.

Next we proceed to a more detailed description of our results. In Section 1 we begin with an observation of some independent interest. We show that stable solutions, in the linearized sense, of a broad class of equations on domains with radial symmetry have to be radial. In this way we reduce the study of (1) for stable solutions to the study of radial solutions. In Section 2 we establish that any sequence of stable solutions of (1) with $\varepsilon\to 0$, has a subsequence which converges pointwise and in L^1 to solutions of (2) with finitely many discontinuities. Theorem 2.6 in Section 2 is one of the main results of this work. It states that away from interfaces, any sequence of global minimizers, with $\varepsilon\to 0$, has a subsequence which converges uniformly and provides the rate of convergence. In Section 3 we study the structure of the interior layers; more precisely we show in the layers that the solution is monotone and we give an estimate on the width of the layer. We also give a sharp upper bound for the number of interfaces. In Section 4 we characterize the location of the interfaces by establishing that at these points (for $N \geqq 2$) the following relations hold: If $u_L = u(r^-)$ and $u_R = u(r^+)$ then

$$(u_L - u_R)\,T(r) = \frac{c}{r}, \qquad c = \sqrt{2}(N-1)\int_{-1}^{1}\sqrt{F(\sigma) - F(1)}\,d\sigma \tag{4}$$

and

$$(u_L - u_R)\,(T(r)\,r)' \leqq 0 \tag{5}$$

Relation (4) in the physical context we have in mind is known as the Gibbs-Thompson condition. Relation (5) is a stability condition that seems new even in the context of formal asymptotic expansions. For $N = 1$ the corresponding relations have a different form:

$$T(r) = 0 \tag{6}$$

$$(u_L - u_R)\,T'(r) \leqq 0. \tag{7}$$

In Section 5 we give sufficient conditions for the existence of interfaces for global minimizers of J_ε. In Section 6 we extend some of the results above given for global minimizers to linearly stable solutions of (1). Specifically we show that (5), (6) and (7) continue to hold in this generality. As far as (4) is concerned we have been able to establish that for a given convergent family $\{u_\varepsilon\}$ of linearly stable solutions to (1) the interfaces are characterized by

$$(u_L - u_R)\,T(r) = \frac{\tilde{C}}{r}. \tag{8}$$

Our result as it stands allows the possibility that $\tilde{C}$ depends upon the family. On the other hand we establish the lower bound

$$\tilde{C} \geqq \sqrt{2}(N-1)\int_{-1}^{1}\sqrt{F(\sigma) - F(1)}\,d\sigma \tag{9}$$

After most of this work had been completed we learned of the recent thesis by P. Sternberg [S] done under the direction of R. Kohn. Sternberg, by extending work of Modica and Mortola [MM] on DeGiorgi's Γ-convergence, identifies the first nontrivial term in the asymptotic expansion of J_ε. These results can also be applied to obtain some of our results in the case of *global* minimizers. The method of Γ-convergence does not render the fine structure of solutions as given in this paper. Some recent work of

L. Modica ([M 1] and [M 2]) also addresses the topic of phase transitions using Γ-convergence and gives a "minimal interfacial area" principle for a two-phase medium.

In a recent paper J. J. Mahony and J. Norbury [MN] locate interfaces for the singular limit of a different but related class of problems by employing formal variational arguments.

We wish to express our thanks to P. Fife for bringing to our attention the phase field model and also P. Sternberg for bringing to our attention his work. The authors would like to express thanks to J. Mallet-Paret and S. Muller for useful conversations regarding this work. The first author gratefully acknowledges support from the Lefschetz Center for Dynamical Systems at Brown University where this work was completed.

1. STABILITY IMPLIES SYMMETRY

Let q be an integer, $2 \leqq q \leqq N$, write $x=(x_1, x_2) \in \mathbb{R}^q \times \mathbb{R}^{N-q}$ and let $r=|x_1|$. Let Ω_2 be an open bounded domain in $\mathbb{R}^{N-q}$, let A, $B:\Omega_2 \to \mathbb{R}$ be bounded continuous functions and define $\Omega=\{x: A(x_2)<r<B(x_2), x_2 \in \Omega_2\}$. If $q=N$, then we take $\Omega_2=\mathbb{R}^{N-q}=\{0\}$ so that Ω is an annular region in $\mathbb{R}^N$. Furthermore, in that case if $A(0)<0$, then Ω is a ball in $\mathbb{R}^N$ of radius $B(0)$. In general, the property which we use is that Ω has rotational symmetry about the $\mathbb{R}^{N-q}$-space. Let $\partial\Omega=\Gamma_0 \cup \Gamma_1$, $\Gamma_0 \cap \Gamma_1=\varnothing$ with Γ_0 and Γ_1 both open and closed in $\partial\Omega$.

Consider the boundary value problem

$$\Delta u - f(r, x_2, u, \nabla u)=0 \quad \text{in } \Omega \tag{1.1}$$

$$Bu=0 \quad \text{on } \partial\Omega \tag{1.2}$$

where

$$Bu=\begin{cases} u & \text{on } \Gamma_0 \\ \dfrac{\partial u}{\partial n}+b(r, x_2)u & \text{on } \Gamma_1, \end{cases}$$

n being a C^1, outward pointing, nowhere tangent vector field.

Assume

(i) Ω has a C^2 boundary;

(ii) f and b are of class C^1;

(iii) $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ is a solution to (1.1) and (1.2).

Our symmetry result in Lemma 1.1 below is also true in some settings not allowed by (i)-(iii), however, at this point we do not wish to cloud the issue with other technical considerations.

Define the linear operator L in C $(\bar{\Omega})$ by

$$L v = \Delta v - f_u(r, x_2, u, \nabla u) v - f_p(r, x_2, u, \nabla u) \cdot \nabla v \quad \text{in } \Omega,$$
$$\text{Dom}(L) = \{ v \in W_p^2(\Omega) : B v = 0,\ L v \in C(\bar{\Omega}),\ L u = 0 \text{ on } \Gamma_0 \}.$$

DEFINITION. – We say that u is a (linearly) stable solution if and only if $\text{Re}(\lambda) \leqq 0$ for all $\lambda \in \sigma(L)$, the spectrum of L.

By a variant of the Krein-Rutman Theorem due to Amann (*see* Theorem 12.1 of [Am]), the principal eigenvalue of L, λ_0, is real, simple and corresponds to a nonnegative eigenfunction.

LEMMA 1.1. – *Assume* (i)-(iii) *above. If u is stable then* $u = u(r, x_2)$.

Proof. – Write $u = u(r, \theta_1, \ldots, \theta_{q-1}, x_2)$ where $x_1 = (r, \theta_1, \ldots, \theta_{q-1})$ in generalized polar coordinates in $\mathbb{R}^q$. Consider u_θ where $\theta = \theta_i$ for some $i = 1, \ldots, q-1$. Since Δ is rotationally invariant we have

$$B u_\theta = 0 \quad \text{on } \partial\Omega$$

and

$$\frac{\partial}{\partial\theta}(\Delta u - f(r, x_2, u, \nabla u)) = L u_\theta = 0 \quad \text{in } \Omega.$$

Either $u_\theta \equiv 0$ or u_θ is an eigenfunction of L corresponding to eigenvalue 0. If u_θ is an eigenfunction then $\lambda_0 = 0$ by the stability assumption and hence $u_\theta \geqq 0$ on Ω by Amann's Theorem. But u_θ must have mean value in θ equal to zero. Hence, $u_\theta \equiv 0$ and it follows that $u = u(r, x_2)$ as claimed.

Remark. – The proof above exploits the connectedness of the sphere in dimension $N \geqq 2$. In one dimension symmetry is not necessarily inherited by stable or even asymptotically stable solutions. Indeed, based on Theorem 3 of [FH] one can easily construct an example

$$u_t = u_{xx} + f(x, u), \qquad -1 < x < 1$$
$$u_x(\pm 1) = 0$$

with f of class C^1 and $f(-x, u) = f(x, u)$, which possesses an asymptotically stable, odd, strictly monotone equilibrium. By "asymptotically stable" we mean that the linear operator $(\partial^2/\partial x^2 + f_u(x, u))$ with Neumann boundary conditions, has all its eigenvalues having negative real part. This counterexample was pointed out to us by G. Fusco who in so doing settled a question posed in a previous version of this paper. We note that the asymmetrical solution mentioned above cannot possibly be a global minimizer for $J(u) \equiv \int_{-1}^{1} 1/2(u')^2\, dx - \int_{-1}^{1} F(x, u)\, dx$ over $H^1(-1, 1)$ (here $F_u = f$).

2. CONVERGENCE

Let $\Omega = \{x \in \mathbb{R}^N : A < |x| < B\}$, where $A \geqq 0$ and for $\varepsilon > 0$ consider the family of functionals on $H^1(\Omega)$

$$J_\varepsilon(u) \equiv \frac{\varepsilon^2}{2} \int_\Omega |\nabla u|^2\, dx + \int_\Omega F(u)\, dx - \varepsilon \int_\Omega T(x)\, u\, dx. \tag{2.1}$$

We assume

(H1) $T = T(|x|)$ is of class C^1 on [A, B] and T' has finitely many, n, changes of sign, occurring at points we call $\{r_i\}_{i=1}^{n}$ arranged so that $A \equiv r_0 < r_1 < \ldots < r_n < r_{n+1} \equiv B$.

(H2) $F'(u) \equiv f(u)$ is bounded and of class C^1.

(H3) f has precisely three zeros,

$$f(-1) = f(0) = f(1) = 0 \qquad \text{with} \quad f'(\pm 1) > 0, \quad f'(0) < 0.$$

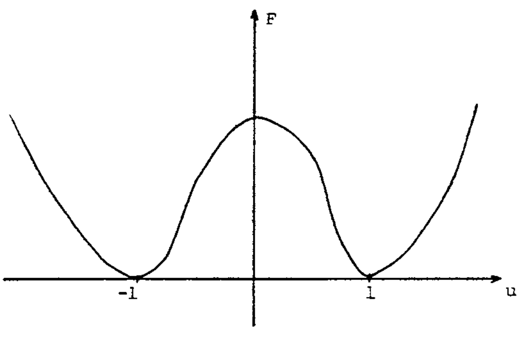

FIG. 1.

We also assume that

$$\text{(H4)} \qquad F(-1)=F(1)$$

but we do not assume that F is even (see *Fig.* 1). There is no loss in assuming $F(1)=0$.

In (H1) the term "changes in sign" is used. We mean this in a rather weak sense as clarified below. In particular T′ is allowed to be zero on nontrivial intervals.

DEFINITION. — A function v has *exactly k changes of sign* in [A, B] provided:

(i) There are points $s_1 < s_2 < \ldots < s_{k+1}$ in [A, B] with $v(s_i)\,v(s_{i+1}) < 0$ for all $i=1, 2, \ldots, k$.

(ii) k is maximal

We will say that v has *sign changes at the points* $t_1 < t_2 < \ldots < t_k$ provided $v(t_i)=0$, the points in (*i*) can be chosen so that $s_i < t_i < s_{i+1}$ and v does not change sign on $[t_i, t_{i+1}]$.

With this definition $\operatorname{sign}(v(t_i^+))$, $\operatorname{sign}(v(t_i^-))$ and sign $(v(c))$ for $c\in(t_i, t_{i+1})$ all may be defined in the obvious way, taking values $+1$ or -1.

