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Concentration of solutions to elliptic equations with critical nonlinearity

by

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ABSTRACT. — We study the asymptotic behavior as ε goes to zero of solutions in $H_0^1(\Omega)$ to the equation: $-\Delta u = |u|^{4/(N-2)} u + \varepsilon f(x)$, where Ω is a bounded domain in $\mathbf{R}^N$. We show the existence of solutions to the problem which blow-up at some well-defined points, depending on f, for $\varepsilon = 0$.

Key words : Nonlinear elliptic equations, variational problems with lack of compactness, limiting Sobolev exponent.

RÉSUMÉ. — Nous étudions le comportement asymptotique quand ε tend vers zéro de solutions dans $H_0^1(\Omega)$ de l'équation :

$$-\Delta u = |u|^{4/(N-2)} u + \varepsilon f(x),$$

où Ω est un ouvert borné de $\mathbf{R}^N$. Nous montrons l'existence de solutions du problème qui explosent en des points caractérisés précisément en fonction de f, pour $\varepsilon = 0$.

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1. INTRODUCTION AND MAIN RESULTS

In this paper, we return to the problems of the form

$$\left.\begin{array}{ll} -\Delta u = u^p + f(x,u) & \text{on } \Omega \\ u>0 & \text{on } \Omega \\ u=0 & \text{on } \partial\Omega \end{array}\right\} \qquad (\mathrm{P})$$

where Ω is a smooth and bounded domain of $\mathbf{R}^N$, $N \geqq 3$, $p=\dfrac{N+2}{N-2}$ and $f(x,u)$ is a term of smaller order than u^p, *i.e.*

$$\frac{f(x,u)}{u^p} \to 0 \quad \text{when } u \to +\infty$$

The exponent p is critical from the viewpoint of Sobolev embeddings, in the sense that the injection of $L^{p+1}(\Omega)$ into $H_0^1(\Omega)$ is continuous but not compact. It follows that the functional:

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{p+1}\int_\Omega |u|^{p+1} - \int_\Omega F(x,u)$$

associated to the problem, with $F(x,u) = \int_0^\infty f(x,t)\,dt$, does not verify the Palais-Smale condition: there exist "critical points at infinity", corresponding to concentration phenomena which may occur at some points of the domain.

In the following, we will focus more specifically on the asymptotic behaviour with respect to ε of the solutions to the problem (P_ε)

$$\left.\begin{array}{ll} -\Delta u = u^p + \varepsilon f(x,u) & \text{on } \Omega \\ u>0 & \text{on } \Omega \\ u=0 & \text{on } \partial\Omega \end{array}\right\} \qquad (\mathrm{P}_\varepsilon)$$

as $\varepsilon>0$ goes to 0.

The existence and multiplicity of solutions to (P_ε) for ε sufficiently small has been proved for some special f – ([BN1], [R1], [R2]). On the other hand, for $\varepsilon=0$, the problem becomes more delicate, and we know for example that if Ω is starshaped, there is no solution [P]. As a consequence, solutions to (P_ε) may disappear for $\varepsilon=0$, either vanishing uniformly, or blowing up at some points of the domain.

In the case where $f(x,u)=u$, for instance, one has the following results:

(i) If $N \geqq 4$, and (u_ε) is a family of solutions of (P_ε) concentrating at a point $x_0 \in \bar{\Omega}$ as $\varepsilon \to 0$ (in the sense: $|\nabla u_\varepsilon|^2 \rightharpoonup S^{N/2}\delta_{x_0}$, δ_{x_0} the Dirac mass at x_0 and S the best Sobolev constant). Then, $x_0 \in \Omega$ and

$$\varphi'(x_0) = 0$$

where

$$\varphi(x)=\mathrm{H}(x,x) \tag{1.1}$$

and H is the regular part of Green's function of the Laplacian on Ω, denoted by G, *i. e.*:

$$\mathrm{H}(x,y)=\frac{1}{|x-y|^{\mathrm{N}-2}}-\mathrm{G}(x,y), \qquad \forall (x,y)\in\Omega^2 \tag{1.2}$$

(ii) Conversely, if $\mathrm{N}\geqq 5$ and if $x_0\in\Omega$ is a non-degenerate critical point of φ, there exists for small enough ε a family of solutions of (P_ε) concentrating at x_0 as $\varepsilon\to 0$.

(iii) Finally, if $\mathrm{N}\geqq 5$, for ε small enough there are at least as many solutions to (P_ε) as the category of Ω, concentrating as $\varepsilon\to 0$ at critical points of φ.

The same results hold for the problem

$$\left.\begin{array}{l} -\Delta u=u^{p-\varepsilon} \quad \text{on } \Omega \\ u>0 \quad \text{on } \Omega \\ u=0 \quad \text{on } \partial\Omega \end{array}\right\} \tag{P'_ε}$$

Furthermore, we have very precise estimates about the shape and the speed of concentration of the solutions of $(\mathrm{P}'_\varepsilon)$ as $\varepsilon\to 0$ – ([H], [R3], [BP]).

Here we will establish what happens in the case

$$f(x,u)=f(x), \qquad f\not\equiv 0$$

We denote by (Q_ε) the problem

$$\begin{array}{l} -\Delta u=|u|^{p-1}u+\varepsilon f \quad \text{on } \Omega \\ u=0 \quad \text{on } \partial\Omega \end{array} \tag{Q_ε}$$

(at this point, we do not impose to the solution u to be positive), and let $\tilde{f}$ be the function defined by

$$\begin{array}{l} -\Delta\tilde{f}=f \quad \text{on } \Omega \\ \tilde{f}=0 \quad \text{on } \partial\Omega \end{array} \qquad i.e. \quad \tilde{f}=-\Delta_\Omega^{-1} f$$

We prove the following results:

THEOREM 1. – (1) *Assume* $\tilde{f}\in\mathrm{C}^2(\Omega)$. *Let* $x_0\in\Omega$ *be such that*

(1) $\tilde{f}(x_0)>0$;

(2) x_0 *is a non-degenerate critical point of* $x\to\dfrac{\tilde{f}(x)}{\varphi(x)^{1/2}}$.

Then there exists a family (u_ε) *of solutions of* (Q_ε) *concentrating at* x_0 *as* $\varepsilon\to 0$, i. e.

$$|\nabla u_\varepsilon|^2\to \mathrm{S}^{\mathrm{N}/2}\,\delta_{x_0}, \qquad |u_\varepsilon|^{p+1}\to \mathrm{S}^{\mathrm{N}/2}\,\delta_{x_0}$$

in the sense of measures, where δ_{x_0} *is the Dirac mass at* x_0 *and*

$$S = \inf_{\substack{u \in H_0^1(\Omega) \\ |u|_{p+1}=1}} \int_\Omega |\nabla u|^2$$

is the Sobolev constant. If $f \geqq 0$ *on* Ω, (i) *is automatically satisfied, and* $u_\varepsilon > 0$ *on* Ω.

