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Conservation laws for the nonlinear Schrödinger equation

by

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ABSTRACT. — We propose a method of calculating the operator densities $\hat{h}_n$ $n = 0, 1, \ldots$ of the conservation laws for the quantum nonlinear Schrödinger equation. It follows from the method that $\hat{h}_n$ are polynomials in fields and their derivatives and in the coupling constant. The densities $\hat{h}_n$ $n \leq 4$ are explicitly calculated. Comparison with the integral densities b_n $n = 0, 1, \ldots$ for the classical nonlinear Schrödinger equation shows that the correspondence between $\hat{h}_n$ and b_n breaks down after $n = 3$.

RÉSUMÉ. — On propose une méthode pour calculer les densités opératoires $\hat{h}_n$ $n = 0, 1, \ldots$ pour les intégrales de l'équation de Schrödinger non linéaire quantique. Il s'ensuit que les $\hat{h}_n$ sont des fonctions polynomiales des champs, de leurs dérivées et de la constante de couplage. Les densités $\hat{h}_n$, $n \leqslant 4$, sont calculées explicitement. En les comparant avec les densités intégrales b_n $n = 0, 1, \ldots$ pour l'équation de Schrödinger non linéaire classique, on voit que la correspondance entre b_n et $\hat{h}_n$ n'est plus valable pour $n > 3$.

1. INTRODUCTION

We consider the quantum nonlinear Schrödinger equation (NLSE) in 1 + 1 space-time dimensions

$$i\Psi_t = -\Psi_{xx} + 2c\Psi^{\dagger}\Psi^2 . \tag{1.1}$$

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Its Hamiltonian

$$\hat{H}_2 = -\int dx(\Psi^\dagger\Psi_{xx} - c\Psi^{\dagger 2}\Psi^2) \tag{1.2}$$

is the second quantized form of the many body Hamiltonian

$$H_2^{(N)} = -\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2} + c\sum_{i\neq j}\delta(x_i - x_j) \tag{1.3}$$

Hamiltonian (1.3) describes the interaction of N identical particles on the line via elastic collisions and c is the strength of interaction. The famous « Bethe Ansatz » [*1*] [*2*] exhibits the system of generalized eigenstates $|\Psi_N(k_1, \ldots, k_N)\rangle = |\Psi_N(k)\rangle$ of $H_2^{(N)}$ which is complete if $c \geq 0$. We have

$$H_2^{(N)}|\Psi_N(\vec{k})> = \left(\sum_{i=1}^{N} k_i^2\right)|\Psi_N(\vec{k})> . \tag{1.4}$$

Since Bethe Ansatz eigenstates depend on N quantum numbers $k_1, \ldots, k_N$ the Hamiltonian (1.3) must be completely integrable. This means that there are N independent operators $H_n^{(N)}$ $n = 1, \ldots, N$ such that

$$H_n^{(N)}|\Psi_N(\vec{k})> = \left(\sum_{i=1}^{N} k_i^n\right)|\Psi_N(\vec{k})> \tag{1.5}$$

$H_1^{(N)} = (-i)\sum_{i=1}^{N}\partial/\partial x_i$ is of course the total momentum and $H_2^{(N)}$ is the Hamiltonian (1.3). Existence of $H_n^{(N)}$ should imply the infinite sequence of independent conservation laws $\hat{H}_n$ $n = 1, 2, \ldots$ for the NLSE given by their operator densities $\hat{h}_n$

$$\hat{H}_n = \int dx\,\hat{h}_n(x) . \tag{1.6}$$

We have

$$\hat{h}_1 = (-i)\Psi^\dagger\Psi_x \tag{1.7}$$

$$\hat{h}_2 = (-i)^2(\Psi^\dagger\Psi_{xx} - c\Psi^{\dagger 2}\Psi^2) . \tag{1.8}$$

Operators $\hat{H}_n$ are completely characterized by the property that for any N

$$\hat{H}_n|\Psi_N(\vec{k})> = \left(\sum_{i=1}^{N} k_i^n\right)|\Psi_N(\vec{k})> . \tag{1.9}$$

It is desirable to have explicit expressions for the operator densities $\hat{h}_n$. In this paper I suggest a method for calculating $\hat{h}_n$ for any n. Using this method I calculate $\hat{h}_3$ and $\hat{h}_4$. In section 4 I compare $\hat{h}_n$ with the functional densities b_n of the integrals of motion for the classical NLSE

$$i\varphi_t = -\varphi_{xx} + 2c\,|\,\varphi\,|^2\varphi \tag{1.10}$$

Thacker [3] has obtained $\hat{h}_3$ using a completely different approach. Kulish and Sklyanin [4] and Thacker [4] have integrated (1.1) using the quantum inverse scattering method. Their method however does not yield explicit formulas for $\hat{h}_n$ in terms of the fields (*).