The functional, J_ε, in (2.1) gives the "energy" of solutions to the equations

$$\begin{gathered}\varepsilon^2\,\Delta u - f(u) + \varepsilon\,T(r) = 0 \quad \text{in } \Omega \\ \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega.\end{gathered} \tag{2.2}$$

By Lemma 1.1, stable solutions of (2.2) are radially symmetric if $N \geqq 2$. Since we will be concerned only with such solutions we can rewrite $J_\varepsilon(u)$ for $u=u(r)$ as

$$J_\varepsilon(u) = \frac{\varepsilon^2}{2}\int_A^B u'^2\, r^{N-1}\,dr + \int_A^B F(u)\, r^{N-1}\,dr - \varepsilon\int_A^B T(r)\, u r^{N-1}\,dr \tag{2.3}$$

and the associated Euler-Lagrange equation as

$$\begin{gathered}\varepsilon^2\left(u'' + \frac{(N-1)}{r}u'\right) - f(u) + \varepsilon\,T(r) = 0, \qquad A < r < B \\ u'(A) = u'(B) = 0.\end{gathered} \tag{2.4}$$

We will be considering the convergence, as ε approaches zero, of local minimizers u_ε of J_ε. The first result of this section will be useful for bounding the number of interfaces and thereby allowing us to determine the fine structure of these minimizers for small values of ε.

LEMMA 2.1. — *Assume* (H1) *and* (H2). *Let u be a linearly stable solution to* (2.2), *then $u=u(r)$ and u' has at most n changes of sign.*

Proof. — Lemma 1.1 shows that $u=u(r)$ and satisfies (2.4).

Suppose that u' has exactly $M > n$ sign changes in [A, B] at points $t_1 < t_2 < \ldots < t_M$. The crux of the lemma lies in the following observation:

STEP I. — For $1 \leqq i \leqq n+1$ if $a, b \in (r_{i-1}, r_i]$, $a < b$ and $u'(a)=u'(b)=0$ then $u'T'$ cannot be nonpositive (and $u' \not\equiv 0$) on $[a, b]$.

Proof. — Suppose $u'T' \leqq 0$, $u' \not\equiv 0$ on $[a, b]$. Let

$$v=\begin{cases} u' & \text{on } [a, b] \\ 0 & \text{otherwise.} \end{cases}$$

Differentiating in (2.4), multiplying the result by v and integrating gives

$$\int_\Omega [\varepsilon^2 \Delta v - f'(u) v] v \, dx = \varepsilon^2 (N-1) \int_\Omega \left(\frac{v}{|x|}\right)^2 dx - \varepsilon \int_a^b T'(r) u'(r) r^{N-1} dr > 0,$$

contradicting the stability of u.

An immediate consequence of Step I is that no more than two of the t_j's can lie in any of the intervals $[r_i, r_{i+1}]$, $0 \leqq i \leqq n$.

STEP II. — Suppose that $t_{j-1}, t_j \in (r_{i-1}, r_i]$ for some i and j. Then for any k, $t_{j+k+1} > r_{i+k}$.

Proof. — By Step I, $u'T' \geqq 0$ on $[t_{j-1}, t_j]$. We induct on k: Suppose that $t_{j+2} \leqq r_{i+1}$, then by Step I and the subsequent observation we have $r_i < t_{j+1} < t_{j+2} \leqq r_{i+1}$ and $u'T' \geqq 0$ on $[t_{j+1}, t_{j+2}]$. Since

$$\operatorname{sign}(T'(t_{j-1})) \operatorname{sign}(T'(t_{j+1})) < 0$$

it follows that

$$\operatorname{sign}(u'(t_{j-1}^+)) \operatorname{sign}(u'(t_{j+1}^+)) < 0,$$

and hence that

$$\operatorname{sign}(u'(t_j^+))\operatorname{sign}(u'(t_{j+1}^-)) < 0,$$

a contradiction. The statement of Step II holds for $k=1$.

Now suppose that $t_{j+k} > r_{i+k-1}$ but that $t_{j+k+1} \leqq r_{i+k}$ for some k. We have

$$\begin{aligned}\operatorname{sign}(u'(t_{j+k}^-)) &= (-1)^k \operatorname{sign}(u'(t_j^-)) \\ &= (-1)^k \operatorname{sign}(T'(r_i)) \\ &= \operatorname{sign}(T'(r_{i+k}^-)) \\ &= \operatorname{sign}(T'(t_{j+k})).\end{aligned}$$

But then $u'T' \leqq 0$ on $[t_{j+k}, t_{j+k+1}]$, a contradiction to Step I. This completes Step II.

STEP III. — By Step I if $u'(r_i)\operatorname{sign}(T'(r_i^+)) \geqq 0$ then at most one of the t_j's lies in $(r_i, r_{i+1}]$. Furthermore, u' has at most one sign change in $(A, r_1]$ and if a change occurs then sign $(u'T')(A^+) \geqq 0$. From this it follows that if exactly one of the t_j's lies in $(r_{m-1}, r_m]$ for each m, $1 \leqq m \leqq i$ then u' can have at most one sign change in $(r_i, r_{i+1}]$.

STEP IV. — $t_{k+1} > r_k$ for each $k \leqq n$.

Proof. — Suppose not, then there exists $i \leqq k$ such that exactly two of the t_j's lie in $(r_{i-1}, r_i]$ and at most one lies in each $(r_{m-1}, r_m]$ for all $1 \leqq m < i$. By the observations in Step III, $i > 1$ and there cannot be one sign change in each $(r_{m-1}, r_m]$, $1 \leqq m < i$. So $t_{i+1} > r_i$ and $t_j \in (r_{i-1}, r_i]$ only if $j \leqq i$. By Step II, $t_{j+(k-i)+1} > r_k$ and so $t_{k+1} = t_{i+(k-i)+1} > r_k$. This contradiction completes the proof of Step IV.

Finally we have

$$\operatorname{sign}(u'(A^+)) = \operatorname{sign}(u'(t_1)) = (-1)^{n+1} \operatorname{sign}(u'(t_{n+1}^+))$$

and

$$\operatorname{sign}(T'(A^+)) = \operatorname{sign}(T'(r_1)) = (-1)^n \operatorname{sign}(T'(r_n^+)).$$

Since at most one of the t_j's can lie in $[r_n, B)$ this implies that $u'T' \leqq 0$ on either $[A, t_1]$ or on $[t_{n+1}, B]$. This contradicts Step I and hence completes the proof of the lemma.

We now give some useful estimates for the rate of convergence, as $\varepsilon \to 0$, of stable solutions, of (2.4) away from interfaces.

LEMMA 2.2. — *Assume* (H1)-(H3). *For* $\varepsilon > 0$ *let* u_ε *be a stable solution to* (2.4) *and suppose that* $\{u_\varepsilon\}$ *converges to the number* $z \in \{\pm 1\}$ *on some interval* $[a, b] \subset [A, B]$ *uniformly as* $\varepsilon \to 0$. *If* $[c, d] \subset (a, b)$ *is fixed, then the following estimates hold as* $\varepsilon \to 0$

(i) $$|u_\varepsilon - z| = O(\varepsilon) \quad \text{on } [c, d],$$

(ii) $$\int_c^d (u_\varepsilon')^2 r^{N-1}\, dr = O(1),$$

(iii) $$\int_c^d |u_\varepsilon'| r^{N-1}\, dr = O(\varepsilon).$$

Proof. — Without loss of generality assume $u_\varepsilon \to 1$ on $[a, b]$ as $\varepsilon \to 0$. Fix points $\bar{c} \in (a, c)$ and $\bar{d} \in (d, b)$. By the Mean Value Theorem, there are points $a_\varepsilon \in (a, \bar{c})$ and $b_\varepsilon \in (\bar{d}, b)$ such that $u_\varepsilon'(a_\varepsilon) = (u_\varepsilon(\bar{c}) - u_\varepsilon(a))/(\bar{c} - a)$ and $u_\varepsilon'(b_\varepsilon) = (u_\varepsilon(b) - u_\varepsilon(\bar{d}))/(b - \bar{d})$. These approach zero with ε. Now fix $C_1 > 0$ such that $f'(1) > C_1$ and choose $\varepsilon_0 > 0$ such that $f'(u_\varepsilon(r)) > C_1$ for all $0 < \varepsilon \leqq \varepsilon_0$ and $r \in [a, b]$. Multiplying (2.4) by $(u_\varepsilon - 1) r^{N-1}$ and integrating gives

$$\varepsilon^2 \int_{a_\varepsilon}^{b_\varepsilon} (u_\varepsilon - 1)[(u_\varepsilon - 1)' r^{N-1}]'\, dr - \int_{a_\varepsilon}^{b_\varepsilon} [f(u_\varepsilon) - f(1)](u_\varepsilon - 1) r^{N-1}\, dr + \varepsilon \int_{a_\varepsilon}^{b_\varepsilon} T(r)(u_\varepsilon - 1) r^{N-1}\, dr = 0.$$

Integration by parts applied to the first term and the Mean Value Theorem applied to the second yields

$$\varepsilon^2 (u_\varepsilon - 1)(u_\varepsilon - 1)' r^{N-1} \Big|_{a_\varepsilon}^{b_\varepsilon} + \varepsilon \|T\|_\infty \int_{a_\varepsilon}^{b_\varepsilon} |u_\varepsilon - 1| r^{N-1}\, dr \geqq \varepsilon^2 \int_{a_\varepsilon}^{b_\varepsilon} [(u_\varepsilon - 1)']^2 r^{N-1}\, dr + C_1 \int_{a_\varepsilon}^{b_\varepsilon} (u_\varepsilon - 1)^2 r^{N-1}\, dr. \quad (2.5)$$

It follows that for all $\varepsilon \leqq \varepsilon_0$

$$C_1 \int_{a_\varepsilon}^{b_\varepsilon} (u_\varepsilon - 1)^2 r^{N-1}\, dr \leqq \varepsilon \|T\|_\infty \int_{a_\varepsilon}^{b_\varepsilon} |u_\varepsilon - 1| r^{N-1}\, dr + O(\varepsilon^2)$$

and consequently that

$$\int_{a_\varepsilon}^{b_\varepsilon} (u_\varepsilon - 1)^2 r^{N-1} dr \leqq C_2 \varepsilon^2 \tag{2.6}$$

for some constant $C_2 > 0$ independent of $\varepsilon \leqq \varepsilon_0$. Now, $[a_\varepsilon, b_\varepsilon] \supset (\bar{c}, \bar{d}) \supset [c, d]$ so for some constant $C_3 > 0$ there are points $c_\varepsilon \in (\bar{c}, c)$ and $d_\varepsilon \in (d, \bar{d})$ such that for all $0 < \varepsilon \leqq \varepsilon_0$

$$|u_\varepsilon(c_\varepsilon) - 1|, \quad |u_\varepsilon(d_\varepsilon) - 1| \leqq C_3 \varepsilon. \tag{2.7}$$

But $(u_\varepsilon - 1)$ satisfies

$$\varepsilon^2 r^{1-N} (r^{N-1} (u_\varepsilon - 1)')' - (f(u_\varepsilon) - f(1)) = -\varepsilon T(r)$$

and $|f(u_\varepsilon) - f(1)| \geqq C_1 |u_\varepsilon - 1|$, for $0 < \varepsilon \leqq \varepsilon_0$ and all $r \in [c_\varepsilon, d_\varepsilon]$. Using a simple comparison argument combined with (2.7) one can see that

$$|u_\varepsilon - 1| \leqq \varepsilon \max\{\|T\|_\infty / C_1, C_3\} \quad \text{on } [c_\varepsilon, d_\varepsilon] \supset [c, d].$$

This proves part (i) of the lemma. To establish part (ii) we note that (2.5) and (2.6) give

$$\varepsilon^2 \int_c^d [u_\varepsilon']^2 r^{N-1} dr \leqq C_4 \varepsilon^2 + O(\varepsilon^2)$$

for some $C_4 > 0$, independent of ε.