(2) *Assume* $f \geqq 0$, *and* $\tilde{f} \in C^1(\Omega)$. *For* ε *small enough*, (Q_ε) *has at least as many strictly positive solutions as the category of* Ω, *each one concentrating at a critical point of the function* $x \to \dfrac{\tilde{f}(x)}{\varphi(x)^{1/2}}$ *as* $\varepsilon \to 0$.

Remarks:

• If $f \in W^{l,p}(\Omega)$, $lp > N$, then $\tilde{f} \in C^2(\Omega)$. If $f \in W^{l,p}(\Omega)$, $(l+1)p > N$, then $\tilde{f} \in C^1(\Omega)$.

• The results of the theorem provide us with equivalents to results (ii), (iii) in the case where $f(x, u) = u$. One can conjecture that an equivalent to result (i) is also true, with φ replaced by the function $x \to \dfrac{\tilde{f}(x)}{\varphi(x)^{1/2}}$.

Concerning the existence of solutions to (Q_ε) with minimum regularity assumptions on f, *i.e.* $f \in H^{-1}(\Omega)$, one deduces from a result of Brézis and Nirenberg [BN2] the following proposition, whose proof is given in appendix:

PROPOSITION 1. — *For* $f \geqq 0$ *and* ε *sufficiently small*, (Q_ε) *has at least two solutions. One of these solutions converges uniformly to* 0 *as* $\varepsilon \to 0$.

COROLLARY. — *From the proposition and Theorem* 1-(2) *we deduce that if* $\tilde{f}$ *is positive and regular* [i.e. $\tilde{f} \in C^1(\Omega)$], (Q_ε) *has for* ε *sufficiently small at least* $\operatorname{cat}(\Omega) + 1$ *solutions, one of them converging uniformly to* 0 *and the others concentrating at critical points of the function* $x \to \dfrac{\tilde{f}(x)}{\varphi(x)^{1/2}}$ *as* $\varepsilon \to 0$.

The study of problem (Q_ε) allows us to state the same type of results concerning the problem

$$\left.\begin{aligned} -\Delta u &= |u|^{p-1} u \quad \text{on } \Omega, \\ u &= \varepsilon g \quad \text{on } \partial\Omega, \end{aligned} \qquad g \not\equiv 0 \right\} \qquad (Q'_\varepsilon)$$

We get:

THEOREM 2. — *The results from Theorem* 1 *and Proposition* 1 *are valid for problem* (Q'_ε), *provided that in all statements* f *is replaced by* g *and* $\tilde{f}$ *by* $\tilde{g}$, *where* $\tilde{g}$ *is the function defined by:*

$$\begin{aligned} \Delta \tilde{g} &= 0 \quad \text{on } \Omega \\ \tilde{g} &= g \quad \text{on } \partial\Omega \end{aligned}$$

Indeed, if we change the variable in (Q_ε), writing:

$$u=\tilde{u}+\varepsilon\tilde{f}$$

we are led to consider the equivalent problem

$$\left.\begin{aligned}-\Delta\tilde{u}&=(\tilde{u}+\varepsilon\tilde{f})^p \quad \text{on } \Omega\\ \tilde{u}&=0 \quad \text{on } \partial\Omega\end{aligned}\right. \qquad (\tilde{Q}_\varepsilon)$$

(where for sake of simplicity we write v^p instead of $|v|^{p-1}v$).

In a similar way, writing

$$u=\tilde{u}+\varepsilon\tilde{g}$$

(Q'_ε) turns out to be equivalent to the problem

$$\left.\begin{aligned}-\Delta\tilde{u}&=(\tilde{u}+\varepsilon\tilde{g})^p \quad \text{on } \Omega\\ \tilde{u}&=0 \quad \text{on } \partial\Omega\end{aligned}\right. \qquad (\tilde{Q}'_\varepsilon)$$

which is exactly $(\tilde{Q}_\varepsilon)$ with $\tilde{f}$ replaced by $\tilde{g}$. Hence the results for (Q'_ε) follow immediately from those for (Q_ε).

We turn now to the proof of the theorems.

2. PROOF OF THE THEOREMS

2.1. Notations

We introduce the functional

$$J(\tilde{u})=\frac{1}{2}\int_\Omega|\nabla\tilde{u}|^2-\frac{1}{p+1}\int_\Omega|\tilde{u}+\varepsilon\tilde{f}|^{p+1} \tag{2.1}$$

on $H_0^1(\Omega)$ whose critical points are solutions of $(\tilde{Q}_\varepsilon)$.

For $x\in\Omega$ and $\lambda>0$ we consider the functions

$$\delta_{\lambda,x}(y)=\frac{\lambda^{(N-2)/2}}{(1+\lambda^2|y-x|^2)^{(N-2)/2}} \tag{2.2}$$

and their projections $P\delta_{\lambda,x}$ on $H_0^1(\Omega)$, defined by

$$\left.\begin{aligned}\Delta P\delta_{\lambda,x}&=\Delta\delta_{\lambda,x} \quad \text{on } \Omega\\ P\delta_{\lambda,x}&=0 \quad \text{on } \partial\Omega\end{aligned}\right\} \tag{2.3}$$

so that

$$P\delta_{\lambda,x}=\delta_{\lambda,x}-\varphi_{\lambda,x} \tag{2.4}$$

with

$$\left.\begin{aligned}\Delta\varphi_{\lambda,x}&=0 \quad \text{on } \Omega\\ \varphi_{\lambda,x}&=\delta_{\lambda,x} \quad \text{on } \partial\Omega\end{aligned}\right\} \tag{2.5}$$

Expanding $\delta_{\lambda,x}$ on $\partial\Omega$ for λd large, we conclude from the maximum principle that

$$\varphi_{\lambda,x}(y)=\frac{1}{\lambda^{(N-2)/2}}H(x,y)+O\left(\frac{1}{\lambda^{(N+2)/2}d^N}\right) \tag{2.6}$$

where $d=d(x,\partial\Omega)$ and H denotes the regular part of Green's function.

