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Closed orbits of fixed energy for a class of N-body problems (*)

by

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ABSTRACT. — We prove the existence of periodic solutions with prescribed energy for a class of N-body type problems.

Key words : Singular Hamiltonian systems, N-body problem, critical point theory.

RÉSUMÉ. — Nous démontrons l'existence de solutions périodiques à énergie fixée pour une classe de problèmes de type N-corps.

1. MAIN RESULTS

The aim of this paper is to prove the existence of periodic solutions with prescribed energy for a class of second order Hamiltonian systems, including the N-body problem. Precisely, we set $\Omega = \mathbf{R}^k \setminus \{0\}$ and consider

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a potential V of the form

$$V(x)=V(x_1, \ldots, x_N)=\frac{1}{2} \sum_{1 \leq i \neq j \leq N} V_{ij}(x_i - x_j) \tag{1.1}$$

where $x_i \in \mathbf{R}^k$, $x=(x_1, \ldots, x_N) \in \mathbf{R}^{Nk}$ and $V_{ij} \in C^1(\Omega, \mathbf{R})$ $(i,j=1, \ldots, N)$. Given $m_i > 0$ $(i=1, \ldots, N)$ and $h \in \mathbf{R}$, we seek for periodic solutions of

$$\text{(Ph)} \quad \begin{cases} m_i x_i'' + \nabla_{x_i} V(x_1, \ldots, x_N)=0 \qquad (1 \leq i \leq N) & \text{(Ph.1)} \\ \dfrac{1}{2}\sum_i m_i |x_i'(t)|^2 + V(x_1(t), \ldots, x_N(t))=h & \text{(Ph.2)} \end{cases}$$

Here ∇ (resp. ∇_{x_i}) denotes the gradient (resp. the gradient with respect x_i). We will use the notation $x.y$, or simply xy (resp. $|x|$) to denote the Euclidean scalar product of any two vectors $x, y \in \mathbf{R}^m$ (resp. the Euclidean norm of x).

We assume $V(x)$ is in the form (1.1) with V_{ij} satisfying:

(V1) $V_{ij}(\xi)=V_{ji}(\xi)$, $\forall \xi \in \Omega$;

(V2) $\exists \alpha \in [1, 2[$ *such that* $\nabla V_{ij}(\xi)\, \xi \geqq -\alpha V_{ij}(\xi) > 0$, $\forall \xi \in \Omega$;

(V3) $\exists \delta \in]0, 2[$ *and* $r>0$ *such that* $\nabla V_{ij}(\xi)\, \xi \leqq -\delta V_{ij}(\xi)$ *for all* $0<|\xi| \leqq r$;

(V4) $V_{ij}(\xi) \to 0$ *as* $|\xi| \to \infty$.

Remarks. – For future references let us note explicitely some consequences of the preceding assumptions. First of all, (V2)-(V3) imply, respectively:

$$V_{ij}(\xi) \leqq -\frac{c_1}{|\xi|^\alpha}, \qquad \forall |\xi|>0 \tag{1.2}$$

$$V_{ij}(\xi) \geqq -\frac{c_2}{|\xi|^\delta}, \qquad \forall\, 0<|\xi| \leqq r \tag{1.3}$$

Here and always in the sequel c, c_1, c_2, etc. denote positive constants.

Moreover, since $\nabla V(x)\, x=\frac{1}{2}\sum_{i \neq j} \nabla V_{ij}(x_i - x_j)(x_i - x_j)$, then from (V2)-(V3) it follows:

$$\nabla V(x)\, x \geqq -\alpha V(x) > 0, \qquad \forall x=(x_1, \ldots, x_N),\ x_i \neq x_j; \tag{1.4}$$

$$\nabla V(x)\, x \leqq -\delta V(x), \qquad \forall x=(x_1, \ldots, x_N),\ 0<|x_i - x_j| \leqq r. \tag{1.5}$$

By a solution of (Ph) we mean an $x(t)=(x_i(t))_{1 \leq i \leq N}$ such that x is periodic with period $T>0$ and for all $i, j=1, \ldots, N$ there results

(i) $x_i \in H^{1,2}(0, T; \mathbf{R}^k)$;

(ii) the set $\mathscr{C}=\{t \in [0, T] : x_i(t)=x_j(t)\}$ has measure zero;

(iii) x_i is C^2 on $[0, T] \setminus \mathscr{C}$ and satisfies (Ph.1)-(Ph.2) therein.

A solution x such that $\mathscr{C} \neq \varnothing$ (resp. $=\varnothing$) is called a *collision* (resp. *non-collision*). We anticipate that our solutions are possibly collisions, found as limit of non-collisions.

