

Gain of regularity for equations of KdV type

by

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ABSTRACT. — This article discusses questions of the smoothness of solutions to nonlinear dispersive evolution equations. We consider equations of KdV type, that is, of the general form

$$\partial_t u + f(\partial_x^3 u, \partial_x^2 u, \partial_x u, u; x, t) = 0$$

with $x \in \mathbf{R}$. The hypothesis on the nonlinear function f is principally that $\partial_{\partial_x^3 u} f \geq C \geq 0$, so that dispersive effects are dominant. We show that if the function $u(0, x)$ decays faster than polynomially on \mathbf{R}^+ , and possesses certain minimal regularity, then *a priori* the solution $u(t, x) \in C^\infty$ for $t > 0$. Furthermore, the relationship between the rate of decay and the amount of gain of regularity is quantified; if

$$\int_{-\infty}^{\infty} u^2(0, x)(1 + |x_+|^k) dx < \infty$$

then $u(t, \cdot) \in H_{\text{loc}}^k(\mathbf{R})$ for all $0 < t \leq T$, and $u(t, x) \in L^1([0, T]; H_{\text{loc}}^{k+1}(\mathbf{R}))$ where T is the existence time.

Key words : Dispersive equations, local smoothing effects, gain of regularity, KdV, dispersive smoothing.

Classification A.M.S. : 35, 76.

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RÉSUMÉ. — Cet article démontre certaines propriétés de régularité des solutions des équations d'évolution dispersives non linéaires. Nous considérons les équations du type KdV, de la forme

$$\partial_t u + f(\partial_x^3 u, \partial_x^2 u, \partial_x u, u; x, t) = 0$$

où $x \in \mathbf{R}$. L'hypothèse la plus importante sur la fonction f est que $\partial_{\partial_x^2 u} f \geq C > 0$, pour que les effets dispersifs soient dominants. Nous montrons que, si la fonction $u(0, x)$ décroît vers zéro plus vite que les polynômes sur \mathbf{R}^+ , et si elle est de minimale régularité, *a priori* la solution $u(t, x) \in C^\infty$ pour $t > 0$. En plus, la relation entre la vitesse de décroissance et l'ordre de régularité supplémentaire est quantifiée; si

$$\int_{-\infty}^{\infty} u^2(0, x)(1 + |x_+|^k) dx < \infty$$

alors $u(t, \cdot) \in H_{\text{loc}}^k(\mathbf{R})$ pour tout $0 < t \leq T$, et $u(t, x) \in L^1([0, T]; H_{\text{loc}}^{k+1}(\mathbf{R}))$ où T est le temps d'existence.

1. INTRODUCTION

An evolution equation enjoys a *gain* of regularity if its solutions are smoother for $t > 0$ than its initial data. An equation need not be hypoelliptic for this to happen provided the initial data vanish at spatial infinity. For instance, for the Schrödinger equation in \mathbb{R}^n , it is clear from the explicit formula that a solution is C^∞ if the initial data decay faster than any polynomial. For the Korteweg-de Vries equation on the line, T. Kato [Ka], motivated by work of A. Cohen [Co], showed that the solutions are C^∞ for any data in L^2 with a weight function $1 + e^{\sigma x}$. Kruzhkov and Faminski [KF] related the order of vanishing at infinity of the initial data to the gain of regularity of the solution for $t > 0$. Corresponding work for some special nonlinear Schrödinger equations was done by Hayashi *et al.* ([HNT1], [HNT2]) and Ponce [Po]. While the proof of Kato appears to depend on special *a priori* estimates, some of its mystery has been resolved by the recent results of local gain of finite regularity for various other linear and nonlinear dispersive equations due to Constantin and Saut [CS], Sjölin [Sj], Ginibre and Velo [GV] and others. However, all of them require growth conditions on the nonlinear term.

In this paper we separate the question of regularity from the question of global existence. We quantify the gain of each derivative by the degree of vanishing of the initial data at infinity. We prove that there is an infinite gain of regularity within the existence interval for solutions with

sufficient decay at infinity. We use a technique of nonlinear multipliers, generalizing Kato's original method, together with ideas of Craig and Goodman [CG]. This is true for a general class of dispersive equations, including those of Schrödinger type and KdV type, without any growth conditions on the nonlinear terms.

We limit ourselves in the present paper to a fully nonlinear equation of KdV type in one space dimension. Namely, we consider the equation

$$u_t + f(u_{xxx}, u_{xx}, u_x, u, x, t) = 0 \quad (-\infty < x < \infty) \quad (1)$$

where $f \in C^\infty$ and

$$\partial f / \partial u_{xxx} \geq c \geq 0, \quad \text{and} \quad \partial f / \partial u_{xx} \leq 0. \quad (2)$$

In addition there is a technical hypothesis on f in cases in which it depends explicitly on x and t ; these are stated precisely in Section 2.

In order to measure the decay, we use one-sided weight functions $w_{\sigma, k}(x)$ which behave roughly like x^k as $x \rightarrow +\infty$ and like $e^{\sigma x}$ as $x \rightarrow -\infty$, where $k \geq 0$ and $\sigma > 0$. That is, they grow to the right and decay to the left. Let $H^j(w_{\sigma, k})$ denote the Sobolev space with respect to the measure $w_{\sigma, k}(x) dx$. We use the notation $\partial / \partial_x = \partial$, $\partial / \partial t = \partial_t$. Our main result is the following.

Gain of Regularity Theorem

Let $u(x, t)$ be a solution of (1) in $\mathbb{R} \times (0, T)$ with u_{xxx} , u_{xx} , u_x and u bounded. Assume that there is an integer $K \geq 2$ such that

$$u \in L^\infty((0, T); H^5(w_{0, K})). \quad (3)$$

Then

$$\int_0^T t^k \int |\partial^{6+k} u|^2 w_{\sigma, K-k-1}(x) dx dt < \infty \quad (4)$$

and

$$\sup_{0 \leq t \leq T} t^k \int |\partial^{5+k} u|^2 w_{\sigma, K-k}(x) dx < \infty \quad (5)$$

for all $0 \leq k \leq K$ and all $\sigma > 0$. In the special case $k = K$, the weight function $w_{\sigma, -1}(x)$ in (4) is required to be a positive integrable function on $(0, \infty)$.

Thus we have a gain of K derivatives at the expense of $K/2$ powers of x as $x \rightarrow +\infty$. By Sobolev's inequality, (5) implies that $t^k w_{\sigma, K-k}(x) |\partial^{4+k} u|^2$ is bounded in $\mathbb{R} \times (0, T)$. Since K can be arbitrarily large, we see that the solution is C^∞ in $\mathbb{R} \times (0, T)$ if (3) is true for all K .

We also show that the hypotheses of the preceding theorem are satisfied under natural initial conditions.

Existence Theorem

If $u(x, 0) = \varphi(x)$ where $\varphi \in H^5(w_{0,K})$ [$K \geq 2$] then there is a maximal existence time $T^* \leq \infty$ such that the hypotheses of the preceding theorem are valid in each time interval $0 \leq t \leq T < T^*$.

Informally speaking, the two theorems together imply that if $\varphi \in H^5$ is rapidly decreasing as $x \rightarrow +\infty$, then $\varphi \in C^\infty$ within the existence interval.

The number five can be reduced if the equation is quasilinear or semilinear. Note that f only satisfies the structure conditions (2). There are no growth conditions on f at all. The first condition in (2) means that the equation is uniformly dispersive. The second condition in (2) has the effect of avoiding a backwards parabolic term.

In particular, the Existence Theorem asserts that five derivatives in the initial data lead to six derivatives in the solution. In fact, we prove that if $k \geq 7$ and φ belongs to H^k without a weight, then the solution belongs to H_{loc}^{k+1} for a.e. $t \in (0, T^*)$. This is one of the results proved by Kato [Ka] for the KdV.

The idea of the proof of gain of regularity is as follows.

(i) Differentiating the equation many times leads to an equation which looks almost linear. Incidentally, for a linear equation this first step may be omitted and the results take a simpler form.

(ii) The simplest equation of type (1) is the Airy equation $u_t + u_{xxx} = 0$. It is invariant under the transformation $\mathfrak{G} : u \rightarrow xu - 3tu_{xx}$. Since the L^2 norm is an invariant, so are

$$\|\mathfrak{G}u\|^2 = \int |xu - 3tu_{xx}|^2 dx = \int x^2 \varphi^2 dx$$

and

$$(\mathfrak{G}u, u) = \int (xu^2 + 3tu_x^2) dx = \int x \varphi^2 dx.$$

They indicate a gain of one or two derivatives solely at the expense of one or two powers of x . For higher derivatives, we need only take higher powers of \mathfrak{G} . However, the idea of invariance does not readily generalize. For instance, consider the equation $u_t + a(x)u_{xxx} = 0$. We would like to

find a nontrivial operator $\Gamma = \sum_{j=1}^N \xi_j(x, t) \partial^j$ of order N which commutes

with $\partial_t + a(x)\partial^3$. It cannot be found in general because Γ has only $N+1$ coefficients ξ_j but the commutator $[\Gamma, a\partial^3/\partial x^3]$ has order $N+2$.

(iii) On the other hand, it is well known that such equations are intrinsically *one-sided*. The solitons travel to the right, not the left. Moreover, the solution of the Airy equation is $u(x, t) = t^{-1/3} A(xt^{-1/3}) * \varphi(x)$ where A is the Airy function. The Airy function decays exponentially as

$x \rightarrow +\infty$ but only very slowly with oscillations as $x \rightarrow -\infty$. Therefore, in order to get the regularity, it ought to be sufficient to assume that $\varphi(x)$ decays fast as $x \rightarrow +\infty$ but only slowly as $x \rightarrow -\infty$. This consideration motivates the use of the one-sided weight functions.

(iv) Our actual procedure is to multiply the Airy equation by ξu for an appropriate weight function $\xi(x, t)$, obtaining the identity

$$(\xi u^2)_t + 3\xi_x u_x^2 - (\xi_{xxx} + \xi_t) u^2 = (3\xi u_x^2 - \xi(u^2)_{xx} + \xi_x(u^2)_x - \xi_{xx} u^2)_x.$$

This identity is integrated over $x \in \mathbb{R}$, whereupon the right hand side vanishes and we obtain through the Gronwall inequality our main energy estimate. If we choose $\xi > 0$ and $\xi_x > 0$, then the second term provides a gain of one derivative relative to the initial data. If $\xi(x, 0) = 0$, then the initial data does not appear explicitly in the estimate at all. We repeat this estimate for the higher derivatives of the solution. For the fully nonlinear equation (1), ξ will depend nonlinearly on the solution itself. This method can be considered as an approximate one-sided version of (ii) with an operator $\Gamma_N = \xi_N \partial^N u$ which changes at each induction step.

The plan of the paper is as follows. In Section 2 we prove the Gain of Regularity Theorem (in slightly greater generality). The first induction steps are delicate, the later ones quite routine.

In Section 3 we prove a first existence theorem, locally in time, for $\varphi \in H^7(\mathbb{R})$ without use of a weight function. In Section 4 we show that the solutions belong to weighted classes provided the initial data do. We also reduce the existence assumptions to only five derivatives on the data with weights, thereby completing the proof of the Existence Theorem. Finally, the appendix contains several versions of technical weighted interpolation lemmas.

It is straightforward to extend these results to equations of odd order in the spatial variable, with $m \geq 5$. If m is even the equation is parabolic, and the properties of smoothness of solutions reflect different phenomena. For the odd order case, the dispersive character of the equation leads to gain of regularity, with higher regularity obtained for more localized initial data. Again initial data which decay rapidly on a half axis lead to C^∞ solutions, locally in time. These results are contained in unpublished notes of the authors.

Additionally, it is clear that the demands on the initial data φ can be considerably weakened if the nonlinearity has special structure. For example if the equation is semilinear, such as the classical KdV equation, the starting differentiability required by our method can be reduced to 3 derivatives.

In a subsequent paper the authors treat equations of Schrödinger type, where similar results are true.

2. GAIN OF REGULARITY

This section contains the main theorem, which we present in a slightly more general form than in the introduction.

We write $\partial = \partial/\partial x$ and $\partial_t = \partial/\partial t$. We abbreviate $u_j = \partial^j u$ and $\partial_j = \partial/\partial u_j$. Our nonlinear function is $f(u_3, u_2, u_1, u_0, x, t)$ whose derivatives are written as $\partial_x^i \partial_t^j \partial_0^{p_0} \dots \partial_3^{p_3} f$. The equation is

$$\partial_t u + f(\partial^3 u, \partial^2 u, \partial u, u, x, t) = 0 \tag{1}$$

where $x \in \mathbb{R}$, $t \in [0, T]$, and T is an arbitrary positive time.

The assumptions on f are as follows. $f: \mathbb{R}^5 \times [0, T] \rightarrow \mathbb{R}$ is C^∞ in all its variables.

(A1) There exists $c > 0$ such that

$$\partial_3 f(y, x, t) \geq c > 0$$

for all $y = (u_3, u_2, u_1, u_0) \in \mathbb{R}^4$, $x \in \mathbb{R}$ and $t \in [0, T]$.

(A2) $\partial_2 f(y, x, t) \leq 0$.

(A3) All the derivatives of $f(y, x, t)$ are bounded for $x \in \mathbb{R}$, for $t \in [0, T]$ and y in a bounded set.

(A4) $x^N \partial_x^j f(0, x, t)$ is bounded for all $N \geq 0, j \geq 0$, and $x \in \mathbb{R}, t \in (0, T]$.

Remark. – These assumptions imply that f has the form

$$f = u_3 g_3 + u_2 g_2 + u_1 g_1 + u_0 g_0 + h \tag{2}$$

where $g_j = g_j(u_j, \dots, u_0, x, t)$ and $h = h(x, t)$ have the properties

$$g_3(u_3, u_2, u_1, u_0, x, t) \geq c > 0, \tag{3}$$

$$g_2(u_2, u_1, u_0, x, t) \leq 0, \tag{4}$$

g_3, g_2, g_1, g_0 and h are C^∞ and each of their derivatives

is bounded for y bounded $x \in \mathbb{R}$ and $t \in [0, T]$. (5)

Indeed, we define

$$g_3 = \begin{cases} [f(y_3, y_2, \dots, t) - f(0, y_2, \dots, t)]/y_3 & \text{for } y_3 \neq 0 \\ \partial_3 f(0, y_2, \dots, t) & \text{for } y_3 = 0, \end{cases}$$

$$g_2 = \begin{cases} [f(0, y_2, y_1, \dots, t) - f(0, 0, y_1, \dots, t)]/y_2 & \text{for } y_2 \neq 0 \\ \partial_2 f(0, 0, y_1, \dots, t) & \text{for } y_2 = 0, \end{cases}$$

and similarly for g_1 and g_0 and $h = f(0, 0, 0, 0, x, t)$. Then (A1) implies (3), (A2) implies (4) and (A3) implies (5).

We now specify the weight functions to be used. A function $\xi(x, t)$ belongs to the *weight class* $\mathbf{W}_{\sigma ik}$ if it is a positive C^∞ function on $\mathbb{R} \times [0, T]$ and there exist constants c_1, c_2, c_3 such that

$$0 < c_1 \leq t^{-k} e^{\sigma |x|} \xi(x, t) \leq c_2 \quad \text{for } x < -1. \tag{6}$$

$$0 < c_1 \leq t^{-k} x^{-i} \xi(x, t) \leq c_2 \quad \text{for } x > 1, \tag{7}$$

$$(t |\partial_t \xi| + |\partial^j \xi|) / \xi \leq c_3 \text{ in } \mathbb{R} \times [0, T] \quad \text{for all } j. \tag{8}$$

We shall always take $\sigma \geq 0$ and $k \geq 0$.