We note that for all x and for all λ

$$\left.\begin{aligned} &-\Delta(c_N\delta_{\lambda,x})=(c_N\delta_{\lambda,x})^p \quad \text{on } \mathbf{R}^N, \\ \text{with}\quad & \\ &c_N=(N(N-2))^{(N-2)/4} \end{aligned}\right. \tag{2.7}$$

Define for $\eta>0$ the subset F_η of $H_0^1(\Omega)$ by

$$F_\eta=\left\{\alpha P\delta_{\lambda,x}/|\alpha-c_N|<\eta, \lambda d(x,\partial\Omega)>\frac{1}{\eta}\right\}$$

It is proved in [BC] that if $u\in H_0^1(\Omega)$ is such that $\operatorname{dist}_{H_0^1(\Omega)}(u,F_\eta)<\eta$, and η is small enough, the problem

$$\text{Minimize}\,|u-\alpha P\delta_{\lambda,x}|_{H_0^1}\quad\text{with respect to } \alpha,\lambda,\alpha$$

has a unique solution in the open set defined by

$$|\alpha-c_N|<4\eta,\qquad \lambda d(x,\partial\Omega)>\frac{1}{4\eta}$$

Then we can look for critical points of J studying those of the functional

$$\left.\begin{aligned} K:\quad M\to\mathbf{R} \\ (\alpha,\lambda,x,v)\mapsto J(\alpha P\delta_{\lambda,x}+v) \end{aligned}\right\} \tag{2.8}$$

where

$$M=\left\{(\alpha,\lambda,x,v)\in\mathbf{R}\times\mathbf{R}_+^*\times\Omega\times H_0^1(\Omega)/v\in E_{\lambda,x},\right.$$
$$\left.|\alpha-c_N|<\eta_0,\ \lambda d(x,\partial\Omega)>\frac{1}{\eta_0},\ |v|_{H_0^1}<\nu_0\right\} \tag{2.9}$$

with η_0 and ν_0 some strictly positive constants, and

$$E_{\lambda,x}=\left\{v\in H_0^1(\Omega)\,|\,\langle v,P\delta_{\lambda,x}\rangle_{H_0^1}\right.$$
$$\left.=\left\langle v,\frac{\partial P\delta_{\lambda,x}}{\partial\lambda}\right\rangle_{H_0^1}=\left\langle v,\frac{\partial P\delta_{\lambda,x}}{\partial x}\right\rangle_{H_0^1}=0\right\} \tag{2.10}$$

Finally, $(\alpha, \lambda, x, v) \in M$ is a critical point for K if and only if there exists $(A,B,C) \in \mathbf{R} \times \mathbf{R} \times \mathbf{R}^N$ such that the following equations are satisfied:

$$\left.\begin{aligned}
&\frac{\partial K}{\partial \alpha} = 0 && \text{(E.1)}\\
&\frac{\partial K}{\partial \lambda} = B \int_\Omega \nabla \frac{\partial^2 P \delta_{\lambda, x}}{\partial \lambda^2} \nabla v + C . \int_\Omega \nabla \frac{\partial^2 P \delta_{x, \lambda}}{\partial \lambda \, \partial x} \nabla v && \text{(E.2)}\\
&\frac{\partial K}{\partial x} = B \int_\Omega \nabla \frac{\partial^2 P \delta_{x, \lambda}}{\partial \lambda \, \partial x} \nabla v + C . \int_\Omega \nabla \frac{\partial^2 P \delta_{x, \lambda}}{\partial x^2} \nabla v && \text{(E.3)}\\
&\frac{\partial K}{\partial v} = A P \delta_{\lambda, x} + B \frac{\partial P \delta_{\lambda, x}}{\partial \lambda} + C . \frac{\partial P \delta_{\lambda, x}}{\partial x} && \text{(E.4)}
\end{aligned}\right\} \text{(E)}$$

The proof of the theorem requires some computations. Only the main steps will be given here, the details being exposed in [B], [R1], [R4].

2.2. Analysis of (E)

This analysis will provide us with the first result of Theorem 1. We consider the last equation (E.4) of (E). Expanding K in a neighbourhood of $v=0$, we obtain

$$J(\alpha P \delta_{\lambda, x} + v) = J(\alpha P \delta_{\lambda, x}) + F_{\alpha, \lambda, x}(v) + Q_{\alpha, \lambda, x}(v) + R_{\alpha, \lambda, x}(v)$$

with F linear in v, Q quadratic, and R collecting the higher order terms, *i.e.*

$$F_{\alpha, \lambda, x}(v) = -\int_\Omega (\alpha P \delta_{\lambda, x} + \varepsilon \bar{f})^p v$$

$$Q_{\alpha, \lambda, x}(v) = \frac{1}{2} \int_\Omega |\nabla v|^2 - \frac{p}{2} \int_\Omega (\alpha P \delta_{\lambda, x} + \varepsilon \bar{f})^{p-1} v^2$$

and

$$R_{\alpha, \lambda, x}(v) = O(|v|_{H_0^1}^{\min(3, p+1)}), \qquad R'_{\alpha, \lambda, x}(v) = O(|v|_{H_0^1}^{\min(2, p)})$$
$$R''_{\alpha, \lambda, x}(v) = O(|v|_{H_0^1}^{\min(1, p-1)})$$

(where for simplicity we denote by θ^q the function $|\theta|^{q-1}\theta$).

In [B], [R1] it is proved that the quadratic form $Q_{\alpha, \lambda, x}$ is positive definite, with a modulus of coercivity independent of α, λ, x, ε, if we assume that η_0 and ε are small enough. So we may write

$$F_{\alpha, \lambda, x}(v) = \langle f_{\alpha, \lambda, x}, v \rangle_{H_0^1}, \qquad f_{\alpha, \lambda, x} \in E_{\lambda, x}$$

$$Q_{\alpha, \lambda, x}(v) = \frac{1}{2} \langle A_{\alpha, \lambda, x} v, v \rangle_{H_0^1}, \qquad A_{\alpha, \lambda, x} \in \mathscr{L}(E_{\lambda, x})$$

where $A_{\alpha, \lambda, x}$ is coercive, with modulus of coercivity independent of α, λ, x, ε. From this, starting from the point $(f, v) = (0, 0)$ and applying the implicit function theorem, we infer: there exists a smooth map which to each $(\alpha, \lambda, x, \varepsilon)$ such that

$$|\alpha - c_N| < \eta_0, \qquad \lambda d(x, \partial\Omega) > \frac{1}{\eta_0}, \qquad \varepsilon < \varepsilon_0$$

(η_0 and ε_0 small enough) associates an element $\bar{v} \in E_{\lambda, x}$, $|\bar{v}|_{H_0^1} < \nu_0$, such that (E.4) is satisfied for certain real numbers A, B, C. Furthermore

$$|\bar{v}|_{H_0^1} = O(|f_{\alpha, \lambda, x}|_{H_0^1}) \tag{2.12}$$

Now, let us remark that

$$\forall v \in E_{\lambda, x}, \quad \langle f_{\alpha, \lambda, x}, v \rangle_{H_0^1} = -\int_\Omega [(\alpha P \delta_{\lambda, x} + \varepsilon \bar{f})^p - \alpha^p \delta_{\lambda, x}^p] v$$

since

$$\int_\Omega \delta_{\lambda, x}^p v = \frac{1}{N(N-2)} \int_\Omega \nabla \delta_{\lambda, x} \nabla v = 0$$

