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On nonhomogeneous elliptic equations involving critical Sobolev exponent

by

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ABSTRACT. – Let $p=\frac{2N}{N-2}$, $N \geqq 3$ be the limiting Sobolev exponent and $\Omega \subset \mathbb{R}^N$ open bounded set.

We show that for $f \in H^{-1}$ satisfying a suitable condition and $f \neq 0$, the Dirichlet problem:

$$\begin{cases} -\Delta u = |u|^{p-2} u + f & \text{on } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

admits *two* solutions u_0 and u_1 in $H_0^1(\Omega)$.

Also $u_0 \geqq 0$ and $u_1 \geqq 0$ for $f \geqq 0$.

Notice that, in general, this is not the case if $f=0$ (*see* [P]).

Key words : Semilinear elliptic equations, critical Sobolev exponent.

RÉSUMÉ. – Soit $p=\frac{2N}{N-2}$ l'exposant de Sobolev critique et $\Omega \subset \mathbb{R}^N$ un domaine borné.

Classification A.M.S. : 35 A 15, 35 J 20, 35 J 65.

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On montre que si $f \in H^{-1}$, $f \neq 0$ satisfait une certaine condition alors le problème de Dirichlet : $\Delta u = |u|^{p-2} u + f$ dans Ω et $u=0$ dans $\partial\Omega$, admet deux solutions u_0 et u_2 dans $H_0^1(\Omega)$. De plus $u_0 \geqq 0$ et $u_1 \geqq 0$ si $f \geqq 0$.

On remarque que ce n'est pas le cas, en général, si $f=0$ (*voir* [P]).

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1. INTRODUCTION AND MAIN RESULTS

In a recent paper Brezis-Nirenberg (B.N.1] have considered the following minimization problem:

$$\inf_{u \in H, \|u\|_p = 1} \int_\Omega (|\nabla u|^2 - fu) \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$, is a bounded set, $H = H_0^1(\Omega)$, $f \in H^{-1}$ and $p = \frac{2N}{N-2}$, $N \geqq 3$ is the limiting exponent in the Sobolev embedding.

It is well known that the infinum in (1.1) is never achieved if $f=0$ (*cf.* [B]). In contrast, in [B.N.1] it is shown that for $f \neq 0$ this infinum is always achieved. (*See* also [C.S.] for previous related results.)

Motivated by this result we consider the functional:

$$I(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{p}\int_\Omega |u|^p - \int_\Omega fu, \qquad u \in H;$$

whose critical points define weak solutions for the Dirichlet problem:

$$\left.\begin{array}{l} -\Delta u = |u|^{p-2} u + f \quad \text{on } \Omega \\ u = 0 \quad \text{on } \partial\Omega. \end{array}\right\} \tag{1.2}$$

We investigate suitable minimization and minimax principles of mountain pass-type (*cf.* [A.R.]), and show how, for suitable f's, they produce critical values for I in spite of a possible failure of the Palais-Smale condition.

To start, notice that I is bounded from below in the manifold:

$$\Lambda = \{u \in H : \langle I'(u), u \rangle = 0\}$$

[here $\langle\ ,\ \rangle$ denotes the usual scalar product in $H = H_0^1(\Omega)$]. Thus a natural question to ask is whether or not I achieves a minimum in Λ.

We show that this is the case if f satisfies the following:

$$\int_\Omega fu \leqq c_N (\|\nabla u\|_2)^{(N+2)/2} \tag{$*$}_0$$

$\forall u \in \mathrm{H}$, $\|u\|_p = 1$, where $c_{\mathrm{N}} = \dfrac{4}{\mathrm{N}-2}\left(\dfrac{\mathrm{N}-2}{\mathrm{N}+2}\right)^{(\mathrm{N}+2)/4}$. More precisely we have:

THEOREM 1. — *Let $f \neq 0$ satisfies $(*)_0$. Then*

$$\inf_{\Lambda} \mathrm{I} = c_0 \tag{1.3}$$

is achieved *at a point $u_0 \in \Lambda$ which is a critical point for* I *and $u_0 \geqq 0$ for $f \geqq 0$.*

In addition if f satisfies the more restrictive assumption:

$$\int_{\Omega} fu < c_{\mathrm{N}} (\|\nabla u\|_2)^{(\mathrm{N}+2)/2} \tag{$*$}$$

$\forall u \in \mathrm{H}$, $\|u\|_p = 1$, then u_0 is a *local minimum* for I. □

Notice that assumption $(*)$ certainly holds if

$$\|f\|_{\mathrm{H}^{-1}} \leqq c_{\mathrm{N}} \mathrm{S}^{\mathrm{N}/4}$$

where S is the best Sobolev constant (*cf.* [T]).

Also if $f = 0$ Theorem 1 remains valid and gives the trivial solution $u_0 = 0$.

