

Some remarks on the number of solutions of some nonlinear elliptic problems

by

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ABSTRACT. — This paper studies the multiplicity of solutions of the problem:

$$\begin{aligned} -\Delta u &= f(x, u) + t\phi && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega \end{aligned}$$

depending on the parameter t for certain terms ϕ . The main hypothesis on f is that, setting $f_{\pm} = \lim_{s \rightarrow \pm\infty} \frac{f(x, s)}{s}$, in the interval $]f_-, f_+[$ there are eigenvalues of $-\Delta$ with the Dirichlet homogeneous boundary conditions on $\partial\Omega$. Studying the « bifurcation from infinity » of the solutions of the problem, multiplicity or sharp multiplicity results are obtained in the case that such eigenvalues are the first two or the first three or only one and simple. In such a way we improve or sharpen previous results of Lazer and McKenna, Hofer, Gallouet and Kavian and of the author.

RÉSUMÉ. — Nous étudions le nombre de solutions du problème :

$$\begin{aligned} -\Delta u &= f(x, u) + t\phi && \text{dans } \Omega \\ u &= 0 && \text{sur } \partial\Omega, \end{aligned}$$

où ϕ est fixée et t est un paramètre. L'hypothèse principale sur f est la suivante. Posons $f_{\pm} = \lim_{s \rightarrow \pm\infty} \frac{f(x, s)}{s}$; alors l'intervalle $]f_-, f_+[$ contient au moins une valeur propre de l'opérateur $(-\Delta)$ avec condition de Dirichlet au bord. Nous étudions la « bifurcation de l'infini » des solutions du problème pour obtenir une estimation (dans certains cas exacte) du nombre

de solutions. Nous considérons principalement les cas où $]f_-, f_+[$ contient les deux premières valeurs propres, les trois premières, ou seulement une valeur propre simple. Nos résultats améliorent ou précisent certains travaux de Lazer et McKenna, Hofer, Gallouët et Kavian, et de nous-même.

1. INTRODUCTION

Let Ω be a given open bounded domain, $\Omega \subset \mathbb{R}^n$ with smooth boundary $\partial\Omega$. The quasi-linear Dirichlet problem:

$$(1.1) \quad \begin{aligned} -\Delta u &= f(u) + h && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega \end{aligned}$$

has recently been studied by many authors under the assumption:

$$(F_1) \quad \text{There exist } f_{\pm} = \lim_{s \rightarrow \pm\infty} \frac{f(s)}{s} \in \mathbb{R}, \quad f_- < f_+.$$

We denote by $\lambda_1, \lambda_2, \dots, \lambda_j, \dots$ the sequence of the eigenvalues of $-\Delta$ on $H_0^1(\Omega)$, repeating each one as many times as its multiplicity. We also denote by ϕ_k a given eigen-function corresponding to the k -th eigenvalue λ_k and which we suppose is normalized in $L^2(\Omega)$. It is well known that the first eigenvalue λ_1 is simple and that ϕ_1 has a constant sign, we choose $\phi_1 > 0$.

The problem (1.1) has a particular interest when there are some eigenvalues λ_k in $]f_-, f_+[$. It has been first studied when $f_- < \lambda_1 < f_+ < \lambda_2$ in [1] and subsequently in [2] [3] [4]. Less sharp results have been obtained in [5] [6] [7] when the last inequality: $f_+ < \lambda_2$ is dropped. Actually in [6], the nonlinearity f is allowed to grow more than linearly. The growth restrictions have also been subsequently relaxed in [8] [9]. In these papers the problem (1.1) has been substantially studied taking $h = t\phi_1$ and showing that (1.1) has no solution for large positive values of t and has at least two solutions for large negative t .

More recently higher multiplicity results have been obtained, beginning with [10], assuming $f_- < \lambda_1 < \lambda_2 < f_+ < \lambda_3$. The hypotheses in [10] have been relaxed in [11] [12]. Stronger results were obtained in [13] [14]. These prove a particular case (i. e. for $k = 2$) of a conjecture formulated in [15] and proved therein for the ordinary differential equation.

This conjecture states that if $f_- < \lambda_1$ and $\lambda_k < f_+ < \lambda_{k+1}$ then problem (1.1) has at least $2k$ solutions for $h = t\phi_1$ and large negative t . One can also suppose this estimate to be somehow optimal. In this paper we prove

a result in this direction stating a sharp estimate on the number of solutions for $k = 2$, moreover we shall prove the existence of exactly six solutions in some cases where $\lambda_3 < f_+$. In this situation in [14] the existence of at least five solutions is proved. We also study (1.1) when in $[f_-, f_+]$ there is exactly one eigenvalue λ_k and it is simple. This problem was first studied in a particular case in [16] and subsequently more generally in [17] [18].