Part (iii) follows from part (i) using the fact that u_ε' changes sign at most n times.

In the following observation, part (i) follows from the Maximum Principle and part (ii) follows from (i) and equation (2.2).

LEMMA 2.3. — *Assume* (H1)-(H3). *Let* u_ε *be a solution of* (2.2). *Then*

(i) $\overline{\lim_{\varepsilon \to 0}} \|u_\varepsilon\|_\infty \leqq 1$

and for $0 < \varepsilon < \varepsilon_0$

(ii) $\int_\Omega |\nabla u_\varepsilon|^2 dx \leqq C/\varepsilon^2$ *for some constant* $C > 0$.

We are now prepared to prove our first convergence result.

LEMMA 2.4. — *Assume* (H1)-(H3). *Let* ε_0 *be as in the previous lemma and for each* $\varepsilon \in (0, \varepsilon_0]$ *let* u_ε *be a stable solution to* (2.2). *Then for any* *sequence $\{\varepsilon_n\}$ approaching zero, there exists a subsequence with corresponding u_ε's converging pointwise to a piecewise constant function $\bar{u}$. Furthermore, $\bar{u}$ has at most $(n+1)$ points of discontinuity in* (A, B) *and* $|\bar{u}|=1$ *except possibly at points of discontinuity.*

Proof. — By Lemma 1.1 each u_ε is radially symmetric. It follows that the total variation is given by

$$\| u_\varepsilon \|_{\mathrm{BV}} = \int_{\mathrm{A}}^{\mathrm{B}} | u'_\varepsilon | \, dr.$$

But Lemmas 2.1 and 2.3 (i) now imply

$$\overline{\lim_{\varepsilon \to 0}} \| u_\varepsilon \|_{\mathrm{BV}} \leqq 2(n+1).$$

Helly's Theorem (*see* e. g. [N]) gives a pointwise convergent subsequence of any sequence of the u_ε's. We suppose that $\{\varepsilon_m\}$ is a sequence approaching zero, and that the corresponding u_ε's, call them $\{u_m\}$, converge pointwise to a function $\bar{u}$ on [A, B].

By lower semicontinuity

$$\| \bar{u} \|_{\mathrm{BV}} \leqq 2(n+1).$$

If we multiply (2.2) by a test function $\varphi \in \mathrm{C}_0^\infty(\Omega)$ and integrate we obtain

$$-\varepsilon_m^2 \int_\Omega \nabla u_m . \nabla \varphi \, dx - \int_\Omega f(u_m) \varphi \, dx + \varepsilon_m \int_\Omega \mathrm{T} \varphi \, dx = 0.$$

Using Lemma 2.3 (ii) and Lebesque's Theorem, letting $m \to \infty$ yields

$$\int_\Omega f(\bar{u}) \varphi \, dx = 0$$

and hence, $f(\bar{u}) = 0$ a. e. This in turn implies that

$$\bar{u}(r) \in \{-1, 0, 1\} \text{ a. e.}$$

If we assume that $\bar{u}$ takes the value 0 on a nonempty open interval I, we may take a smooth function h with support in I and compute

$$\int_\Omega [\varepsilon_m^2 \Delta h - f'(u_m) h] h \, dx = -\varepsilon_m^2 \int_\Omega |\nabla h|^2 \, dx - \int_\Omega f'(u_m) h^2 \, dx > 0$$

for m sufficiently large. This contradicts the stability of u_m and so such an interval I does not exist. It now follows that $\bar{u}$ has at most $(n+1)$ points of discontinuity in (A, B) and the proof is complete.

Remarks. — A similar application of Helly's Theorem occurred in [ASi]. The pointwise convergence together with the L^∞ bounds for $\{u_m\}$ immediately gives L^p convergence for all $p < \infty$.

So far we have been concerned with solutions which are linearly stable. If we restrict our attention to minimizers of J_ε it is easy to obtain stronger estimates as the following shows.

LEMMA 2.5. — *Assume* (H1)-(H4). *Let u be a global minimizer of* J_ε, *then for some constant* $C > 0$

$$\frac{\varepsilon^2}{2}\int_\Omega |\nabla u|^2\,dx + \int_\Omega F(u)\,dx \leqq C\,\varepsilon. \tag{2.8}$$

Proof. — $J_\varepsilon(u) \leqq J_\varepsilon(\pm 1)$ and so $J_\varepsilon(u) \leqq 0$. Lemma 2.3 (i) implies (2.8).
We now come to the main result of this section.

THEOREM 2.6. — *Assume* (H1)-(H4). *Let $\{\varepsilon_m\}$ converge to zero and let $\{u_m\}$ be corresponding global minimizers, converging pointwise to a function $\bar{u}$ as given by Lemma 2.4. Let $\{s_j\}_{j=1}^n$ be the interior points of discontinuity of $\bar{u}$ and set $s_0 = A$, $s_{n+1} = B$. Given $\delta > 0$ small, there exists a constant $K > 0$ such that on* $S \equiv \bigcup_{j=0}^{n} [s_j+\delta,\ s_{j+1}-\delta]$

$$|u_m - \bar{u}| \leqq K\,\varepsilon_m.$$

Proof. — Suppose not, then by Lemma 2.2 (i) convergence cannot be uniform on S, for some $\delta > 0$. Thus we can select an interval $[a, b] = [s_j+\delta,\ s_{j+1}-\delta]$ for some $j \leqq n$, a subsequence $\{u_{m_i}\}$, a sequence $\{t_i\} \subset [a, b]$ converging to some point t_0, and a number $\beta > 0$ such that

$$|u_{m_i}(t_i) - \bar{u}(t_i)| \geqq \beta \quad \text{for all } i \geqq 1. \tag{2.9}$$

Without loss of generality we may take $\bar{u} = +1$ on $[a, b]$. We will also consider a subinterval of (s_j, s_{j+1}), relabelled as $[a, b]$, containing t_0 in its interior. We may assume that $|u_{m_i}(a) - 1| < \beta$ and $|u_{m_i}(b) - 1| < \beta$ for all i. Note that by Lemma 2.1 we can find subintervals P and Q of $[a, b]$ on either side of t_0 and a further subsequence of $\{u_{m_i}\}$ with all terms monotone on P and on Q (*see* the proof of Lemma 6.1). Thus we may assume that $\{u_{m_i}\}$ converges uniformly on neighborhoods of a and b, with a rate of $O(\varepsilon)$ by Lemma 2.2 (i). We need a more detailed description of the lack of uniformity. We pause to give the following lemma which indicates that the only nonuniformities are of large amplitude.

LEMMA 2.7. – *Assume* (H1)-(H4). *Let* $\alpha, \gamma \in (0, 1/4)$ *and* $\varepsilon_1 > 0$ *be such that*

$$F(z) - \varepsilon T(r) z \text{ is decreasing in } z \in [1-2\alpha, 1-\alpha] \quad \text{for all } r \in [A, B] \text{ and } \varepsilon \in (0, \varepsilon_1], \tag{2.10}$$

and

$$F(z) - F(1-\alpha) \begin{cases} \geqq 2\varepsilon_1 \|T\|_\infty & \text{for } z \in [-1+\gamma, 1-2\alpha] \\ < 2\varepsilon_1 \|T\|_\infty & \text{for } z \in [-1, -1+\gamma). \end{cases} \tag{2.11}$$

Let u *be a global minimizer of* J_ε *for some* $\varepsilon \in (0, \varepsilon_1]$. *Suppose that there are points* $a_1 < a_3 < a_2$ *such that*

$$u(a_1) = u(a_2) = 1-\alpha > u(a_3). \tag{2.12}$$

Then there is a point $c \in (a_1, a_2)$ *with*

$$u(c) < -1+\gamma \quad \text{and} \quad F(u(c)) < F(1-\alpha) + 2\varepsilon_1 \|T\|_\infty.$$

Proof. – By (2.11) it suffices to show that there is a point $c \in (a_1, a_2)$ such that

$$u(c) \leqq 1-2\alpha \quad \text{and} \quad F(u(c)) < F(1-\alpha) + 2\varepsilon_1 \|T\|_\infty.$$

Suppose not. We may assume without loss of generality that $u \leqq 1-\alpha$ and $u \not\equiv 1-\alpha$ on $[a_1, a_2]$.

Define a comparison function (see *Fig.* 2) by

$$\hat{u} = \begin{cases} u & \text{outside } [a_1, a_2] \\ 1-\alpha & \text{on } [a_1, a_2] \end{cases}$$

Let $Q = \{r \in [a_1, a_2] : u(r) \leqq 1-2\alpha\}$ and $R = [a_1, a_2] \setminus Q$. Since $J_\varepsilon(u) \leqq J_\varepsilon(\hat{u})$ we have

$$\int_Q ([F(u) - F(1-\alpha)] - \varepsilon T(r)[u-1+\alpha]) r^{N-1} dr + \int_R ([F(u) - \varepsilon T(r) u] - [F(1-\alpha) - \varepsilon T(r)(1-\alpha)]) r^{N-1} dr < 0$$

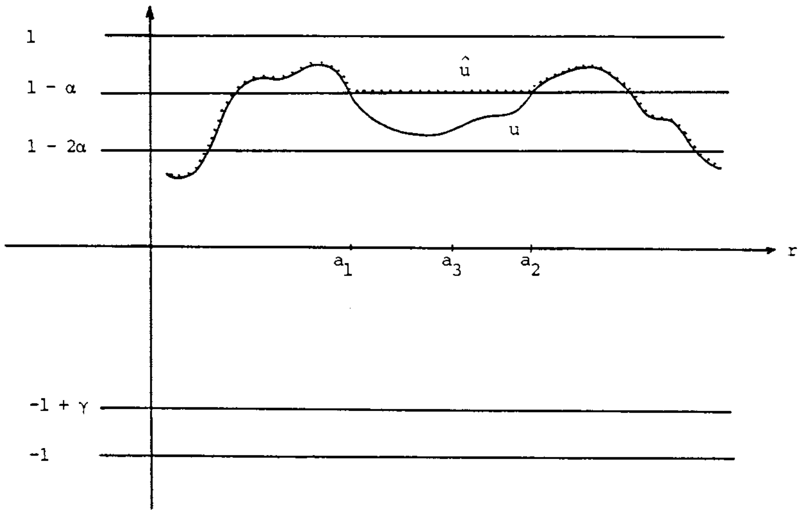

FIG. 2.

But the first integral is nonnegative by supposition and so is the second, by (2.10). This establishes the lemma.

Remark. — We can ensure that (2.10) and (2.11) hold with γ arbitrarily close to zero by making α and ε_1 sufficiently small.

Returning to the proof of Theorem 2.6 now that we know that the nonuniformities must be of large amplitude we will be able to show that the gradient contribution to $J_\varepsilon(u)$ is too costly if u has such a nonuniformity. Let $\beta \in (0, 1/4)$ be a small positive number. Choose the interval (a, b) to contain t_0 and such that

$$b-a < \frac{F(1-\beta)}{\|T\|_\infty C} \frac{(A+\delta)^{2N-2}}{(B-\delta)^{N-1}} \tag{2.13}$$

holds, where C is given in (2.8). We will drop the subscripts on ε and u for notational convenience. By the remarks before the Lemma, there is some cosntant $D > 0$ such that $u \geqq 1 - D\varepsilon$ in neighborhoods of a and b. We will take ε to be so small that $D\varepsilon < \beta$ and the conditions (2.10) and (2.11) hold with $\varepsilon_1 = \varepsilon$, $\alpha = \beta$ and $1/4 > \gamma > 0$. Note that for ε small the two local minima $v(r) > w(r)$ of $F(z) - \varepsilon T(r) z$ satisfy $|v-1| = O(\varepsilon)$ and $|w+1| = O(\varepsilon)$. Because of this we may assume that $|u| \leqq 1 + D\varepsilon$ on $[a, b]$.