By $H^s(W_{\sigma ik})$ we denote the Sobolev space on \mathbb{R} with a weight; that is, with the norm

$$\|v\|^2 = \sum_{j=0}^s \int_{-\infty}^{\infty} |\partial^j v(x)|^2 \xi(x, t) dx$$

for any $\xi \in W_{\sigma ik}$ and $0 < t < T$. Even though the norm depends on ξ , all such choices lead to equivalent norms. By $L^p(H^s(W_{\sigma ik}))$ we denote the space of functions $v(x, t)$ with the norm

$$\|v\|^p = \int_0^T \left\{ \sum_{k=0}^s \int_{-\infty}^{\infty} |\partial_x^k v(x, t)|^2 \xi(x, t) dx \right\}^p dt.$$

The traditional Sobolev space is $H^s = H^s(W_{000})$, without a weight. We define

$$\bar{W}_{\sigma ik} = \bigcup_{j < i} W_{\sigma jk}$$

and

$$L^p(H^s(\bar{W}_{\sigma ik})) = \bigcup_{j < i} L^p(H^s(W_{\sigma jk})).$$

We shall use the last spaces only in the case $i = -1$. With this notation we can succinctly state the two main theorems. The distinction between them resides only in the order of differentiability assumed for the solution and in the nature of the weight functions.

THEOREM 2.1. — *Let $T > 0$ and let $u(x, t)$ be a solution of (1) in the region $\mathbb{R} \times [0, T]$ such that*

$$u \in L^\infty(H^7(W_{0L0})) \quad (9)$$

for some $L \geq 1$ and all $\sigma > 0$. Then

$$u \in L^\infty(H^{7+l}(W_{\sigma, L-l, l})) \cap L^2(H^{8+l}(W_{\sigma, L-l, l})) \quad (10)$$

for all $0 \leq l \leq L$ and all $\sigma > 0$, with the exception that if $l = L$ then $W_{\sigma, -1, l}$ is replaced by $\bar{W}_{\sigma, -1, l}$.

Thus we have a gain of L derivatives, at the expense of L powers of x in the weight functions as $x \rightarrow +\infty$, which means $L/2$ powers of x for the solution $u(x, t)$. Therefore if the assumption (9) holds for all $L \geq 1$, the solution is infinitely differentiable in the x -variable. From the equation (1) itself the solution is C^∞ in both of its variables. The next theorem shows that the assumed number of derivatives can be reduced by two, with only a minor change in the weight functions.

THEOREM 2.2. — *Let $T > 0$ and let $u(x, t)$ be a solution of (1) in the region $\mathbb{R} \times [0, T]$ such that*

$$u \in L^\infty(H^5(W_{0K0})) \quad (11)$$

for some $K \geq 2$ and all $\sigma > 0$. Then

$$u \in L^\infty (H^{5+k} (W_{\sigma, K-k, k})) \cap L^2 (H^{6+k} (W_{\sigma, K-k-1, k})) \tag{12}$$

for all $0 \leq k \leq K$ and all $\sigma > 0$, with the exception that $W_{\sigma, -1, k}$ is replaced by $\bar{W}_{\sigma, -1, k}$.

The rest of this section is devoted to a proof of these two theorems. It is based entirely on a sequence of *a priori* estimates, as mentioned in the introduction. We shall derive the *a priori* estimates assuming that the solution is C^∞ , is bounded as $x \rightarrow -\infty$, and is rapidly decreasing as $x \rightarrow +\infty$, together with all of its derivatives. At the end of the proof we shall use an approximation argument for the general case. In Theorems 4.1 and 4.5 it is proved that (9) implies $u \in L^2 (H^8 (W_{\sigma, L-1, 0}))$ and that (11) implies $u \in L^2 (H^6 (W_{\sigma, K-1, 0}))$.

The *a priori* estimate

We begin the proof by taking α x -derivatives of the equation where $\alpha = 7 + l = 5 + k$ with $6 \leq \alpha \leq 7 + L = 5 + K$. Thus

$$\begin{aligned} \partial_t u_\alpha + \partial_3 f \cdot u_{\alpha+3} + \alpha \partial [\partial_3 f] \cdot u_{\alpha+2} + \partial_2 f \cdot u_{\alpha+2} \\ + \binom{\alpha}{2} \partial^2 [\partial_3 f] \cdot u_{\alpha+1} + \alpha \partial [\partial_2 f] \cdot u_{\alpha+1} + \partial_1 f \cdot u_{\alpha+1} \\ + O(u_\alpha, u_{\alpha-1}, \dots) = 0 \end{aligned} \tag{13}$$

where only the highest-order terms have been written explicitly. We have also omitted the arguments of f and its derivatives and have denoted

$$\partial [\partial_3 f] = \frac{\partial}{\partial x} [(\partial_3 f) (\partial^3 u, \partial^2 u, \partial u, u, x, t)].$$

Then we let $\xi \in W_{\sigma, K+5-\alpha, \alpha-5}$, multiply (13) by $\xi \partial^\alpha u = \xi u_\alpha$ and integrate over $x \in \mathbb{R}$. Each term in (13) is treated separately. The first two terms yield

$$\int \partial_t u_\alpha \cdot \xi u_\alpha = \frac{1}{2} \partial_t \int \xi u_\alpha^2 - \frac{1}{2} \int (\partial_t \xi) u_\alpha^2$$

and

$$\begin{aligned} \int \partial_3 f \cdot u_{\alpha+3} \cdot \xi u_\alpha &= - \int \xi \partial_3 f u_{\alpha+2} u_{\alpha+1} - \int \partial [\xi \partial_3 f] u_{\alpha+2} u_\alpha \\ &= \frac{3}{2} \int \partial [\xi \partial_3 f] u_{\alpha+1}^2 - \frac{1}{2} \int \partial^3 [\xi \partial_3 f] u_\alpha^2. \end{aligned}$$

The third and fourth terms in (13) are treated similarly, integrating by parts twice. Indeed,

$$\alpha \int \partial [\partial_3 f] \cdot u_{\alpha+2} \cdot \xi u_\alpha = \frac{1}{2} \alpha \int \partial^2 [\xi \partial [\partial_3 f]] u_\alpha^2 - \alpha \int \xi \partial [\partial_3 f] u_{\alpha+1}^2$$

and

$$\int \partial_2 f \cdot u_{\alpha+2} \cdot \xi u_\alpha = - \int \xi \partial_2 f u_{\alpha+1}^2 + \frac{1}{2} \int \partial^2 [\xi \partial_2 f] u_\alpha^2.$$

The fifth, sixth and seventh terms in (13) are also treated similarly, integrating by parts once. By this procedure we obtain the *main identity*

$$\partial_t \int \xi u_\alpha^2 + \int (\eta - 2\xi \partial_2 f) u_{\alpha+1}^2 + \int \mathfrak{G} u_\alpha^2 + \int \mathbf{R} = 0. \quad (14)$$

where

$$\eta = 3 \partial [\xi \partial_3 f] - 2 \alpha \xi \partial [\partial_3 f] = 3 \partial_3 f \cdot \partial \xi + (3 - 2\alpha) \partial [\partial_3 f] \cdot \xi, \quad (15)$$

$$\mathfrak{G} = -\partial_t \xi - 2 \partial^3 [\xi \partial_3 f] + \alpha \partial^2 [\xi \partial [\partial_3 f]] + \binom{\alpha}{2} \partial [\xi \partial^2 [\partial_3 f]], \quad (16)$$

and where $\mathbf{R} = O(u_\alpha, u_{\alpha-1}, \dots)$ has coefficients containing ξ but no derivatives of ξ . Our main estimate comes from the first two terms in (14) if η can be chosen to be positive. The second part of the second term in (14) is non-negative by (A2).

Choice of weight function

If η is an arbitrary weight function in $W_{\sigma ik}$, then there exists $\xi \in W_{\sigma, i+1, k}$ which satisfies (15). Indeed, letting $a(x, t) = \partial_3 f = \partial_3 f(u_{xxx}, \dots, x, t)$, equation (15) takes the form $3 a \xi_x - (2\alpha - 3) a_x \xi = \eta$. It has a solution

$$\xi = \frac{1}{3} a^{-1+2a/3} \int_{-\infty}^x a^{-2a/3} \eta. \quad (17)$$

Since $a(x, t)$ is bounded below by (A1) and is bounded above by (A3), it follows that ξ inherits the properties (6), (7) and (8) of η with an increase of growth at $+\infty$ by one power.

In the case of Theorem 2.1, we take $8 \leq \alpha = l + 7 \leq L + 7$. If $\alpha < L + 7$, we take any

$$\eta \in W_{\sigma, L-\alpha+6, \alpha-7} \quad \text{to get} \quad \xi \in W_{\sigma, L-\alpha+7, \alpha-7}. \quad (18)$$

In case $\alpha = L + 7$, we take any $\eta \in \bar{W}_{\sigma, -1, L}$ to get $\xi \in W_{\sigma, 0, L}$. Due to the full nonlinearity of equation (1) the multiplier is nonlinear; that is, ξ depends on $\partial^3 u, \dots$ though $a = \partial_3 f$. So ξ and its derivatives must be treated carefully.

LEMMA 2.3 (Estimate of error terms). – If $8 \leq \alpha \leq L + 7$ and the weight functions are chosen as in (18), then

$$\left| \int_0^T \int (\mathfrak{R} u_x^2 + \mathbf{R}) dx dt \right| \leq C \tag{19}$$

where C depends only on the norms of u in

$$L^\infty(H^\beta(W_{\sigma, L-\beta+7, \beta-7})) \cap L^2(H^{\beta+1}(W_{\sigma, L-\beta+6, \beta-7})) \tag{20}$$

for $7 \leq \beta \leq \alpha - 1$ and on the norm of u in $L^\infty(H^7(W_{0L0}))$.

Proof of Theorem 2.1 assuming Lemma 2.3. – We will use induction on α , beginning with $\alpha = 8$. The estimate will be applied to a smooth approximation of the solution. Let u be a solution satisfying (9). The equation itself implies that $\partial_t u \in L^\infty(H^4(W_{0L0}))$. Hence u is a weakly continuous function of t with values in $H^7(W_{0L0})$. In particular, $u(\cdot, t) \in H^7(W_{0L0})$ for every t . Let $t_0 \in (0, T)$ and let $\{\varphi^{(n)}(\cdot)\}$ be a sequence of functions in $C_0^\infty(\mathbb{R})$ which converges to $u(\cdot, t_0)$ strongly in $H^7(W_{0L0})$. Let $u^{(n)}(x, t)$ be the unique solution of (1) with the initial data $\varphi^{(n)}(x)$ at time $t = t_0$. By Theorem 3.2 it is guaranteed to exist in a time interval $[t_0, t_0 + \delta]$ where $\delta > 0$ does not depend on n . By Theorem 4.1,

$$u^{(n)} \in L^\infty(H^7(W_{0L0})) \cap L^2(H^8(W_{\sigma, L-1, 0})) \tag{21}$$

with a bound that depends only on the norm of $\varphi^{(n)}$ in $H^7(W_{0L0})$. Furthermore, Theorem 4.1 guarantees the non-uniform bounds

$$\sup_{[t_0, t_0 + \delta]} \sup_x (1 + x_+)^k |\partial^{\alpha} u^{(n)}(x, t)| < \infty \tag{22}$$

for each n, k and α .

The main identity (14) and the estimate (19) are therefore valid for each $u^{(n)}$ in the interval $[t_0, t_0 + \delta]$. The multiplier η may be chosen arbitrarily in its weight class (18) and then $\xi = \xi^{(n)}$ is defined by (17) and depends on n . However the constants c_1 and c_2 in (7) are independent of n . From (14) and (19) we have

$$\sup_{[t_0, t_0 + \delta]} \int \xi^{(n)} (u_x^{(n)})^2 dx + \int_{t_0}^{t_0 + \delta} \eta (u_{x+1}^{(n)})^2 dx dt \leq C. \tag{23}$$

By (19), C is independent of n . This estimate (23) is proved by induction for $\alpha = 8, 9, 10, \dots$. Thus $u^{(n)}$ is also bounded in

$$L^\infty(H^\alpha(W_{\sigma, L-\alpha+7, \alpha-7})) \cap L^2(H^{\alpha+1}(W_{\sigma, L-\alpha+6, \alpha-7})) \tag{24}$$

for $\alpha \geq 8$. Since $u^{(n)} \rightarrow u$ in $L^\infty(H^7)$ by Section 3, it follows that u belongs to the space (24). Since δ is fixed, this result is valid over the whole interval $[0, T]$. In the last step of the induction we have $\alpha = L + 7$, $\xi \in W_{\sigma 0 L}$ and $\eta \in \tilde{W}_{\sigma, -1, L}$. This completes the proof of Theorem 2.1.

LEMMA 2.4. — *The expression R in the main identity (4) is a sum of terms of the form*

$$\xi \partial_3^{p_3} \partial_2^{p_2} \partial_1^{p_1} \partial_0^{p_0} \partial_x^\gamma f \cdot u_{v_1} u_{v_2} \dots u_{v_{p-1}} u_{v_p} u_\alpha \tag{25}$$

where $1 \leq v_1 \leq \dots \leq v_p \leq \alpha$,

$$p = p_3 + p_2 + p_1 + p_0 \geq 0 \tag{26}$$

$$v_1 + v_2 + \dots + v_p + \alpha + \gamma = 2\alpha + 3p_3 + 2p_2 + p_1 \tag{27}$$

$$p + v_{p-1} + v_p \leq \alpha + 8 \quad \text{if } p \geq 2. \tag{28}$$

Proof. — The terms of R come from differentiating the equation and multiplying by ξu_α . They clearly have the form (25), where p is the total number of factors pulled out of f by differentiation, with p_j derivatives of f with respect to u_j ($j = 3, 2, 1, 0$). The total number of derivatives which occur in (25) is given by (27). Each differentiation of f with respect to u_3 contributes an extra 3 derivatives, which leads to the term $3p_3$ in (27); similarly for u_2 and u_1 . Thus at least p_3 among $\{v_1, \dots, v_p\}$ are ≥ 4 , at least p_2 of them are ≥ 3 , at least p_1 of them are ≥ 2 and they are all ≥ 1 . Therefore

$$v_1 + \dots + v_{p-2} \geq 4(p_3 - 2) + 3p_2 + 2p_1 + p_0.$$

Combining this with (27), we get

$$4(p_3 - 2) + 3p_2 + 2p_1 + p_0 + v_p + v_{p-1} + \alpha \leq 2\alpha + 3p_3 + 2p_2 + p_1.$$

Using (26) we get (28).

Proof of Lemma 2.3. — We must estimate both R and \mathcal{G} . We begin with a term of R of the form (25), assuming that $v_{p-1} \leq \alpha - 2$ ($p \geq 2$) or else $p = 1$. By the induction hypothesis, u is bounded in $L^\infty(W_{\sigma, L - (\beta - 7)^+, (\beta - 7)^+})$ for all $\sigma > 0$ and $0 \leq \beta \leq \alpha - 1$. By the appendix,

$$\sup_t \sup_x \zeta u_p^2 < \infty \tag{30}$$

for $0 \leq \beta \leq \alpha - 2$ and $\zeta \in W_{\sigma, L - (\beta - 6)^+, (\beta - 6)^+}$. In the term (25) we estimate $u_{v_1}, \dots, u_{v_{p-1}}$ using (30). We estimate u_{v_p} and u_α using the weighted L^2 bounds

$$\int_0^T \int \zeta u_{v_p}^2 dx dt < \infty \quad \text{for } \zeta \in W_{\sigma, L - (v_p - 7)^+, (v_p - 8)^+} \tag{31}$$

and the same with v_p replaced by α . It suffices to check the powers of t , the powers of x as $x \rightarrow +\infty$ and the exponentials as $x \rightarrow -\infty$.