We also give these results in a sharp form, therefore we generalize those in [19]. Finally further multiplicity results under this last assumption have been given in [20] [21]. In this case we also compute the exact number of the solutions.

The paper is divided in two sections. In the first we make a general study of the problem, in the second we give the applications stated above.

In the following we shall denote by E the Hilbert space $L^2(\Omega)$ with norm and scalar product respectively denoted by $\| \cdot \|$ and (\cdot, \cdot) . E is ordered by the positive cone P of the functions a. e. positive in Ω . It is well known that this ordering makes E a lattice. We adopt the usual notation:

$$u^+ = \sup(u, 0) \quad u^- = (-u)^+.$$

2. GENERAL REMARKS

We assume (F_1) in the stronger form:

$$(F_2) \quad \begin{array}{l} a) \quad g(u) = f(u) - f_+u^+ + f_-u^- \quad \text{is a bounded function} \\ b) \quad f \in C^1, \quad f_{\pm} = \lim_{s \rightarrow \pm \infty} f'(s) \end{array}$$

We point out that $(F_2)(a)$ is assumed in order to simplify the computations while $(F_2)(b)$ is a relevant condition which cannot be completely dropped. This condition has been assumed previously so as to determine the exact number of solutions of some elliptic problems in [19] and in [21]. Moreover, in the case $f_- < \lambda_1 < f_+ < \lambda_2$ the exact number of solutions has been studied assuming that f is a smooth convex function, see e. g. [4], and, of course, this implies $(F_2)(b)$.

In order to study (1.1) we also consider the problem

$$(2.1) \quad \begin{array}{ll} -\Delta u = f_+u^+ - f_-u^- + \phi & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{array}$$

where ϕ is a given term in E .

In view of our applications we always suppose that the set $\{x \in \Omega; \phi(x) = 0\}$ has zero Lebesgue measure, however this will not be necessary in all cases but will avoid some difficulties. This condition implies, see [22], page 53 and [19], prop. 1, that, if u is a solution of (2.1), also the set $\{x \in \Omega; u(x) = 0\}$

has zero measure. We shall denote the Lebesgue measure by μ . Given u , we also denote by $\chi(u)$ the characteristic function of the positive set of u , i. e.

$$[\chi(u)](x) = \begin{cases} 1 & \text{if } u(x) > 0 \\ 0 & \text{if } u(x) \leq 0 \end{cases}$$

PROPOSITION 2.2. — Given $p, q > 1$, $q < p$ let us consider the operator $(\cdot)^+$ from $L^p(\Omega)$ into $L^q(\Omega)$ which sends u into u^+ . If $u \in L^p$ is such that $\mu(\{x \in \Omega; u(x) = 0\}) = 0$, then, given $\varepsilon > 0$, there exists a neighbourhood U of u such that, if we write: $v^+ - \chi(u)v = z(v)$ for $v \in U$, then the function z is a Lipschitz mapping with Lipschitz constant less or equal than ε from U into L^q .

Proof. — Let $\frac{1}{r} = 1 - \frac{q}{p}$. Given ε and having that the convergence in L^p implies the convergence in measure, we choose U in such a way that $\forall v \in U$.

$$(2.3) \quad \mu(\{x: |v(x) - u(x)| > a\}) < \frac{1}{4} \left(\frac{\varepsilon}{2}\right)^{qr}$$

and the constant a is such that:

$$(2.4) \quad \mu(\{x: |u(x)| < a\}) < \frac{1}{4} \left(\frac{\varepsilon}{2}\right)^{qr}$$

This can be done by using the assumption $\mu(\{x: u(x) = 0\}) = 0$. We denote by $\Omega(\varepsilon, v)$ the union of the two sets appearing on the left-hand side of (2.3) and (2.4). One has:

$$(2.5) \quad \forall x \notin \Omega(\varepsilon, v): \quad v^+(x) = [\chi(u)](x) \cdot v(x)$$