Let $[c, d] \subset [a, b]$ be such that $u \leqq 1 - D\varepsilon$ on $[c, d]$ with equality holding at $c = c(\varepsilon)$ and $d = d(\varepsilon)$ and such that $u(r) < 1 - \beta$ at some point of

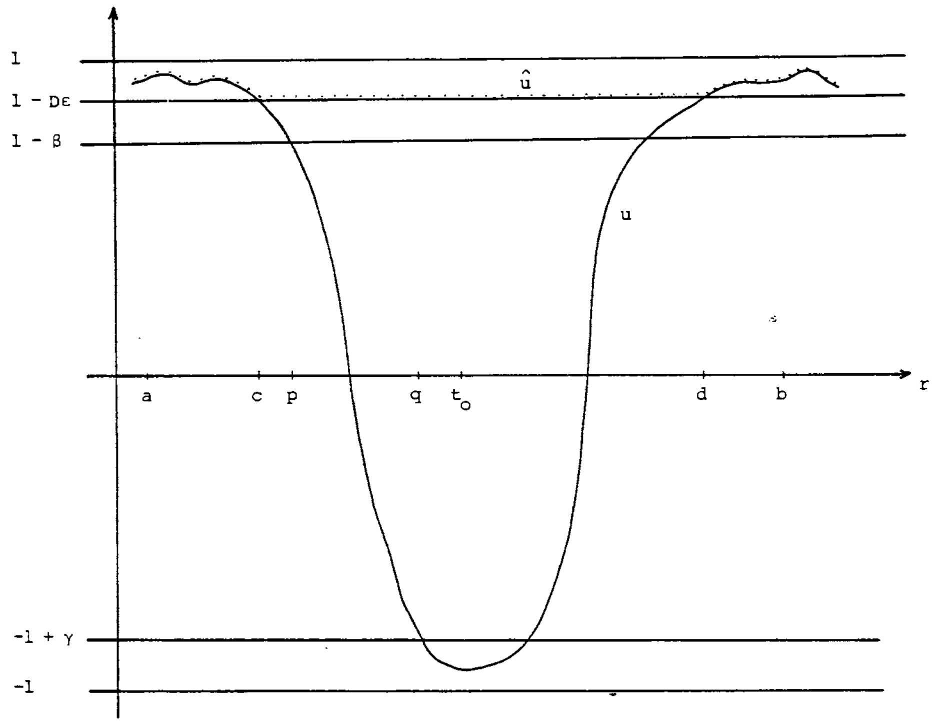

FIG. 3

$[c, d]$. Let p and q be the first points in $[c, d]$ such that $u(p)=1-\beta$ and $u(q)=-1+\gamma$, respectively (see *Fig.* 3).

Now define a comparison function

$$\hat{u}=\begin{cases} u & \text{outside } [c, d] \\ 1-\mathrm{D}\varepsilon & \text{on } [c, d] \end{cases}$$

We have the estimates

$$0 \geqq \mathrm{J}_\varepsilon(u)-\mathrm{J}_\varepsilon(\hat{u}) \geqq \frac{\varepsilon^2}{2}\int_c^d (u')^2 r^{\mathrm{N}-1}\,dr$$

$$-\varepsilon\int_c^d |\mathrm{T}(r)(u-\hat{u})|\, r^{\mathrm{N}-1}\,dr-\int_c^d \mathrm{F}(\hat{u})\, r^{\mathrm{N}-1}\,dr, \quad (2.14)$$

$$\int_c^d \mathrm{F}(1-\mathrm{D}\varepsilon)\, r^{\mathrm{N}-1}\,dr=O(\varepsilon^2), \quad (2.15)$$

$$\varepsilon \int_c^d |T(r)(u-1+D\varepsilon)| r^{N-1} dr \leqq 2\varepsilon \|T\|_\infty (B-\delta)^{N-1}(d-c). \quad (2.16)$$

To estimate the gradient term we first note that (2.8) implies

$$q-p \leqq C\varepsilon/[F(1-\beta)(A+\delta)^{N-1}]. \quad (2.17)$$

We also have

$$\frac{\varepsilon^2}{2}\int_p^q (u')^2 r^{N-1} dr \geqq \frac{\varepsilon^2}{2} \min \int_p^q (v')^2 r^{N-1} dr,$$

where the minimum is taken over all $v \in H^1$ such that $v(p)=1-\beta$ and $v(q)=-1+\gamma$. A short calculation gives this minimum, which we then estimate from below to obtain

$$\frac{(2-\beta-\gamma)^2(A+\delta)^{N-1}}{(q-p)} \leqq \int_p^q (u')^2 r^{N-1} dr. \quad (2.18)$$

A similar estimate can be obtained for the gradient term as u increases from $-1+\gamma$ to $1-\beta$. Dividing by ε in (2.14) and letting ε approach zero shows that (2.13)-(2.18) are incompatible. This completes the proof of Theorem 2.6.

Estimates (2.17) and (2.18) for the width and gradient of the interior layers of global minimizers of J_ε will be useful in giving a more detailed description of these layers in the following section. Note that the proof of this theorem also shows that $\bar{u}$ cannot have removable discontinuities in (A, B), that is the only interior discontinuities are jumps (between ± 1) with different right and left hand limits. The same gradient estimates can be used to show that there is no discontinuity or boundary layer at B nor at A provided $A > 0$.

3. INTERIOR LAYERS

We assume (H1)-(H4) hold throughout the remainder of this paper. Consider some interval $[a, b] \subset [A, B]$ and suppose that for some sequence $\{\varepsilon_m\}$ approaching zero the corresponding sequence of global minimizers

$\{u_m\}$ converges pointwise on $[a, b]$ to the function $\bar{u}$ where

$$\bar{u}(r)=\begin{cases} u_L, & a \leqq r < c \\ u_R, & c < r \leqq b \end{cases}$$

For definiteness we take $u_L=-1$ and $u_R=+1$. In Section 4 it is shown that if $N=1$, then $T(c)=0 \leqq T'(c)$ and if $N \geqq 2$, then $T(c) < 0$ and $T'(c) > 0$. Fix $\beta \in (0, 1/4)$ and consider the solvability in $[a, b]$ of

$$u_m(r)=-1+\beta, \qquad u_m(r)=1-\beta. \tag{3.1}$$

LEMMA 3.1. — *If* $N=1$ *assume that the zeros of* T *are nondegenerate. For all* $N \geqq 1$ *if* n *is sufficiently large then* (3.1) *has unique solutions in* $[a, b]$, r_L *and* r_R, *respectively with* $r_L < r_R$. *Furthermore,* $r_R - r_L = O(\varepsilon)$ *and* u_m *is monotone on* $[r_L, r_R]$.

Proof. — By Theorem 2.6, the solutions to (3.1)in $[a, b]$ can be taken to be arbitrarily close to c by making n sufficiently large. So we may assume that all roots of (3.1) which lie in $[a, b]$ are located in an interval on which $T' > 0$. Let r_L be the first point in $[a, b]$ where $u_m=-1+\beta$ and r_R the last point in $[a, b]$ where $u_m=1-\beta$. If u_m were not monotone in $[r_L, r_R]$ there would be points $s < t$ in $[r_L, r_R]$ such that $u'_m(s)=u'_m(t)=0$ and $u'_m \leqq 0$ on $[s, t]$. This violates Step I in the proof of Lemma 2.1. Finally, the estimate $r_R - r_L = O(\varepsilon)$ is given by (2.8) as was (2.17).

Now we give bounds for the number of interior layers for global minimizers in terms of the number of sign changes of T. In fact we have local information.

PROPOSITION 3.2. — *Let* $\{\varepsilon_m\}$ *converge to zero and let* $\{u_m\}$ *be corresponding global minimizers converging pointwise to a function* $\bar{u}$. *Let* $[a, b]$ *be an interval on which* T *has constant sign. Then* $\bar{u}$ *has at most one interface in* $[a, b]$ *if* $N \geqq 2$ *and at most two if* $N=1$. *Furthermore, if there is a partition of* $[A, B]$ *into* k *subintervals with* T *one signed on each subinterval then* $\bar{u}$ *has at most* k *interfaces in* $[A, B]$.

Proof. — The statement concerning the case $N \geqq 2$ follows trivially, from the Gibbs-Thompson relation (4.18 *a*). The case $N=1$ is also trivial if we assume that T has only nondegenerate zeros. For the case $N=1$ without this nondegeneracy assumption we must work disproportionately harder. We first show that if there are points $c < d$ in $[a, b]$ such that $\bar{u}$ has interfaces at c and d then

$$0 \not\equiv T\bar{u} \geqq 0 \quad \text{on } (c, d). \tag{3.2}$$

Suppose (3.2) fails. Without loss of generality assume that $T \geqq 0$ and $\bar{u} = -1$ on (c, d). Choose $\gamma > 0$ small (specified below). There is a number $D = D(\gamma)$ such that $u_m(c-\gamma)$ and $u_m(d+\gamma)$ lie above $1 - D\varepsilon_m$ and $u_m(r)$ lies below $-1 + D\varepsilon_m$ for $r \in [c+\gamma, d-\gamma]$ for all m sufficiently large. From now on we drop the subscripts for notational convenience.

There exist points $s \in [c-\gamma, c+\gamma]$ and $t \in [d-\gamma, d+\gamma]$ such that $u(s) = u(t) = 1 - D\varepsilon$. Define a comparison function

$$\hat{u} = \begin{cases} u & \text{outside } [s, t] \\ 1 - D\varepsilon & \text{in } [s, t] \end{cases}$$

One has

$$0 \geqq J_\varepsilon(u) - J_\varepsilon(\hat{u}) \geqq \frac{\varepsilon^2}{2} \int_s^t (u')^2 r^{N-1} \, dr$$

$$- \varepsilon \left| \int_s^{c+\gamma} T(u - \hat{u}) r^{N-1} \, dr \right| - \varepsilon \left| \int_{d-\gamma}^t T(u - \hat{u}) r^{N-1} \, dr \right| - O(\varepsilon^2) \quad (3.3)$$

Here we have used the facts that $F \geqq 0$, $F(w) = O(\varepsilon^2)$ when $|w - 1| < D\varepsilon$ and when $|w + 1| < D\varepsilon$ and also $T(u - \hat{u}) \leqq 0$ on $[c+\gamma, d-\gamma]$. By (2.17) and (2.18) there is a constant C, independent of γ and ε, such that

$$\frac{\varepsilon^2}{2} \int_s^t (u')^2 r^{N-1} \, dr \geqq C\varepsilon \quad \text{for all } \varepsilon \text{ sufficiently small.} \quad (3.4)$$

The other integral terms in (3.3) can be estimated below by

$$-C_1 \varepsilon\gamma \quad \text{for all } \varepsilon \text{ sufficiently small} \quad (3.5)$$

and for some constant C_1 which can be taken to be independent of γ and ε. Choosing $\gamma < C/C_1$ and letting $\varepsilon \to 0$ in the estimates (3.3)-(3.5) provides a contradiction and so (3.2) must hold. This shows that $\bar{u}$ has at most two interfaces in $[a, b]$.