Then, a computation using (2.2), (2.4), (2.6), the Hölder inequality and the Sobolev embedding theorem yields (as in [B], [R1])

$$|f_{\alpha, \lambda, x}|_{H_0^1} = \begin{cases} O\left(\dfrac{1}{(\lambda d)^{N-2}} + \dfrac{\varepsilon}{\lambda^{(N-2)/2}} + \varepsilon^p\right) \\ \text{if } N < 6 \\ O\left(\dfrac{(\operatorname{Log} \lambda d)^{2/3}}{(\lambda d)^4} + \dfrac{\varepsilon (\operatorname{Log} \lambda d)^{2/3}}{\lambda^2} + \varepsilon^2\right) \\ \text{if } N = 6 \\ O\left(\dfrac{1}{(\lambda d)^{(N+2)/2}} + \varepsilon^p\right) \\ \text{if } N > 6 \end{cases} \tag{2.13}$$

From (2.12), the same estimate holds for $|\bar{v}|_{H_0^1}$.

Now we are left with the remaining equations, namely the system formed by (E.1), (E.2), (E.3). Setting

$$\rho = c_N - \alpha = (N(N-2))^{(N-2)/4} - \alpha \tag{2.14}$$

using (2.2), (2.4), (2.6) and the estimate that we obtained for $|\bar{v}|_{H_0^1}$, we get from (2.1), (2.8) the expansions:

$$\frac{\partial K}{\partial \alpha}(\alpha, \lambda, x, \bar{v}) = -4 N \Gamma_1 \rho + V_\alpha(\alpha, \lambda, x) \tag{2.15}$$

where V_α is a smooth function which satisfies the estimate

$$V_\alpha(\alpha, \lambda, x) = O\left[\rho^2 + \frac{1}{(\lambda d)^{N-2}} + \frac{\varepsilon}{\lambda^{(N-2)/2}} + \left(\frac{\varepsilon^2 \operatorname{Log}\lambda}{\lambda^2} \text{ if } N=4;\ \frac{\varepsilon^2}{\lambda^2} \text{ if } N>4\right) + \varepsilon^{2p}\right]$$

In the same way we obtain

$$\frac{\partial K}{\partial \lambda}(\alpha, \lambda, x, \bar{v}) = -\frac{(N-2)^2}{2}\sigma_{N-1} \times \left[\frac{\alpha\varepsilon \tilde{f}(x)}{\lambda^{N/2}} - \frac{\alpha^2 \varphi(x)}{\lambda^{N-1}}\right] + V_\lambda(\alpha, \lambda, x) \quad (2.16)$$

with V_λ a smooth function which satisfies

$$V_\lambda(\alpha, \lambda, x) = O\left[\frac{1}{\lambda^N d^{N-1}} + \frac{\varepsilon \operatorname{Log}\lambda}{\lambda^{(N+4)/2}} + \frac{|\rho|}{\lambda^{N-1} d^{N-2}} + \frac{|\rho|\varepsilon}{\lambda^{N/2}} + \frac{\varepsilon^p}{\lambda^{N/2}} + \frac{\varepsilon^{2p}}{\lambda} + \left(\frac{\varepsilon^2 \operatorname{Log}\lambda}{\lambda^3} \text{ if } N=4;\ \frac{\varepsilon^2}{\lambda^3} \text{ if } N>4\right) + \left(\frac{\varepsilon}{\lambda^{N-1} d^{N-2}} \text{ if } N<6;\ \frac{\varepsilon(\operatorname{Log}\lambda d)^{2/3}}{\lambda^5 d^2} \text{ if } N=6;\ \frac{\varepsilon}{\lambda^{(N+4)/2} d^{(N-2)/2}} \text{ if } N>6\right)\right]$$

A last computation provides us with the expansion

$$\frac{\partial K}{\partial x}(\alpha, \lambda, x, \bar{v}) = N(N-2)\sigma_{N-1}\left[\frac{\alpha^2 \varphi'(x)}{2\lambda^{N-2}} - \varepsilon\alpha \frac{\tilde{f}'(x)}{\lambda^{(N-2)/2}}\right] + V_x(\alpha, \lambda, x) \quad (2.17)$$

where V_x is a smooth function verifying

$$V_x(\alpha, \lambda, x) = O\left[\left(\frac{1}{\lambda^3 d^4} \text{ if } N=4;\ \frac{1}{\lambda^N d^{N+1}} \text{ if } N>4\right) + \frac{|\rho|}{\lambda^{N-2} d^{N-1}} + \frac{\varepsilon|\rho|}{\lambda^{(N-2)/2}} + \lambda\varepsilon^{2p} + \left(\frac{\varepsilon}{\lambda^2 d^2} + \frac{\varepsilon \operatorname{Log}\lambda}{\lambda^3 d^3} + \frac{\varepsilon^2}{\lambda} \text{ if } N=4;\ \frac{\varepsilon \operatorname{Log}\lambda}{\lambda^{7/2} d^3} + \frac{\varepsilon^2 \operatorname{Log}\lambda}{\lambda^2} + \frac{\varepsilon^{7/3}}{\lambda^{3/2}} \text{ if } N=5;\ \frac{\varepsilon}{\lambda^{(N+2)/2} d^{\sup(4, N/2)}} + \frac{\varepsilon^p}{\lambda^{(N-2)/2}} \text{ if } N \geqq 6\right)\right]$$

We used the notations

$$\Gamma_1=\int_{\mathbf{R}^N}\delta_{0,1}^{p+1}=\frac{\pi^{(N+1)/2}}{2^{N-1}\Gamma(N/2)}\qquad\text{and}\qquad\sigma_{N-1}=\operatorname{mes}(S^{N-1})=\frac{2\pi^{N/2}}{\Gamma(N/2)}$$

Suppose now that $x_0=0\in\Omega$, $\tilde{f}(0)>0$, and that 0 is a non-degenerate critical point of the function $x\to\dfrac{\tilde{f}(x)}{\varphi(x)^{1/2}}$.

Then we can write

$$2\varphi(x)\tilde{f}'(x)-\tilde{f}(x)\varphi'(x)=Mx+o(|x|)\tag{2.18}$$

where M is an invertible matrix.