Therefore, for given $v, w \in U$, the following holds:

$$(2.6) \quad \begin{aligned} \|(v^+ - \chi(u)v) - (w^+ - \chi(u)w)\|_{L^q}^q &= \int_{\Omega} |v^+ - \chi(u)v - w^+ + \chi(u)w|^q dx \leq \\ &\leq \int_{\Omega(\varepsilon, v) \cup \Omega(\varepsilon, w)} (|v^+ - w^+| + |\chi(u)(v - w)|)^q dx \leq \int_{\Omega(\varepsilon, v) \cup \Omega(\varepsilon, w)} 2^q |v - w|^q dx. \end{aligned}$$

Using Hölder inequality and taking into account that

$$\mu(\Omega(\varepsilon, v) \cup \Omega(\varepsilon, w)) < \left(\frac{\varepsilon}{2}\right)^{qr}, \quad \text{one has:}$$

$$(2.7) \quad \begin{aligned} \|(v^+ - \chi(u)v) - (w^+ - \chi(u)w)\|_{L^q}^q &\leq \\ &\leq \left(\int_{\Omega(\varepsilon, v) \cup \Omega(\varepsilon, w)} 2^{rq} dx\right)^{1/r} \left(\int_{\Omega(\varepsilon, v) \cup \Omega(\varepsilon, w)} |v - w|^p dx\right)^{q/p} \leq \varepsilon^q \|v - w\|_{L^p}^q. \end{aligned}$$

that is:

$$\|z(v) - z(w)\|_{L^q} \leq \varepsilon \|v - w\|_{L^p} \quad \blacksquare$$

REMARK 2.8. — The previous proposition states in particular that $(.)^+$ is Frechet differentiable in u . However, since it cannot be C^1 in a neighbourhood of u , one needs all of proposition 2.2 in order to apply the local inversion theorem.

Now we set: $a(u) = f_+\chi(u) - f_-\chi(-u)$ when $\mu(\{x : u(x) = 0\}) = 0$. By the previous remark u is a nondegenerate solution of (2.1) iff the problem

$$(2.9) \quad \begin{aligned} -\Delta v &= a(u)v && \text{in } \Omega \\ v &= 0 && \text{on } \partial\Omega \end{aligned}$$

has only the trivial solution $v = 0$.

By the local inversion theorem, every nondegenerate solution of (2.1) is an isolated solution. Moreover the above results imply that it has non-zero local degree.

We denote by Σ the set of the pairs (f_+, f_-) such that (2.1) has a solution $u \neq 0$ for $\phi = 0$. A complete general description of Σ is not known. However some results in this direction are known, see [23] [17] [18], and they cover all the cases which we treat in this paper.

We denote by K the operator $(-\Delta)^{-1}$ from $H^{-1}(\Omega)$ into $H_0^1(\Omega)$ and we consider it as a compact operator on E in view of Sobolev's imbedding theorems. The following proposition generates some interest in finding nondegenerate solutions of (2.1).

PROPOSITION 2.10. — Assume that j is a positive integer and that (2.1) has j nondegenerate solutions. Also assume $(F_2)(a)$ holds and $h = t\phi$ in (1.1). Then there exist $t_0 \in \mathbb{R}$ such that (1.1) has at least j solutions if $t > t_0$.

Proof. — Let u_1, u_2, \dots, u_j be j nondegenerate solutions of (2.1). Since the operator on T_0 on E which sends u into $u - K(f_+u^+ - f_-u^-)$ has in u_i a derivative with a continuous inverse, we can choose a neighbourhood U_i of u_i in such a way that the norm of $T_0(u) - K\phi$ is bounded from below by a positive constant c if $u \in \partial U_i$. We can also suppose that U_1, U_2, \dots, U_j are disjoint sets and that $\forall i$:

$$(2.11) \quad \text{deg}(T_0, U_i, K\phi) \neq 0$$

as shown in [24], Theorem 4.7.

We denote by g_* the Nemytskii operator induced by g and we set $T = T_0 - Kg_*$.

It is easy to verify that, if $t > t_0 = c^{-1} \|K\| (\mu(\Omega))^{\frac{1}{2}} \sup |g|$ the homotopy $H(s, u) = (1-s)T_0u + sTu$ for $s \in [0, 1]$ is admissible in tU_i in order to compute the Leray-Schauder topological degree. Therefore we get:

$$(2.12) \quad \text{deg}(T, tU_i, Kt\phi) = \text{deg}(T_0, tU_i, Kt\phi) = \text{deg}(T_0, U_i, K\phi) \neq 0$$

using the positive homogeneity of T_0 .

Since (2.12) holds for any $i = 1, 2, \dots, j$ and the sets tU_i are disjoint, the statement easily follows. ■

In order to give a sharp estimate of the number of solutions of (1.1) we need the following lemmas.