Next, let $R_1 > A$ be such that T has constant sign on $[A, R_1]$. By an argument similar to that given above one can show that there are only two alternatives:

$$
\begin{array}{l}
\text{Either} \\
\bar{u} \text{ has no interface in } [A, R_1] \\
\text{or} \\
\bar{u} \text{ has exactly one interface, } I_1 > A, \\
\text{in } [A, R_1] \text{ and } 0 \not\equiv T\bar{u} \geqq 0 \text{ on } [A, I_1].
\end{array}
\tag{3.6}
$$

Suppose T changes sign at $\{R_i\}_{i=1}^{k-1}$ and has constant sign on $[R_i, R_{i+1}]$, $1 \leqq i \leqq k-1$, where $R_k \equiv B$. Suppose that for some $j \leqq k-1$, $\bar{u}$ has no more than j interfaces in $[A, R_j]$ but more than $j+1$ interfaces in $[A, R_{j+1}]$. Choose j to be minimal with this property. Then, by (3.2), there are exactly j interfaces $\{I_j\}_{i=1}^{j}$ in $[A, R_j]$ and two, $I_{j+1} < I_{j+2}$, in $(R_j, R_{j+1}]$. Necessarily $T\bar{u} \geqq 0$ on $[I_{j+1}, I_{j+2}]$ and so $T\bar{u} \leqq 0$ on $[A, \min\{I_1, R_1\}]$. Statement (3.6) implies $I_1 > R_1$. Now we can deduce that there is a first $l < j$ such that $\bar{u}$ has two interfaces in $(R_l, R_{l+1}]$. This in turn implies that there are integers m and n with $l \leqq m < n \leqq j$ and such that $\bar{u}$ has two interfaces in $[R_m, R_{m+1}]$, two in $[R_n, R_{n+1}]$ and exactly $n-m-1$ in (R_{m+1}, R_n). The possibility that $n=m+1$ is not excluded. A counting argument shows that (3.2) is violated between one of these pairs of interfaces. The Principle of Mathematical Induction completes the proof of Proposition 3.2.

Remark. — What has been proved is the stronger statement that the number of interfaces of $\bar{u}$ in $[A, R_j]$ is no more than j, the number of sign changes of T in $[A, R_j]$. A similar statement holds with respect to the interval $[R_j, B]$.

4. LOCATIONS OF INTERFACES: THE GIBBS-THOMPSON AND STABILITY CONDITIONS

We begin with the case $N=1$ which unlike the higher dimensional case, does not require a sharp constant C in estimate (2.8).

THEOREM 4.1. — *Suppose* $N=1$. *Let* $\{u_\varepsilon\}$ *be a family of global minimizers converging pointwise to* $\bar{u}$ *as* $\varepsilon \to 0$. *Suppose* $\bar{u}$ *has an interface at* x_0:

$$
\bar{u} = \begin{cases} u_L & \textit{on}\,(x_0 - 2\delta, x_0) \\ u_R & \textit{on}\,(x_0, x_0 + 2\delta) \end{cases}
$$

for some $\delta > 0$, *fixed, where* u_L, $u_R \in \{\pm 1\}$.
Then

$$\begin{aligned} &(a) \qquad T(x_0)=0 \quad (\textit{Gibbs-Thompson relation}) \\ &(b) \qquad (u_L-u_R)\,T'(x_0) \leqq 0 \quad (\textit{Stability condition}) \end{aligned} \tag{4.1}$$

Proof. – We begin by establishing (*a*). Let the continuous function ρ be given by

$$\rho(x)=\begin{cases} 0 & \text{for } x \text{ outside } [x_0-2\delta, x_0+2\delta] \\ 1 & \text{for } x \text{ in } [x_0-\delta, x_0+\delta] \\ \text{linear} & \text{otherwise} \end{cases}$$

Define for small $h > 0$

$$v_\varepsilon^h(x)=u_\varepsilon(x+\rho(x)h) \qquad \text{and} \qquad \xi_\varepsilon(h)=J_\varepsilon(v_\varepsilon^h)/\varepsilon.$$

Since u_ε is in particular a critical point of J_ε

$$\xi_\varepsilon'(0)=0 \tag{4.2}$$

From now on we drop the subscripts on ξ_ε and u_ε for notational convenience.

An easy calculation reveals

$$0=\xi'(0)=\frac{\varepsilon}{2}\int_A^B (u')^2\rho'\,dx-\frac{1}{\varepsilon}\int_A^B F(u)\rho'\,dx+\int_A^B (T\rho)'u\,dx \tag{4.3}$$

This last term can be rewritten as

$$\int_A^B (T\rho)'u\,dx=\int_A^B (T\rho)'(u-\bar{u})\,dx+u_L\int_A^{x_0}(T\rho)'\,dx+u_R\int_{x_0}^B (T\rho)'\,dx. \tag{4.4}$$

Since $\|u-\bar{u}\|_\infty$ is bounded independently of ε and since $u \to \bar{u}$ pointwise as $\varepsilon \to 0$, Lebesgue's Theorem implies that the first integral on the right hand side of (4.4) approaches zero with ε. Now consider the first two integrals in (4.2). Theorem 2.6 and Lemma 2.2 (i) and (ii) imply that if $S \equiv \operatorname{supp} \rho'$, then

$$\frac{\varepsilon}{2}\left|\int_A^B (u')^2\rho'\,dx\right| \leqq \frac{\varepsilon}{2}\|\rho'\|_\infty \int_S (u')^2\,dx = O(\varepsilon)$$

and

$$\frac{1}{\varepsilon}\left|\int_A^B F(u)\,\rho'\,dx\right| \leqq \|\rho'\|_\infty \int_S \frac{[F(u)-F(\bar{u})]}{\varepsilon}\,dx = O(\varepsilon)$$

since $u \to \bar{u} \in \{\pm 1\}$ uniformly on S. Letting ε approach zero in (4.3) and (4.4) gives

$$u_L \int_A^{x_0} (T\rho)'\,dx + u_R \int_{x_0}^B (T\rho)'\,dx = 0$$

and (4.1) (*a*) is established.

For part (*b*) we choose ρ as before except that we smooth the corners. Since u is a local minimum we have $\xi''(0) \geqq 0$ and so after a short computation we find

$$0 \leqq -\varepsilon \int_A^B (u')^2\,\rho''\,\rho\,dx - \frac{1}{\varepsilon}\int_A^B 2\,f(u)\,u'\,\rho\rho'\,dx - \int_A^B (T\rho^2)''\,u\,dx. \quad (4.5)$$

As before, this first integral is $O(\varepsilon)$. The second integral is $O(\varepsilon)$ since $\sup\{f(u(r)): r \in S\} = O(\varepsilon)$ and $\int_S |u'|\,dx \leqq C\varepsilon$ by (H3) and Lemma 2.2 parts (i) and (ii). The third integral can be rewritten as

$$\int_A^B (T\rho^2)''\,u\,dx = \int_A^B (T\rho^2)''\,(u-\bar{u})\,dx + u_L \int_A^{x_0} (T\rho^2)''\,dx + u_R \int_{x_0}^B (T\rho^2)''\,dx.$$

This clearly converges to $(u_L - u_R)\,T'(x_0)$ as $\varepsilon \to 0$ and so (*b*) is established.

Remark. — The reader will notice that the above proof seems to require more smoothness on T than was assumed. However, now the result is established for $T \in C^2$, a density argument allows us to draw the same conclusions for T of class C^1. This reasoning will also be used in the proof of Theorem 4.5 below.

The analysis when $N \geqq 2$ requires a sharpening of estimate (2.8). Suppose that $\{u_\varepsilon\}$ is a family of global minimizers converging pointwise as $\varepsilon \to 0$ to the piecewise constant function $\bar{u}$, as given by Lemma 2.4. Let $r_0 \in (A, B)$ be a point of discontinuity of $\bar{u}$ and let $\delta > 0$ be so small that $[r_0 - 2\delta, r_0 + 2\delta]$ contains no other points of discontinuity. For definiteness we assume $\bar{u}(r_0^-) = -1$ and $\bar{u}(r_0^+) = +1$.

LEMMA 4.2. — *Under the above assumptions*

$$\lim_{\varepsilon \to 0} \left(\frac{\varepsilon}{2} \int_{r_0-\delta}^{r_0+\delta} (u_\varepsilon')^2 r^{N-1} \, dr + \frac{1}{\varepsilon} \int_{r_0-\delta}^{r_0+\delta} F(u_\varepsilon) r^{N-1} \, dr \right)$$

$$= r_0^{N-1} \int_{-1}^{1} \sqrt{2 F(z)} \, dz. \quad (4.6)$$

Proof. — We first obtain an upper bound for the $\limsup_{\varepsilon \to 0}$ and for this purpose we need to choose an almost optimal energy comparison function. This we do by finding a function which is almost the pointwise minimum of the sum of the integrands in the left hand side of (4.6). Let λ and μ be positive constants to be specified later, let $\varphi = \varphi_\varepsilon$ be a monotone increasing function satisfying $\varphi(r_0 - \lambda\varepsilon) = -\varepsilon$, $\varphi(r_0 + \mu\varepsilon) = \varepsilon$ and $|\varphi(r)| \leqq \varepsilon$ for all r. A particular φ having these properties will be chosen later. Define the continuous function $\hat{u} = \hat{u}_\varepsilon$ by

$$\hat{u}(r) = \begin{cases} u_\varepsilon & \text{for } r \in [A, r_0 - 2\delta] \cup [r_0 + 2\delta, B] \\ -1 & \text{for } r \in [r_0 - \delta, r_0 - \lambda\varepsilon] \\ 1 & \text{for } r \in [r_0 + \mu\varepsilon, r_0 + \delta] \\ \varphi(r)/\varepsilon & \text{for } r \in [r_0 - \lambda\varepsilon, r_0 + \mu\varepsilon] \\ \text{linear} & \text{for } r \in [r_0 - 2\delta, r_0 - \delta] \quad \text{and} \quad r \in [r_0 + \delta, r_0 + 2\delta]. \end{cases}$$

By Theorem 2.6 we have $|\hat{u} - \bar{u}| = O(\varepsilon)$ and $|\hat{u}'| = O(\varepsilon)$ on $[r_0 - 2\delta, r_0 - \delta] \cup [r_0 + \delta, r_0 + 2\delta]$. From now on we will drop all ε subscripts. Since $J(u) \leqq J(\hat{u})$, after dividing by ε we have

$$\frac{\varepsilon}{2} \int_{r_0-\delta}^{r_0+\delta} (u')^2 r^{N-1} \, dr + \frac{1}{\varepsilon} \int_{r_0-\delta}^{r_0+\delta} F(u) r^{N-1} \, dr$$

$$- \int_{r_0-\delta}^{r_0+\delta} T u r^{N-1} \, dr \leqq \frac{\varepsilon}{2} \int_{r_0-\delta}^{r_0+\delta} (\hat{u}')^2 r^{N-1} \, dr$$

$$+ \frac{1}{\varepsilon} \int_{r_0-\delta}^{r_0+\delta} F(\hat{u}) r^{N-1} - \int_{r_0-\delta}^{r_0+\delta} T \hat{u} r^{N-1} \, dr + O(\varepsilon) \quad (4.7)$$

Since $u \to \bar{u}$ and $\bar{u} = \hat{u}$ pointwise on $[r_0 - \delta, r_0 - \lambda\varepsilon] \cup [r_0 + \mu\varepsilon, r_0 + \delta]$ the difference between the integrals involving T is $o(1)$. Taking this and the definition of $\hat{u}$ into account we may rewrite (4.7) as

$$\frac{\varepsilon}{2}\int_{r_0-\delta}^{r_0+\delta}(u')^2 r^{N-1}\,dr+\frac{1}{\varepsilon}\int_{r_0-\delta}^{r_0+\delta}F(u)\,r^{N-1}\,dr$$

$$\leqq \frac{\varepsilon}{2}\int_{r_0-\lambda\varepsilon}^{r_0+\mu\varepsilon}\frac{\varphi'}{\varepsilon}\hat{u}'\,r^{N-1}\,dr+\frac{1}{\varepsilon}\int_{r_0-\lambda\varepsilon}^{r_0+\mu\varepsilon}F(\hat{u})\,r^{N-1}\,dr+o(1) \quad (4.8)$$