We will assume in the sequel that x is restricted to a neighbourhood $\mathcal{O}$ of 0, such that

$$\forall x\in\bar{\mathcal{O}},\quad \tilde{f}(x)>0 \text{ and dist }(\mathcal{O},\partial\Omega)=d_0>0$$

As suggested by (2.16), we perform the change of variable

$$\frac{1}{\lambda^{(N-2)/2}}=\frac{\varepsilon\tilde{f}(x)}{\alpha\varphi(x)}(1+\xi)\tag{2.19}$$

assuming *a priori* that $|\xi|<1/2$. We note that we then have

$$\frac{\partial K}{\partial\lambda}(\alpha,\lambda,x,\bar{v})=\frac{(N-2)^2\sigma_{N-1}}{2}\frac{\alpha\varepsilon\tilde{f}(x)}{\lambda^{N/2}}\xi+V_\lambda(\alpha,\lambda,x)$$

and

$$\frac{\partial K}{\partial x}(\alpha,\lambda,x,\bar{v})=-\frac{N(N-2)\sigma_{N-1}}{2}\frac{\varepsilon^2\tilde{f}(x)}{\varphi^2(x)}Mx+O(\varepsilon^2|\xi|)+V_x(\alpha,\lambda,x)$$

together with the estimates

$$\left.\begin{aligned}\frac{\partial K}{\partial\alpha}&=O[|\rho|+\varepsilon^2]\\ \frac{\partial K}{\partial\lambda}&=O[\varepsilon^{(2N-2)/(N-2)}(|\xi|+|\rho|)+\varepsilon^{2N/(N-2)}]\\ \frac{\partial K}{\partial x}&=O[\varepsilon^2(|\xi|+|\rho|+|x|)\\ &\qquad+(\varepsilon^3\text{ if }N=4;\ \varepsilon^{2N/(N-2)}|\operatorname{Log}\varepsilon|\text{ if }N>4)\end{aligned}\right\}\tag{2.20}$$

and

$$|\bar{v}|_{H_0^1}=O[\varepsilon^2\text{ if }N<6;\ \varepsilon^2|\operatorname{Log}\varepsilon|^{2/3}\text{ if }N=6;\ \varepsilon^p\text{ if }N>6]\tag{2.21}$$

These estimates will allow us to estimate the numbers A, B, C which were determined by (E.4). Indeed, if we take the scalar product in $H_0^1(\Omega)$ of equation (E.4) respectively with $P\delta_{\lambda,x}$, $\dfrac{\partial P\delta_{\lambda,x}}{\partial\lambda}$, $\dfrac{\partial P\delta_{\lambda,x}}{\partial x}$, we obtain a quasi-diagonal system of linear equations in A, B, C – the coefficients being estimated by a direct computation using (2.2), (2.4), (2.6):

$$\left.\begin{aligned}
&\langle P\delta_{\lambda,x}, P\delta_{\lambda,x}\rangle_{H_0^1} = N(N-2)\Gamma_1 + O\left(\frac{1}{\lambda^{N-2}}\right)\\
&\left\langle P\delta_{\lambda,x}, \frac{\partial P\delta_{\lambda,x}}{\partial\lambda}\right\rangle_{H_0^1} = O\left(\frac{1}{\lambda^{N-1}}\right)\\
&\left\langle P\delta_{\lambda,x}, \frac{\partial P\delta_{\lambda,x}}{\partial x_i}\right\rangle_{H_0^1} = O\left(\frac{1}{\lambda^{N-2}}\right)\\
&\left\langle \frac{\partial P\delta_{\lambda,x}}{\partial\lambda}, \frac{\partial P\delta_{\lambda,x}}{\partial\lambda}\right\rangle_{H_0^1} = \frac{N(N+2)\Gamma_2}{\lambda^2} + O\left(\frac{1}{\lambda^{N}}\right)\\
&\left\langle \frac{\partial P\delta_{\lambda,x}}{\partial\lambda}, \frac{\partial P\delta_{\lambda,x}}{\partial x_i}\right\rangle_{H_0^1} = O\left(\frac{1}{\lambda^{N-1}}\right)\\
&\left\langle \frac{\partial P\delta_{\lambda,x}}{\partial x_i}, \frac{\partial P\delta_{\lambda,x}}{\partial x_j}\right\rangle_{H_0^1} = N(N+2)\Gamma_3\lambda^2\delta_{ij} + O\left(\frac{1}{\lambda^{N-2}}\right)
\end{aligned}\right\} \quad (2.22)$$

$$\Gamma_2 = \int_{\mathbb{R}^N} \delta_{0,1}^{p-1}\left(\frac{\partial\delta_{0,1}}{\partial\lambda}\right)^2, \qquad \Gamma_3 = \int_{\mathbb{R}^N} \delta_{0,1}^{p-1}\left(\frac{\partial\delta_{0,1}}{\partial x_i}\right)^2$$

and the right-hand side being given by

$$\left\langle \frac{\partial K}{\partial v}, P\delta_{\lambda,x}\right\rangle_{H_0^1} = \frac{\partial K}{\partial\alpha}$$

$$\left\langle \frac{\partial K}{\partial v}, \frac{\partial P\delta_{\lambda,x}}{\partial\lambda}\right\rangle_{H_0^1} = \frac{1}{\alpha}\frac{\partial K}{\lambda}$$

$$\left\langle \frac{\partial K}{\partial v}, \frac{\partial P\delta_{\lambda,x}}{\partial x}\right\rangle_{H_0^1} = \frac{1}{\alpha}\frac{\partial K}{\partial x}$$

The solution of this system yields

$$A = O\left[\left|\frac{\partial K}{\partial\alpha}\right| + \frac{1}{\lambda^{N-3}}\left|\frac{\partial K}{\partial\lambda}\right| + \frac{1}{\lambda^N}\left|\frac{\partial K}{\partial x}\right|\right]$$

$$B = O\left[\frac{1}{\lambda^{N-3}}\left|\frac{\partial K}{\partial\alpha}\right| + \lambda^2\left|\frac{\partial K}{\partial\lambda}\right| + \frac{1}{\lambda^N}\left|\frac{\partial K}{\partial x}\right|\right]$$

$$C = O\left[\frac{1}{\lambda^{N}}\left|\frac{\partial K}{\partial\alpha}\right| + \frac{1}{\lambda^{N-1}}\left|\frac{\partial K}{\partial\lambda}\right| + \frac{1}{\lambda^2}\left|\frac{\partial K}{\partial x}\right|\right]$$

which allows us to estimate the expressions

$$\mathrm{B}\int_\Omega \nabla \frac{\partial^2 \mathrm{P}\,\delta_{\lambda,x}}{\partial\lambda^2}\nabla\bar{v}+\mathrm{C}\int_\Omega \nabla\frac{\partial^2 \mathrm{P}\,\delta_{\lambda,x}}{\partial\lambda\,\partial x}\nabla\bar{v}=O\left[\left(\frac{|\mathrm{B}|}{\lambda^2}+|\mathrm{C}|\right)|\bar{v}|_{\mathrm{H}_0^1}\right]$$