LEMMA 2.13. — *Let $u \in E$ be such that $\mu(\{x \in \Omega : u(x) = 0\}) = 0$ and let $(u_n)_{n \in \mathbb{N}}$ be a sequence of elements of E converging in measure to u . Moreover assume (F_2) (b) holds and denote by f'_* the Nemytskii operator induced by f' .*

Then for any given sequence $(t_n)_{n \in \mathbb{N}}$ of positive real numbers such that $t_n \rightarrow +\infty : f'_(t_n u_n)$ converges to $a(u)$ in $L^p(\Omega)$ for any $1 < p < +\infty$.*

Proof. — Let $\varepsilon > 0$ be given and fix $M > 0$ such that for $|s| > M$:

$$(2.14) \quad |f'(s) - f_{\pm}| < \varepsilon$$

(with the \pm sign taken according to the sign of s). Furthermore let us choose $\alpha > 0$ in order to have: $\mu\{x : |u(x)| < 2\alpha\} < \varepsilon$. Finally we find $v \in \mathbb{N}$ such that, for $n > v$, there exists a subset $\Omega_n \subset \Omega$ such that $\mu(\Omega_n) < \varepsilon$ and $\sup_{\Omega \setminus \Omega_n} |u - u_n| < \alpha$. These conditions imply that, out of a set of measure less or equal than $2\varepsilon : |u| > \alpha, |u_n| > \alpha, uu_n > 0$. Moreover if we also take v such that: $t_n > \alpha^{-1}M$ for $n > v$, this will imply that, out of the same set $|f'_*(u_n) - au| < \varepsilon$. Therefore, by easy computations:

$$\|f'_*(u) - a(u)\|_{L^p} < (2\varepsilon)^{1/p} 2 \sup |f'| + (\mu(\Omega))^{1/p} \varepsilon. \quad \blacksquare$$

Given $a \in L^{\frac{n}{2}}(\Omega)$ one can consider the eigenvalue problem

$$(2.15) \quad \begin{aligned} -\Delta v &= \lambda v && \text{in } \Omega \\ v &= 0 && \text{on } \partial\Omega \end{aligned}$$

It is well known that, if $a > 0$ in a set of positive measure, then the positive numbers λ for which (2.15) has a nontrivial solution are the terms of a sequence $\lambda_1(a), \lambda_2(a), \dots, \lambda_j(a), \dots$ diverging to $+\infty$. Since each eigenvalue λ_j has finite multiplicity, we can repeat it in the sequence as many times as its multiplicity.

It is also known (see [25]) that, for any j, λ_j is a continuous and strictly decreasing function in $L^{n/2}(\Omega)$. Therefore, for $p \geq \frac{n}{2}$, lemma 2.13 yields the following:

COROLLARY 2.16. — *Let $u \in E$ be such that $\mu(\{x \in \Omega : u(x) = 0\}) = 0$ and suppose $\lambda_j(a(u)) \neq 1 \forall j \in \mathbb{N}$. Then there exists a neighbourhood U of u and a positive real number t_0 such that $\forall t > t_0, \forall v \in tU, \forall j \in \mathbb{N} : \lambda_j(f'_*(v)) \neq 1$ and $\text{sign}[(\lambda_j(f'_*(v)) - 1)] = \text{sign}[(\lambda_j(a(u)) - 1)]$.*

We also need the following lemmas.

LEMMA 2.17. — Assume $(F_2)(a)$ and that $(f_+, f_-) \notin \Sigma$ and let $\phi \in E$ be given. Assume that U is a neighbourhood of all the solutions of (2.1). Then there exist $t_0 \in \mathbb{R}$ such that (1.1) has no solution $u \in E \setminus U$ for $h = t\phi$, $t > t_0$.

Proof. — Choose by contradiction a sequence $(u_n)_{n \in \mathbb{N}}$ in $E \setminus U$ and a sequence of real numbers $(t_n)_{n \in \mathbb{N}}$ diverging to $+\infty$ in such a way that $t_n u_n$ is a solution of (1.1) for $h = t_n \phi$. Dividing both the sides of (1.1) by t_n one gets:

$$(2.18) \quad \begin{aligned} -\Delta u_n &= f_+ u_n^+ - f_- u_n^- + t_n^{-1} g_{**}(u_n) + \phi && \text{in } \Omega \\ u_n &= 0 && \text{on } \partial\Omega \end{aligned}$$

The assumption $(f_+, f_-) \notin \Sigma$ implies that the sequence u_n is bounded in E and this implies by (2.18) that it is also bounded in $H^2(\Omega)$. By the compactness of the Sobolev's imbedding it has a subsequence converging to some $u \in E \setminus U$. Taking the limit in (2.18) one finds that u is a solution of (1.1), which is a contradiction since $u \notin U$. ■

LEMMA 2.19. — Assume $(f_+, f_-) \notin \Sigma$ and that all the possible solutions of (1.1) are nondegenerate. Then (1.1) has a finite number of solutions.