2. N-PARTICLE SECTOR

In this section we fix N and omit the superscript N in formulas. The Hamiltonian H_2 is equal to the Laplacean $-\sum_{i=1}^{N} \partial^2/\partial x_i^2$ with the boundary conditions

$$(\partial/\partial x_j - \partial/\partial x_i)F = cF \tag{2.1}$$

on hyperplanes $\{ x_i - x_j = 0 \}$ $i, j = 1, \ldots, N$.

Because of the symmetry of function F it suffices to restrict it to $R_\dagger^N = \{ x_1 \leq x_2 \leq \ldots \leq x_N \}$ and to impose boundary conditions

$$(\partial/\partial x_{k+1} - \partial/\partial x_k)F = cF \tag{2.2}$$

on hyperplanes $x_k = x_{k+1}$ $k = 1, \ldots, N-1$.

I will use the following fact. There is an operator P on symmetric functions in R^N that intertwines Laplacean with the Neumann boundary conditions

$$(\partial/\partial x_{k+1} - \partial/\partial x_k)F = 0 \tag{2.3}$$

and Laplacean with boundary conditions (2.2) for $c \geq 0$. The operator P constructed as follows. For any $i \neq j$ let P_{ij} be given by

$$(P_{ij}f)(x_1, \ldots, x_N) = \int_0^\infty dt e^{-ct} f(x_1, \ldots, x_i - t, \ldots, x_j + t, \ldots, x_N). \tag{2.4}$$

Denote by S the operator from all functions f on $\mathbb{R}^N$ into symmetric functions on R^N obtained by restricting f to $R_\dagger^N$ and then extending it to $\mathbb{R}^N$ by symmetry. Then [5]

$$P = S \prod_{i<j} (1 - cP_{ij}). \tag{2.5}$$

(*) *Added in proofs:* in a forthcoming paper I show that the formulas for integrals of the NLSE obtained in [4] via the quantum scattering method are false.

Denoting by Δ_2 the Laplacean with boundary conditions (2.3) we express the intertwining property of P by

$$H_2P = P\Delta_2 . \tag{2.6}$$

Let Δ_n be given by $(-i)^n \sum_{i=1}^{N} \partial^n/\partial x_i^n$ with « higher » Neumann boundary conditions

$$(\partial/\partial x_{k+1} - \partial/\partial x_k)^{2i+1} f = 0 \tag{2.7}$$

for $i = 0, 1, \ldots, [n/2] - 1$ on hyperplanes $\{ x_k = x_{k+1} \}$ $k = 1, \ldots, N-1$. Let H_n be defined from

$$P\Delta_n = H_nP \tag{2.8}$$

for $n = 1, \ldots$. Since operators Δ_n commute, H_n also commute. It follows from (2.5) that P takes boundary conditions (2.7) into boundary conditions

$$(\partial/\partial x_{k+1} - \partial/\partial x_k)^{2i+1} f = c(\partial/\partial x_{k+1} - \partial/\partial x_k)^{2i} f \tag{2.9}$$

So H_n is equal to $(-i)^n \sum_{i=1}^{N} \partial^n/\partial x_i^n$ with boundary conditions (2.9) for $i = 0, \ldots, [n/2] - 1$. It remains to obtain formulas for H_n similar to the formula (1.3) for H_2.