The main results of this paper are:

THEOREM A. – *Suppose* (V1)-(V4) *hold. Then for all* $h<0$ *problem* (Ph) *has a periodic solution.*

THEOREM B. – *Suppose* V *satisfies* (V1), (V3), (V4) *and*
(V2′) $\exists\, \alpha \in]0, 2[$ *such that* $\nabla V_{ij}(\xi)\,\xi \geqq -\alpha V_{ij}(\xi) > 0$, $\forall\, \xi \in \Omega$;
(V5) $V_{ij} \in C^2(\Omega, \mathbf{R})$ *and* $3 \nabla V_{ij}(\xi)\,\xi + V''_{ij}(\xi)\,\xi \,.\, \xi > 0$.
Then for all $h<0$ (Ph) *has a periodic solution.*

It is worth pointing out that Theorems A and B above cover the case of the N-body problem, namely when $V_{ij}(\xi) = -\dfrac{m_i m_j}{|\xi|}$, $x \in \mathbf{R}^3$, and (Ph.1) is nothing but the equation of motion of N bodies in $\mathbf{R}^3$ of position $x_1, \ldots, x_N$ and masses $m_1, \ldots, m_N$ subjected to their mutual gravitational attraction. In fact, it is immediate to verify that the potentials $V_{ij}(\xi) = -\dfrac{m_i m_j}{|\xi|}$ satisfy both the assumptions (V1)-(V4) with $\alpha = \delta = 1$, as well as (V5).

Theorems A and B must be related with the results of [1] where problem (Ph) has been studied for potentials of the form $V(x) \cong -\dfrac{1}{|x|^{\alpha}}$, $\alpha > 0$. Actually, Theorem B extends Theorem 4.12 of [1] to problems of the N-body type under quite similar assumptions, in particular (V2′) and (V5). On the contrary, in Theorem A we eliminate (V5) but require that (V2) holds for $\alpha \geqq 1$.

Both the proofs of theorem A and B are based upon critical point theory. In the latter we employ the same techniques of [1]: roughly, (V5) allows us to find solutions of (Ph) looking for critical points of a functional f constrained on a suitable manifold M, where the Palais-Smale condition (PS) holds true.

The proof of Theorem A is more direct and relies on an application of the Mountain-Pass theorem to f. Actually, when (V2′) is substituted by the stronger (V2) it is possible to prove that (PS) holds for f without constraints. An example shows that indeed the lack of (PS) arises when $V_{ij}(\xi) = -|\xi|^{-\alpha}$ with $\alpha < 1$.

Existence of periodic solutions with prescribed period for some classes of N-body problems has been proved in [3], [4], [5]. On the contrary, we do not know any result *in the large* concerning the existence of trajectories with prescribed energy.

2. APPROXIMATE PROBLEMS

Let us introduce the following notation:

$$H = H^{1,2}(S^1, \mathbf{R}^k)$$
$$H_{\#} = \left\{ u \in H : u\left(t + \frac{1}{2}\right) = -u(t) \right\}$$
$$E = \{ u = (u_1, \ldots, u_N) : u_i \in H_{\#} \ (i = 1, \ldots, N) \}$$
$$\Lambda_0 = \{ u \in E : u_i(t) \neq u_j(t), \forall t, i \neq j \}$$
$$(u \mid v) = \int u' v', \qquad \| u \|^2 = \int | u' |^2 \qquad (u, v \in H_{\#}).$$

Here and always in the sequel $\int$ stands for $\int_0^1 dt$. It is well known that $\| u_i \|$ is a norm on $H_{\#}$ equivalent to the usual one and one has:

$$\| u_i \| \geqq 4 \, | u_i |_\infty$$

As an immediate consequence, for all $u = (u_1, \ldots, u_N) \in E$ setting

$$\| u \|_E^2 = \sum_i m_i \| u_i \|^2$$

there results

$$\| u \|_E \geqq c \, | u(t) |, \qquad \forall t \tag{2.1}$$

Define the following functionals on Λ_0:

$$f(u) = \frac{1}{2} \| u \|_E^2 \cdot \int [h - V(u)]$$

Formally, it is known (*cf.* [1], *see* also Lemma 2 below) that critical points of f on Λ_0 give rise, after a rescaling of time, to periodic solutions of (Ph). Actually, since Λ_0 is an open subset of E, critical point theory cannot be employed directly. A device to overcome this problem has been used in [1] (*see* also [3], [5]) and consists in substituting V with

$$V_\varepsilon(x) = V(x) - \varepsilon W(x), \qquad W(x) = \sum_{1 \leq i < j \leq N} \frac{1}{| x_i - x_j |^2} \qquad (\varepsilon > 0)$$

Note that from (1.4) it follows:

$$\nabla V_\varepsilon(x) \, x = \nabla V(x) \, x + 2 \varepsilon W(x) > 0 \tag{2.2}$$

Let us set $f_\varepsilon(u) = \frac{1}{2} \| u \|_E^2 \cdot \int [h - V_\varepsilon(u)])$. Since $h - V_\varepsilon(u) > h + \varepsilon W(u)$, one can show (*see*, for ex. [5]) that f_ε is suitable for the critical point theory because there results

$$u_n \to u, \quad \text{weakly in E}, \qquad \text{and} \qquad u \in \partial \Lambda_0 \quad \Rightarrow \quad \int V_\varepsilon(u_n) \to -\infty \tag{2.3}$$

The procedure to find solutions of (Ph) will consists in two steps: first, critical points of f_ε are found, giving rise to solutions x_ε of corresponding approximate problems; second, we show that x_ε coverge, as $\varepsilon \to 0$, to a solution of (Ph).