Proof. — The statement follows from a simple compactness argument since remark 2.8 implies that any solution is isolated. ■

Collecting the results in this section we get the following:

THEOREM 1. — Let (F_2) hold and let $(f_+, f_-) \notin \Sigma$. Moreover suppose that (2.1) has no degenerate solution. Then (2.1) has a finite number j of solutions and (1.1) has exactly j solutions for $h = t\phi$ and t large positive, and they are nondegenerate.

Proof. — By the previous lemma (1.1) has a finite number of solutions u_1, u_2, \dots, u_j . For any $i = 1, 2, \dots, j$ we choose a neighbourhood U_i in such a way that the results of proposition 2.10 and Corollary 2.16 hold. By proposition 2.10, (1.1) has at least a solution in each U_i for $h = t\phi$ and large positive t . Using Corollary 2.16 we find that these solutions are not degenerate since the u_i are not degenerate. Moreover for any possible solution $u \in tU_i$ we get:

$$(2.20) \quad \text{deg loc } (T, u) = \text{deg } (T, tU_i, Kt\phi)$$

and this implies that one has exactly a solution u in each U_i . Finally lemma 2.17 states that (1.1) has no solution $u \notin \bigcup_{i=1}^j tU_i$. ■

REMARK 2.21. — In order to apply the previous theorem one has in general to prove « *a priori* » that $(f_+, f_-) \notin \Sigma$, and that (2.1) has no degenerate solution. This does not seem to be a simple matter in the general case. However we can prove this in some cases of particular interest, as we shall do in the next section.

3. APPLICATIONS

We treat first the case $f_- < \lambda_1 < \lambda_2 < f_+ < \lambda_3$. Multiplicity results under this assumption have been given in several recent papers, see e. g. [13], [14]. In the last two papers it is proved that, if one assumes (F_1) , then (1.1) has at least four solutions for $h = t\phi_1$ and large negative t . In order to apply Theorem 1, we prove:

LEMMA 3.1. — Assume $f_- < \lambda_1$ and $f_+ \leq \lambda_k$ for a given integer $k > 2$, take $\phi = -\phi_1$. Then if u is a solution of (2.1) which changes sign in Ω , one has

$$(3.2) \quad v_1(a(u)) < 1 < v_{k-1}(a(u))$$

Proof. — We use in a relevant way that the functions v_j are strictly decreasing in $L^{n/2}(\Omega)$. Let us point out that (2.1) has the positive solution $\hat{u} = (f_+ - \lambda_1)^{-1}\phi_1$ and the negative solution $\hat{u} = (f_- - \lambda_1)^{-1}\phi_1$. Writing (2.1) for u and \hat{u} and subtracting we get:

$$(3.3) \quad -\Delta(\hat{u} - u) = f_+(\hat{u} - u^+) - f_-u^-$$

Let us use the notation $\hat{a} = \frac{f_+(\hat{u} - u^+) + f_-u^-}{\hat{u} - u}$.

We have the inequalities:

$$(3.4) \quad f_- < a(u) < \hat{a} < f_+$$

By (3.3) $v_j(\hat{a}) = 1$ for some j and by (3.4) this j belongs to $\{1, 2, \dots, k-1\}$. One makes similar computations with \tilde{u} and finds a function \tilde{a} such that $v_{j'}(\tilde{a}) = 1$ for some $j' \in \{1, 2, \dots, k-1\}$ and

$$(3.5) \quad f_- < \tilde{a} < a(u) < f_+$$

Since:

$$(3.6) \quad 1 = v_j(\hat{a}) \leq v_{k-1}(\hat{a}) < v_{k-1}(a(u))$$

$$(3.7) \quad v_1(a(u)) < v_1(\tilde{a}) \leq v_{j'}(\tilde{a}) = 1$$

the statement follows. ■

LEMMA 3.8. — Assume $f_- < \lambda_1 < \lambda_2 < f_+ < \lambda_3$. Then, for $\phi = -\phi_1$, (2.1) has exactly four solutions and they are nondegenerate.