With $Q(z)\equiv\varphi^{-1}(z)$ and $z=\hat{u}(r)$ we can estimate the right hand side of (4.8) and change variables to obtain

$$\frac{\varepsilon}{2}\int_{r_0-\delta}^{r_0+\delta}(u')^2 r^{N-1}\,dr+\frac{1}{\varepsilon}\int_{r_0-\delta}^{r_0+\delta}F(u)\,r^{N-1}\,dr$$

$$\leqq (r_0+\mu\varepsilon)^{N-1}\int_{-1}^{1}\left(\frac{1}{2\,Q'(\varepsilon z)}+F(z)\,Q'(\varepsilon z)\right)dz+o(1). \quad (4.9)$$

At this point the choice of $Q'(\varepsilon z)$ that minimizes the right hand side can be seen to be $1/\sqrt{2F(z)}$. This is equivalent to choosing

$$\varphi^{-1}(s)=r_0+\varepsilon\int_0^{s/\varepsilon}\frac{dt}{\sqrt{2F(t)}},$$

however this cannot satisfy the requirements that

$$\varphi^{-1}(-\varepsilon)=r_0-\lambda\varepsilon \qquad \text{and} \qquad \varphi^{-1}(\varepsilon)=r_0+\mu\varepsilon$$

for finite values of λ and μ. Instead of the above we take $\gamma>0$ and small but otherwise arbitrary and define

$$Q'(\varepsilon z)=1/\sqrt{2F(z)+\gamma}$$

or

$$\varphi^{-1}(s)=r_0+\varepsilon\int_0^{s/\varepsilon}\frac{dt}{\sqrt{2F(t)+\gamma}}.$$

The matching conditions become

$$\int_0^{-1}\frac{dt}{\sqrt{2F(t)+\gamma}}=-\lambda \qquad \text{and} \qquad \int_0^{1}\frac{dt}{\sqrt{2F(t)+\gamma}}=\mu \quad (4.10)$$

For γ fixed, (4.10) determines λ and μ. Note that for $\varepsilon > 0$ small we still have $\lambda\varepsilon < \delta$ and $\mu\varepsilon < \delta$. Substituting into (4.9) we obtain

$$\limsup_{\varepsilon \to 0} \left\{ \frac{\varepsilon}{2} \int_{r_0-\delta}^{r_0+\delta} (u')^2 r^{N-1} dr + \frac{1}{\varepsilon} \int_{r_0-\delta}^{r_0+\delta} F(u) r^{N-1} dr \right\} \leqq r_0^{N-1} \int_{-1}^{1} \sqrt{2F(t)+\gamma}\, dt \quad (4.11)$$

Since γ is arbitrary (4.11) gives the right hand side of (4.6) as an upper bound. Next we show that this is a lower bound for the $\liminf_{\varepsilon \to 0}$. For this purpose we introduce the family of stretched variables.

$$\eta \equiv \eta_\varepsilon \equiv (r - r_\varepsilon)/\varepsilon$$

where r_ε is the point closest to r_0 such that $u_\varepsilon(r_\varepsilon) = 1/2$ and $u'_\varepsilon(r_\varepsilon) \geqq 0$. The value 1/2 is not important, any number v satisfying $0 < |v| < 1$ would do. Note that $r_\varepsilon \to r_0$ as $\varepsilon \to 0$. Define

$$U(\eta) = u(\varepsilon\eta + r_\varepsilon).$$

It turns out that r_ε is unique for ε sufficiently small, by Lemma 3.1, however, we cannot use that fact here since the Lemma relies on Theorem 4.5, below. Changing variables we see that U satisfies

$$U'' + \frac{\varepsilon(N-1)}{\varepsilon\eta + r_\varepsilon} U' - f(U) + \varepsilon T(\varepsilon\eta + r_\varepsilon) = 0,$$

Using a diagonal selection process, since $\|U\|_\infty$ has a bound independent of ε, one can find a subsequence $\varepsilon_m \to 0$ such that the corresponding sequence of U's converges in C^2 on compact sets to $\bar{U}$, the unique (monotone) solution to

$$w'' - f(w) = 0, \qquad w(\pm\infty) = \pm 1. \quad (4.12)$$

The boundary conditions at $\pm\infty$ follow from the observations that $\bar{U}$ is bounded, $\bar{U}(0) = 1/2$, $\bar{U}'(0) \geqq 0$ and $\bar{U}'$ has at most finitely many changes in sign by Lemma 2.1.

The uniqueness of the candidate for the limit $\bar{U}$ means that the whole family $\{u_\varepsilon\}$ converges to $\bar{U}$ as $\varepsilon \to 0$. We have

$$\int_{r_0-\delta}^{r_0+\delta} \left[\frac{\varepsilon}{2}(u')^2 + \frac{1}{\varepsilon}F(u)\right] r^{N-1}\, dr$$
$$= \int_{(r_0-r_\varepsilon-\delta)/\varepsilon}^{(r_0-r_\varepsilon+\delta)/\varepsilon} \left[\frac{1}{2}(U'(\eta))^2 + F(U)\right](\varepsilon\eta + r_\varepsilon)^{N-1}\, d\eta,$$

and so by Fatou's Lemma,

$$\liminf_{\varepsilon\to 0} \int_{r_0-\delta}^{r_0+\delta} \left[\frac{\varepsilon}{2}(u')^2 + \frac{1}{\varepsilon}F(u)\right] r^{N-1}\, dr$$
$$\geqq r_0^{N-1} \int_{-\infty}^{\infty} \left[\frac{1}{2}(\bar{U}')^2 + F(\bar{U})\right] d\eta \quad (4.13)$$

Note that from (4.12)

$$\int_{-\infty}^{\infty} F(\bar{U})\, d\eta = \int_{-\infty}^{\infty} \frac{F(\bar{U})\,\bar{U}'}{\bar{U}'}\, d\eta = \frac{1}{\sqrt{2}} \int_{-1}^{1} \sqrt{F(z)}\, dz \quad (4.14)$$

and

$$\int_{-\infty}^{\infty} \frac{1}{2}(\bar{U}')^2\, d\eta = \int_{-\infty}^{\infty} F(\bar{U})\, d\eta = \frac{1}{\sqrt{2}} \int_{-1}^{1} \sqrt{F(z)}\, dz. \quad (4.15)$$

These combine with (4.13) to give the desired inequality.

More can be said about the limit in (4.6). The following equipartition of energy result is a consequence of (4.14) and (4.15).

COROLLARY 4.3. — *Under the hypotheses of Lemma* 4.2

$$\lim_{\varepsilon\to 0} \frac{\varepsilon}{2} \int_{r_0-\delta}^{r_0+\delta} (u_\varepsilon')^2 r^{N-1}\, dr = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_{r_0-\delta}^{r_0+\delta} F(u_\varepsilon)\, r^{N-1}\, dr$$
$$= \frac{r_0^{N-1}}{\sqrt{2}} \int_{-1}^{1} \sqrt{F(z)}\, dz. \quad (4.16)$$

We shall use a modification of the previous result.

COROLLARY 4.4. — *Under the hypotheses of Lemma 4.2, for any number* α *we have*

$$\lim_{\varepsilon\to 0}\frac{\varepsilon}{2}\int_{r_0-\delta}^{r_0+\delta}(u_\varepsilon')^2 r^\alpha\,dr=\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{r_0-\delta}^{r_0+\delta}\mathrm{F}(u_\varepsilon)\,r^\alpha\,dr=\frac{r_0^\alpha}{\sqrt{2}}\int_{-1}^{1}\sqrt{\mathrm{F}(z)}\,dz \quad (4.17)$$

Proof. — The result follows from (4.16) using some simple inequalities, noting that by Lemma 2.2 the limits do not depend upon the choice of δ. For instance, in the case $\alpha \geqq \mathrm{N}-1$ we have

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{r_0-\delta}^{r_0+\delta}\mathrm{F}(u_\varepsilon)\,r^\alpha\,dr\leqq \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left(\int_{r_0-\delta}^{r_0+\delta}\mathrm{F}(u_\varepsilon)\,r^{\mathrm{N}-1}\,dr\right)(r_0+\delta)^{\alpha-N+1}$$

$$=\frac{r_0^\alpha}{\sqrt{2}}\left(\int_{-1}^{1}\sqrt{\mathrm{F}(z)}\,dz\right)(1+\delta/r_0)^{\alpha-\mathrm{N}+1}.$$

The remaining parts of the proof are just as obvious and are omitted.

We are now ready to prove

THEOREM 4.5. — *Suppose* $\mathrm{N} \geqq 2$. *Let* $\{u_\varepsilon\}$ *be a family of global minimizers converging pointwise to* $\bar{u}$ *as* $\varepsilon \to 0$. *Suppose* $\bar{u}$ *has an interface at* $r=r_0$:

$$\bar{u}=\begin{cases} u_{\mathrm{L}} & \text{on } (r_0-2\delta,\ r_0)\\ u_{\mathrm{R}} & \text{on } (r_0,\ r_0+2\delta)\end{cases}$$

for some $\delta > 0$, *fixed, where* $u_{\mathrm{L}}, u_{\mathrm{R}} \in \{\pm 1\}$. *Then*

$$(a)\qquad (u_{\mathrm{L}}-u_{\mathrm{R}})\,\mathrm{T}(r_0)\,r_0=\sqrt{2}(\mathrm{N}-1)\int_{-1}^{1}\sqrt{\mathrm{F}(z)}\,dz$$

(*Gibbs-Thompson relation*) (4.18)

$$(b)\qquad (u_{\mathrm{L}}-u_{\mathrm{R}})\,(\mathrm{T}(r)\,r)'\big|_{r=r_0}\leqq 0$$

(*Stability Condition*)

Proof. — Define ρ and $\xi_\varepsilon(h)$ as in the proof of Theorem 4.1. Again dropping the subscripts we compute

$$0=\xi'(0)=\frac{\varepsilon}{2}\int_{\mathrm{A}}^{\mathrm{B}}(u')^2\,[\rho' r-(\mathrm{N}-1)\,\rho]\,r^{\mathrm{N}-2}\,dr$$

$$-\frac{1}{\varepsilon}\int_{\mathrm{A}}^{\mathrm{B}}\mathrm{F}(u)\,[\rho' r+(\mathrm{N}-1)\,\rho]\,r^{\mathrm{N}-2}\,dr+\int_{\mathrm{A}}^{\mathrm{B}}(\mathrm{T}\,\rho\,r^{\mathrm{N}-1})'\,u\,dr. \quad (4.19)$$

Notice that, as in (4.4), this last integral may be written as

$$\int_A^B (\mathrm{T}\,\rho\, r^{N-1})'\, u\, dr = \mathrm{T}(r_0)\, r_0^{N-1}\,[u_L - u_R] + o(1) \qquad (4.20)$$

The contributions from the first two integrals in (4.19) corresponding to $[A, r_0-\delta] \cup [r_0+\delta, B]$ are of order $o(1)$. Using this observation and (4.20) in (4.19) we find

$$(u_L - u_R)\,\mathrm{T}(r_0)\, r_0^{N-1}$$
$$= (N-1)\frac{\varepsilon}{2}\int_{r_0-\delta}^{r_0+\delta} (u')^2\, r^{N-2}\, dr + \frac{1}{\varepsilon}\int_{r_0-\delta}^{r_0+\delta} F(u)\, r^{N-2}\, dr + o(1) \qquad (4.21)$$

Letting $\varepsilon \to 0$ and using Corollary 4.4 yields (4.18 *a*).