Let $g(x_1, \ldots, x_N)$ be an infinitely differentiable function and let f satisfy the boundary conditions (2.9). Then

$$\langle g \,|\, H_3 \,|\, f \rangle = (-i)^3 \int d^N x \bar{g} \left(\sum_{i=1}^{N} \frac{\partial^3}{\partial x_i^3} f \right). \tag{2.10}$$

Integrating by parts and taking (2.9) into account we get

$$\begin{aligned} \langle g \,|\, H_3 \,|\, f \rangle = & -(-i)^3 \int d^N x \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \bar{g} \frac{\partial^2}{\partial x_i^2} f \\ & - c(-i)^3 \sum_{i \neq j} \int d^N x \delta(x_i - x_j) \bar{g} \left(\frac{\partial}{\partial x_i} + \frac{\partial}{\partial x_j} \right) f . \end{aligned} \tag{2.11}$$

Integrating by parts again

$$\langle g \,|\, \mathrm{H}_3 \,|\, f \rangle = (-i)^3 \int d^{\mathrm{N}}x \sum_{i=1}^{\mathrm{N}} \frac{\partial^2}{\partial x_i^2} \bar{g} \frac{\partial}{\partial x_i} f$$
$$+ (-i)^3 c \int d^{\mathrm{N}}x \sum_{i \neq j} \delta(x_i - x_j) \frac{\partial}{\partial x_i} \bar{g} f$$
$$+ (-i)^3 c \int d^{\mathrm{N}}x \sum_{i \neq j} \left(\frac{\partial}{\partial x_i} + \frac{\partial}{\partial x_j} \right) \delta(x_i - x_j) \bar{g} f. \quad (2.12)$$

After one more integration by parts and obvious transformations (2.12) becomes

$$\langle g \,|\, \mathrm{H}_3 \,|\, f \rangle = i^3 \int d^{\mathrm{N}}x \sum_{i=1}^{\mathrm{N}} \frac{\partial^3}{\partial x_i^3} \bar{g} f$$
$$+ (-i)^3 \frac{3}{2} c \int d^{\mathrm{N}}x \sum_{i \neq j} \delta(x_i - x_j) \left(\frac{\partial}{\partial x_i} + \frac{\partial}{\partial x_j} \right) \bar{g} f \quad (2.13)$$

which yields

$$\mathrm{H}_3 = (-i)^3 \left(\sum_{i=1}^{\mathrm{N}} \frac{\partial^3}{\partial x_i^3} - \frac{3}{2} c \sum_{i \neq j} \delta(x_i - x_j) \left(\frac{\partial}{\partial x_i} + \frac{\partial}{\partial x_j} \right) \right). \quad (2.14)$$

For H_4 we have

$$\langle g \,|\, \mathrm{H}_4 \,|\, f \rangle = \int d^{\mathrm{N}}x \bar{g} \sum_{i=1}^{\mathrm{N}} \frac{\partial^4}{\partial x_i^4} f. \quad (2.15)$$

Integrating by parts the right hand side

$$- \int d^{\mathrm{N}}x \sum_{i=1}^{\mathrm{N}} \frac{\partial}{\partial x_i} \bar{g} \frac{\partial^3}{\partial x_i^3} f$$
$$- c \int d^{\mathrm{N}}x \sum_{i \neq j} \delta(x_i - x_j) \bar{g} \left(\frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial x_i \partial x_j} + \frac{\partial^2}{\partial x_j^2} \right) f. \quad (2.16)$$

Integrating by parts the first term in (2.16) we get

$$\int d^{\mathrm{N}}x \sum_{i=1}^{\mathrm{N}} \frac{\partial^2}{\partial x_i^2} \bar{g} \frac{\partial^2}{\partial x_i^2} f$$
$$+ \frac{1}{2} c \int d^{\mathrm{N}}x \sum_{i \neq j} \delta(x_i - x_j) \left(\frac{\partial}{\partial x_i} + \frac{\partial}{\partial x_j} \right) \bar{g} \left(\frac{\partial}{\partial x_i} + \frac{\partial}{\partial x_j} \right) f. \quad (2.17)$$

Integrating (2.17) by parts again and remembering the second term in (2.16) we have

$$\langle g \mid \mathrm{H}_4 \mid f \rangle = \int d^{\mathrm{N}}x \sum_{i=1}^{\mathrm{N}} \frac{\partial^4}{\partial x_i^4}\, \bar{g} f$$
$$- \frac{c}{2} \int d^{\mathrm{N}}x \sum_{i \neq j} \delta(x_i - x_j)\left(\frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial x_j^2}\right)\bar{g} f$$
$$- \frac{c}{2} \int d^{\mathrm{N}}x \sum_{i \neq j} \delta(x_i - x_j)\left(\frac{\partial}{\partial x_i} + \frac{\partial}{\partial x_j}\right)^2 \bar{g} f$$
$$- c \int d^{\mathrm{N}}x \sum_{i \neq j} \delta(x_i - x_j)\bar{g}\left(\frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial x_i \partial x_j} + \frac{\partial^2}{\partial x_j^2}\right) f. \quad (2.18)$$