Let us start with:

LEMMA 1. – *For any* $\varepsilon > 0$, *let* $u_\varepsilon \in \Lambda_0$ *be such that* $f'_\varepsilon(u_\varepsilon) = 0$ *and* $\| u_\varepsilon \| > 0$ *and set*

$$\omega_\varepsilon^2 = \frac{\displaystyle\int \nabla V_\varepsilon(u_\varepsilon)\, u_\varepsilon}{\| u_\varepsilon \|_E^2} > 0. \tag{2.4}$$

Then $x_\varepsilon(t) := u_\varepsilon(\omega_\varepsilon t)$ *is a non-collision solution of*

$$m_i x_i'' + \nabla_{x_i} V_\varepsilon(x_1, \ldots, x_N) = 0 \qquad \text{(Ph.1}\,\varepsilon)$$

$$\frac{1}{2} \sum_i m_i | x_i'(t) |^2 + V_\varepsilon(x_1(t), \ldots, x_N(t)) = h \qquad \text{(Ph.2}\,\varepsilon)$$

Proof. – The proof is similar to that of Lemma 2.3 of [1] and therefore we will be sketchy. From $f'_\varepsilon(u_\varepsilon) = 0$ it follows:

$$\| u_\varepsilon \|_E^2 \int [h - V_\varepsilon(u_\varepsilon)] - \frac{1}{2} \| u_\varepsilon \|_E^2 \int \nabla V_\varepsilon(u_\varepsilon)\, u_\varepsilon = 0$$

and hence [*cf.* (2.2)]:

$$\int [h - V_\varepsilon(u_\varepsilon)] = \frac{1}{2} \int \nabla V_\varepsilon(u_\varepsilon)\, u_\varepsilon > 0 \tag{2.5}$$

Moreover $u_\varepsilon = (u_{\varepsilon,i})_{1 \leqq i \leqq N}$ satisfies:

$$\sum_i m_i \int u'_{\varepsilon,i} v'_i \cdot \int [h - V_\varepsilon(u_\varepsilon)] - \frac{1}{2} \| u_\varepsilon \|_E^2 \int \nabla V_\varepsilon(u_\varepsilon)\, v = 0$$
$$\forall v = (v_1, \ldots, v_N) \in E$$

and hence, dividing by $\frac{1}{2} \| u_\varepsilon \|_E^2$ and using (2.5):

$$\omega_\varepsilon^2 \sum_i m_i \int u'_{\varepsilon,i} v'_i - \int \nabla V_\varepsilon(u_\varepsilon)\, v = 0, \qquad \forall v = (v_1, \ldots, v_N) \in E \tag{2.6}$$

Next, since $V_{ij}(x) = V_{ji}(x)$, one shows as in [5], Thm. 1.1, that (2.6) holds not only for all $v \in E$ but also for all $v \in H^N = H \times H \times \ldots \times H$ (N-times). Thus u_ε satisfies

$$\omega_\varepsilon^2 m_i u''_{\varepsilon,i} + \nabla_{x_i} V_\varepsilon(u_\varepsilon) = 0 \tag{2.7}$$

Rescaling the time, one finds that $x_\varepsilon(t) = u_\varepsilon(\omega_\varepsilon t)$ satisfies (Ph.1 ε). Integrating (2.7) the conservation of the energy (Ph.2 ε) holds, too. ■

3. EXISTENCE OF CRITICAL POINTS OF f_ε

Critical points of f_ε on Λ_0 will be found by means of the Mountain-Pass Theorem. Let us begin proving:

LEMMA 2. – *There exist* ρ, $\beta>0$ *such that*
(i) $f_\varepsilon(u)\geqq\beta$ *for all* $\varepsilon>0$ *and all* $u\in\Lambda_0$, $\|u\|_E=\rho$;
(ii) *there exist* $\varepsilon_0>0$, u_0, $u_1\in\Lambda_0$ *with* $\|u_0\|_E<\rho<\|u_1\|_E$, *such that* $f_\varepsilon(u_0)$, $f_\varepsilon(u_1)<\beta$, $\forall\, 0<\varepsilon\leqq\varepsilon_0$.