In the term (25), the factor ξ contributes the power $t^{\alpha-7}$, according to (18). Each factor u_{v_j} uses up the power $t^{(v_j-6)^+/2}$ in the estimate (30), for $1, \dots, p-1$. The last two factors in (25) use up the powers $t^{(v_p-8)^+/2}$

and $t^{(\alpha-8)/2}$ in the estimate (31). Thus the difference in the powers of t is

$$M = (\alpha - 7) - \frac{1}{2} \sum_{j=1}^{p-1} (v_j - 6)^+ - \frac{1}{2} (v_p - 8)^+ - \frac{1}{2} (\alpha - 8). \quad (32)$$

We claim that $M \geq 0$, so that the extra power of t can be thrown away. To prove the claim, we argue as follows. Among the indices $\{1, \dots, p\}$, let q be the largest index $\leq p-1$ such that $v_q \leq 6$. Each derivative u_{v_1}, \dots, u_{v_q} comes from differentiating $f(u_3, u_2, u_1, u_0, x, t)$ with respect to its first four arguments. Let q_3 of the derivatives come from u_3 , q_2 from u_2 , etc. Then

$$q = q_3 + q_2 + q_1 + q_0 \quad \text{and} \quad v_1 + \dots + v_q \geq 4q_3 + 3q_2 + 2q_1 + q_0,$$

as in the proof of Lemma 2.4. Now by (27)

$$\begin{aligned} \alpha &\geq \alpha - \gamma = v_1 + \dots + v_q + \dots + v_p - 3p_3 - 2p_2 - p_1 \\ &\geq 4q_3 + 3q_2 + 2q_1 + q_0 + v_{q+1} + \dots + v_p - 3p_3 - 2p_2 - p_1. \end{aligned} \quad (33)$$

Hence, if $q < p$,

$$\begin{aligned} 2M &= \alpha - 6 - \sum_{j=1}^{p-1} (v_j - 6)^+ - (v_p - 8)^+ \\ &\geq 4q_3 + 3q_2 + 2q_1 + q_0 - 3p_3 - 2p_2 - p_1 + (v_p - 6) + 6(p-1-q) - (v_p - 8)^+ \\ &= p + 2(p_3 - q_3) + 3(p_2 - q_2) + 4(p_1 - q_1) + 5(p_0 - q_0) - 12 + v_p - (v_p - 8)^+ \\ &\geq p + 2(p - q) - 12 + v_p - (v_p - 8)^+ \\ &\geq p + 4 - 12 + v_p - (v_p - 8)^+ \geq p - 1 > 0 \end{aligned}$$

if $0 \leq q \leq p-2$. The remaining cases are as follows. If $2 \leq q = p-1$, then $2M \geq p-10 + v_p - (v_p - 8)^+ \geq p-3 \geq 0$. If $1 = q = p-1$, then $2M \geq \alpha - 7 > 0$ from (32). If $p=1$, then $2M \geq \alpha - 6 - (v_p - 8)^+ \geq 0$ because $v_p \leq \alpha + 2$. Finally if $v_p \leq 6$, then $p = q$ and $2M = \alpha - 6 > 0$.

Similarly, as $x \rightarrow +\infty$, the difference in the powers of x is

$$\begin{aligned} N &= (L - \alpha + 7) - \frac{1}{2} \sum_{j=1}^{p-1} (L - (v_j - 6)^+) \\ &\quad - \frac{1}{2} (L - (v_p - 7)^+) - \frac{1}{2} (L - \alpha + 7). \end{aligned} \quad (34)$$

We claim that $N \leq 0$, which means that the extra power x^N can be thrown away. To prove this claim assuming $v_p \geq 7$, we write

$$\begin{aligned} -2N &= \alpha + (p-1)L - \sum_{j=1}^{p-1} (v_j - 6)^+ - v_p \\ &= \alpha + (p-1)L + 6(p-1-q) + \sum_{j=q+1}^{p-1} v_j - v_p. \end{aligned}$$

By (33),

$$\begin{aligned}
 -2N &\geq p - 4(p_3 - q_3) - 3(p_2 - q_2) - 2(p_1 - q_1) \\
 &\quad - (p_0 - q_0) + (p - 1)L + 6(p - 1 - q) \\
 &= 2(p_3 - q_3) + 3(p_2 - q_2) + 4(p_1 - q_1) + 5(p_0 - q_0) + (p - 1)L - 6 + p \\
 &\geq 2(p - q) + (p - 1)L - 6 + p \geq 4 + (p - 1) - 6 + p = 2p - 3 > 0
 \end{aligned}$$

in case $q \leq p - 2$. On the other hand, if $q = p - 1$, then

$$-2N = \alpha + (p - 1)L - v_p \geq (p - 1)L \geq 0.$$

Finally if $v_p \leq 6$, then

$$-2N = \alpha - 7 + (p - 1)L \geq 1 + (p - 1)L > 0.$$

The behavior as $x \rightarrow -\infty$ is easier, since each factor u_{v_j} must grow slower than an exponential $e^{\sigma' |x|}$ and ξ decays like an exponential $e^{-\sigma |x|}$. We simply need to choose the appropriate relationship between σ and σ' at each induction step.

Now consider the term of R with $p = 0$, namely the term $\xi u_\alpha \partial_x^\alpha f$ where all the differentiation acts on the explicit x variable of f . By (2) this term has the integral

$$\iint \xi u_\alpha \partial_x^\alpha f \, dx \, dt = \iint \xi u_\alpha \left(\sum_{j=0}^3 u_j \partial_x^\alpha g_j + \partial_x^\alpha h \right) \, dx \, dt.$$

By (5) each $\partial_x^\alpha g_j$ is bounded. Hence

$$\begin{aligned}
 &\iint \xi u_\alpha \partial_x^\alpha f \, dx \, dt \\
 &\leq c \left\{ \iint \xi u_\alpha^2 \, dx \, dt \right\}^{1/2} \left\{ \sum_{j=0}^3 \iint \xi u_j^2 \, dx \, dt + \iint \xi (\partial_x^\alpha h)^2 \, dx \, dt \right\}^{1/2}
 \end{aligned}$$

is finite because of (A4).

The analysis of all the terms of R will be completed with the case of $v_{p-1} \geq \alpha - 1$ with $p \geq 2$. In this case $p + 2(\alpha - 1) \leq \alpha + 8$ by (28), or $p + \alpha \leq 10$. Since $\alpha \geq 8$ and $p \geq 2$, this requires $\alpha = 8$ and $p = 2$. By (27), $v_1 + v_2 \leq 14$. That is, $v_1 = v_2 = 7$ is the only possibility. Thus the only term of this form is

$$\xi \partial_3^2 f \cdot u_7^2 u_8.$$

This term is integrated by parts once, leading to a term of the form

$$\{ c_1 \xi \partial [\partial_3^2 f] + c_2 \partial \xi \cdot \partial_3^2 f \} u_7^3$$

where $\xi \in W_{\sigma, L-1, 1}$. For this term we use the interpolation inequality $|v'|_3 \leq |v|_6^{1/2} |v''|_2^{1/2}$. For $x > 1$ this leads to the estimate

$$\int_0^T \int_1^\infty t x^{L-1} |u_7|^3 dx dt \leq T (\sup_{t, x} |u_6|) \left(\iint x^{L-1} u_6^2 dx dt \right)^{1/4} \left(\iint x^{L-1} u_8^2 dx dt \right)^{3/4}, \quad (35)$$

which is bounded. For $x < 1$ the estimate is similar except for exponential weights. This completes the estimate of R .

Now we estimate the terms $\mathfrak{G} u_\alpha^2$ where \mathfrak{G} is given by (16). We claim that \mathfrak{G} involves derivatives of u only up to order six and hence that $\mathfrak{G} u_\alpha^2$ is a sum of terms of the same type we have already encountered in R , so that its integral can be bounded in the same manner. Indeed (16) shows that \mathfrak{G} depends on $\partial_t \xi$, $\partial^3 \xi$ and derivatives of lower order. Formula (17) expresses ξ in terms of η and derivatives of u up to order three. The first term of \mathfrak{G} , namely $-\partial_t \xi$, is given by

$$3 \partial_t \xi = 3 \partial_t \{ a^{\beta-1} I(a^{-\beta} \eta) \} = a^{\beta-1} I(a^{-\beta} \partial_t \eta) + (\beta-1) a^{\beta-2} \partial_t a I(a^{-\beta} \eta) - \beta a^{\beta-1} I(a^{-\beta-1} (\partial_t a) \eta)$$

where $I = \int_{-\infty}^x$, $a = \partial_3 f$ and

$$\partial_t a = \partial_3^2 f \cdot \partial^3 [f] + \text{terms with fewer derivatives of } u.$$

This shows explicitly that \mathfrak{G} depends only on derivatives of u up to order six. This completes the proof of Lemma 2.3 and therefore of Theorem 2.1.

In order to prove Theorem 2.2, we use the main identity (14) with $6 \leq \alpha = k + 5 \leq K + 5$. If $\alpha < K + 5$, we take any weight function

$$\eta \in W_{\sigma, K-\alpha+4, \alpha-5} \quad \text{to get } \xi \in W_{\sigma, K-\alpha+5, \alpha-5}. \quad (37)$$

In case $\alpha = K + 5$, we take any $\eta \in \bar{W}_{\sigma, -1, K}$ to get $\xi \in W_{\sigma 0 K}$.

LEMMA 2.5. — *If $K \geq 3$ and $8 \leq \alpha \leq K + 5$ and the weight functions are chosen as in (37), then*

$$\left| \int_0^T \int (\mathfrak{G} u_\alpha^2 + R) dx dt \right| \leq C. \quad (38)$$

where C depends only on the norms of u in

$$L^\infty(H^\beta(W_{\sigma, K-\beta+5, \beta-5})) \cap L^2(H^{\beta+1}(W_{\sigma, K-\beta+4, \beta-5})) \quad (39)$$

for $5 \leq \beta \leq \alpha - 1$ and on the norm of u in $L^\infty(W_{0 K 0})$.

Proof. — The proof is similar to that of Lemma 2.3. In fact (30) is true for $0 \leq \beta \leq \alpha - 2$ and $\xi \in W_{\sigma, K-(\beta-4)^+, (\beta-4)^+}$. Also (31) is true for $\xi \in W_{\sigma, K-(\nu_p-5)^+, (\nu_p-6)^+}$ and the same with ν_p replaced by α . Following

the proof of Lemma 2.3, the difference in the powers of t is

$$M = (\alpha - 5) - \frac{1}{2} \sum_{j=1}^{p-1} (v_j - 4)^+ - \frac{1}{2} (v_p - 6)^+ - \frac{1}{2} (\alpha - 6). \quad (40)$$

The difference in the powers of x as $x \rightarrow +\infty$ is

$$N = (K - \alpha + 5) - \frac{1}{2} \sum_{j=1}^{p-1} (K - (v_j - 4)^+) - \frac{1}{2} (K - (v_p - 5)^+) - \frac{1}{2} (K - \alpha + 5). \quad (41)$$

It is left to the reader to verify that $M \geq 0$ and $N \leq 0$, as in Lemma 2.3. The special term of R with $p=0$ is treated as before. In case $\alpha=8$, the special term of R of the form $\xi \partial_3^2 f \cdot u_7^2$ is treated as in (35) with $L-1$ replaced by $K-3$. Such a term occurs on the third induction step $k-3$, that is, only in case $K \geq 3$.

It remains to estimate the terms $\mathfrak{G} u_\alpha^2$ where \mathfrak{G} is given by (16). They are estimated in the same way as before in case $\alpha \geq 8$, but for the sake of our future discussion of the cases $\alpha=6$ and 7 , we provide a more detailed analysis of them. Because \mathfrak{G} involves up to six derivatives of u , some of the $\mathfrak{G} u_\alpha^2$ terms will be major terms when we take $\alpha=6$ and 7 in the next lemma. We begin with the second term in (16), $\partial^3 [\xi \partial_3 f]$. We recall the formula (17), which we write in the form

$$3 \xi a = a^\beta I(a^{-\beta} \eta)$$

where $a = \partial_3 f$, $\beta = 2\alpha/3$, and $I = \int_{-\infty}^x$, for brevity. Then

$$3 \partial^3 (\xi a) = \beta a^{\beta-1} \cdot \partial^3 a \cdot I(a^{-\beta} \eta) + 3 \beta (\beta - 1) a^{\beta-2} \cdot \partial^2 a \cdot \partial a \cdot I(a^{-\beta} \eta) + 2 \beta a^{-1} \cdot \partial^2 a \cdot \eta + m \quad (42)$$

where m involves at most one derivative of a . In (42) the factors $I(a^{-\beta} \eta)$ have the same behavior as ξ . The factors a and ∂a involve at most four derivatives of u , which are harmless because $u, \partial u, \partial^2 u, \partial^3 u$ and $(1 + e^{\sigma x}) \partial^4 u$ are bounded [by (39)] for all $\sigma > 0$. So only the factors $\partial^3 a$ and $\partial^2 a$ in (42) require careful analysis. Now $\partial^3 a = b_1 u_6 + b_2 u_5 + b_3$ and $\partial^2 a = b_4 u_5 + b_5$, where the coefficients b_j are harmless. Therefore the second term in (16) contributes terms to (14) of the form

$$b_6 u_6 u_\alpha^2 + b_7 u_5 u_\alpha^2 + b_8 u_\alpha^2. \quad (43)$$

If $\alpha \geq 8$, these terms are of the same kinds as the regular ones in R . If $\alpha=6$ or 7 , they will be analyzed separately in Lemma 2.6. The 3rd-7th terms in (16) are of the same type as the second term, or better, and are analyzed in the same way.

It remains to estimate the first term in (16), namely $-\partial_t \xi \cdot u_\alpha^2$. For convenience we choose $\eta(x, t)$ of the form $t^k \eta(x)$ where $k = \alpha - 5$. Then $3\xi = t^k a^{\beta-1} I(a^{-\beta} \eta)$ and

$$3 \partial_t \xi = k t^{k-1} a^{\beta-1} I(a^{-\beta} \eta) + (\beta - 1) t^k a^{\beta-2} \partial_t a \cdot I(a^{-\beta} \eta) - \beta t^k a^{\beta-1} I(a^{-\beta-1} (\partial_t a) \eta). \quad (44)$$

The first term in (44) is regular (like the terms of R which we have already analyzed). Now

$$\partial_t a = \partial_t [\partial_3 f] = \partial_3^2 f \cdot \partial_t u_3 + \partial_3 \partial_2 f \cdot \partial_t u_2 + \partial_3 \partial_1 f \cdot \partial_t u_1 + \partial_3 \partial_0 f \cdot \partial_t u + \partial_3 \partial_t f.$$

But, differentiating equation (1), we have $\partial_t u_j = -\partial^j [f]$ so that

$$-\partial_t a = \partial_3^2 f \cdot \partial_3 f \cdot u_6 + (\partial_3^2 f)^2 \cdot u_4 u_5 + 2 \partial_3^2 f \cdot \partial [\partial_3 f] \cdot u_5 + \partial_3^2 f \cdot \partial_2 f \cdot u_5 + \partial_3 \partial_2 f \cdot \partial_3 f \cdot u_5 + O(u_4, \dots).$$

This appears in the second term of (44) multiplied by some harmless factors. If $\alpha \geq 8$, all these terms (multiplied by u_α^2) are regular.