Proof. — The statement follows from the previous lemma which ensures that any solution which changes sign is nondegenerate and has local degree -1 . Since we know that the solutions of constant sign only are \tilde{u} and \hat{u} and that they have local degree 1, by using the equality:

$$(3.9) \quad \deg(T_0, B(0, r), -K\phi_1) = 0$$

which is proved in [23] for large positive r , we complete the proof. ■

REMARK 3.10. — The solutions of unconstant sign also have a variational characterization since they can be found by using the mountain-pass theorem, [26], Theorem 2.1. Therefore the computation of the local degree can be also made by using the results in [13].

REMARK 3.11. — In the case $f_+ = \lambda_k$ one has a branch of positive solutions which are given by $\hat{u} + \psi$ for ψ belonging to a closed convex bounded and absorbing neighbourhood A of zero in the eigenspace V_k corresponding to λ_k . In the following lemma we study some properties of this set of solutions.

LEMMA 3.12. — *Suppose $f_+ = \lambda_k$, $\phi = -\phi_1$. Then $\hat{u} + A$ is an isolated set of solutions and, if one also assumes $\lambda_{k-1} < \lambda_k$, it has local degree $(-1)^{k-1}$.*

Proof. — By continuity choose a neighbourhood C of $\hat{u} + A$ such that if $u \in \bar{C}$ and $\mu(\{x: u(x) = 0\}) = 0$ then $v_{k-1}(a(u)) < c < 1$. Of course lemma 3.1 implies that in \bar{C} there is no solution which changes sign. This situation is preserved if we make a little perturbation of the data taking $f_+ < \lambda_k$. In this case lemma (3.1) tells that in \bar{C} there is only the positive solution and, since it has local degree $(-1)^{k-1}$ the lemma is proved.

REMARK 3.13. — If $k = 3$, by applying the previous lemma and lemma 3.1 and arguing as in lemma 3.8 we find that, for $f_- < \lambda_1$ and $\lambda_2 < f_+ = \lambda_3$ and $\phi = -\phi_1$, problem (2.1) has exactly two solutions of unconstant sign.

LEMMA 3.14. — *Suppose λ_2 and λ_3 be simple and $f_- < \lambda_1$ be given. Then there exist $\varepsilon > 0$ such that if $\lambda_3 < f_+ < \lambda_3 + \varepsilon$ then (2.1) has exactly six solutions for $\phi = -\phi_1$ and they are nondegenerate.*

Proof. — We know that for $f_+ = \lambda_3$ there are exactly two solutions of unconstant sign and that they are nondegenerate. Therefore they are preserved for small perturbations of f_+ . Moreover, since the preceding lemma states that $\deg(T_0, C, -K\phi_1) = 1$ and, by continuity, for ε small enough all the solutions which are out of C and which change sign have local degree -1 , we have exactly two solutions which change sign out of C ; finally for any solution $u \in C$, which changes sign, by lemma 3.1 and a conti-

nunity argument one gets: $v_2(a(u)) < 1 < v_3(a(u))$. Since the positive solution \tilde{u} belongs to C and it has local degree -1 using again that

$$\text{deg}(\mathbb{T}_0, C, -K\phi_1) = 1$$

one concludes that also in C there are exactly three solutions and that they are not degenerate. ■

Using these lemmas we can now give some results concerning the problem (1.1). We shall always choose $h = t\phi_1$ and we shall refer to (2.1) assuming $\phi = \phi_1$.

THEOREM 2. — *Assume (F_2) hold and let: $f_- < \lambda_1 < \lambda_2 < f_+$. Then there exist $t_0 \in \mathbb{R}$ such that (1.1) has at least four solutions for $t < t_0$, Moreover if we also assume $f_+ < \lambda_3$, then the solutions are exactly four and they are nondegenerate.*