Proof. – First of all let us remark that from (1.2) it follows

$$\left.\begin{aligned} -V(x)=-\frac{1}{2}\sum_{i\neq j}V_{ij}(x_i-x_j)\geqq\frac{c_1}{2}\sum_{i\neq j}\frac{1}{|x_i-x_j|^\alpha}\geqq\frac{c_2}{|x|^\alpha},\\ \forall\, 0<|x_i-x_j|\leqq r\end{aligned}\right\}\quad(3.1)$$

Using (3.1) joinly with (2.1) one deduces:

$$f_\varepsilon(u)\geqq\frac{1}{2}\|u\|_E^2\int[h-V(u)]\geqq\frac{1}{2}\|u\|_E^2\int\left[h+\frac{c_2}{|u|^\alpha}\right]\geqq\frac{h}{2}\|u\|_E^2+c_3\|u\|_E^{2-\alpha}$$

proving (i).

To complete the proof we take $u=(u_1(t),\ldots,u_N(t))$, with

$$u_i(t)=\xi\cos\left(2\pi\left(t+\frac{i}{N}\right)\right)+\eta\sin\left(2\pi\left(t+\frac{i}{N}\right)\right)\qquad(i=1,\ldots,N)$$

where ξ, $\eta\in\mathbf{R}^k$ satisfy: $|\xi|=|\eta|=1$, $\xi\eta=0$.

For $R>0$ we consider

$$f_\varepsilon(R\,u)=\frac{1}{2}R^2\|u\|_E^2\int[h-V_\varepsilon(R\,u)]$$

Note that $|u_i(t)-u_j(t)|=a_{ij}$ is independent on t and hence

$$\sum_{i<j}\frac{1}{|u_i(t)-u_j(t)|^2}=c_4.$$

From this it follows:

$$f_\varepsilon(R\,u)=\frac{1}{2}R^2\|u\|_E^2\int\left[h-V(R\,u)+\frac{c_4\,\varepsilon}{R^2}\right]$$

Since $|R\,u_i(t)-R\,u_j(t)|=R\,a_{ij}$, then $h<0$ and (V4) imply

$$\limsup\left[h-\int V(R\,u)\right]<0$$

and hence $f_\varepsilon(R\,u)\to-\infty$ as $R\to\infty$, proving the existence of $u_i\in\Lambda_0$, such that $\|u_1\|>\rho$ and $f_\varepsilon(u_1)<\beta$.

Lastly, let $R>0$ be small enough and recall that $|u_i(t)-u_j(t)|=a_{ij}$ is constant. Then using (1.3) one finds

$$-V(R\,u(t))\leqq \frac{c_5}{R^\delta}\sum_{i<j} a_{ij}^{-\delta}\leqq \frac{c_6}{R^\delta}$$

Hence

$$f_\varepsilon(R\,u)\leqq \frac{h}{2}R^2\|u\|_E^2+c_7\,R^{2-\delta}+c_8\,\varepsilon$$

Since $0<\delta<2$, then the existence of $\varepsilon_0>0$ and u_0 satisfying (ii) follows. ■

Next, we investigate the Palais-Smale [in short (PS)] condition. For this, some lemmas are in order.

LEMMA 3. — *Let* $u_n\in\Lambda_0$ *be such that*

$$(*)\qquad \begin{cases} f_\varepsilon(u_n)\leqq c\\ f'_\varepsilon(u_n)\to 0.\end{cases}$$

Then $\|u_n\|_E\leqq c'$.

Proof. — Since $f(u)\leqq f_\varepsilon(u)$, from $f_\varepsilon(u_n)\leqq c$ we infer

$$-\frac{1}{2}\|u_n\|_E^2\int V(u_n)\leqq c-\frac{1}{2}h\|u_n\|_E^2 \qquad (3.2)$$

Setting $\sigma_{\varepsilon,n}=\sigma_n=(f'_\varepsilon(u_n)\,|\,u_n)$ one has:

$$\sigma_n=\|u_n\|_E^2\int\left[h-V_\varepsilon(u_n)-\frac{1}{2}\nabla V_\varepsilon(u_n)\,u_n\right]$$

Using (1.4) we deduce:

$$\sigma_n=\|u_n\|_E^2\int\left[h-V(u_n)-\frac{1}{2}\nabla V(u_n)\,u_n\right]$$
$$\leqq\|u_n\|_E^2\int\left[h-\left(1-\frac{\alpha}{2}\right)V(u_n)\right]. \qquad (3.3)$$

From (3.2) and (3.3) it follows

$$\sigma_n\leqq h\|u_n\|_E^2+\left(1-\frac{\alpha}{2}\right)(2c-h\|u_n\|_E^2)=\frac{\alpha}{2}h\|u_n\|_E^2+c_1$$

and thus

$$-\frac{\alpha}{2}h\|u_n\|_E^2\leqq c_2+\|f'_\varepsilon(u_n)\|\,\|u_n\|_E.$$

Since h is negative we infer $\|u_n\|_E\leqq c'$. ■

LEMMA 4. — *Let* u_n *be a sequence satisfying* (*). *If* $|u_n|_\infty\to 0$ *then* lim up $f_\varepsilon(u_n)\leqq 0$.