In the last term of (44), the factor $\partial_t a$ appears inside the integral. We write it as $-\partial_3^2 f \cdot \partial^3 [f]$ plus terms with fewer derivatives of u . Thus the last term in (44) is

$$\beta t^k [\partial_3 f]^{\beta-1} I \{ [\partial_3 f]^{-\beta-1} \cdot \partial_3^2 f \cdot \partial^3 [f] \cdot \eta + \dots \}.$$

The main term in this integral is integrated by parts to give

$$\eta [\partial_3 f]^{\beta-1} \cdot \partial_3^2 f \cdot \partial^2 [f] - \int_{-\infty}^x \partial \{ \eta [\partial_3 f]^{-\beta-1} \cdot \partial_3^2 f \} \cdot \partial^2 [f]. \quad (45)$$

The first term in (45) is $\eta (\partial_3 f)^{-\beta} \cdot \partial_3^2 f \cdot u_5 + O(u_4, \dots)$. It is regular if $\alpha \geq 7$. The second term in (45) is

$$\int_{-\infty}^x (g \cdot \eta u_4 u_5 + h \cdot \eta_x u_5 + j)$$

where g and h involve u_3, \dots, u and j involves u_4, \dots, u . Since $u_4 u_5 = \partial [u_4^2/2]$, another integration by parts eliminates the fifth derivative, leaving us with terms of the previous types. This completes the estimates of all the ∂u_α^2 terms and thereby the proof of Lemma 2.5.

LEMMA 2.6. — *If $\alpha = 6$ or 7 and $\varepsilon > 0$, there exists a positive constant C , which depends only on ε and on the norms of u in $L^\infty(H^5(W_{0, K_0}))$ and in (39) for $5 \leq \beta \leq \alpha - 1$, such that*

$$\left| \int_0^T \int (\partial u_\alpha^2 + R) dx dt \right| \leq C + \varepsilon \int_0^T \int \zeta u_{\alpha+1}^2 dx dt$$

for some $\zeta \in W_{\alpha, L^{-\alpha+4}, \alpha-5}$.

Proof. — We begin with the first induction step, $\alpha = 6$, which is the most delicate. The main identity (14) then takes the following explicit

form

$$\begin{aligned} \partial_t(\xi u_6^2) - (\partial_t \xi) u_6^2 + \{ & \partial_3 f \cdot u_9 + 6 \partial[\partial_3 f] \cdot u_8 \\ & + c_1 \partial_3^2 f \cdot u_7 u_5 + c_2 \partial_3^2 f \cdot u_6^2 + c_3 \partial_3^3 f \cdot u_7 u_4^2 \\ & + c_4 \partial_3^3 f \cdot u_6 u_5 u_4 + c_5 \partial_3^3 f \cdot u_5^3 + c_6 \partial_3^4 f \cdot u_6 u_4^3 \\ & + c_7 \partial_3^4 f \cdot u_5^2 u_4^2 + c_8 \partial_3^5 f \cdot u_5 u_4^4 + c_9 \partial_3^6 f \cdot u_4^6 \\ & + \partial_2 f \cdot u_8 + c_{10} \partial_2^2 f \cdot u_7 u_3 + \dots \} 2 \xi u_6 = 0 \end{aligned} \quad (46)$$

(with various combinatorial constants c_j), where we have written all the terms with $\partial_3^k f$, and the dots represent terms involving $\partial_2 f$, $\partial_1 f$ and $\partial_0 f$. Integration by parts leads to the identity (14) where there are a number of regular remainder terms as before, as well as four kinds of exceptional remainder terms, namely terms involving u_6^3 , $u_5 u_6^2$, u_5^4 and u_5^3 . Clearly the estimation of $u_5 u_6^2$ will be easier than u_6^3 , and the estimation of u_5^3 will be easier than u_5^4 ; we limit the discussion here to these two kinds of terms.

The u_5^4 Terms

$$\begin{aligned} |\zeta v'|_4^4 &= \int (\zeta v')^3 \zeta v' dx \\ &= - \int [(\zeta^4 (v')^3)]' v dx = -3 \int \zeta^4 (v')^2 v'' v dx - 4 \int \zeta^3 \zeta' (v')^3 v dx \\ &\leq c |\zeta v''|_2 |\zeta v'|_4 |\zeta v|_\infty + c |\zeta v'|_4^3 |\zeta v|_4. \end{aligned}$$

Assuming $|\zeta'| \leq c \zeta$, we have $|\zeta' v|_4 \leq c |\zeta v|_2^{1/2} |\zeta v|_\infty^{1/2}$. Thus

$$|\zeta v'|_4 \leq (|\zeta v''|_2 + |\zeta v|_2)^{1/2} |\zeta v|_\infty^{1/2}, \quad (47)$$

which is a standard interpolation inequality with a weight function. We apply it first to $v = u_5$ and $\zeta^4 =$ an upper bound for the coefficient of the u_5^4 term. So ζ^4 is a linear combination of upper bounds of ξ and $\partial \xi$. Because of (8), ζ^4 is a weight function of the same class as ξ . Therefore the sum of the u_5^4 terms is bounded by

$$\iint \zeta^4 u_5^4 dx dt \leq c \left\{ \iint \zeta^2 (u_6^2 + u_4^2) dx dt \right\} \left\{ \sup_{t,x} \zeta^2 u_4^2 \right\},$$

which is bounded by (11).

The u_6^3 Terms

A typical such term is $|\partial^2 f \cdot u_6^3| \leq c |u_6|^3$. A standard interpolation inequality is

$$|v'|_3 \leq c |v|_2^{5/12} |v''|_2^{7/12} \quad (48)$$

If we disregarded the weight functions, we would estimate as follows. Letting $v = u_5$,

$$\int |u_6|^3 dx \leq c \left\{ \int u_5^2 dx \right\}^{5/8} \left\{ \int u_7^2 dx \right\}^{7/8}.$$

Hence

$$\begin{aligned} \iint |u_6|^3 dx dt &\leq \left\{ \iint \left(\int u_5^2 dx \right)^5 dt \right\}^{1/8} \left\{ \iint u_7^2 dx dt \right\}^{7/8} \\ &\leq \varepsilon \iint u_7^2 dx dt + c_\varepsilon \iint \left(\int u_5^2 dx \right)^5 dt \end{aligned}$$

for any $\varepsilon > 0$. The idea is to subtract the ε -term from the ηu_7^2 and to estimate the c_ε term using a known bound of u_5 .

Taking account of the weight functions, we recall that $\eta \in W_{\sigma, K-2, 1}$ and $\xi \in W_{\sigma, K-1, 1}$. By assumption (11), $u_5 \in L^\infty(L^2(W_{0, K, 0}))$ and $u_6 \in L^2(L^2(W_{\sigma, K-1, 0}))$. In the appendix it is proven that

$$\int_2^\infty x^{K-1} |u_6|^3 dx \leq c \left\{ \int_1^\infty x^{K-2} (u_7^2 + u_6^2) dx \right\}^{7/8} \left\{ \int_1^\infty x^{(K+6)/5} u_5^2 dx \right\}^{5/8}.$$

Since $K \geq 3/2$, we may replace $(K+6)/5$ by the larger power K . Multiplying by t and integrating we get

$$\begin{aligned} \int_0^T t \int_2^\infty x^{K-1} |u_6|^3 dx dt \\ \leq \varepsilon \int_0^T \int_1^\infty t x^{K-2} (u_7^2 + u_6^2) dx dt + c_\varepsilon \int_0^T t \left\{ \int_1^\infty x^K u_5^2 dx \right\}^5 dt \end{aligned}$$

for any $\varepsilon > 0$. This implies the required estimate for positive x . As for $x \rightarrow -\infty$, a similar argument applied to $e^{\sigma x} v$ yields

$$\begin{aligned} \int_{-\infty}^{-1} e^{3\sigma x} |u_6|^3 dx &\leq c \left\{ \int_{-\infty}^{-1} e^{2\sigma x} u_5^2 dx \right\}^{5/8} \left\{ \int_{-\infty}^{-1} e^{2\sigma x} (u_7^2 + u_6^2 + u_5^2) dx \right\}^{7/8} \\ &\quad + c \left\{ \sup e^{\sigma x} |u_5| \right\} \left\{ \int_{-\infty}^{-1} e^{2\sigma x} u_5^2 dx \right\}. \end{aligned}$$

Multiplication by t and integration over $(0, T)$ yields an inequality similar to the previous ones but over the interval $(-\infty, -1)$ with exponential weight functions. This completes the estimate of the u_6^3 term.

The second induction step, $\alpha = 7$, leads to the identity (14) where

$$R = c_1 \xi u_4 u_6^2 u_7 + c_2 \xi u_6 u_7^2 + \text{regular terms.} \tag{49}$$

We integrate the first term by parts, using $3 u_6 u_7^2 = \partial(u_6^3)$. The terms ∂u_7^2 lead to some exceptional terms of the same types as we have discussed earlier. Therefore the only exceptional terms which remain are $u_6 u_7^2$ and

u_6^3 . In this step, $\eta \in W_{\sigma, K-3, 2}$ and $\xi \in W_{\sigma, K-2, 2}$ where $K \geq 2$. In case $K=2$, $\eta \in \bar{W}_{\sigma, -1, 2}$. We require bounds in terms of the norms of u_5 in $L^\infty(L^2(W_{0K0}))$, u_6 in $L^\infty(L^2(W_{\sigma, K-1, 0}))$, u_7 in $L^2(L^2(W_{\sigma, K-2, 1}))$ and u_8 in $L^2(L^2(W_{\sigma, K-3, 2}))$, the last one with a small coefficient.

The u_6^3 term is estimated (for $x > 1$) as in (35), namely

$$\int_0^T \int_1^\infty t^2 x^{K-2} |u_6|^3 dx dt \leq c_T (\sup |u_5|) \left(\iint x^K u_5^2 dx dt \right)^{1/4} \left(\iint t x^{K-2} u_7^2 dx dt \right)^{3/4}$$

as desired. The main term to be estimated has the form $\xi u_6 u_7^2$. Again we explicitly write only the case of positive x . By Hölder's inequality,

$$\int x^{K-2} |u_6 u_7^2| dx \leq \left(\int x^{K-1} |u_6|^3 dx \right)^{1/3} \left(\int x^{K-5/2} |u_7|^3 dx \right)^{2/3}. \tag{50}$$

The first factor in (50) is estimated by the appendix as

$$\left(\int_2^\infty x^{K-1} |u_6|^3 dx \right)^{1/3} \leq c \left\{ \int_1^\infty x^{K-2} (u_7^2 + u_6^2) dx \right\}^{7/24} \left\{ \int_1^\infty x^{(K+6)/5} u_5^2 dx \right\}^{5/24}.$$

Replacing $(K+6)/5$ by K as before, we get

$$\left(\int x^{K-1} |u_6|^3 dx \right)^{1/3} \leq \varepsilon \int x^{K-2} (u_7^2 + u_6^2) dx + c_\varepsilon \left\{ \int x^K u_5^2 dx \right\}^{5/17},$$

where we can choose $\varepsilon = 1$. The last factor in (50) is

$$\left(\int_2^\infty x^{K-5/2} |u_7|^3 dx \right)^{2/3} \leq c \left\{ \int_1^\infty x^{K-3} (u_8^2 + u_7^2) dx \right\}^{7/12} \left\{ \int_1^\infty x^{(K+1)/5} u_6^2 dx \right\}^{5/12}.$$

Replacing $(K+1)/5$ by the larger power $K-1$ since $K \geq 3/2$, we get

$$\left(\int x^{K-5/2} |u_7|^3 dx \right)^{2/3} \leq \varepsilon \int x^{K-3} (u_8^2 + u_7^2) dx + c_\varepsilon \int x^{K-1} u_6^2 dx.$$

This is placed into (50), the result is multiplied by t^2 and integrated over t . This completes the proof of Lemma 2.6.

Proof of Theorem 2.2. — By Theorems 4.5 and 4.7, $u \in L^2(H^6(W_{\sigma, K-1, 0}))$. Now the induction begins with $\alpha = 6$. The proof is similar to that of Theorem 2.1 but the space $H^7(W_{0L0})$ is replaced by $H^5(W_{0K0})$. Since $K \geq 2$ we can use Theorem 4.2 to guarantee that the

approximate solution $u^{(n)}(x, t)$ exists in a uniform time interval $[t_0, t_0 + \delta]$, that $u^{(n)}$ is uniformly bounded in $L^\infty(H^5(W_{0\kappa 0})) \cap L^2(H^6(W_{\sigma, \kappa-1, 0}))$, and that (22) holds for each n . This estimate, which is valid in the interval $[t_0, t_0 + \delta]$, is not uniform in n , however the bound (23) is valid for $\xi^{(n)}$ and η in the weight classes (37) because of Lemmas 2.5 and 2.6. Thus $u^{(n)}$ is a bounded sequence in

$$L^\infty(H^\alpha(W_{\alpha, \kappa-\alpha+5, \alpha-5})) \cap L^2(H^{\alpha+1}(W_{\alpha, \kappa-\alpha+4, \alpha-5})) \tag{51}$$

for $\alpha \geq 6$. Since $u^{(n)} \rightarrow u$ in $L^\infty(H^5(W_{0\kappa 0}))$ by Section 4, it follows that u belongs to the space (51).

3. EXISTENCE AND UNIQUENESS

In this section we prove the basic local-in-time existence and uniqueness result. This is used in Sections 2 and 4, but is also of independent interest. In Section 4 the condition for existence and uniqueness will be improved. First, we address the question of uniqueness:

THEOREM 3.1. (Uniqueness). — *Let $0 < T < \infty$. Assume f satisfies (A.1)-(A.4). Then for $\varphi \in H^7(\mathbb{R})$ there is at most one solution $u \in L^\infty([0, T]; H^7)$ of (2.1) with initial data φ .*

Proof. — Assume u, v are two solutions of (2.1) on $L^\infty([0, T], H^7(\mathbb{R}))$ with $\partial_t u, \partial_t v$ in $L^\infty([0, T], H^4(\mathbb{R}))$ and with the same initial data. Then

$$\begin{aligned} \partial_t(u-v) + f(u_3, u_2, u_1, u_0, x, t) - f(v_3, u_2, u_1, u_0, x, t) \\ + f(v_3, u_2, u_1, u_0, x, t) - f(v_3, v_2, u_1, u_0, x, t) \\ + f(v_3, v_2, u_1, u_0, x, t) - f(v_3, v_2, v_1, u_0, x, t) \\ + f(v_3, v_2, v_1, u_0, x, t) - f(v_3, v_2, v_1, v_0, x, t) = 0. \end{aligned} \tag{1}$$

By the mean value theorem there are smooth functions $d^{(j)} (0 \leq j \leq 3)$, depending smoothly on u_3, u_2, u_1, u_0, x, t and v_3, v_2, v_1, v_0 , such that (1) takes the form

$$\partial_t(u-v) + \sum_{j=0}^3 d^{(j)}(u_j - v_j) = 0. \tag{2}$$

Moreover, using (A.1) and (A.2) respectively we conclude by the mean value theorem, that $d^{(3)}(y, x, t) \geq c_1 > 0$ and $d^{(2)}(y, x, t) \leq 0$ for $0 \leq t \leq T, x \in \mathbb{R}, y \in \mathbb{R}^4$.

Multiplying (2) by $2\xi(u-v)$ and integrating in x where $\xi = 1/d^{(3)}$, we obtain by partial integration,

$$\begin{aligned} \partial_t \int \xi (u-v)^2 dx + \int (3 \partial [\xi d^{(3)}] - 2 \xi d^{(2)})(u_1 - v_1)^2 dx \\ \leq \int \xi (u-v)^2 \{ \partial_t \xi / \xi - 2 d^{(0)} + \partial [d^{(1)} \xi] / \xi \\ - \partial^2 [d^{(2)} \xi] / \xi + \partial^3 [d^{(3)} \xi] / \xi \} d\xi. \end{aligned} \quad (3)$$

Observe that all the partial integrations can be justified, as u and v are in $L^\infty([0, T], H^7(\mathbb{R}))$ and $\xi_r \in L^\infty([0, T]; H^1)$. Since $d^{(2)} \leq 0$ and $\partial [\xi d^{(3)}] = 0$ we conclude that, for a suitably chosen constant $c > 0$,

$$\partial_t \int \xi (u-v)^2 dx \leq c \int \xi (u-v)^2 dx.$$

By Gronwall's inequality and the fact that $u-v$ vanishes at $t=0$ it follows that $u \equiv v$. This proves uniqueness.