In order to establish (4.18 *b*) we take ξ and ρ as before and impose the stability condition that $\xi''(0) \geqq 0$. A short computation yields

$$\xi''(0) = \varepsilon \int_A^B \{(u''^2 + u'\, u''')\,\rho^2 + u'^2\,\rho'^2 + 4\, u'\, u''\,\rho\rho'\}\, r^{N-1}\, dr$$
$$+ \int_A^B \{f'(u)\, u'^2 + f(u)\, u''\}\,\rho^2\, r^{N-1}\, dr/\varepsilon$$
$$- \int_A^B \mathrm{T}(r)\, u''\,\rho^2\, r^{N-1}\, dr.$$

Integration by parts with the $u'\,u'''$ term and the fu'' term and then with the $u'\,u''\,\rho\rho'$ term produces

$$\xi''(0) = -\varepsilon \int_A^B \{u'^2\,\rho\rho''\, r^{N-1} + u'^2\,\rho\rho'\,(N-1)\, r^{N-2} + u'\, u''\,\rho^2\,(N-1)\, r^{N-2}\}\, dr$$
$$- \int_A^B f(u)\, u'\,\{2\,\rho\rho'\, r^{N-1} + (N-1)\,\rho^2\, r^{N-2}\}\, dr/\varepsilon - \int_A^B \mathrm{T}(r)\, u''\,\rho^2\, r^{N-1}\, dr.$$

Now,

$$\int_A^B (u'^2\,\rho\rho' + u'\, u''\,\rho^2)\, r^{N-2}\, dr = -\frac{(N-2)}{2}\int_A^B u'^2\,\rho^2\, r^{N-3}\, dr.$$

So setting

$$D = [r_0 - 2\,\delta,\ r_0 - \delta] \cup [r_0 + \delta,\ r_0 + 2\,\delta]$$

we may write

$$\xi''(0) = -\varepsilon \int_D u'^2 \rho r^{N-3} \left(\rho'' r^2 - \frac{(N-1)(N-2)}{2} \rho \right) dr$$
$$+ \varepsilon \frac{(N-1)(N-2)}{2} \int_{r_0-\delta}^{r_0+\delta} u'^2 r^{N-3} \, dr$$
$$- \int_D f(u) u' \rho \{ 2\rho' r + (N-1)\rho \} r^{N-2} \, dr/\varepsilon$$
$$-(N-1) \int_{r_0-\delta}^{r_0+\delta} f(u) u' r^{N-2} \, dr/\varepsilon - \int_{r_0-2\delta}^{r_0+2\delta} (T \rho^2 r^{N-1})'' (u - \bar{u}) \, dr$$
$$- \int_{r_0-2\delta}^{r_0+2\delta} (T \rho^2 r^{N-1})'' \bar{u} \, dr. \quad (4.22)$$

The first term approaches zero with ε by part (ii) of Lemma 2.2. The third term approaches zero with ε since $\max \{ | f(u(r)) | : r \in D \} = O(\varepsilon)$ as $\varepsilon \to 0$ by part (i) of Lemma 2.2 and $\int_D |u'| \, dr \leqq C\varepsilon$ by part (iii) of Lemma 2.2. The fifth term also approaches zero with ε, by Lebesgue's Theorem. The fourth term can be rewritten as

$$-(N-1) \frac{F(u(r))}{\varepsilon} r^{N-2} \Big|_{r_0-\delta}^{r_0+\delta} + (N-1)(N-2) \int_{r_0-\delta}^{r_0+\delta} F(u) r^{N-3} \, dr/\varepsilon.$$

Note that $F(u(r_0 \pm \delta))/\varepsilon \to 0$ as $\varepsilon \to 0$. Letting $\varepsilon \to 0$ and using the stability criterion $\xi''(0) \geqq 0$ in (4.22) yields

$$(N-1)(N-2) \lim_{\varepsilon \to 0} \left[\int_{r_0-\delta}^{r_0+\delta} \left(\varepsilon \frac{u'^2}{2} + \frac{F(u)}{\varepsilon} \right) r^{N-3} \, dr \right]$$
$$-(u_L - u_R)(T(r) r^{N-1})' \big|_{r=r_0} \geqq 0 \quad (4.23)$$

Observe that for $N=1$ and $N=2$ we do not need to know if the above limit even exists in order to arrive at (4.1 *b*) and (4.18 *b*), respectively.

For $N > 2$ we use (4.17) and (4.18 *a*) to rewrite (4.23) as

$$(u_R - u_L) \{ -(N-2) T(r_0) r_0^{N-2} + T'(r_0) r_0^{N-1} + (N-1) T(r_0) r_0^{N-2} \} \geqq 0.$$

Combining terms after dividing by r_0^{N-2} gives (4.18 *b*).

5. SUFFICIENT CONDITIONS FOR INTERIOR INTERFACES

Let $\varepsilon_1 > 0$ be fixed. In Lemma 2.4 we showed that if u_ε was a global minimizer of J_ε then $\{u_\varepsilon\}=\{u_\varepsilon: 0<\varepsilon \leqq \varepsilon_1\}$ is relatively compact in L^1. We also showed that limit points are simple functions taking on the values ± 1 except possibly at the points of discontinuity, which is at most a finite set. In the remarks following the proof of Theorem 2.6, we pointed out that if $\bar{u}$ is such a limit and if $\bar{u}$ is discontinuous at $r_0 \in (A, B)$, then $\bar{u}(r_0^+) \neq \bar{u}(r_0^-)$. For this reason and in consideration of the physical motivation for our problem, we have called these points of discontinuity *interfaces*. In this section we give conditions which guarantee that any limit, $\bar{u}$, of $\{u_\varepsilon\}$ has an interface. We also indicate how one may ensure that multiple interfaces occur.

THEOREM 5.1. — *Let*

$$W_1 \equiv \max\left\{\sup_s\left(-\frac{1}{s^{N-1}}\int_A^s T(r)\, r^{N-1}\,dr\right),\ \sup_s\left(-\frac{1}{s^{N-1}}\int_s^B T(r)\, r^{N-1}\,dr\right)\right\}$$

and

$$W_2 \equiv \max\left\{\sup_s\left(\frac{1}{s^{N-1}}\int_A^s T(r)\, r^{N-1}\,dr\right),\ \sup_s\left(\frac{1}{s^{N-1}}\int_s^B T(r)\, r^{N-1}\,dr\right)\right\}$$

$$\text{If}\quad \min\{W_1, W_2\} > \frac{1}{\sqrt{2}}\int_{-1}^{1}\sqrt{F(z)}\,dz \tag{5.1}$$

then at least one interface exists, that is, any limit point of $\{u_\varepsilon\}$ *in* L^1 *has at least one point of discontinuity in* (A, B).

Proof. — We assume the conclusion fails and proceed to obtain a contradiction. So, suppose that for some sequence $\{\varepsilon_m\}$ converging to zero, the corresponding sequence of minimizers $\{u_{\varepsilon_m}\}$ converges to $\bar{u}=1$ in L^1. We will show that if (5.1) holds then for n sufficiently large u_{ε_m} is not a minimizer of J_{ε_m}. This is done by considering a comparison function similar to that used in the proof of Lemma 4.2. From now on we will drop the subscripts on ε_m and u_{ε_m}. Define

$$\hat{u}(r) = \begin{cases} -1 & \text{for}\quad A \leqq r \leqq s-\lambda\varepsilon \\ 1 & \text{for}\quad s+\mu\varepsilon \leqq r \leqq B \\ \varphi(r)/\varepsilon & \text{for}\quad s-\lambda\varepsilon \leqq r \leqq s+\mu\varepsilon \end{cases} \tag{5.2}$$

where $s\in(A, B)$ is fixed but arbitrary, λ and μ are positive numbers to be determined and $\varphi=\varphi_\varepsilon$ is a monotone function to be specified later but which will satisfy $|\varphi(r)|\leqq\varepsilon$, $\varphi(s-\lambda\varepsilon)=-\varepsilon$ and $\varphi(s+\mu\varepsilon)=\varepsilon$. By definition of $u=u_\varepsilon$ we have $J_\varepsilon(u)\leqq J_\varepsilon(\hat{u})$ and so, after dropping some positive terms from the left-hand side and dividing by ε,

$$-\int_A^B T(r)\,u(r)\,r^{N-1}\,dr\leqq\frac{\varepsilon}{2}\int_A^B(\hat{u}')^2\,r^{N-1}\,dr+\frac{1}{\varepsilon}\int_A^B F(\hat{u})\,r^{N-1}\,dr-\int_A^B T(r)\,\hat{u}(r)\,r^{N-1}\,dr.\quad(5.3)$$

Using the L^1-convergence of u and $\hat{u}$ as $\varepsilon\to 0$, we see that

$$-2\int_A^s T(r)\,r^{N-1}\,dr\leqq\frac{\varepsilon}{2}\int_A^B(\hat{u}')^2\,r^{N-1}\,dr+\frac{1}{\varepsilon}\int_A^B F(\hat{u})\,r^{N-1}\,dr+o(1).\quad(5.4)$$

Changing variables as in the proof of Lemma 4.2 [*cf.* (4.9)] we obtain

$$-2\int_A^s T(r)\,r^{N-1}\,dr\leqq\left(\int_{-1}^{1}\frac{1}{2\,Q'(\varepsilon z)}+F(z)\,Q'(\varepsilon z)\,dz\right)(s+\mu\varepsilon)^{N-1}+o(1)\quad(5.5)$$

where $Q=\varphi^{-1}$. For the same reasons as before we make the nearly optimal choice

$$Q'(\varepsilon z)=1/\sqrt{2F(z)+\gamma}\quad\text{for small fixed }\gamma>0.$$

This gives $\varphi^{-1}(t)=\varepsilon\int_0^{t/\varepsilon}\frac{dz}{\sqrt{2F(z)+\gamma}}$ and the matching conditions become

$$\int_0^{-1}\frac{dz}{\sqrt{2F(z)+\gamma}}=-\lambda,\qquad\int_0^{1}\frac{dz}{\sqrt{2F(z)+\gamma}}=\mu$$

and determine λ and μ.

Substituting into (5.5), letting ε approach zero and finally letting γ approach zero, we obtain

$$-2\int_A^s T(r)\,r^{N-1}\,dr\leqq s^{N-1}\int_{-1}^{1}\sqrt{2F(z)}\,dz.\quad(5.6)$$

Dividing by s^{N-1} and maximizing over [A, B] yields a contradiction to (5.1) provided W_1 is given by the first quantity in its definition.

If W_1 is given by the second quantity in its definition, we reach a contradiction in a similar way by using $-\hat{u}$ as a comparison function. Likewise, a contradiction is reached if we suppose that $\bar{u}=-1$ a. e.

The above criterion guarantees an interface provided that T has sufficient positive mass near one end of the interval and sufficient negative mass near the other. It is most suited to the case when T has only one sign change. An independent criterion can be obtained by considering comparison functions with multiple interfaces. The proof is essentially the same as that given above so we omit it and only record the result for the convenience of the reader.