Proof. – Let us set

$$r_n = \min\{|u_n(t)| : 0 \leqq t \leqq 1\}, \qquad R_n = \max\{|u_n(t)| : 0 \leqq t \leqq 1\}.$$

We claim that $R_n/r_n \leqq c_1$. To see this we argue by contradiction. Suppose that (without relabeling) $\frac{R_n}{r_n} \to \infty$, and let t_n and s_n be such that $R_n = |u_n(t_n)|$ and $r_n = |u_n(s_n)|$. One has

$$\log \frac{R_n}{r_n} = \log \frac{|u_n(t_n)|}{|u_n(s_n)|} = \int_{s_{nn}}^{t_n} \frac{d}{d\tau} \log |u_n(\tau)| \leqq \int_{s_n}^{t_n} \frac{|u_n'|}{|u_n|}$$
$$\leqq \left[\int |u_n'|^2\right]^{1/2} \left[\int \frac{1}{|u_n|^2}\right]^{1/2} \leqq c_2 \| u_n \|_E \left[\int \frac{1}{|u_n|^2}\right]^{1/2}.$$

Since $\log \frac{R_n}{r_n} \to \infty$, then

$$\| u_n \|_E \left[\int \frac{1}{|u_n|^2}\right]^{1/2} \to \infty \tag{3.4}$$

Furthermore, from $|u_n|_\infty \to 0$ and (3.1) it follows $\int h - V(u_n) \to \infty$. In particular, $\int [h - V(u_n)] > 0$ for n large and hence, using (3.4) we infer

$$f_\varepsilon(u_n) = \frac{1}{2} \| u_n \|_E^2 \cdot \int [h - V(u_n) + \varepsilon W(u_n)] \geqq \frac{\varepsilon}{2} \| u_n \|_E^2 \int \frac{1}{|u_n|^2} \to \infty,$$

a contradiction with $f_\varepsilon(u_n) \leqq c$, proving the claim.

Next, let us set

$$\gamma_n = -\int V(u_n)$$
$$A_n = \frac{1}{2} \| u_n \|_E^2 [h + \gamma_n]$$
$$B_n = \| u_n \|_E^2 \int W(u_n)$$

From [*see* (3.2)]

$$\sigma_n = \| u_n \|_E^2 \left[h + \gamma_n - \frac{1}{2} \int \nabla V(u_n) u_n\right] \tag{3.5}$$

it follows that

$$A_n = \frac{1}{2} \frac{\sigma_n}{\left[h + \gamma_n - (1/2) \int \nabla V(u_n) u_n\right]} [h + \gamma_n].$$

Using (1.5) one has $\int \nabla V(u_n)\, u_n \leqq \delta\gamma_n$ and hence

$$A_n \leqq \frac{1}{2}\,\frac{\sigma_n [h+\gamma_n]}{[h+(1-(\delta/2))\,\gamma_n]}$$

Since $\sigma_n \to 0$ and $\gamma_n \to \infty$ then $\limsup A_n \leqq 0$.

To estimate B_n we use again (3.1) and (3.5) yielding, respectively:

$$\left[h+\gamma_n-\frac{1}{2}\int \nabla V(u_n)\, u_n\right] > h+\left(1-\frac{\delta}{2}\right)\gamma_n > h+c_3\int |u_n|^{-\alpha} \qquad (>0)$$

$$\|u_n\|_E\left[h+\gamma_n-\frac{1}{2}\int \nabla V(u_n)\, u_n\right] \leqq \|f'_\varepsilon(u_n)\|$$

These two inequalities imply

$$\|u_n\|_E \leqq \frac{\|f'_\varepsilon(u_n)\|}{h+c_3\int |u_n|^{-\alpha}}$$

and hence

$$B_n \leqq \|f'_\varepsilon(u_n)\|^2\,\frac{c_4\int |u_n|^{-2}}{\left(h+c_3\int |u_n|^{-\alpha}\right)^2}$$

From $r_n \leqq |u_n(t)| \leqq R_n$ we deduce

$$B_n \leqq \|f'_\varepsilon(u_n)\|^2\,\frac{c_4\, r_n^{-2}}{(h+c_3\, R_n^{-\alpha})^2}$$

Since $R_n/r_n \leqq c_1$, $\alpha \geqq 1$ and $\|f'_\varepsilon(u_n)\| \to 0$, it follows that $B_n \to 0$. Finally, from

$$f_\varepsilon(u_n) = A_n + \frac{\varepsilon}{2} B_n$$

we infer that $\limsup f_\varepsilon(u_n) \leqq 0$. This completes the proof of the lemma. ■