Proof. — The first part of the statement is proved in [13] when $f_+ \neq \lambda_j \forall j$. Therefore we assume $f_+ = \lambda_k$ and choose k in such a way that $\lambda_{k-1} < \lambda_k$. We use the ideas in lemma 2.10 to find a neighbourhood U of \tilde{u} and $t_0 \in \mathbb{R}$ such that (1.1) has a solution in tU for $t < t_0$ (one can observe that this is the negative solution found in [11] Theorem 1). After we use lemma 2.13 to state that this negative solution is nondegenerate and that for any $u \in tC$: $v_{k-1}(f'_*(u)) < 1$. In [13], it is proved that (1.1) has also a solution \bar{u} , found by using the mountain pass theorem and that it has local degree -1 . Since $v_2(f'_*(\bar{u})) \geq 1$, as shown in [13] and $k \geq 3$, it follows that $\bar{u} \notin tC$. We argue as in Lemma 2.10 to prove that: $\text{deg}(\mathbb{T}, tC, Kt\phi_1) \neq 0$. Therefore one gets the existence of a third solution in tC and since $\text{deg}(\mathbb{T}, B(0, r), Kt\phi_1) = 0$ if r is large enough (depending on t), see [23], one also has:

$$\begin{aligned} \text{deg}(\mathbb{T}, B(o, r) \setminus (tC \cup \{ \tilde{u}, \bar{u} \}), Kt\phi_1) &= \text{deg}(\mathbb{T}, B(o, r), Kt\phi_1) \\ (3.15) \quad &- \text{deg}(\mathbb{T}, tC, Kt\phi_1) - \text{deg} \text{loc}(\mathbb{T}, \tilde{u}) - \text{deg} \text{loc}(\mathbb{T}, \bar{u}) = \\ &= - \text{deg}(\mathbb{T}, tC, Kt\phi_1) - 1 + 1 \neq 0. \end{aligned}$$

So we finally find a fourth solution in $B(o, r) \setminus tC$.

The last part of the statement immediately follows from Theorem 1 and lemma 3.8. ■

REMARK 3.16. — In the previous proof we used $(F_2)(b)$ also to prove an existence result. It is not difficult to observe that assumption is not completely essential in this case. However if one wants to drop it, one can work as in [11] § 3 in order to prove $\bar{u} \notin tC$. This requires a variational approach to (1.1) and we do not want to introduce it here.

The following is a straightforward consequence of Theorem 1 and lemma 3.14.

THEOREM 3. — *Suppose λ_2 and λ_3 be simple and $f_- < \lambda_1$ be given. Then there exist $\varepsilon > 0$ such that if $\lambda_3 < f_+ < \lambda_3 + \varepsilon$ then (1.1) has exactly six distinct solutions for large negative t and they are nondegenerate.*

In the following we shall always suppose that λ_k is a given simple eigenvalue, $k \neq 1$, and that $\lambda_{k-1} < f_- < \lambda_k < f_+ < \lambda_{k+1}$. Problem (1.1) has been studied in this case in [17]. In that paper it is proved that there exists a unique constant $C(f_+, f_-)$ such that (2.1) has a solution u , $(u, \phi_k) = 1$, for $\phi = C(f_+, f_-)\phi_k$. The function C defined in this way on $]\lambda_{k-1}, \lambda_{k+1}[^2$, turns out to be continuous and strictly decreasing in each of the two variables, moreover: $C(\lambda_k, \lambda_k) = 0$. Using the positive homogeneity of (2.1), one easily sees that $C(f_+, f_-) > 0$ is equivalent to ask that (2.1) has a solution u , $(u, \phi_k) \geq 0$ for $\phi = \phi_k$. We refer to [17] for more informations on the function C and on the possibility to compute its sign. We use it now to give the following result.

THEOREM 4. — *Suppose (F_2) holds and $C(f_+, f_-) > 0$, take $h = t\phi_k$. Then for large positive t (1.1) has exactly a solution u such that $(u, \phi_k) \geq 0$ and it is nondegenerate, for large negative t (1.1) has no solution u such that $(u, \phi_k) \geq 0$.*

Proof. — The theorem follows from a version of Theorem 1 on the half space $\{u : (u, \phi_k) \geq 0\}$, which can be proved in the same way, if we show that any solution u of (2.1) is nondegenerate if $\phi = \phi_k$. Suppose by contradiction that it is not true and let v be a nonzero function such that:

$$\begin{aligned} -\Delta v &= a(u)v && \text{in } \Omega \\ v &= 0 && \text{on } \partial\Omega \end{aligned}$$

By (2.1) and the Fredholm alternative we get $(v, \phi_k) = 0$. Now we denote by K' the inverse of $-\Delta - \frac{\lambda_{k-1} + \lambda_{k+1}}{2} I$ on ϕ_k^\perp , with the homogeneous Dirichlet boundary condition, and by L the mapping which sends $u \in \phi_k^\perp$ in

$$u - K'(I - P_k)\left(f_+u^+ - f_-u^- - \frac{\lambda_{k-1} + \lambda_{k+1}}{2} u\right).$$

where we denote by P_k the orthogonal projection onto $V_k = \mathbb{R} \cdot \phi_k$.