THEOREM 5.2. — *Suppose that* T *changes sign at* $\{R_i\}_{i=1}^{m}$ *and that for some* $0 \leqq j,\ k \leqq m$

$$\frac{1}{\sqrt{2}}\left(\int_{-1}^{1} \sqrt{F(z)}\,dz\right) \sum_{i=1}^{j} R_i^{N-1} < \sum_{\substack{i \text{ even} \\ 0 \leqq i \leqq j}} \int_{R_i}^{R_{i+1}} |T(r)|\, r^{N-1}\, dr \tag{5.6}$$

and

$$\frac{1}{\sqrt{2}}\left(\int_{-1}^{1} \sqrt{F(z)}\,dz\right) \sum_{i=1}^{k} R_i^{N-1} < \sum_{\substack{i \text{ odd} \\ 1 \leqq i \leqq k}} \int_{R_i}^{R_{i+1}} |T(r)|\, r^{N-1}\, dr \tag{5.7}$$

where $R_0=A$ *and* $R_{m+1}=B$, *then at least one interface exists.*

Sufficient conditions for multiple interfaces can be obtained by making use of the Gibbs-Thompson and stability relations and constructing appropriate energy comparison functions with multiple interior layers. We omit this analysis.

6. LINEARLY STABLE SOLUTIONS

Here we consider the limits of sequences of stable solutions to (2.2). It was shown in Lemma 2.4 that from any sequence $\{u_m\}$ of stable solutions to (2.2) corresponding to $\varepsilon=\varepsilon_m$ approaching zero, one can extract a subsequence converging pointwise to a function $\bar{u}$ taking values ± 1 except for possibly a finite number of discontinuities. Now we show that these discontinuities can only occur at points where a Gibbs-Thompson relation holds, similar to the case for limits of global minimizers, and that the stability condition must hold as before. We also show that interior layers are monotone as in the case of global minimizers.

Throughout this section we assume that $\{u_\varepsilon\}$ is a family of stable solutions to (2.2) converging pointwise on [A, B], as $\varepsilon \to 0$, to the function $\bar{u}$ as given by Lemma 2.4. We assume that for some numbers $a < r_0 < b$

$$\bar{u}(r) = \begin{cases} u_L & \text{for} \quad a \leqq r < r_0 \\ u_R & \text{for} \quad r_0 < r \leqq b \end{cases}$$

where u_L, $u_R \in \{\pm 1\}$.

LEMMA 6.1. — *Let* $\delta > 0$ *be fixed. There exist points* $a_1 < b_1$ *in* $(r_0 - \delta, r_0)$, *points* $a_2 < b_2$ *in* $(r_0, r_0 + \delta)$ *and a sequence* $\{u_{\varepsilon_m}\} \subset \{u_\varepsilon\}$ *converging uniformly to* $\bar{u}$ *on* $[a_1, b_1] \cup [a_2, b_2]$. *The rate of convergence is* $O(\varepsilon_m)$.

Proof. — Divide each of the intervals $(r_0 - \delta, r_0)$ and $(r_0, r_0 + \delta)$ into $n+2$ subintervals of equal length. By Lemma 2.1, from each of these partitions we can find an interval, call them $[a_1, b_1]$ and $[a_2, b_2]$, such that for some sequence, $\{u_{\varepsilon_m}\}$, all terms are monotone on each of these intervals. Uniform convergence now follows from pointwise convergence and the rate of convergence is given by Lemma 2.2 (i).

Remark. — The proof of this lemma actually shows that from any sequence $\{\varepsilon_m\}$ converging to 0 one can extract a subsequence $\{\varepsilon_{m_j}\}$ such that the corresponding u's converge uniformly on some intervals each side of and arbitrarily close to r_0.

One can show that if u is a stable solution to (2.2) then $\xi'(0) = 0 \leqq \xi''(0)$ where ξ is given in Section 4. In the definition of ξ function ρ may be taken to be any smooth function with support in (a, b). Let $P = [a_1, b_1]$ and $Q = [a_2, b_2]$ be intervals given by Lemma 6.1 where $\delta > 0$ is such that $a < r_0 - \delta$ and $r_0 + \delta < b$. Choose $\rho \in C^2$ so that $S \equiv \operatorname{supp} \rho' \subset P \cup Q$ and

$$\rho = \begin{cases} 0 & \text{outside } [a_1, b_2] \\ 1 & \text{in } [b_1, a_2]. \end{cases}$$

Now consider the limit as $\varepsilon \to 0$ in (4.3) along the sequence given by Lemma 6.1. The uniform convergence on $P \cup Q$ gives the limit of $\xi'(0)$ exactly as before. Similarly, for (4.5) and we have the following.

THEOREM 6.2. — *Suppose* $N=1$. *Suppose that* $\bar{u}$ *has an interface at* x_0:

$$\bar{u}=\begin{cases} u_L & on\ (x_0-\delta,\ x_0) \\ u_R & on\ (x_0,\ x_0+\delta) \end{cases}$$

for some $\delta > 0$, *where* u_L, $r_R \in \{\pm 1\}$. *Then*

$$\begin{aligned} &(a) \qquad (u_L-u_R)\,T(x_0)=0 \\ &(b) \qquad (u_L-u_R)\,T'(x_0) \leqq 0 \end{aligned} \tag{6.1}$$

The analysis for $N \geqq 2$ is more difficult and we are not able to duplicate the results obtained for global minimizers. This is because the first half of the proof of Lemma 4.2 relied heavily upon energy comparisons. We can, however, achieve a partial result which still gives a weak form of the Gibbs-Thompson relation.

LEMMA 6.3. — *Under the above assumptions, for any sufficiently small* $\delta > 0$

$$\liminf_{\varepsilon \to 0} \left(\int_{r_0-\delta}^{r_0+\delta} \left[\frac{\varepsilon}{2}(u_\varepsilon')^2 + \frac{1}{\varepsilon} F(u_\varepsilon) \right] r^{N-1}\, dr \right) \geqq r_0^{N-1} \int_{-1}^{1} \sqrt{2\,F(z)}\, dz. \tag{6.2}$$

In fact, for any number α

$$\liminf_{\varepsilon \to 0} \left(\int_{r_0-\delta}^{r_0+\delta} \left[\frac{\varepsilon}{2}(u_\varepsilon')^2 + \frac{1}{\varepsilon} F(u_\varepsilon) \right] r^{\alpha}\, dr \right) \geqq r_0^{\alpha} \int_{-1}^{1} \sqrt{2\,F(z)}\, dz. \tag{6.3}$$

Proof. — The second half of the proof of Lemma 4.2 still applies and leads to (6.3).

Remark. — The $\liminf\limits_{\varepsilon \to 0}$ in (6.3) can be taken to mean along any sequence $\varepsilon_m \to 0$ as $m \to \infty$. This can be seen by the uniqueness of $\bar{U}$ in the proof of Lemma 4.2.

We are now ready to consider the analogue of Theorem 4.5 for the case of stable solutions to (2.2). Using the choice of ρ given before Theorem 6.2 we can again establish (4.21) for $\varepsilon=\varepsilon_m$ and $u=u_{\varepsilon_m}$ given by Lemma 6.1. Hence taking the lim inf in (4.21) along this sequence yields

$$(u_L-u_R)\,T(r_0)\,r_0^{N-1} \geqq (N-1)\,r_0^{N-2} \int_{-1}^{1} \sqrt{2\,F(z)}\, dz. \tag{6.4}$$

Again, when considering the stability condition $\xi''(0) \geqq 0$ we compute (4.22) and argue as before to get a new version of (4.23), namely,

$$(\mathrm{N}-1)(\mathrm{N}-2) \liminf_{m \to \infty} \left[\int_{r_0-\delta}^{r_0+\delta} \left(\varepsilon_m \frac{u'^2}{2} + \frac{1}{\varepsilon_m} \mathrm{F}(u) \right) r^{\mathrm{N}-3} \, dr \right] \geqq (u_\mathrm{L} - u_\mathrm{R}) (\mathrm{T}(r) r^{\mathrm{N}-1})' \big|_{r=r_0} \quad (6.5)$$

where we have dropped the ε_m subscript from u. Now we see from (4.21) that the limit

$$\lim_{m \to \infty} (\mathrm{N}-1) \int_{r_0-\delta}^{r_0+\delta} \left(\varepsilon_m \frac{u'^2}{2} + \frac{1}{\varepsilon_m} \mathrm{F}(u) \right) r^{\mathrm{N}-2} \, dr$$

exists, is independent of δ, and is equal to

$$(u_\mathrm{L} - u_\mathrm{R}) \, \mathrm{T}(r_0) \, r_0^{\mathrm{N}-1}.$$

It follows that

$$\lim_{m \to \infty} (\mathrm{N}-1) \int_{r_0-\delta}^{r_0+\delta} \left(\varepsilon_m \frac{u'^2}{2} + \frac{1}{\varepsilon_m} \mathrm{F}(u) \right) r^{\mathrm{N}-3} \, dr = (u_\mathrm{L} - u_\mathrm{R}) \, \mathrm{T}(r_0) (r_0) \, r_0^{\mathrm{N}-2} \quad (6.6)$$

Combining (6.5) and (6.6) gives

$$(\mathrm{N}-2)(u_\mathrm{L} - u_\mathrm{R}) \, \mathrm{T}(r_0) \, r_0^{\mathrm{N}-2} \geqq (u_\mathrm{L} - u_\mathrm{R}) (\mathrm{T}'(r_0) \, r_0^{\mathrm{N}-1} + (\mathrm{N}-1) \, \mathrm{T}(r_0) \, r_0^{\mathrm{N}-2})$$

or, after combining terms,

$$0 \geqq r_0^{\mathrm{N}-2} (u_\mathrm{L} - u_\mathrm{R}) (\mathrm{T}(r) \, r)' \big|_{r=r_0}. \quad (6.7)$$

In establishing (6.4) and (6.7) we have proved

THEOREM 6.4. — *Suppose* $\mathrm{N} \geqq 2$. *Suppose that* $\bar{u}$ *has an interface at* r_0:

$$\bar{u} = \begin{cases} u_\mathrm{L} & on \ (r_0 - \delta, r_0) \\ u_\mathrm{R} & on \ (r_0, r_0 + \delta) \end{cases}$$

for some $\delta > 0$, *where* $u_\mathrm{L}, u_\mathrm{R} \in \{\pm 1\}$. *Then*

$$\begin{aligned} &(a) \qquad (u_\mathrm{L} - u_\mathrm{R}) \, \mathrm{T}(r_0) \, r_0 \geqq \sqrt{2} (\mathrm{N}-1) \int_{-1}^{1} \sqrt{\mathrm{F}(z)} \, dz \\ &(b) \qquad (u_\mathrm{L} - u_\mathrm{R}) (\mathrm{T}(r) \, r)' \big|_{r=r_0} \leqq 0 \end{aligned} \quad (6.8)$$

Finally, using Lemma 6.1 and the subsequent remark in place of Theorem 2.6 in the proof of Lemma 3.1 one can show the following

THEOREM 6.5. — *If* N = 1, *we assume that the zeros of* T *are nondegenerate. For* $\beta > 0$ *fixed consider the solvability of*

$$u_\varepsilon(r) = \pm(1-\beta) \tag{6.9}$$

in a neighborhood of any interface r_0 *such that* $u_L \neq u_R$. *There is a* $\delta > 0$ *such that equations* (6.9) *have unique solutions in* $[r_0-\delta, r_0+\delta]$ *and* u_ε *is monotone between these points.*

Remark. — For N > 1, (6.8 *a*) shows that $u_L \neq u_R$ always holds. The remarks following the proof of Theorem 2.6 point out that $u_L \neq u_R$ always holds even for N = 1 when $\bar{u}$ is the limit of global minimizers.

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(*Manuscrit reçu le* 1988.)