We are now in position to prove:

LEMMA 5. – *The functional f_ε satisfies*:

(PS$^+$) *If $u_n \in \Lambda_0$ is such that $0 < \beta \leqq f_\varepsilon(u_n) \leqq c$, and $f'_\varepsilon(u_n) \to 0$, then (up to a subsequence) $u_n \to u^* \in \Lambda_0$.*

Proof. – From lemma 3 it follows that $\|u_n\|_E \leqq c'$ and $\exists\, u^* \in E$ such that (up to a subsequence) $u_n \to u^*$, weakly and uniformly in [0, 1]. From lemma 4 we infer that $u^* \not\equiv 0$, otherwise $\limsup f_\varepsilon(u_n) \leqq 0$, in contradiction with $f_\varepsilon(u_n) \geqq \beta > 0$. If $u^* \in \partial\Lambda_0$, then (2.3) implies $h - \int V_\varepsilon(u_n) \to +\infty$. This and (3.6) would contradict $f_\varepsilon(u_n) \leqq c$, proving that $u^* \in \Lambda_0$. Hence:

$$\liminf \| u_n \|_E \geqq \| u^* \|_E > 0 \tag{3.6}$$

as well as

$$V(u_n) \to V(u^*), \qquad W(u_n) \to W(u^*), \qquad \nabla V(u_n) u_n \to \nabla V(u^*) u^* \tag{3.7}$$

Moreover from

$$\sigma_n = \| u_n \|_E^2 \int \left[h - V_\varepsilon(u_n) - \frac{1}{2} \nabla V_\varepsilon(u_n) u_n \right]$$

we infer

$$\int [h - V_\varepsilon(u_n)] = \frac{1}{2} \int \nabla V_\varepsilon(u_n) u_n + \frac{\sigma_n}{\| u_n \|_E^2} \tag{3.8}$$

Taking into account (3.6), (3.7) and since $\sigma_n \to 0$ we can pass to the limit into (3.8) yielding

$$\int [h - V_\varepsilon(u_n)] \to \frac{1}{2} \int \nabla V_\varepsilon(u^*) u^* > 0 \tag{3.9}$$

Finally, from $f'_\varepsilon(u_n) \to 0$ it follows:

$$(u_n \mid v) \int [h - V_\varepsilon(u_n)] - \frac{1}{2} \| u_n \|_E^2 \int \nabla V_\varepsilon(u_n) v \to 0, \qquad \forall v \in H^N$$

Then (3.9) and $\int \nabla V_\varepsilon(u_n) v \to \int \nabla V_\varepsilon(u^*) v$ imply that $u_n \to u^*$ strongly in E. ■

LEMMA 6. — *Let* (V1)-(V4) *hold. Then* $\exists\, \varepsilon_0 > 0$ *such that* $\forall\, 0 < \varepsilon \leqq \varepsilon_0$ *there is* $u_\varepsilon \in \Lambda_0$ *such that* $f'_\varepsilon(u_\varepsilon) = 0$. *Moreover* $\exists\, a, b > 0$ *such that* $0 < a \leqq \| u_\varepsilon \|_E \leqq b$, $\forall\, 0 < \varepsilon \leqq \varepsilon_0$.

Proof. — Lemmas 2 and 5 allow us to apply the Mountain-Pass Theorem [2] yielding a critical point $u_\varepsilon \in \Lambda_0$ of f_ε. From the min-max characterization of $f_\varepsilon(u_\varepsilon)$ it follows:

$$f_\varepsilon(u_\varepsilon) \leqq \max_{R>0} f_\varepsilon(R\, u) \leqq \max_{R>0} f_{\varepsilon_0}(R\, u) \equiv c. \tag{3.10}$$

Since $f'_\varepsilon(u_\varepsilon) = 0$, then the arguments of lemma 3 imply the existence of $b > 0$ such that $\| u_\varepsilon \|_E \leqq b$. Furthermore from (2.5) we infer readily

$$h = \int \left[V_\varepsilon(u_\varepsilon) + \frac{1}{2} \nabla V_\varepsilon(u_\varepsilon) u_\varepsilon \right] = \int \left[V(u_\varepsilon) + \frac{1}{2} \nabla V(u_\varepsilon) u_\varepsilon \right].$$

If $\| u_\varepsilon \|_E \to 0$ as $\varepsilon \to 0$, then $| u_\varepsilon |_\infty \to 0$ and (1.4) implies

$$h \leqq \left(1 - \frac{\delta}{2} \right) \int V(u_\varepsilon), \tag{3.11}$$

while (2.7) yields $\int V(u_\varepsilon) \to -\infty$. This and (3.11) lied to a contradiction, proving the lemma. ■