$$\mathrm{B}\int_\Omega \nabla \frac{\partial^2 \mathrm{P}\,\delta_{\lambda,x}}{\partial\lambda\,\partial x}\nabla\bar{v}+\mathrm{C}\int_\Omega \nabla\frac{\partial^2 \mathrm{P}\,\delta_{\lambda,x}}{\partial x^2}\nabla\bar{v}=O\,[(|\mathrm{B}|+\lambda^2|\mathrm{C}|)\,|\bar{v}|_{\mathrm{H}_0^1}]$$

using (2.20) and (2.21). We therefore conclude that the system of equations (E.1), (E.2), (E.3) is equivalent to

$$\left.\begin{aligned}\rho&=\mathrm{V}_1(\varepsilon,\rho,\xi,x)\\ \xi&=\mathrm{V}_2(\varepsilon,\rho,\xi,x)\\ x&=\mathrm{V}_3(\varepsilon,\rho,\xi,x)\end{aligned}\right\}\qquad (\mathrm{E}')$$

where V_1, V_2, V_3 are some continuous functions satisfying the estimates

$$\mathrm{V}_1=O(\rho^2+\varepsilon^2)$$

$$\mathrm{V}_2=O(|\rho|+\xi^2+\varepsilon^{2\,\mathrm{N}/(\mathrm{N}-2)})$$

$$\mathrm{V}_3=O\left(|\rho|+|\xi|+\begin{cases}\varepsilon \text{ if } \mathrm{N}=4\\ \varepsilon^{4/(\mathrm{N}-2)}|\operatorname{Log}\varepsilon| \text{ if } \mathrm{N}>4\end{cases}\right)+o(|x|)$$

The Brouwer fixed point theorem shows that for ε sufficiently small (E′) has a solution $(\rho_\varepsilon,\xi_\varepsilon,x_\varepsilon)$, which further satisfies:

$$\left.\begin{aligned}\rho_\varepsilon&=O(\varepsilon^2)\\ \xi_\varepsilon&=O(\varepsilon^{2/(\mathrm{N}-2)})\\ x_\varepsilon&=O(\varepsilon^{2/(\mathrm{N}-2)})\end{aligned}\right\}\qquad (2.23)$$

One easily checks that

$$u_\varepsilon=\alpha_\varepsilon\,\mathrm{P}\,\delta_{\lambda_\varepsilon,x_\varepsilon}+\bar{v}_\varepsilon+\varepsilon\bar{f}\qquad (2.24)$$

with $\alpha_\varepsilon=c_\mathrm{N}-\rho_\varepsilon$, $\dfrac{1}{\lambda_\varepsilon^{(\mathrm{N}-2)/2}}=\dfrac{\varepsilon\bar{f}(x_\varepsilon)}{\alpha_\varepsilon\,\varphi(x_\varepsilon)}(1+\xi_\varepsilon)$, and $\bar{v}_\varepsilon=\bar{v}_{\varepsilon,\alpha_\varepsilon,\lambda_\varepsilon,x_\varepsilon}$, a solution to (Q_ε) by construction, is such that

$$|\nabla u_\varepsilon|^2\to \mathrm{S}^{\mathrm{N}/2}\,\delta_0,\qquad |u_\varepsilon|^{p+1}\to\mathrm{S}^{\mathrm{N}/2}\,\delta_0\quad\text{when } \varepsilon\to 0$$

Moreover, if $f\geqq 0$, $u_\varepsilon>0$ on Ω. Indeed, multiplying the equation $-\Delta u_\varepsilon=|u_\varepsilon|^{p-1}u_\varepsilon+\varepsilon f$ by $u_\varepsilon^-=\max(0,-u_\varepsilon)$ and integrating on Ω, we get

$$\int_\Omega|\nabla u_\varepsilon^-|^2=\int_\Omega(u_\varepsilon^-)^{p+1}-\varepsilon\int_\Omega f u_\varepsilon^-\leqq\int_\Omega(u_\varepsilon^-)^{p+1}$$

On the other hand, the Sobolev inequality yields

$$\int_\Omega|\nabla u_\varepsilon^-|^2\geqq \mathrm{S}\left(\int_\Omega(u_\varepsilon^-)^{p+1}\right)^{2/(p+1)}$$

so that we have either $\int_\Omega (u_\varepsilon^-)^{p+1} \geqq S^{N/2}$, or $u_\varepsilon^- \equiv 0$. Remember that $u_\varepsilon^- \leqq |\bar{v}_\varepsilon|$, and $|\bar{v}_\varepsilon|_{p+1} \to 0$ as $\varepsilon \to 0$. Therefore, for ε sufficiently small, $u_\varepsilon^- \equiv 0$, and the strong maximum principle shows that $u_\varepsilon > 0$ on Ω. This concludes the proof of the first part of Theorem 1.

2.3. Category of the domain and multiplicity of the solutions

We are going to show the multiplicity of the solutions to (Q_ε) in relation to the category of the domain Ω, looking for solutions in the same form as before.

More precisely, for θ a positive constant to be chosen later on, we define for $\varepsilon > 0$, $d > 0$ the open set:

$$\mathcal{M}_{d,\varepsilon} = \left\{ (\rho, \lambda, x) \in \mathbf{R} \times \mathbf{R}_+^* \times \Omega \,/\, |\rho| < \theta\varepsilon^2. \right.$$
$$\left. d(x, \partial\Omega) > d, \frac{1}{\lambda^{(N-2)/2}} \in \left] \frac{\varepsilon \mathcal{J}(x)}{2 c_N \varphi(x)}, \frac{3 \varepsilon \mathcal{J}(x)}{2 c_N \varphi(x)} \right[\right\} \tag{2.25}$$

and the function

$$\mathcal{K}(\rho, \lambda, x) = K(\alpha, \lambda, x, \bar{v}) = J(\alpha P \delta_{\lambda, x} + \bar{v}) \tag{2.26}$$

on $\mathcal{M}_{d,\varepsilon}$, with $\alpha = c_N - \rho$, whose critical points provide us with solutions of (Q_ε). The first order derivatives of $\mathcal{K}$ are given by

$$\left.\begin{aligned} \frac{\partial \mathcal{K}}{\partial \rho} &= -\frac{\partial K}{\partial \alpha} \\ \frac{\partial \mathcal{K}}{\partial \lambda} &= \frac{\partial K}{\partial \lambda} + \left\langle \frac{\partial K}{\partial v}, \frac{\partial \bar{v}}{\partial \lambda} \right\rangle_{H_0^1} \\ \frac{\partial \mathcal{K}}{\partial x} &= \frac{\partial K}{\partial x} + \left\langle \frac{\partial K}{\partial v}, \frac{\partial \bar{v}}{\partial x} \right\rangle_{H_0^1} \end{aligned}\right\} \tag{2.27}$$

The first order derivatives of K having already been estimated, we are left with the evaluation of the products $\left\langle \frac{\partial K}{\partial v}, \frac{\partial \bar{v}}{\partial \lambda} \right\rangle_{H_0^1}$ and $\left\langle \frac{\partial K}{\partial v}, \frac{\partial \bar{v}}{\partial x} \right\rangle_{H_0^1}$.