The next result deals with existence.

THEOREM 3.2. — *Assume f satisfies (A.1)-(A.4). Let N be an integer ≥ 7 and let $c_0 > 0$. Then there exists a time $0 < T < \infty$ depending only on c_0 such that for all $\varphi \in H^N(\mathbb{R})$ with $\|\varphi\|_{H^7} \leq c_0$ there exists a solution of (2.1), $u \in L^\infty([0, T]; H^N(\mathbb{R}))$ with $u(x, 0) = \varphi(x)$.*

The conclusion implies that $u \in L^\infty([0, T]; C^{N-1}(\mathbb{R}))$ and, by the equation, $\partial_t u \in L^\infty([0, T]; C^{N-4}(\mathbb{R}))$. The proof proceeds in several stages. First we differentiate the equation to make it quasilinear, and define a sequence of approximations as solutions of a linearized equation. Then a version of the estimate of Section 2 is obtained and is shown to be valid uniformly. Hence a passage to the limit succeeds.

We saw in Section 2 that differentiating the equation 6 times leads to

$$\begin{aligned} \partial_t u_6 + \partial_3 f u_9 + \{ 6 \partial [\partial_3 f] + \partial_2 f \} u_8 \\ + \left\{ \binom{6}{2} \partial^2 [\partial_3 f] + 6 \partial [\partial_2 f] + \partial_1 f \right\} u_7 + O(u_6, \dots) = 0. \end{aligned} \quad (4)$$

Upon substitution of $u = \Lambda v$ where $\Lambda = (I - \partial^6)^{-1}$ this equation takes the form

$$\begin{aligned} -\partial_t v + \partial_3 f \Lambda v_9 + \{ 6 \partial [\partial_3 f] + \partial_2 f \} \Lambda v_8 \\ + \{ 15 \partial^2 [\partial_3 f] + 6 \partial [\partial_2 f] + \partial_1 f \} \Lambda v_7 + O(\Lambda v_6, \dots) = 0 \end{aligned} \quad (5)$$

where v_j denotes $\partial^j v$ and $\partial_t u_6 = -\partial_t v - f$ is used. (5) is linearized by substituting a new variable w in each coefficient;

$$\begin{aligned}
 & -\partial_t v + (\partial_3 f)(\Lambda w_3, \dots) \Lambda v_9 \\
 & \quad + \{6 \partial[\partial_3 f](\Lambda w_3, \dots) + \partial_2 f(\Lambda w_3, \dots)\} \Lambda v_8 \\
 & \quad + \left\{ \binom{6}{2} \partial^2[\partial_3 f](\Lambda w_3, \dots) + 6 \partial[\partial_2 f](\Lambda w_3, \dots) \right. \\
 & \quad \left. + \partial_1 f(\Lambda w_3, \dots) \right\} \Lambda v_7 + O(\Lambda w_6, \dots) = 0. \quad (6)
 \end{aligned}$$

The main estimate is stated in the next lemma.

LEMMA 3.3. — *Let v, w be a pair of functions in $C^k([0, \infty), H^N(\mathbb{R}))$ for all k, N which satisfy (6). Define $\xi_\alpha \equiv [\partial_3 f(\Lambda w_3, \dots)]^{3+2\alpha/3}$. For each integer $\alpha \geq 0$ there exist positive, nondecreasing functions $g^{(\alpha)}, h^{(\alpha)}$ and $k^{(\alpha)}$ such that for all $t \geq 0$*

$$\begin{aligned}
 \partial_t \int \xi v_\alpha^2 dx \leq & (g^{(\alpha)}(\|w_1\|) + \|\partial_t \xi(\Lambda w_3, \dots)\|_1) \|v\|_\alpha^2 \\
 & + h^{(\alpha)}(\|w\|_1) \|w\|_\alpha^2 + k^{(\alpha)}(\|w\|_{\alpha-1}) \quad (7)
 \end{aligned}$$

where $\|w\|_\alpha$ is the norm in $H^\alpha(\mathbb{R})$.

Proof of Lemma 3.3. — Taking ∂^α of equation (6) for some $\alpha \geq 0$, we obtain an equation of the form

$$\begin{aligned}
 -\partial_t v_\alpha = & \partial_3 f \Lambda v_{9+\alpha} + \{(6+\alpha) \partial[\partial_3 f] + \partial_2 f\} \Lambda v_{8+\alpha} \\
 & + \sum_{7 \leq j \leq 7+\alpha} h^{(j)} \Lambda v_j + q(\Lambda w_6, \dots) \Lambda w_{6+\alpha} + p(\Lambda w_{5+\alpha}, \dots) \quad (8)
 \end{aligned}$$

where $h^{(j)}$ is a smooth function depending on $\Lambda w_{3+i}, \Lambda w_{2+i}, \dots$ with $i = 12 + \alpha - j$. For $\alpha \geq 3$, $p(\Lambda w_{5+\alpha}, \dots)$ depends at most linearly on $\Lambda w_{5+\alpha}$, while for $\alpha = 2$, $p(\Lambda w_{5+\alpha}, \dots)$ depends at most quadratically on $\Lambda w_{5+\alpha}$.

Following the basic estimate we multiply equation (8) by $2 \xi v_\alpha$ to obtain

$$\begin{aligned}
 \partial_t (\xi v_\alpha^2) - \partial_t \xi v_\alpha^2 = & 2 \xi \partial_3 f v_\alpha \Lambda v_{9+\alpha} \\
 & + 2 \xi \{(6+\alpha) \partial[\partial_3 f] + \partial_2 f\} v_\alpha \Lambda v_{8+\alpha} \\
 & + 2 \xi \sum_{7 \leq j \leq 7+\alpha} h^{(j)} v_\alpha \Lambda v_j + 2 \xi q \Lambda w_{6+\alpha} v_\alpha + 2 \xi p v_\alpha. \quad (9)
 \end{aligned}$$

The first term on the right side of (9) is, when integrated,

$$\begin{aligned}
 \int 2 \xi \partial_3 f \Lambda v_{9+\alpha} v_\alpha dx & = \int 2 \xi \partial_3 f \Lambda v_{9+\alpha} (\Lambda v_\alpha - \Lambda v_{6+\alpha}) dx \\
 & = \int -3 \partial[\xi \partial_3 f](\Lambda v_{7+\alpha})^2 dx + S_1
 \end{aligned}$$

where

$$S_1 = \int (\partial^3 [\xi \partial_3 f](\Lambda v_{6+\alpha})^2 - 2 \partial^3 [\xi \partial_3 f \cdot \Lambda v_\alpha] \Lambda v_{6+\alpha}) dx.$$

The second term on the right side of (9) is, when integrated,

$$\int 2\xi c_1 \Lambda v_{8+\alpha} (-\Lambda v_{6+\alpha} + \Lambda v_\alpha) dx = \int 2\xi c_2 (\Lambda v_{7+\alpha})^2 dx + S_2$$

where

$$S_2 = \int (2\partial^2 [\xi c_3 \Lambda v_\alpha] \Lambda v_{6+\alpha} - \partial [\xi c_4] (\Lambda v_{6+\alpha})^2) dx,$$

for some coefficients c_j which only depend upon lower derivatives of w .

The term in the sum in (9) with $j=7+\alpha$ is integrated by parts just once and the remaining terms are left as is. The resulting identity is

$$\begin{aligned} \partial_t \int \xi v_\alpha^2 dx = & \int (-\eta + 2\xi \partial_2 f) (\Lambda v_{7+\alpha})^2 dx \\ & + \int \partial_t \xi v_\alpha^2 dx + S_1 + S_2 + \int \partial [\xi h^{(7+\alpha)}] (\Lambda v_{6+\alpha})^2 dx \\ & - \int \partial [\xi h^{(7+\alpha)} \Lambda v_\alpha] \Lambda v_{6+\alpha} - \int \partial [2\xi h^{(6+\alpha)}] (\Lambda v_{6+\alpha})^2 dx \\ & + \int 2\xi h^{(6+\alpha)} \Lambda v_\alpha \Lambda v_{6+\alpha} dx + \sum_{7 \leq j \leq 7+\alpha} \int 2\xi h^{(j)} \Lambda v_j v_\alpha dx \\ & + \int 2\xi (q \Lambda w_{6+\alpha} + p) v_\alpha dx. \quad (10) \end{aligned}$$

The new weight function η is given by

$$\eta = 3 \partial [\xi \partial_3 f] - 2\xi (6+\alpha) \partial [\partial_3 f].$$

With our choice of $\xi = (\partial_3 f)^{3+2\alpha/3}$ one obtains $\eta \equiv 0$.

Furthermore, by assumption (A2), $\partial_2 f \leq 0$; thus the first term in the righthand side of (10) is nonpositive. By standard estimates, the lemma now follows. If $\alpha=2$, $h^{(2)}$ is chosen to be 0.

Proof of Theorem 3.2. — Using the same approximation procedure as in Section 2 it suffices to prove that for φ in $\bigcap_{k \geq 0} H^k(\mathbb{R})$, there exists a

solution u in $L^\infty([0, T]; H^N(\mathbb{R}))$ with initial data φ and with a time of existence $T > 0$ which only depends on $\|\varphi\|_{H^7}$. As a first step we prove that there exists a solution u in $L^\infty([0, T]; H^7(\mathbb{R}))$ of (2.1). We define a sequence of approximations to equation (5) as

$$\begin{aligned} \partial_t v^{(n)} = & \partial_3 f \Lambda v_9^{(n)} + \{6 \partial [\partial_3 f] + \partial_2 f\} \Lambda v_8^{(n)} \\ & + \left\{ \binom{6}{2} \partial^2 [\partial_3 f] + 6 \partial [\partial_2 f] + \partial_1 f \right\} \Lambda v_7^{(n)} + O(\Lambda v_6^{(n-1)}, \dots) \quad (11) \end{aligned}$$

where $f = f(\Lambda v_3^{(n-1)}, \dots)$ and where the initial condition is given by $v^{(n)}(x, 0) = \varphi(x) - \partial^6 \varphi(x)$. The first approximation is given by

$v^{(0)}(x, t) \equiv \varphi(x) - \partial^6 \varphi(x)$. Equation (11) is a linear equation at each iteration which can be solved in any interval of time in which the coefficients are defined. This is shown in Lemma 3.5 below. By Lemma 3.3,

$$\partial_t \int_{-\infty}^{\infty} \xi_{\alpha}^{(n-1)} (v_{\alpha}^{(n)})^2 dx \leq (g^{(\alpha)}(\|v^{(n-1)}\|) + \|\partial_t \xi_{\alpha}^{(n-1)}\|) \|v^{(n)}\|_{\alpha}^2 + h^{(\alpha)}(\|v^{(n-1)}\|_1) \|v^{(n-1)}\|_{\alpha}^2 + k^{(\alpha)}(\|v^{(n-1)}\|_{\alpha-1}) \quad (12)$$

where $\xi_{\alpha}^{(n)} = \partial_3 f(\Lambda v_3^{(n)}, \dots)^{3+2\alpha/3}$.

By (A.1) we have $\xi_{\alpha}^{(n-1)}(x, t) \geq c_1^{3+2\alpha/3} > 0$, for $x \in \mathbb{R}$, $0 \leq t \leq T$. Observe that $\xi_{\alpha}^{(n-1)}(x, 0) = [\partial_3 f(\varphi_3, \dots)]^{3+2\alpha/3}$ is independent of n and $[\partial_3 f(\varphi_3, \dots)]^{3+2\alpha/3} \leq c_2^{3+2\alpha/3} < \infty$ where c_2 depends only on $\|\varphi\|_4$ and is, in particular, independent of α .

First choose $\alpha = 1$ and let $c_0 \geq \|\varphi - \partial^6 \varphi\|_7 \geq \|\varphi\|_7$. Observe that for each iterate $v^{(n)}$, $\|v^{(n)}(\cdot, t)\|$ is continuous in $t \in [0, T]$ and $\|v^{(n)}(\cdot, 0)\|_1 \leq c_0$.

Define $c_3 \equiv 1 + c_0 \left(\frac{c_2}{c_1}\right)^2$ and let $T^{(n)}$ be the maximum time such that $\|v^{(k)}(\cdot, t)\|_1 \leq c_3$ for $0 \leq t \leq T^{(n)}$, $0 \leq k \leq n$. By integrating (12) over the time interval $[0, t]$ we obtain for $0 \leq t \leq T^{(n)}$ and $j = 0, 1, \dots$,

$$c_1^{3+2j/3} \|v_j^{(n)}(\cdot, t)\|^2 \leq c_0^2 c_2^{3+2j/3} + g^{(j)}(c_3) c_3^2 t + l^{(j)}(c_3) c_3^2 t + h^{(j)}(c_3) c_3^2 t + k^{(j)}(c_3) t$$

where $\|\partial_t \xi_{\alpha}^{(n-1)}\|_1 \leq l^{(j)}(c_3)$.

Choosing T sufficiently small, depending on c_0, c_1 and c_2 but *not* on n , one concludes that

$$\|v^{(n)}(\cdot, t)\|_1 \leq 1 + c_0 \left(\frac{c_2}{c_1}\right) = c_3 \quad (0 \leq t \leq T). \quad (13)$$

This shows that $T^{(n)} \geq T$. Estimate (13) implies that there exists a subsequence, still denoted by $v^{(n)}$, such that $v^{(n)} \rightharpoonup v$ weak* in $L^{\infty}([0, T]; H^1(\mathbb{R}))$. We claim that $u = \Lambda v$ is the solution we are looking for. By equation (11), $\partial_t v^{(n)}$ is a sum of terms each of which is the product of a coefficient, bounded uniformly in n , and a function in $L^2([0, T]; H^{-2}(\mathbb{R}))$ bounded uniformly in n , so that the sequence $\partial_t v^{(n)}$ is bounded $L^2([0, T]; H^{-2}(\mathbb{R}))$. By Aubin's compactness theorem, there is a subsequence such that $v^{(n)} \rightarrow v$ strongly in $L^2([0, T]; H_{loc}^{1/2}(\mathbb{R}))$. Hence, for a subsequence, $v^{(n)} \rightarrow v$ a.e. in x and t . It follows that $\partial_3 f(\Lambda v_3^{(n-1)}, \dots) \rightarrow \partial_3 f(\Lambda v_3, \dots)$ strongly in $L^2([0, T]; H_{loc}^3(\mathbb{R}))$. Thus the first term on the right hand side of (11), $\partial_3 f(\Lambda v_3^{(n-1)}, \dots) \Lambda v_9^{(n)}$ converges in $L^2([0, T]; L_{loc}^1(\mathbb{R}))$ to $\partial_3 f(\Lambda v_3, \dots) \Lambda v_9$ as $\Lambda v_9^{(n)} \rightharpoonup \Lambda v_9$ weakly in $L^2([0, T]; H^{-2}(\mathbb{R}))$ and $\partial_3 f(\Lambda v_3^{(n-1)}, \dots) \rightarrow \partial_3 f(\Lambda v_3, \dots)$ strongly in $L^2([0, T]; H_{loc}^3(\mathbb{R}))$. Similarly all other terms in (11) converge to their correct limits, implying $\partial_t v^{(n)} \rightarrow \partial_t v$ in $L^2([0, T]; L_{loc}^1)$, and $\partial_t v + (I - \partial^6) f(\Lambda v_3, \dots) = 0$. Applying Λ to both sides of this equation we find that equation (2.1) is satisfied by $u = \Lambda v$.