4. PROOF OF THEOREM A

Let u_ε the Mountain-Pass critical point of f_ε given by Lemma 6. Since $\|u_\varepsilon\|_E > 0$, from Lemma 1 it follows that $x_\varepsilon(t) = u_\varepsilon(\omega_\varepsilon t)$ is a solution of (Ph.1 ε)-(Ph.2 ε). Furthermore, again from lemma 6 one has that $\|u_\varepsilon\|_E \leqq b$, and $u_\varepsilon \to u$ $(\varepsilon \to 0)$ uniformly in [0, 1]. In order to show that $u = (u_i)_{1 \leqq i \leqq N}$ gives rise to a solution of (Ph) we follow the same procedure as in [1]. For completeness we outline these arguments referring to [1] for more details. First, one proves that

(i) $\exists\, t : V(u(t)) \neq h$.

In fact, otherwise, $V(u(t)) \equiv h$, hence $u \in \Lambda_0$ and $V(u_\varepsilon) \to V(u)$, $\nabla V(u_\varepsilon) u_\varepsilon \to \nabla V(u) u$, uniformly in [0, 1]. Then

$$h = \int V(u_\varepsilon) + \frac{1}{2} \nabla V(u_\varepsilon) u_\varepsilon \to \int V(u) + \frac{1}{2} \nabla V(u) u = h + \frac{1}{2} \int \nabla V(u) u$$

implies $\int \nabla V(u) u = 0$, a contradiction because $\nabla V(x) x > 0$.

Next, one shows:

(ii) $\exists\, t : u_i(t) \neq u_j(t)$ for some $i \neq j$.

Otherwise, the components $u_{\varepsilon,i}$ of u_ε are such that $|u_{\varepsilon,i} - u_{\varepsilon,j}| \to 0$ uniformly in [0, 1] for all i, j and (1.2) implies $\int V(u_\varepsilon) \to -\infty$. On the other side, using (1.5) one finds

$$h = \int V(u_\varepsilon) + \frac{1}{2} \nabla V(u_\varepsilon) u_\varepsilon \leqq \left(1 - \frac{\delta}{2}\right) \int V(u_\varepsilon),$$

a contradiction.

Next, we claim that for the ω_ε given by (2.4) the following estimate holds:

(iii) $\exists\, 0 < \Omega_0 < \Omega_1$ such that $\Omega_0 \leqq \omega_\varepsilon \leqq \Omega_1$.

To prove this fact, let us take a closed interval $I \subset [0, 1]$, with measure $|I| > 0$, such that $u_i(t) \neq u_j(t)$, $V(u(t)) \neq h$, $\forall\, t \in I$. Such an interval exists because of (i) and (ii) above. Since $h - V_\varepsilon(u_\varepsilon) = \frac{1}{2} \nabla V_\varepsilon(u_\varepsilon) u_\varepsilon > 0$ and

$\|u_\varepsilon\|_E \leqq b$, it follows

$$\omega_\varepsilon^2 = \frac{\int \nabla V_\varepsilon(u_\varepsilon) u_\varepsilon}{\|u_\varepsilon\|_E^2} = \frac{2\int h - V_\varepsilon(u_\varepsilon)}{\|u_\varepsilon\|_E^2} \geqq \frac{2\int_I h - V_\varepsilon(u_\varepsilon)}{b^2} \tag{4.1}$$

Furthermore, from $V_\varepsilon(u_\varepsilon(t)) \to V(u(t))$ (uniformly on I), $h - V_\varepsilon(u_\varepsilon) > 0$ and (i) it follows that $h - V(u) > 0$ on I. Then, taking also into account that $|I| > 0$, we infer:

$$\omega_\varepsilon^2 \geqq \frac{2\int_I [h - V_\varepsilon(u_\varepsilon)]}{b^2} \to \frac{2\int_I [h - V(u)]}{b^2} > 0 \tag{4.2}$$

From (4.1) and (4.2) it follows immediately that $\omega_\varepsilon \geqq \Omega_0 > 0$.

In a similar way, using lemma 6 and (3.10) we find:

$$\omega_\varepsilon^2 = \frac{2\int h - V_\varepsilon(u_\varepsilon)}{\|u_\varepsilon\|_E^2} = \frac{4 f_\varepsilon(u_\varepsilon)}{\|u_\varepsilon\|_E^4} \leqq \frac{4c}{a^4} \equiv \Omega_1^2.$$

As a consequence of (iii) one has that $\omega_\varepsilon \to \omega$. Letting $x(t) = u(\omega t)$, a standard argument shows that x solves (Ph) (*see* the proof of theorem 4.12 of [1] and [5]). This completes the proof of the theorem A. ■

5. PROOF OF THEOREM B

The proof of Theorem B requires different arguments, because when (V2) is replaced by the weaker (V2′) the (PS$^+$) condition can fail (*see* Example below). The difficulty can be overcome, as in [1], by looking for critical points of f_ε constrained on a suitable manifold.