To this end, we write $\frac{\partial \bar{v}}{\partial \lambda}$ in the form

$$\frac{\partial \bar{v}}{\partial \lambda} = w + a P \delta_{\lambda, x} + b \frac{\partial P \delta_{\lambda, x}}{\partial \lambda} + c \frac{\partial P \delta_{\lambda, x}}{\partial x} \tag{2.28}$$

with $(a b, c) \in \mathbf{R} \times \mathbf{R} \times \mathbf{R}^N$ and $w \in E_{\lambda, x}$. Thus we have

$$\left\langle \frac{\partial K}{\partial v}, \frac{\partial \bar{v}}{\partial \lambda} \right\rangle_{H_0^1} = a \left\langle \frac{\partial K}{\partial v}, P\delta_{\lambda, x} \right\rangle_{H_0^1} + b \left\langle \frac{\partial K}{\partial v}, \frac{\partial P\delta_{\lambda, x}}{\partial \lambda} \right\rangle_{H_0^1} + c \left\langle \frac{\partial K}{\partial v}, \frac{\partial P\delta_{\lambda, x}}{\partial x} \right\rangle_{H_0^1} = a \frac{\partial K}{\partial \alpha} + \frac{b}{\alpha} \frac{\partial K}{\partial \lambda} + \frac{c}{\alpha} \frac{\partial K}{\partial x} \quad (2.29)$$

Furthermore, if we take the scalar product in $H_0^1(\Omega)$ of (2.28) with respectively $P\delta_{\lambda, x}, \frac{\partial P\delta_{\lambda, x}}{\partial \lambda}, \frac{\partial P\delta_{\lambda, x}}{\partial x}$, we get a quasi-diagonal linear system in a, b, c, whose coefficients are given by (2.22) and the right-hand side by

$$\left\langle \frac{\partial \bar{v}}{\partial \lambda}, P\delta_{\lambda, x} \right\rangle_{H_0^1} = -\left\langle \bar{v}, \frac{\partial P\delta_{\lambda, x}}{\partial \lambda} \right\rangle_{H_0^1} = 0$$

$$\left\langle \frac{\partial \bar{v}}{\partial \lambda}, \frac{\partial P\delta_{\lambda, x}}{\partial \lambda} \right\rangle_{H_0^1} = -\left\langle \bar{v}, \frac{\partial^2 P\delta_{\lambda, x}}{\partial \lambda^2} \right\rangle_{H_0^1} = O\left(\frac{|\bar{v}|_{H_0^1}}{\lambda^2} \right)$$

$$\left\langle \frac{\partial \bar{v}}{\partial \lambda}, \frac{\partial P\delta_{\lambda, x}}{\partial x} \right\rangle_{H_0^1} = -\left\langle \bar{v}, \frac{\partial^2 P\delta_{\lambda, x}}{\partial \lambda \partial x} \right\rangle_{H_0^1} = O(|\bar{v}|_{H_0^1})$$

since $\left| \frac{\partial^2 P\delta_{\lambda, x}}{\partial \lambda^2} \right|_{H_0^1} = O\left(\frac{1}{\lambda^2} \right)$ and $\left| \frac{\partial^2 P\delta_{\lambda, x}}{\partial \lambda \partial x} \right|_{H_0^1} = O(1)$.

The solution of this system then yields

$$\left. \begin{aligned} a &= O\left(\frac{|\bar{v}|_{H_0^1}}{\lambda^{N-1} d^{N-2}} \right) \\ b &= O(|\bar{v}|_{H_0^1}) \\ c &= O\left(\frac{|\bar{v}|_{H_0^1}}{\lambda^2} \right) \end{aligned} \right\} \quad (2.30)$$

Using the fact that $\varphi(x) \sim \frac{1}{(2 d(x, \partial\Omega))^{N-2}}$ as $d(x, \partial\Omega) \to 0$ [R2], one sees that on $\mathcal{M}_{d, \varepsilon}$ we have $\frac{1}{\lambda^{(N-2)/2}} = O(\varepsilon d^{N-1})$, so that from (2.15), (2.16), (2.17) we get

$$\left.\begin{aligned}\frac{\partial K}{\partial \alpha} &= O(\varepsilon^2)\\ \frac{\partial K}{\partial \lambda} &= O\big(\varepsilon^{(2N-2)/(N-2)} d^{(N^2-2)/(N-2)} + \varepsilon^{(2N+2)/(N-2)} d^{(6N-6)/(N-2)} \\ &\qquad + \varepsilon^{(2N+6)/(N-2)} d^{(2N-2)/(N-2)}\big)\\ \frac{\partial K}{\partial \lambda} &= O\left(\varepsilon^2 d^{N-1} + \frac{\varepsilon^{(2N+2)/(N-2)}}{d^{(2N-2)/(N-2)}}\right)\end{aligned}\right\} \tag{2.31}$$

From (2.29), (2.30), (2.31) we conclude finally that

$$\left\langle \frac{\partial K}{\partial v}, \frac{\partial \bar{v}}{\partial \lambda} \right\rangle_{H_0^1} = O\,[|\bar{v}|_{H_0^1}(\varepsilon^{(2(N-1))/(N-2)} d^{(N(N-1))/(N-2)} + \varepsilon^{(2(N+1))/(N-2)} d^{(6(N-1))/(N-2)} + \varepsilon^{(2(N+3))/(N-2)} d^{(2(N-1))/(N-2)})] \tag{2.32}$$

In a similar way we find

$$\left\langle \frac{\partial K}{\partial v}, \frac{\partial \bar{v}}{\partial x} \right\rangle_{H_0^1} = O\left[|\bar{v}|_{H_0^1}\left(\varepsilon^{(2(N-3))/(N-2)} d^{((N-4)(N-1))/(N-2)} + \varepsilon^{(2(N-1))/(N-2)} d^{(2(N-1))/(N-2)} + \frac{\varepsilon^{(2(N+1))/(N-2)}}{d^{(2(N-1))/(N-2)}}\right)\right] \tag{2.33}$$

with (2.12) and (2.13) giving us the estimate

$$|\bar{v}|_{H_0^1} = O\,[\varepsilon^2 d^{N-1} + \varepsilon^p \text{ if } N<6;\ \varepsilon^2(1+|\operatorname{Log}\varepsilon|^{2/3} d^5) \text{ if } N=6;\ \varepsilon^p \text{ if } N>6] \tag{2.34}$$