As a second step we prove that there exists a solution u in $L^\infty([0, T]; H^N(\mathbb{R}))$ of (2.1) with $N \geq 8$, where T depends only on $\|\varphi\|_7$. We already know that there is a solution u in $L^\infty([0, T]; H^7(\mathbb{R}))$. It suffices to prove that the approximating sequence $v^{(n)}$ is bounded in $L^\infty([0, T]; H^{N-6}(\mathbb{R}))$. Take $\alpha = N - 6$, and consider (12) for $\alpha \geq 2$. By the same arguments as for $\alpha = 1$ we conclude that there exists $T^{(\alpha)} > 0$, depending on $\|\varphi\|_\alpha$, c_1 and c_2 , but independent of n such that $\|v^{(n)}(\cdot, t)\|_\alpha \leq 1 + \|\varphi\|_\alpha \left(\frac{c_2}{c_1}\right)^{(3+2\alpha/3)/2}$ for $0 \leq t \leq T^{(\alpha)}$. Thus v is in $L^\infty([0, T^{(\alpha)}]; H^\alpha(\mathbb{R}))$. Now denote by $0 \leq T^{*(\alpha)} \leq \infty$ the maximal number such for all $0 < T < T^{*(\alpha)}$, $u = \Lambda v$ is in $L^\infty([0, T]; H^N(\mathbb{R}))$. We claim that $T^{(1)} \leq T^{*(\alpha)}$ for all $\alpha \geq 2$. Thus a time of existence T can be chosen depending only on $\|\varphi\|_7$.

As a consequence of Theorem 3.1 and Theorem 3.2 and its proof one obtains the following

COROLLARY 3.4. — *Let $\varphi \in H^N(\mathbb{R})$ with $N \geq 7$ and $\varphi^{(y)}$ be a sequence converging to φ in $H^N(\mathbb{R})$. Let u and $u^{(y)}$ be the corresponding unique solutions given by Theorems 3.1 and 3.2 in $L^\infty([0, T]; H^N(\mathbb{R}))$ with T depending only on $\sup\|\varphi^{(y)}\|_{H^7}$. Then $u^{(y)} \rightharpoonup u$ weak* in $L^\infty([0, T]; H^N(\mathbb{R}))$ and strongly in $L^2([0, T]; H^{N-1}(\mathbb{R}))$.*

The linear equation (11) which is to be solved at each iteration has the form

$$\partial_t v = b^{(3)} \wedge v_9 + b^{(2)} \wedge v_8 + b^{(1)} \wedge v_7 + b^{(0)} \tag{14}$$

with smooth bounded coefficients which satisfy

$$b^{(3)} = b^{(3)}(x, t) \geq c_1 > 0 \quad \text{and} \quad b^{(2)} = b^{(2)}(x, t) \leq 6 \partial [b^{(3)}].$$

LEMMA 3.5. — *Given initial data in $\bigcap_{N \geq 0} H^N(\mathbb{R})$ there exists a unique solution of (14). The solution is defined in any time interval in which the coefficients are defined.*

Proof. — The proof is standard so we only sketch it. We fix an arbitrary time $T > 0$ and a constant $M > 0$. Let

$$\mathcal{L} = \xi (\partial_t - b^{(3)} \wedge \partial^9 - b^{(2)} \wedge \partial^8 - b^{(1)} \wedge \partial^7)$$

where $\xi = (b^{(3)})^3$. Introduce the bilinear form

$$\langle g, h \rangle = \int_0^T \int e^{-Mt} gh \, dx \, dt \tag{15}$$

defined on $C_0^\infty(\mathbb{R} \times [0, T])$, the set of smooth functions with compact support in \mathbb{R} , which vanish for $t = 0$.

Our estimates from Lemma 3.3 show that

$$\int \mathcal{L} v \cdot v \, dx \geq \partial_t \int \xi v^2 \, dx - c \int \xi v^2 \, dx. \tag{16}$$

Multiply by e^{-Mt} and integrate in time to obtain for $v \in C_0^\infty(\mathbb{R} \times [0, T])$ with $v(x, 0) \equiv 0$,

$$\langle \mathcal{L} v, v \rangle \geq e^{-MT} \int \xi(x, T) v^2(x, T) \, dx + (M - c) \int_0^T \int e^{-Mt} \xi v^2 \, dx \tag{17}$$

Thus $\langle \mathcal{L} v, v \rangle \geq \langle v, v \rangle$ provided M is chosen large enough.

Similarly $\langle \mathcal{L}^* w, w \rangle \geq \langle w, w \rangle$ for all $w \in C_0^\infty(\mathbb{R} \times [0, T])$ with $w(x, T) \equiv 0$ where \mathcal{L}^* denotes the formal adjoint of \mathcal{L} . Therefore $\langle \mathcal{L}^* w, \mathcal{L}^* v \rangle$ is an inner product on $\mathcal{D} = \{w \in C_0^\infty : w(x, T) \equiv 0\}$. Denote by X the completion of \mathcal{D} with respect to this inner product. By the Riesz representation theorem, there exists a unique solution $V \in X$, such that for any $w \in \mathcal{D}$, $\langle \mathcal{L}^* V, \mathcal{L}^* w \rangle = \langle \xi b^{(0)}, w \rangle$, where we used that $\xi b^{(0)}$ is in X . Then $v = \mathcal{L}^* V$ is a weak solution of $\mathcal{L} v = \xi f$, with $v \in L^2(\mathbb{R} \times [0, T])$. To obtain higher regularity of the solution, we repeat the proof with higher derivatives included in the inner product. It is a standard approximation procedure to obtain a result for general initial data.

4. WEIGHTED SOBOLEV ESTIMATES

In this section we develop a series of estimates for solutions of equation (2.1) in weighted Sobolev norms. These provide both a starting point for the *a priori* gain of regularity results that are discussed in Section 2, and they lead to an improved existence and uniqueness result, with fewer demands upon the regularity of the initial data φ than in Section 3. The existence of these weighted estimates is often called the “persistence” of a property of the initial data. We show that if $\varphi \in H^L(W_{0,i,0}) \cap H^7(\mathbb{R})$, for $L \geq 0, i \geq 0$, then the solution $u(\cdot, t)$ evolves in $H^L(W_{0,i,0})$ for $t \in [0, T]$. The time interval of such persistence is at least as long as the interval guaranteed by the Existence Theorem 3.2, and depends only upon $\|\varphi\|_{H^7}$. Using these weighted norms, we give an improved theory of existence and uniqueness, for initial data $\varphi \in H^5(W_{0,2,0})$. The guaranteed time of existence is estimated in Theorem 4.5; it depends only on a norm of φ in $H^5(W_{0,2,0})$. Additionally, with weights greater than $i=2$, the persistence property is shown to start with $L \geq 5$.

THEOREM 4.1. – *Let i and L be non-negative integers and $0 < T < \infty$. Assume that u is the solution to (2.1) in $L^\infty([0, T]; H^7(\mathbb{R}))$ with initial data*

$\varphi(x) = u(x, 0)$ in H^7 . If, in addition, φ is in $H^L(W_{0i0})$ then

$$u \in L^\infty([0, T]; H^7 \cap H^L(W_{0i0})) \tag{1}$$

$$\int_0^T \int |\partial^{L+1} u(x, t)|^2 \eta \, dx \, dt < \infty \tag{2}$$

where $\sigma > 0$ is arbitrary, and η is a weight function in $W_{\sigma, i-1, 0}$ for $i \geq 1$, and in $\bar{W}_{\sigma, -1, 0}$ for $i = 0$.

Remark. — In case $i = 0$, the conclusion (2) can be viewed as a generalization of the local gain of regularity results as established by Kato [Ka], Constantin and Saut [CS] and Ponce [Po] in various special settings.

Proof. — We prove this result by induction on α ,

$$u \in L^\infty([0, T]; H^7 \cap H^\alpha(W_{0i0})) \quad \text{for } 0 \leq \alpha \leq L.$$

As in the proofs of Theorems 2.1 and 2.2 we first derive formally some *a priori* estimates for the solution where the bounds involve only the norms of u in $L^\infty([0, T], H^7(\mathbb{R}))$ and of φ in $H^7(W_{0i0})$. One has to justify these estimates; we do it by approximating $u(x, t)$ by smooth solutions and weight functions by smooth bounded functions.

According to the existence result in Section 3 the solution $u(x, t)$ evolves in $L^\infty([0, T]; H^N)$ with $N = \max(L, 7)$. In particular, $\partial^j u(x, t)$ is in $L^\infty(\mathbb{R} \times [0, T])$ for $0 \leq j \leq N - 1$.

To obtain estimates (1) and (2) we use a procedure similar to those presented in Section 2. In the rigorous derivation of these estimates there are two approximations performed; we approximate general solutions by smooth solutions, and we approximate general weight functions by bounded weight functions. The first of these procedures has already been discussed in Section 3, so we will concentrate on the second.

Given a smooth weight function $\eta(x) \in W_{\sigma, i-1, 0}$ with $\sigma > 0$, we take a sequence $\eta_\delta(x)$ of smooth bounded weight functions approximating $\eta(x)$ from below, uniformly on any half line $(-\infty, c)$. Define the weight functions for the α -th induction step

$$\xi_{\alpha\delta} = \frac{1}{3} a^{2\alpha/3-1} \left(1 + \int_{-\infty}^x a^{-2\alpha/3}(y) \eta_\delta(y) \, dy \right)$$

where $a = \partial_3 f$. Then $\xi_{\alpha\delta}$ are bounded weight functions which approximate a desired weight function $\xi_\alpha \in W_{0i0}$ from below, uniformly on compact sets. For $\alpha = 0$, ξ_0 and $\xi_{0\delta}$ are defined slightly differently;

$$\xi_{0\delta} = \frac{1}{3g_3} \left(1 + \int_{-\infty}^x \eta_\delta \, dy \right)$$

where g_3 is defined in (2.2). These weight functions are designed to satisfy the usual relations;

$$\left. \begin{aligned} 0 < \eta_\delta \leq 3 \partial (\xi_{\alpha\delta} a) - 2 \alpha \xi_{\alpha\delta} \partial a \\ 0 < \eta_\delta \leq 3 \partial (\xi_{0\delta} g_3) \end{aligned} \right\} \tag{3}$$

for $\alpha \geq 1$ and $\alpha = 0$ respectively.

The first induction step is to obtain a weighted estimate for $\alpha = 0$. Multiply equation (2.1) by $2u \xi_{0\delta}$ and integrate, to obtain

$$\partial_t \int \xi_{0\delta} u^2 dx + 2 \int \xi_{0\delta} u f dx - \int \partial_t \xi_{0\delta} u^2 dx = 0. \tag{4}$$

Using the form (2.2) of the nonlinear term,

$$\int \xi_{0\delta} u f dx = \sum_{j=0}^3 \int \xi_{0\delta} g_j u_j u dx + \int \xi_{0\delta} h u dx.$$

We write $\xi_\delta = \xi_{0\delta}$ and note that $|\partial^3 \xi_\delta| \leq c \xi_\delta$. Using (3) with $\alpha = 0$, $0 < \eta_\delta \leq 3 \partial [\xi_\delta g_3]$, we find that

$$2 \int \xi_\delta g_3 u_3 u dx = 3 \int \partial [\xi_\delta g_3] u_1^2 dx - \int \partial^3 [\xi_\delta g_3] u^2 dx \geq -c \int \xi_\delta u^2 dx \tag{5}$$

where c depends only on $\|\varphi\|_{H^7}$ and, in particular, is independent of δ . Similarly we obtain

$$\sum_{j=0}^2 \int \xi_\delta g_j u_j u dx \geq -c \int \xi_\delta u^2 dx \tag{6}$$

and

$$\int \xi_\delta h u dx \geq -c' \left(\int \xi_\delta^2 h^2 \right)^{1/2} dx \geq -c. \tag{7}$$

Combining (5)-(7) with (4) yields

$$\partial_t \int \xi_\delta u^2 dx \leq c \int \xi_\delta u^2 dx + c.$$

One applies Gronwall's lemma to conclude

$$\int \xi_\delta u^2 dx \leq c \int \xi_\delta \varphi^2 dx + c \tag{8}$$

for $0 \leq t \leq T$. As c does not depend on $\delta > 0$, the weighted estimate (8) remains true in the limit for $\delta \rightarrow 0$.

To prove the α -th induction step we start from formula (2.14) with η and ξ given by η_δ and $\xi_{\alpha,\delta} = \xi_\delta$.

$$\partial_t \int \xi_\delta u_\alpha^2 dx + \int (\eta_\delta - 2 \xi_\delta \partial_2 f) u_{\alpha+1}^2 dx + \int \mathfrak{G} u_\alpha^2 dx + \int R dx = 0. \tag{9}$$

The expressions \mathfrak{Q} and \mathbf{R} are given as in Section 2.

By (3) and (A.2) the second term in (9) is non-negative. Moreover it follows from (2.16) that \mathfrak{Q}/ξ_δ is a function bounded in x and t by a constant c , which depends only on $\|\varphi\|_{H^7}$ and T and which is in particular independent of δ . Thus (9) yields

$$\partial_t \int \xi_\delta u_\alpha^2 dx + \int \eta_\delta u_{\alpha+1}^2 dx \leq c \int \xi_\delta u_\alpha^2 dx + \int |\mathbf{R}| dx.$$

To use Gronwall's lemma to conclude the induction step it thus suffices to prove

$$\int |\mathbf{R}| dx \leq c \left(\int \xi_\delta u_\alpha^2 dx + 1 \right)$$

where the constant c depends upon T , $\|\varphi\|_{H^7}$ and the norm of φ in $H^{\alpha-1}(W_{0i0})$ only. According to (2.25), $\int \mathbf{R}$ contains terms of the form

$$\int \xi_\delta (\partial_3^{q_3} \partial_2^{q_2} \partial_1^{q_1} \partial_0^{q_0} \partial_x^v f) u_{v_1} \dots u_{v_p} u_\alpha dx. \quad (10)$$

If $v_p \leq \alpha - 2$ we perform one integration by parts, and then use the induction hypothesis to bound the resulting terms by the quantity:

$$c \left[\left(\int \xi_\delta u_{v_p+1}^2 dx \right)^{1/2} + \left(\int \xi_\delta u_{v_p}^2 dx \right)^{1/2} \right] \left(\int \xi_\delta u_{\alpha-1}^2 dx \right)^{1/2}. \quad (11)$$

The expression (11) is bounded in the previous induction steps. Suppose that $\alpha - 1 = v_p = v_{p-1+1} > v_{p-1}$, then the term (10) has the estimate

$$\left| \int b \xi_\delta u_{\alpha-1}^l u_\alpha dx \right| \leq \|b u_{\alpha-1}^{l-1}\|_\infty \left(\int u_{\alpha-1}^2 \xi_\delta dx \right)^{1/2} \left(\int u_\alpha^2 \xi_\delta dx \right)^{1/2}.$$

As discussed in Section 2, the multiplicity $l \geq 2$ only when $\alpha \leq 8$. Thus for $\alpha > 8$ a differential inequality of the necessary form is obtained. Furthermore, for $\alpha \leq 7$, $\|u_{\alpha-1}\|_\infty$ is bounded by hypothesis, and again the estimate is complete. Finally, for $\alpha = 8$ we use the Sobolev lemma

$$\begin{aligned} \left| \int b \xi_\delta u_7^2 u_8 dx \right| &\leq \|b u_7 \xi_\delta\|_\infty \|u_7\|_{L^2} \left(\int u_8^2 \xi_\delta dx \right)^{1/2} \\ &\leq \|b\|_\infty \|u_7\|_{L^2} \left(\int u_8^2 \xi_\delta dx \right). \end{aligned}$$

Using these estimates in (9), and applying the Gronwall argument, we obtain for any $0 \leq t \leq T$

$$\int \xi_\delta(t) u_\alpha^2(t) dx + \int_0^t \int \eta_\delta u_{\alpha+1}^2 dx dt \leq c_1 \exp(c_2 t) \left(\int \varphi_\alpha^2 \xi_\delta dx + 1 \right). \quad (12)$$

The constants are independent of δ so that letting the parameter $\delta \rightarrow 0$ the desired estimates (1), (2) are obtained. \square

For the next two theorems we need to take a closer look at the existence results of Section 3. We first prove the following two lemmas.