The last term in (2.18) can again be integrated by parts yielding

$$- c \int d^{\mathrm{N}}x \sum_{i \neq j} \left(\frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial x_i^2 \partial x_j} + \frac{\partial^2}{\partial x_j^2}\right)\delta(x_i - x_j)\bar{g} f$$
$$+ 3c^2 \int d^{\mathrm{N}}x \sum_{i \neq j \neq k} \delta(x_i - x_j)\delta(x_j - x_k)\bar{g} f. \quad (2.19)$$

From (2.18) and (2.19) we have

$$\mathrm{H}_4 = \sum_{i=1}^{\mathrm{N}} \frac{\partial^4}{\partial x_i^4} - c \sum_{i \neq j} \delta(x_i - x_j)\left(\frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial x_i \partial x_j} + \frac{\partial^2}{\partial x_j^2}\right)$$
$$- c \sum_{i \neq j} \left(\frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial x_i \partial x_j} + \frac{\partial^2}{\partial x_j^2}\right)\delta(x_i - x_j) + 3c^2 \sum_{i \neq j \neq k} \delta(x_i - x_j)\delta(x_j - x_k). \quad (2.20)$$

3. SECOND QUANTIZED FORM OF H_n

A standard calculation gives

$$\hat{\mathrm{H}}_3 = (-i)^3 \int dx \Psi^{\dagger}\Psi_{xxx}$$
$$- \frac{3}{2} c(-i)^3 \int dx \int dy \Psi^{\dagger}(x)\Psi^{\dagger}(y)\delta(x-y)\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)\Psi(x)\Psi(y)$$
$$= (-i)^3 \int dx \left[\Psi^{\dagger}\Psi_{xxx} - 3c\Psi^{\dagger 2}\Psi\Psi_x\right]. \quad (3.1)$$

Thus

$$\hat{h}_3(x) = (-i)^3(\Psi^\dagger\Psi_{xxx} - 3c\Psi^{\dagger 2}\Psi\Psi_x). \quad (3.2)$$

Also

$$\begin{aligned}\hat{H}_4 = &\int dx\Psi^\dagger\Psi_{xxxx} \\ &- c\int dx\int dy\Psi^\dagger(x)\Psi^\dagger(y)\delta(x-y)\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial x\partial y}+\frac{\partial^2}{\partial y^2}\right)\Psi(x)\Psi(y) \\ &- c\int dx\int dy\Psi^\dagger(x)\Psi^\dagger(y)\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial x\partial y}+\frac{\partial^2}{\partial y^2}\right)\delta(x-y)\Psi(x)\Psi(y) \\ &+ 3c^2\int dx\int dy\int dz\Psi^\dagger(x)\Psi^\dagger(y)\Psi^\dagger(z)\delta(x-y)\delta(y-z)\Psi(x)\Psi(y)\Psi(z). \quad (3.3)\end{aligned}$$

After obvious integrations by parts we have

$$\begin{aligned}\hat{H}_4 = \int dx[&\Psi^\dagger\Psi_{xxx} - 2c\Psi^{\dagger 2}\Psi\Psi_{xx} - c\Psi^{\dagger 2}\Psi_x^2 - 2c\Psi^\dagger\Psi^\dagger_{xx}\Psi^2 \\ &- c\Psi_x^{\dagger 2}\Psi^2 + 3c^2\Psi^{\dagger 3}\Psi^3]. \quad (3.4)\end{aligned}$$

Thus

$$\begin{aligned}\hat{h}_4(x) = \Psi^\dagger\Psi_{xxxx} &- 2c\Psi^{\dagger 2}\Psi\Psi_{xx} - c\Psi^{\dagger 2}\Psi_x^2 - 2c\Psi^\dagger\Psi_{xx}\Psi^2 \\ &- c\Psi_x^{\dagger 2}\Psi^2 + 3c^2\Psi^{\dagger 3}\Psi^3. \quad (3.5)\end{aligned}$$