It is not difficult to see that L is a contraction on ϕ_k^\perp and $L(0) = 0$. Since (3.17) states that $L(v) = v$, $v = 0$ follows. ■

One can consider the analogous of the previous theorem replacing ϕ_k by $-\phi_k$, which is equivalent to changing the order of the variables of C . Combining these one gets results of zero-two solutions in a sharp form, see [17]. Finally we point out that the analogue of the role of the condition $(f_+, f_-) \notin \Sigma$, which was assumed in Theorem 1, is played by the implicit assumption: $C(f_+, f_-) \neq 0$ ([17]). In the last application of this paper we assume on (f_+, f_-) the same condition as in the previous theorem while we take $h = t\phi_1$ in (1.1) and $\phi = \pm \phi_1$ in (2.1). We use a compa-

riason argument like in lemma 3.1. However (2.1) has now the positive solution only with the $-$ sign and has a negative solution with the $+$ sign.

LEMMA 3.18. — *Let u be a solution of unconstant sign of (2.1) for $\phi = \pm \phi_1$. Then u is nondegenerate and $v_{k-1}(a(u)) < 1 < v_k(a(u))$ where the $-$ sign is taken while $v_k(a(u)) < 1 < v_{k+1}(a(u))$ when the $+$ sign is taken.*

The proof of this lemma is completely similar to that of lemma 3.1 if one takes into account the difference pointed out above.

$+$ Using this lemma we can compute the exact number of solutions of (2.1) arguing as in lemma 3.8. To this aim we recall that when $C(f_+, f_-)C(f_-, f_+) \neq 0$ then it is possible to compute the degree $\deg(T_0, B(o, r), \pm K\phi_1)$ if r is large enough and we have:

$$(3.19) \quad \deg(T_0, B(o, r), \pm K\phi_1) = \begin{cases} (-1)^{k-1} & \text{if } C(f_+, f_-) > 0, C(f_-, f_+) > 0 \\ 0 & \text{if } C(f_+, f_-)C(f_-, f_+) < 0 \\ (-1)^k & \text{if } C(f_+, f_-) < 0, C(f_-, f_+) < 0 \end{cases}$$

LEMMA 3.20. — *Suppose $C(f_+, f_-) > 0, C(f_-, f_+) > 0$. Then (2.1) has exactly one solution for the $+$ sign and exactly three for the $-$ sign. Suppose $C(f_+, f_-) < 0, C(f_-, f_+) < 0$. Then (2.1) has exactly three solutions for the $+$ sign and exactly one for the $-$ sign. Suppose $C(f_+, f_-)C(f_-, f_+) < 0$. Then (2.1) has exactly two solutions for both $+$ and $-$ sign.*

Moreover all these solutions are nondegenerate.

Proof. — We prove the first statement and omit the proof of the last two, since it is completely similar. With $\phi = \phi_1$ one has the negative solution $\tilde{u} = \frac{1}{\lambda_1 - f_-} \phi_1$. Obviously it has local degree $(-1)^{k-1}$. Since we easily see that (2.1) has no positive solution, and therefore we know by lemma (3.18) that any other solution has local degree $(-1)^k$, by the first inequality in (3.19) we see that (2.1) has no more solutions. For $\phi = -\phi_1$ we find the positive solution \hat{u} which has local degree $(-1)^k$ and by lemma (3.18) we know that any other solution has local degree $(-1)^{k-1}$. Therefore, by using the first equality in (3.19), we see that (2.1) has exactly two more solutions. ■

We finally state a straightforward consequence of the previous lemma and Theorem 1, which sharpen similar results in [20] and [21].

THEOREM 5. — *Let (F_2) hold and $\lambda_{k-1} < f_- < \lambda_k < f_+ < \lambda_{k+1}$. If we take $h = t\phi_1$ in (1.1) the following holds.*

If $C(f_+, f_-) > 0$ and $C(f_-, f_+) > 0$ then (1.1) has exactly one solution for t large positive and exactly three for t large negative.

If $C(f_+, f_-) < 0$ and $C(f_-, f_+) < 0$ then (1.1) has exactly three solutions for large positive t and exactly one for large negative t .

If $C(f_+, f_-)C(f_-, f_+) < 0$ then (1.1) has exactly two solutions for large t in modulus.

Moreover all these solutions are nondegenerate.

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