LEMMA 4.2. — *Let $0 < T < \infty$ and $i = 7/4$.*

Assume that $u \in L^\infty([0, T]; H^7(W_{0i0}))$ is a solution of (2.1) with initial data $\varphi \in H^7(W_{0i0})$. Then for $0 \leq t \leq T$, $\alpha = 6$ or 7 and $\xi \in W_{0i0}$

$$\int u_\alpha^2 \xi \, dx \leq c \left(\int \varphi_\alpha^2 \xi \, dx + 1 \right) \tag{13}$$

where c depends only on the norm of u in $L^\infty([0, T]; H^5(W_{0i0}))$.

Proof. — The proofs for $\alpha = 6$ and 7 are very similar and thus we treat the case $\alpha = 6$ only. The point is to derive differential inequalities leading to estimates for (13) which are at most linear in $\int u_\alpha^2 \xi \, dx$. As in the proofs of Theorem 2.1 and 2.2 we first derive formally the *a priori* estimate. In a by now familiar way one then justifies these estimates, approximating $u(x, t)$ by a sequence of solution in $H^N(W_{0i0})$ with N sufficiently large.

Choose a smooth weight function $\eta(x)$ in $W_{\sigma, i-1, 0}$, for $\sigma > 0$ arbitrary, with $\partial^j \eta / \eta$ bounded for all j , and define ξ as earlier by

$$\xi = \frac{1}{3} a^{-1+2\alpha/3} \left(1 + \int_{-\infty}^x a^{-2\alpha/3} \eta \, dx \right)$$

where $a = \partial_3 f$ and $\alpha = 6$. Then $t \partial_t \xi / \xi$ and $\partial^j \xi / \xi$ are bounded for $0 \leq j \leq 3$.

Again we start from formula (2.14);

$$\partial_t \int \xi u_6^2 \, dx + \int (\eta - 2\xi \partial_2 f) u_7^2 \, dx + \int \mathfrak{G} u_6^2 \, dx + \int R \, dx = 0 \tag{14}$$

where \mathfrak{G} and R are described in Section 2. From (14) we will derive a differential inequality of the form

$$\partial_t \int \xi u_6^2 \, dx \leq c \left(\int \xi u_6^2 \, dx + 1 \right) \tag{15}$$

from which (13) follows by applying Gronwall's lemma.

To obtain (15), an estimate of $\int \mathfrak{G} u_6^2 \, dx$ and $\int R \, dx$ is obtained, which uses the presence of the term $\int (\eta - 2\xi \partial_2 f) u_7^2 \, dx$ in a crucial way. The analysis of $\mathfrak{G} u_6^2$ in Section 2 shows that $\int \mathfrak{G} u_6^2 \, dx$ can be brought into the

form

$$\int b_1 \xi u_6^3 dx + \int b_2 \xi u_5 u_6^2 dx + \int b_3 \xi u_6^2 dx$$

where the coefficients b_j depend on $u, u_1, u_2, u_3, u_4, x, t$ only and are thus easily estimated. The analysis of R in Section 2 shows that $\int R$ can be brought into the form of an integral of

$$b_4 \xi u_6^3 + b_5 \xi u_6^2 + b_6 \xi u_5^4 + b_7 \xi u_5^3 + b_8 \xi.$$

It thus suffices to estimate the terms $\int \xi |u_6|^3 dx, \int \xi |u_5| u_6^2 dx, \int \xi u_5^4 dx$ and $\int \xi |u_5|^3 dx$. To accomplish this we use interpolation estimates as in Lemma 2.6. For $\xi_0 \in W_{0 i_0 0}$ with $i_0 = (i + 7)/5$.

$$\begin{aligned} \int \xi |u_6|^3 dx &\leq \left(\int (u_6^2 + u_7^2) \eta dx \right)^{7/8} \left(\int u_5^2 \xi_0 dx \right)^{5/8} \\ &\leq \varepsilon \int u_7^2 \eta dx + \int u_6^2 \xi dx + c(\varepsilon) \left(\int u_5^2 \xi_0 dx \right)^5 \end{aligned} \tag{16}$$

for a certain constant c depending on ε . Since $i \geq 7/4, i_0 \leq i$ and thus $H^7(W_{0 i_0 0}) \subseteq H^7(W_{0 i 0})$, Hölder's inequality gives

$$\int \xi |u_5| u_6^2 dx \leq \left(\int |u_5|^3 \xi dx \right)^{1/3} \left(\int |u_6|^3 \xi dx \right)^{2/3},$$

thus the term $\int \xi |u_5| u_6^2 dx$ is also estimated using (16). Next

$$\int \xi u_5^4 dx \leq \|u_5\|_{L^\infty}^2 \int \xi u_5^2 dx \leq c \left(\int u_5^2 dx + \int u_6^2 \xi dx \right) \int \xi u_5^2 dx.$$

The term $\int \xi |u_5|^3 dx$ is estimated similarly. This completes the proof of (15), and the result follows.

LEMMA 4.3. — *Let $0 < T < \infty$. There exists a strictly increasing positive smooth function $H(s)$ such that if u is a solution of (2.1) in $L^\infty([0, T]; H^7(W_{0 i_0}))$ for some $i \geq 1/2$, then, for $0 \leq t \leq T$ and appropriate weight functions ξ_α in $W_{0 i_0}, 0 \leq \alpha \leq 5$, we have*

$$\partial_t \left(\sum_{\alpha=0}^5 \int u_\alpha^2 \xi_\alpha dx \right) \leq H \left(\sum_{\alpha=0}^5 \int u_\alpha^2 \xi_\alpha dx \right). \tag{17}$$

Proof. — Choose a smooth weight function $\eta(x)$ in $W_{\sigma, i-1, 0}$ ($\sigma > 0$ arbitrary) such that $\partial^j \eta / \eta$ is bounded for $j \geq 1$. For $1 \leq \alpha \leq 5$ define as

before

$$\xi_\alpha = \frac{1}{3} a^{-1+2\alpha/3} \left(1 + \int_{-\infty}^x a^{-2\alpha/3} \eta \, dx \right).$$

In case $\alpha=0$ we define $\xi_0 = \frac{1}{3} g_3^{-1} \left(1 + \int_{-\infty}^x \eta \, dx \right)$ where g_3 is given by (2.2).

First, consider $\partial_t \int u^2 \xi_0 \, dx$. As in (4) we have

$$\partial_t \int \xi_0 u^2 \, dx + 2 \int \xi_0 u f \, dx - \int \partial_t [\xi_0] u^2 = 0. \tag{18}$$

Now by (2.2),

$$\int \xi_0 u f \, dx = \sum_{j=0}^3 \int \xi_0 g_j u_j u \, dx + \int \xi_0 h u \, dx.$$

Moreover

$$\begin{aligned} \partial_t [\xi_0] &= -\frac{1}{3} g_3^{-2} \partial_t [g_3] \left(1 + \int_{-\infty}^x \eta \, dx \right) \\ &= -\frac{1}{3} g_3^{-2} \left(\sum_{j=0}^3 \partial_j g_3 \partial_t u_j + \partial_t g_3 \right) \left(1 + \int_{-\infty}^x \eta \, dx \right). \end{aligned}$$

This implies by a number of integrations by parts that

$$\left| 2 \int \xi_0 u f \, dx - \int (\xi_0)_t u^2 \, dx \right| \leq c \sum_{\alpha=0}^5 \int \xi_\alpha u_\alpha^2 \, dx$$

where c depends only on the norm of u in $L^\infty([0, T], H^5)$. For $1 \leq \alpha \leq 5$ we start again from formula (2.14).

$$\partial_t \int \xi_\alpha u_\alpha^2 \, dx + \int (\eta - 2 \xi_\alpha \partial_2 f) u_{\alpha+1}^2 \, dx + \int \vartheta_\alpha u_\alpha^2 \, dx + \int R_\alpha \, dx = 0 \tag{19}$$

where we have written ϑ_α and R_α instead of ϑ and R for reasons of clarity. The analysis of $\int \vartheta_\alpha u_\alpha^2 \, d\xi$ in Section 2 (cf. 2.16) shows that $\int \vartheta_\alpha u_\alpha^2 \, dx$ ($1 \leq \alpha \leq 5$) can be brought into the form

$$\int \xi_\alpha b_1 u_6 u_\alpha^2 \, dx + \int \xi_\alpha b_2 u_5 u_\alpha^2 \, dx + \int b_3 \xi_\alpha u_\alpha^2 \, dx \tag{20}$$

where the coefficients b_j indicate functions which depend on $u, u_1, u_2, u_3, u_4, x, t$ only and are thus easily estimated by the norm of u in

$L^\infty([0, T]; H^5(\mathbb{R}))$. It suffices thus to estimate $\int \xi_\alpha |u_6| u_\alpha^2 dx$ and $\int \xi_\alpha |u_5| u_\alpha^2 dx$. For $1 \leq \alpha \leq 4$ this is done a straightforward way:

$$\int \xi_\alpha |u_5| u_\alpha^2 dx \leq \|u_\alpha\|_{L^\infty} \left(\int \xi_\alpha u_5^2 dx \right)^{1/2} \left(\int \xi_\alpha u_\alpha^2 dx \right)^{1/2} \leq c \left(\int \xi_5 u_5^2 dx + \int \xi_\alpha u_\alpha^2 dx \right)$$

and

$$\left| \int \xi_\alpha u_6 u_\alpha^2 dx \right| \leq \left| \int \partial [\xi_\alpha u_\alpha^2] u_5 dx \right| \leq c \sum_{j=1}^5 \int \xi_j u_j^2 d\xi.$$

For $\alpha = 5$ we first observe that $\int \xi_5 b_1 u_6 u_5^2 dx$ can be written as $-\frac{1}{3} \int \partial [\xi_5 b_1] u_5^3 dx$ and thus it remains to estimate $\int \xi_5 |u_5|^3 d\xi$. For this purpose we use an interpolation inequality, discussed in the Appendix:

For $0 \leq t \leq T$

$$\int \xi_5 |u_5|^3 dx \leq c \left(\left(\int u_6^2 \eta dx \right)^{1/4} + \left(\int u_5^2 \xi_5 dx \right)^{1/4} \right) \left(\int u_5^2 \xi dx \right)^{5/4} \leq \varepsilon \int u_6^2 \eta dx + c(\varepsilon) \left(\int u_5^2 \xi dx \right)^{2/3} \left(\int u_5^2 \xi_5 dx \right)$$

where ξ is in $W_{0 i_0 0}$ with $i_0 = (3i + 1)/5$.

As for the error term $\int R_\alpha$ we learn from Section 2 that it contains terms of the form (2.25).

$$\int \xi_\alpha \partial_3^{p_3} \partial_2^{p_2} \partial_1^{p_1} \partial_0^{p_0} \partial_x^\nu f u_{v_1} \dots u_{v_p} u_\alpha dx \tag{21}$$

with $1 \leq v_1 \leq \dots \leq v_p \leq \alpha$. This can be easily estimated if $1 \leq \alpha \leq 4$. For $\alpha = 5$, $v_p = 5$ we must have $v_{p-2} < 5$.

Indeed, by an analogue of (2.28) one obtains

$$p - 12 + v_{p-2} + v_{p-1} + v_p \leq \alpha$$

where $p = p_0 + p_1 + p_2 + p_3$. If $v_{p-2} = v_{p-1} = v_p = 5$, this is contradicted.

Thus choosing $\varepsilon = \frac{1}{2}, \sum_{\alpha=1}^5 \int \vartheta_{\alpha} u_{\alpha}^2 dx + \int R_{\alpha} dx$ is estimated by

$$C \left(\sum_{\alpha=0}^5 \int \xi_{\alpha} u_{\alpha}^2 dx + \left(\int \xi_5 u_5^2 dx \right)^{2/3} \int \xi_5 u_5^2 dx \right) + \frac{1}{2} \int \eta u_6^2 d\xi \tag{22}$$

where C does depend only on the norm of u in $L^{\infty}([0, T]; H^5)$. Inequality (17) now follows from (18), (19) and (22), together with the fact that for $i \geq \frac{1}{2}, \xi \in W_{0 i_0 0}$ with $i_0 = (3i + 1)/5 \leq i$.

As a consequence of Lemmas 4.2 and 4.3 we obtain

COROLLARY 4.4. — *Let $\varphi \in H^7(W_{0 i_0 0})$ with $i \geq 7/4$. Let u be the solution of (2.1) with initial data φ . Denote by $0 < T^* \leq \infty$ the life span of this solution, in the space $H^7(W_{0 i_0 0})$. Then there exists $0 < T' \leq T^*$, depending only on the norm of φ in $H^5(W_{0 7/4 0})$ such that*

$$\sup_{0 \leq t \leq T'} \int (\xi_6(x, t) u_6^2 + \xi_7(x, t) u_7^2) dx \leq C \int (1 + \xi_6(x, 0) \varphi_6^2 + \xi_7(x, 0) \varphi_7^2) dx \tag{23}$$

where C depends only on the norm of φ in $H^5(W_{0 i_0 0})$.

Proof. — In view of Lemma 4.2, only the existence of the uniform time T' is left to prove. This is done by induction on i , the strength of the weight class. As in the proof of Theorems 2.1 and 2.2 we approximate $u(x, t)$ by smoother solutions, here by a sequence $u^{(n)}(x, t)$ in $L^{\infty}([0, T]; H^7(W_{0 i_0 0}))$ so that we can apply Lemma 4.3. We then justify the existence of T' , independent of n , in a straightforward way. In fact, as Lemma 4.3 holds for $i \geq \frac{1}{2}$, we may start the induction at $i = 1$.

For the first induction step $i = 1$, we use the differential inequality from Lemma 4.3 with $i = 1$.

$$\partial_t \sum_{\alpha=0}^5 \int \xi_{\alpha} u_{\alpha}^2 dx = H \left(\sum_{\alpha=0}^5 \int \xi_{\alpha} u_{\alpha}^2 dx \right) \tag{24}$$

where H is a strictly increasing, positive, smooth function independent of n . Indeed, let $q(t) = \sum_{\alpha=0}^5 \int \xi_{\alpha} u_{\alpha}^2 dx$. Then the differential inequality is stated $\dot{q}(t) \leq H(q(t))$.

Let $K(s)$ be such that $\frac{d}{ds} K(s) = 1/H(s)$. Then $K(s)$ is increasing and concave. Furthermore $\partial_t K(q(t)) = \dot{q}(t)/H(q(t)) \leq 1$ and thus

$K(q(t)) \leq K(q(0)) + t$. Thus $q(t)$ will be finite as long as $K(q(0)) + t$ remains in the domain of K^{-1} . This provides a lower bound on the existence time T' with $0 < T' \leq T$, depending only on

$$q(0) = \sum_{\alpha=0}^5 \int \varphi_{\alpha}^2 \xi_{\alpha} dx.$$

It remains to prove the subsequent induction steps. We start from (18) and (19) to obtain as in the proof of Lemma 4.3

$$\partial_t \sum_{\alpha=0}^5 \int \xi_{\alpha} u_{\alpha}^2 dx \leq C \left(\sum_{\alpha=0}^5 \int \xi_{\alpha} u_{\alpha}^2 dx + \left(\int \xi u_3^2 dx \right)^{2/3} \int \xi_5 u_5^2 dx \right) \quad (25)$$

where C depends only on the norm of u in $L^{\infty}([0, T]; H^5)$, $\xi_{\alpha} \in W_{0, i_0}$ and $\xi \in W_{0, i_0}$ with $i_0 = (3i + 1)/5$. For $i_0 = 1$, we obtain $i_1 = (5i_0 - 1)/3 = 4/3$.