We are now able to estimate the derivatives of $\mathcal{K}$ on the boundary of $\mathcal{M}_{d,\varepsilon}$. One sees easily using (2.15) and (2.27) that for d sufficiently small, and then ε sufficiently small, we have for all $(\rho, \lambda, x) \in \mathcal{M}_{d,\varepsilon}$

$$\frac{\partial \mathcal{K}}{\partial \rho}(-\theta\varepsilon^2, \lambda, x) < 0 < \frac{\partial \mathcal{K}}{\partial \rho}(\theta\varepsilon^2, \lambda, x) \tag{2.35}$$

for a good choice of θ, independent of d and ε. Likewise, combining (2.16), (2.27) with (2.32), (2.34) one gets under the same conditions

$$\frac{\partial \mathcal{K}}{\partial \lambda}\left(\rho, \frac{\varepsilon \bar{f}(x)}{2 c_N \varphi(x)}, x\right) < 0 < \frac{\partial \mathcal{K}}{\partial \lambda}\left(\rho, \frac{3\varepsilon \bar{f}(x)}{2 c_N \varphi(x)}, x\right) \tag{2.36}$$

Set $\Omega_d = \{x \in \Omega \mid d(x, \partial\Omega) > d\}$ and denote by $n(x)$ the outward normal vector at $x \in \partial\Omega_d$ to the boundary of Ω_d. One has the equivalence $\varphi'(x) \sim \dfrac{N-2}{2^{N-1} d^{N-1}} n(x)$ as $d \to 0$ [R2], and $\bar{f}'(x).n < 0$ for small enough d

by the strong maximum principle (here f is assumed to be positive). Combining then (2.17), (2.27) with (2.33), (2.34) we get, again with the same conditions on d and ε

$$\frac{\partial \mathcal{K}}{\partial n}(\rho, \lambda, x) . n(x) > 0, \qquad \forall x \in \partial \Omega_d \tag{2.37}$$

At this point, we can deduce from the Ljusternik-Schnirelman theory that $\mathcal{K}$ has at least as many critical points in $\mathcal{M}_{d,\varepsilon}$ as the category of $\mathcal{M}_{d,\varepsilon}$. Now cat $\mathcal{M}_{d,\varepsilon} = \operatorname{cat} \Omega_d$ and $\operatorname{cat} \Omega_d = \operatorname{cat} \Omega$ for d sufficiently small, Ω being smooth.

One proves as before that the corresponding solutions of (Q_ε) are strictly positive. By construction, each of them concentrates at a point of Ω as $\varepsilon \to 0$, and (E) shows that these points are critical for the function $x \to \dfrac{\tilde{f}(x)}{\varphi(x)^{1/2}}$. This concludes the proof of the second part of Theorem 1.

Remark. – Combining the estimates that we obtain here with those in [R1], we can prove considering the problem

$$\left.\begin{aligned} -\Delta u = |u|^{p-1} u + \varepsilon u + \varepsilon f \quad \text{on } \Omega \\ u = 0 \quad \text{on } \partial \Omega \end{aligned}\right\} \tag{R_ε}$$

the following results (assuming f is sufficiently regular):

If $N = 4$, 5, and if $x_0 \in \Omega$, $\tilde{f}(x_0) > 0$, is a non-degenerate critical point of the function $x \to \dfrac{\tilde{f}(x)}{\varphi(x)^{1/2}}$.

If $N = 6$, and if $x_0 \in \Omega$, $\tilde{f}(x_0) > -1/2$, is a non-degenerate critical point of the function $x \to \dfrac{\tilde{f}(x) + 1/2}{\varphi(x)^{1/2}}$.

If $N > 6$, and if $x_0 \in \Omega$ is a non-degenerate critical point of the function φ then there exists a family (u_ε) of solutions to (R_ε) concentrating at x_0 as $\varepsilon \to 0$.

If $f \geqq 0$, these solutions are strictly positive.

Finally, let us remark that for $N = 5$, the same result holds if x_0 is a non-degenerate critical point of $\tilde{f}$, with $\tilde{f}(x_0) = 0$.

APPENDIX

Proof of Proposition 1

H. Brézis and L. Nirenberg consider in [BN2] the problem

$$\left.\begin{array}{c} -\Delta u=\mu\,(u+\varphi)^p \quad \text{on } \Omega \\ u\geqq 0 \quad \text{on } \Omega \\ u=0 \quad \text{on } \partial\Omega \end{array}\right\}$$

where μ is a fixed real number, $\varphi\in H^1(\Omega)$, $\varphi\geqq 0$, and $\varphi\not\equiv 0$.

They prove the existence of $\mu^*<+\infty$ such that for all $\mu\in[0,\mu^*]$ there exists a smallest regular solution $\underline{u}(\mu)$ to the problem, whereas there is no solution for $\mu>\mu^*$. This branch of solutions is obtained applying the implicit function theorem starting from the point $\underline{u}(0)=0$. Furthermore, for $\mu\in[0,\mu^*[$, the first eigenvalue of the linearized problem at $\underline{u}(\mu)$ is positive, so that using the saddle lemma it is possible to prove the existence of a second solution – also regular.

Now if we take $f\in H^{-1}(\Omega)$, $f\geqq 0$, and if we define φ by

$$\left.\begin{array}{c} -\Delta\varphi=f \quad \text{on } \Omega \\ \varphi=0 \quad \text{on } \partial\Omega \end{array}\right\}$$

we have: $\varphi\in H_0^1(\Omega)$, $\varphi>0$, and $\varphi\not\equiv 0$ – hence the existence of two solutions to the problem

$$\left.\begin{array}{c} -\Delta v=\mu\, v^p+f \quad \text{on } \Omega \\ v>0 \quad \text{on } \Omega \\ v=0 \quad \text{on } \partial\Omega \end{array}\right\}$$

where we set $v=u+\varphi$, or, writing $w=\mu^{1/(p-1)}v$, the existence of two solutions to the problem

$$\left.\begin{array}{c} -\Delta w=w^p+\mu^{1/(p-1)}f \quad \text{on } \Omega \\ w>0 \quad \text{on } \Omega \\ w=0 \quad \text{on } \partial\Omega \end{array}\right\}$$

for $\mu\in]0,\mu^*[$. This is the announced result.

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