Referring to [1] for more details, let us outline the proof.

Set $g(u) := \int \left[V(u) + \frac{1}{2} \nabla V(u) u \right]$ and note that

$$(f_\varepsilon'(u) \mid u) = \|u\|_E^2 \int \left[h - V(u) - \frac{1}{2} \nabla V(u) u \right] = \|u\|_E^2 (h - g(u))$$

Hence, if u is any possible critical point of f_ε, then $g(u) = h$. Setting $M_h = \{ u \in \Lambda_0 : g(u) = h \}$, it turns out that, under assumptions (V1), (V2′), (V3), (V4), $M_h \neq \varnothing$, $\forall h < 0$. Furthermore, (V5) implies that $(g'(u) \mid u) \neq 0$, $\forall u \in M_h$ and hence M_h is a (smooth) manifold of codimension 1 in E. Moreover, if u is a critical point of f_ε on M_h there results $f_\varepsilon'(u) = \lambda g'(u)$ for some $\lambda \in \mathbf{R}$. From this it follows:

$$(f_\varepsilon'(u) \mid u) = \lambda (g'(u) \mid u)$$

Since $(f'_\varepsilon(u)\,|\,u)=0$ for $u\in M_h$ while $(g'(u)\,|\,u)\neq 0$, then $\lambda=0$ and $f'_\varepsilon(u)=0$. Noticing that $\forall\, u\in M_h$ there results $\|u\|_E>0$, then Lemma 2 implies $x_\varepsilon(t):=u(\omega_\varepsilon t)$ solves (Ph.1 ε)-(Ph.2 ε), with ω_ε given by (2.4). To find critical points of f_ε on M_h we first note that for all $u\in M_h$ there results $f_\varepsilon(u)=\dfrac{1}{4}\|u\|_E^2\displaystyle\int\nabla V_\varepsilon(u)\,u>0$. Moreover, repeating the arguments of Lemmas 4.5-6 of [1] [the fact that now the potential V has the form (1.1) requires minor changes, already indicated in the preceding section] one shows that f_ε satisfies (PS) on M_h. As a consequence f_ε achieves the minimum on M_h. Let us remark explicitely that here we do not need to use min-max arguments, because, in view of the symmetry assumption (V1), we are working in Λ_0. Lemmas 4.9-10-11 of [1] enable us to show that $u_\varepsilon\to u$ and $\omega_\varepsilon\to\omega$ as $\varepsilon\to 0$, yielding a solution $x(t):=u(\omega t)$ of (Ph). ■

The following example shows that the (PS) condition can fail when V (2) is replaced by (V2'). For simplicity we take a potential $V(x)=-|x|^{-\alpha}$ and not in the form (1.1).

Example. – Let us consider

$$f_\varepsilon(u)=\frac{1}{2}\|u\|_E^2\cdot\int\left[h+\frac{1}{|u|^\alpha}+\frac{\varepsilon}{|u|^2}\right]\quad(0<\alpha<1)$$

We claim that for all $k\in\mathbb{N}$ there exists a sequence $u_n=u_{n,k}$ such that

(i) $f_\varepsilon(u_n)\to 2k^2\pi^2\varepsilon$;

(ii) $f'_\varepsilon(u_n)\to 0$.

To see this, we take a sequence $r_n\to 0$ and set (using complex notation) $u_n(t)=r_n e^{i2\pi kt}$.

Since $\alpha<1$ there results:

$$f_\varepsilon(u_n)=2k^2\pi^2 r_n^2(h+r_n^{-\alpha}+\varepsilon r_n^{-2})\to 2k^2\pi^2\varepsilon,$$

proving (i).

Furthermore one has readily:

$$(f'_\varepsilon(u_n)\,|\,v)=(h+r_n^{-\alpha}+\varepsilon r_n^{-2})\int u'_n v'+2k^2\pi^2 r_n^2\left(-\frac{\alpha}{r_n^{\alpha+2}}\int u_n v-2\frac{\varepsilon}{r_n^4}\int u_n v\right).$$

Letting $v=\Sigma\, v_k e^{i2\pi kt}$ it follows:

$$\begin{aligned}(f'_\varepsilon(u_n)\,|\,v)&=4k^2\pi^2 r_n v_k(h+r_n^{-\alpha}+\varepsilon r_n^{-2})-4k^2\pi^2 r_n^2 v_k\left(\frac{\alpha}{2}r_n^{-\alpha-1}+\varepsilon r_n^{-3}\right)\\&=4k^2\pi^2 r_n v_k\left(h+\left(1-\frac{\alpha}{2}\right)r_n^{-\alpha}\right)\to 0,\end{aligned}$$

and (ii) follows.

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