From the first induction step, we know that $\sup_{0 \leq t \leq T'} \int \xi u_3^2 dx \leq C$ where C depends only on the norm of φ in $H^5(W_{0, i_0})$. For $i_0 = 1 \leq i \leq \frac{4}{3} = i_1$ we apply Gronwall's lemma to (25) to conclude that for the whole time interval $[0, T']$, $\sup_{0 \leq t \leq T'} \int \sum_{\alpha=0}^5 \xi u_{\alpha}^2 dx$ stays bounded for all ξ in W_{0, i_0} .

Inductively, given i_{n-1} , define $i_n = (5i_{n-1} - 1)/3$. Then for any $i_{n-1} \leq i \leq i_n$ we conclude as above, that $\sup_{0 \leq t \leq T'} \int \sum_{\alpha=0}^5 \xi u_{\alpha}^2 dx$ stays bounded for ξ in W_{0, i_0} . This proves Corollary 4.4. We now obtain the following main result of this section:

THEOREM 4.5. — *Let φ be in $H^5(W_{0, i_0})$ with $i \geq 7/4$. Then there exists $0 < T < \infty$, depending only on the norm of φ in $H^5(W_{0, 7/4, 0})$, and a solution u of (2.1) in $L^{\infty}([0, T]; H^5(W_{0, i_0}))$ with initial data φ such that*

$$\int_0^T \int |\partial^6 u|^2 \eta dx dt < \infty \quad (30)$$

for η in $W_{\sigma, i-1, 0}$, for $\sigma > 0$ arbitrary.

Proof. — Approximate φ in $H^5(W_{0, i_0})$ -norm by a sequence $\varphi^{(n)}$ in $H^7(W_{0, i_0})$. According to Corollary 4.4 and Theorem 4.1, there exists $0 < T < \infty$ independent of n and a sequence of solutions $u^{(n)}$ in $L^{\infty}([0, T]; H^5(W_{0, i_0}))$ with initial data $\varphi^{(n)}$. According to Lemma 4.3, $(u^{(n)})_{n \geq 1}$ is a bounded sequence in $L^{\infty}([0, T]; H^5(W_{0, i_0}))$. A limiting argument gives a solution u of (2.1) in $L^{\infty}([0, T]; H^5(W_{0, i_0}))$ with initial data φ . (30) is obtained in a by now standard way.

THEOREM 4.6. — *Let φ be in $H^6(W_{0i0})$ with $i \geq 2$. Then the solution $u(x, t)$ of Theorem 4.5 belongs to $L^\infty([0, T]; H^6(W_{0i0}))$ and satisfies*

$$\int_0^T \int |\partial^7 u|^2 \eta \, dx \, dt < \infty \tag{32}$$

with η in $W_{\sigma, i-1, 0}$ ($\sigma > 0$ arbitrary).

Proof. — Again, one approximates φ , in this case in the $H^6(W_{0i0})$ -norm, by a sequence $\varphi^{(n)}$ in $H^7(W_{0i0})$. Again by Corollary 4.4 and Theorem 4.1 there exists $0 < T < \infty$ independent of n and solutions $u^{(n)}(x, t)$ of (2.1) in $L^\infty([0, T]; H^7(W_{0i0}))$ with initial data $\varphi^{(n)}$. One then proceeds using similar arguments as in the proof of Theorem 4.5.

The last result concerns the uniqueness of the solutions that we have constructed.

THEOREM 4.7 (Uniqueness). — *Let $0 < T < \infty$. Assume f satisfies (A.1)-(A.4). Then for $\varphi \in H^5(W_{020})$ there exists at most one solution $u \in L^\infty([0, T]; H^5(W_{020}))$ of (2.1) with initial data φ .*

Proof. — Assume u and v are two solutions of (2.1) in $L^\infty([0, T]; H^5(W_{0i0}))$ (hence $\partial_t u, \partial_t v$ in $L^\infty([0, T]; H^2(W_{020}))$), with the same initial data. As in the proof of Theorem 3.1 we write

$$\partial_t(u - v) + \sum_{j=0}^3 d^{(j)}(u_j - v_j) = 0 \tag{33}$$

where $d^{(j)}$ ($0 \leq j \leq 3$) depend smoothly on $u_3, \dots, u_0, v_3, \dots, v_0, x, t$ and are defined as in the proof of Theorem 3.1. However, because of the lower differentiability of u and v , in order to perform the main estimate, a weight function ξ is used which is independent of t and u and v . Define $\sigma = 1 + \|\partial[d^{(3)}/d^{(3)}]\|_{L^\infty([0, T] \times \mathbb{R})}$. Note that because u is in $L^\infty([0, T]; H^5(W_{0i0}))$, $\sup_{0 \leq t \leq T} x |u_\alpha(x, t)| \rightarrow 0$ as $x \rightarrow +\infty$ for $\alpha \leq 4$. Hence there exists $x_1 \geq 1$ such that for all $x \geq x_1$, $x |(\partial d^{(3)})/d^{(3)}| \leq 1$.

For our purpose it suffices to choose ξ in the following way:

$$\xi(x) = \begin{cases} e^{\sigma x} & \text{for } x \leq x_1 \\ e^{\sigma(x_1+1)} x^2 / (x_1+1)^2 & \text{for } x \geq x_1 + 1, \end{cases}$$

and for $x_1 < x < x_1 + 1$ define ξ such that $\xi \geq 1$, is smooth, does not depend on t and satisfies for some $\beta > 0$

$$\partial(\xi d^{(3)}) \geq \beta \quad \text{for } x_1 < x < x_1 + 1.$$

Now use this weight functions in the main estimate; multiply (33) by $2\xi(u-v)$ and integrate in x to obtain, by partial integrations

$$\begin{aligned} \partial_t \int \xi(u-v)^2 dx + \int (3 \partial [\xi d^{(3)}] - \xi d^{(2)})(u_1 - v_1)^2 dx \\ = -2 \int \xi d^{(0)}(u-v)^2 dx - \int \partial [\xi d^{(1)}](u-v)^2 dx \\ + \int \partial [\xi d^{(2)}](u_1 - v_1)(u-v) dx - \int \partial^2 [\xi d^{(3)}](u_1 - v_1)(u-v) dx. \end{aligned} \quad (35)$$

The second term on the left hand side of (35) can be estimated as follows;

$$\begin{aligned} \int_{-\infty}^{\infty} (3 \partial [\xi d^{(3)}] - \xi d^{(2)})(u_1 - v_1)^2 dx \\ \geq 3 \int_{-\infty}^{x_1} (-|\partial [d^{(3)}]/d^{(3)}| + \sigma) e^{\sigma x} d^{(3)}(u_1 - v_1)^2 dx + \int_{x_1}^{x_1+1} \beta(u_1 - v_1)^2 dx \\ + 3 \int_{x_1+1}^{\infty} (-|\partial [d^{(3)}]/d^{(3)}| x + 2) \frac{e^{\sigma(x_1+1)}}{(x_1+1)^2} x d^{(3)}(u_1 - v_1)^2 dx. \end{aligned} \quad (36)$$

Our choice of the weight function ξ implies that the above integrands are strictly positive. The right hand side of (36) will be needed to balance terms from the righthand side of (35). The two terms which need control are those with the factor $(u_1 - v_1)$. Here are the necessary estimates;

$$\begin{aligned} \left| \int_{-\infty}^{x_1} \partial [\xi d^{(2)}](u-v)(u_1 - v_1) dx \right| \\ \leq \| \partial [\xi d^{(2)}] / \xi \|_{L^\infty} \left(\int_{-\infty}^{x_1} (u-v)^2 e^{\sigma x} dx \right)^{1/2} \left(\int_{-\infty}^{x_1} (u_1 - v_1)^2 e^{\sigma x} dx \right)^{1/2} \\ \leq \| \partial [\xi d^{(2)}] / \xi \|_{L^\infty} \left(c(\varepsilon) \int_{-\infty}^{x_1} e^{\sigma x} (u-v)^2 dx + \varepsilon \int_{-\infty}^{x_1} e^{\sigma x} (u_1 - v_1)^2 dx \right) \end{aligned} \quad (37)$$

and

$$\begin{aligned} \left| \int_{x_1}^{\infty} \partial [\xi d^{(2)}](u-v)(u_1 - v_1) dx \right| \\ \leq \| \partial [\xi d^{(2)}] / \xi \|_{L^\infty} \left(c(\varepsilon) \int_{x_1}^{\infty} \xi(u-v)^2 dx + \varepsilon \int_{x_1}^{\infty} \xi(u_1 - v_1)^2 dx \right) \end{aligned} \quad (38)$$

Later ε will be chosen appropriately small.

The estimate of the last term from the righthand side of (35) is similar; however the factors that are of principal concern include the coefficients,

since they must accept two derivatives.

$$\left| \int_{-\infty}^{x_1} \partial^2 [\xi d^{(3)}] (u-v) (u_1 - v_1) dx \right| \leq c(\varepsilon) \| (u-v) e^{\sigma x/2} \|_{L^\infty(-\infty, x_1)}^2 \\ \times \int_{-\infty}^{x_1} |\partial^2 [\xi d^{(3)}] / \xi|^2 dx + \varepsilon \int_{-\infty}^{x_1} e^{\sigma x} (u_1 - v_1)^2 dx \quad (39)$$

and

$$\left| \int_{x_1}^{\infty} \partial^2 [\xi d^{(3)}] (u-v) (u_1 - v_1) dx \right| \leq c(\varepsilon) \| (u-v) \|_{L^\infty(x_1, \infty)}^2 \\ \times \int_{x_1}^{\infty} |\partial^2 [\xi d^{(3)}]|^2 dx + \varepsilon \int_{x_1}^{\infty} (u_1 - v_1)^2 dx. \quad (40)$$

As $\xi(x) \geq 1$ for $x \geq x_1$ we deduce

$$\| u-v \|_{L^\infty(x_1, \infty)}^2 \leq \varepsilon \int_{x_1}^{\infty} |u_1 - v_1|^2 dx + C(\varepsilon) \int_{x_1}^{\infty} \xi (u-v)^2 dx.$$

Similarly, the L^∞ -norm in (39) is estimated;

$$\| e^{\sigma x/2} (u-v) \|_{L^\infty(-\infty, x_1)}^2 \\ \leq \varepsilon \int_{-\infty}^{x_1} e^{\sigma x} (u_1 - v_1)^2 dx + C(\sigma, \varepsilon) \int_{-\infty}^{x_1} e^{\sigma x} (u-v)^2 dx.$$

To complete the discussion of (39) and (40), we remark that $u, v \in L^\infty([0, T]; H^5(W_{0,2,0}))$, thus the integrals $\int_{-\infty}^{x_1} |\partial^2 (\xi d^{(3)}) \xi|^2 dx$ and $\int_{x_1}^{\infty} |\partial^2 (\xi d^{(3)})|^2 dx$ are finite. Hence

$$\left| \int \partial^2 [\xi d^{(3)}] (u-v) (u_1 - v_1) dx \right| \\ \leq \varepsilon \int (1_{\{x > x_1\}} + e^{\sigma x} 1_{\{x \leq x_1\}}) (u_1 - v_1)^2 dx + C(\varepsilon, \sigma) \int \xi (u-v)^2 dx. \quad (41)$$

Applying (41), and (36)-(38) to (35), we conclude that

$$\partial_t \int (u-v)^2 \xi dx \leq C \int (u-v)^2 \xi dx.$$

Now Theorem 4.7 follows from Gronwall's lemma.

APPENDIX

Interpolation with weights

In Sections 2 and 4, various interpolation estimates with weight functions are used. They are all proved in the same way, so we restrict ourselves to the most delicate of them. For u in $H^1(\mathbb{R})$, one has the standard interpolation $|u|_{L^3} \leq C |\partial u|_{L^2}^{1/6} |u|_{L^2}^{5/6}$, that is

$$\int_{-\infty}^{\infty} u^3 dx \leq C \left(\int_{-\infty}^{\infty} |\partial u|^2 \right)^{1/4} \left(\int_{-\infty}^{\infty} |u|^2 \right)^{5/4}.$$

LEMMA 1. — Let $\beta \geq 0$ and $\delta \geq 0$. Whenever $(4\beta - \delta)/5 = \gamma \geq 0$ there exists $C > 0$ such that for $u \in H^1(\mathbb{R})$

$$\int_2^{\infty} x^\beta |u|^3 dx \leq C \left(\left(\int_1^{\infty} x^\delta |\partial u|^2 \right)^{1/4} + \left(\int_1^{\infty} x^{\delta-2} |u|^2 \right)^{1/4} \right) \left(\int_1^{\infty} x^\gamma |u|^2 \right)^{5/4}.$$

Proof. — Choose a partition of unity $\chi_j \geq 0$ ($j \geq 0$), such that $\chi_j \in C^\infty(\mathbb{R})$, $\text{supp } \chi_j \subseteq (2^{j-1}, 2^{j+1})$, $\sum_{j \geq 0} \chi_j = 1$ in $(2, +\infty)$ with $\text{supp } \chi_j \cap \text{supp } \chi_{j'} = \emptyset$

for $j \neq \{j', j' \pm 1\}$. Moreover we assume that $\frac{1}{2} \leq \sum_{j \geq 0} \chi_j^3(x)$ for $x \geq 2$ and $\sum_{j \geq 0} (2^{2j} |\partial \chi_j|^2 + \chi_j^2) \leq C$.

Then for $j \geq 0$.

$$|x^{\beta/3} \chi_j u|_{L^3}^3 \leq C 2^{(j+1)\beta} (|\chi_j \partial u|_{L^2}^{1/2} + |(\partial \chi_j) u|_{L^2}^{1/2}) |\chi_j u|_{L^2}^{5/2}.$$

Choosing γ such that $2^{j\beta} = (2^{j\delta})^{5/4} (2^{j\delta})^{1/4}$, i. e. $\gamma = (4\beta - \delta)/5$,

$$|x^{\beta/3} \chi_j u|_{L^3}^3 \leq C (|x^{\delta/2} \chi_j \partial u|_{L^2}^{1/2} + |x^{\delta/2} (\partial \chi_j) u|_{L^2}^{1/2}) |x^{\gamma/2} \chi_j u|_{L^2}^{5/2}.$$

Thus

$$\begin{aligned} \int_2^{\infty} x^\beta |u|^3 dx &\leq C \int_1^{\infty} x^\beta \left(\sum_j \chi_j^3 \right) |u|^3 dx \\ &\leq C \left\{ \left(\int_1^{\infty} x^\delta \left(\sum_j \chi_j^2 \right) |\partial u|^2 \right)^{1/4} + \left(\int_1^{\infty} x^{\delta-2} \left(\sum_j |\partial \chi_j|^2 \right) |u|^2 \right)^{1/4} \right\} \\ &\quad \times \left(\int_1^{\infty} x^\gamma \left(\sum_j \chi_j^2 \right) |u|^2 \right)^{5/4} \end{aligned}$$

and the lemma follows.

By the same method, using the standard interpolation

$$|\partial u|_{L^3} \leq C |u|_{L^2}^{5/2} |\partial^2 u|_{L^2}^{1/2},$$

one proves.

LEMMA 2. — Let $\beta \geq 0$, $\delta \geq 0$. Whenever $(8\beta - 7\delta)/5 = \gamma \geq 0$ there exists $C > 0$ such that for u in $H^2(\mathbb{R})$,

$$\int_2^\infty x^\beta |\partial u|^3 dx \leq C \left[\int_1^\infty x^\delta |\partial^2 u|^2 dx + \int_1^\infty x^{\delta-2} |\partial u|^2 d\xi \right]^{7/8} \left(\int_1^\infty x^\gamma u^2 \right)^{5/8}.$$

Finally, for reference, we include the following lemma, whose proof is straightforward.

LEMMA 3. — For ξ in $W_{\sigma, i, 0}$ and $\sigma \geq 0$, $i \geq 0$, there exists a constant C such that, for u in $H^1(W_{\sigma, i, 0})$

$$\sup_{x \in \mathbb{R}} |\xi u| \leq C \int (|u|^2 + |\partial u|^2) \xi dx.$$

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