4. COMPARISON WITH CLASSICAL INTEGRALS

The classical NLSE (or Zakharov-Shabat equation [6])

$$i\varphi_t = -\varphi_{xx} + 2c\,|\,\varphi\,|^2\varphi \quad (4.1)$$

is a completely integrable Hamiltonian system with infinitely many degrees of freedom [7]. In particular (4.1) has an infinite number of integrals of motion $B_n(\overline{\varphi}, \varphi)$. The functionals B_n are determined by the local densities b_n

$$B_n(\overline{\varphi}, \varphi) = \int_{-\infty}^{\infty} dx b_n(\overline{\varphi}(x), \varphi(x)). \quad (4.2)$$

The densities b_n are found from the recurrence relation

$$b_{n+1} = \overline{\varphi}\frac{d}{dx}\left(\frac{b_n}{\overline{\varphi}}\right) - c\sum_{i+j=n-1} b_i b_j \quad (4.3)$$

and

$$b_0 = \overline{\varphi}\varphi. \quad (4.4)$$

From (4.3) and (4.4) we get

$$b_1 = \overline{\varphi}\varphi_x \quad (4.5)$$

$$b_2 = \overline{\varphi}\varphi_{xx} - c\,|\,\varphi\,|^4 \quad (4.6)$$

$$b_3 = \overline{\varphi}\varphi_{xxx} - 2c\overline{\varphi}^2(\varphi^2)_x - c\overline{\varphi}\overline{\varphi}_x\varphi^2 \tag{4.7}$$

$$b_4 = \overline{\varphi}\varphi_{xxxx} - 2c\overline{\varphi}^2(\varphi^2)_{xx} - 2c\overline{\varphi}^2\varphi\varphi_{xx} - c\overline{\varphi}^2\varphi_x^2 - 3c\overline{\varphi}\overline{\varphi}_x(\varphi^2)_x - c\overline{\varphi}\overline{\varphi}_x\varphi^2 + 2c^2 \mid \varphi \mid^6 . \tag{4.8}$$

The local densities h and g that differ by a total derivative are equivalent $h \simeq g$ because they define the same functional $\int_{-\infty}^{\infty} dxh(x) = \int_{-\infty}^{\infty} dxg(x)$. We have

$$b_3 \simeq \overline{\varphi}\varphi_{xxx} - \frac{3}{2} c\overline{\varphi}^2(\varphi^2)_x = b'_3 \tag{4.9}$$

$$b_4 \simeq \overline{\varphi}_{xx}\varphi_{xx} + 2c(\overline{\varphi}^2)_x(\varphi^2)_x + c\overline{\varphi}^2\varphi_x^2 + c\overline{\varphi}_x^2\varphi^2 + 2c^2 \mid \varphi \mid^6 = b'_4 . \tag{4.10}$$

We see from (3.2) that $\hat{h}_3$ differs from b'_3 by the factor $(-i)^3$ only. On the other hand the difference between $\hat{h}_4$ (3.5) and b_4 is essential. Replacing $\hat{h}_4$ by an equivalent operator density h'_4 the closest that we can get to b'_4 is

$$\hat{h}'_4 = \Psi^{\dagger}_{xx}\Psi_{xx} + 2c(\Psi^{\dagger 2})_x(\Psi^2)_x + c\Psi^{\dagger 2}\Psi_x^2 + c\Psi_x^{\dagger 2}\Psi^2 + 3c^2\Psi^{\dagger 3}\Psi^3 . \tag{4.11}$$

The difference corresponds to $c^2 \mid \varphi \mid^6$ which is a nontrivial density.

5. CONCLUSION

The nonlinear Schrödinger equation (1.1) has an infinite sequence of conservation laws $\hat{\text{H}}_n$ given by the operator densities $\hat{h}_n(\Psi^{\dagger}(x), \Psi(x))$. The densities $\hat{h}_n$ can be found using the method of sections 2 and 3. It is clear from the method that $\hat{h}_n$ are polynomials in the fields and their derivatives. Besides $\hat{h}_n$ are polynomials in the coupling constant c. The degree of $\hat{h}_n$ in c is $[n/2]$.

Correspondence between $\hat{h}_n$ and the integral densities b_n of the classical NLSE (4.1) breaks down at $n = 4$.

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