Moreover in the situation where u_0 is a local minimum for I, necessarily:

$$\|\nabla u_0\|_2^2 - (p-1)\|u_0\|_p^p \geqq 0 \tag{1.4}$$

This suggests to look at the following splitting for Λ:

$$\Lambda^+ = \{ u \in \Lambda : \|\nabla u\|_2^2 - (p-1)\|u\|_p^p > 0 \}$$
$$\Lambda_0 = \{ u \in \Lambda : \|\nabla u\|_2^2 - (p-1)\|u\|_p^p = 0 \}$$
$$\Lambda^- = \{ u \in \Lambda : \|\nabla u\|_2^2 - (p-1)\|u\|_p^p < 0 \}.$$

It turns out that assumption $(*)$ implies $\Lambda_0 = \{0\}$ (*see* Lemma 2.3 below). Therefore for $f \neq 0$ and (1.4) we obtain $u_0 \in \Lambda^+$ and consequently

$$c_0 = \inf_{\Lambda} \mathrm{I} = \inf_{\Lambda^+} \mathrm{I}.$$

So we are led to investigate a second minimization problem. Namely:

$$\inf_{\Lambda^-} \mathrm{I} = c_1. \tag{1.5}$$

In this direction we have:

THEOREM 2. — *Let $f \neq 0$ satisfies $(*)$. Then $c_1 > c_0$ and the infinum in* (1.5) *is achieved at a point $u_1 \in \Lambda^-$ which define a critical point for* I.

Furthermore $u_1 \geqq 0$ for $f \geqq 0$. □

Notice that the assumption $f\neq 0$ is *necessary* in Theorem 2. In fact for $f=0$ we have:

$$\operatorname{Inf}_{\Lambda^-} \mathrm{I} = \inf_{u\neq 0} \frac{1}{\mathrm{N}}\left[\frac{\|\nabla u\|_2^2}{\|u\|_p^2}\right]^{\mathrm{N}/2} = \frac{1}{\mathrm{N}}[\inf_{\|u\|_p=1} \|\nabla u\|_2^2]^{\mathrm{N}/2}$$

and the infinum in the right hand side is *never* achieved.

The proofs of Theorem 1 and Theorem 2 rely on the Ekeland's variational principle (*cf.* [A.E.]) and careful estimates inspired by these in [B.N.1].

As an immediate consequence of Theorems 1 and 2 we have the following for the Dirichlet problem (1.2).

THEOREM 3. — Problem (1.2) admits at least *two* weak solutions u_0, $u_1 \in \mathrm{H}_0^1(\Omega)$ for $f\neq 0$ satisfying $(*)$; and at least *one* weak solution for f satisfying $(*)_0$.

Moreover $u_0 \geqq 0$, $u_1 \geqq 0$ for $f \geqq 0$. □

This result for $f\geqq 0$ was also pointed out by Brezis-Nirenberg in [B.N.1]. Their approach however uses in an essential way the fact that f does not change sign. It relies on a result of Crandall-Rabinowitz [C.R.] and techniques developed in [B.N.2].

Furthermore for $f\geqq 0$ it is known that (1.2) cannot admit positive solution when $\|f\|_{\mathrm{H}^{-1}}$ is too large (*see* [C.R.], [M.] and [Z]). So our approach necessarily breaks down when $\|f\|_{\mathrm{H}^{-1}}$ is large. In fact we suspect that assumptions $(*)_0$ and $(*)$ on f are not only sufficient but also *necessary* to guarantee the statements of Theorems 1 and 2.

By a result of Brezis-Kato [B-K] we know that Theorem 3 gives *classical* solutions if f is sufficiently regular and $\partial\Omega$ is smooth; and for $f\geqq 0$, via the strong maximum principle, such solutions are *strictly* positive in Ω.

Obviously an equivalent of Theorem 3 holds for the *subcritical* case where one replaces the power $p=\frac{2\mathrm{N}}{\mathrm{N}-2}$ in (1.2) by $q\in\left(2, \frac{2\mathrm{N}}{\mathrm{N}-2}\right)$. In such a case more standard compactness arguments apply, and the proof can be consistently simplified. The details are left to the interested reader. Finally going back to the functional I, if f satisfies $(*)$ then Theorem 1 suggests a mountain-pass procedure; which will be carried out as follows.

Take:

$$u_\varepsilon(x) = \frac{\varepsilon^{(\mathrm{N}-2)/2}}{(\varepsilon^2+|x|^2)^{(\mathrm{N}-2)/2}} \qquad \varepsilon>0, \quad x\in\mathbb{R}^{\mathrm{N}} \tag{1.6}$$

be an extremal function for the Sobolev inequality in $\mathbb{R}^{\mathrm{N}}$.

For $a\in\Omega$ let $u_{\varepsilon,a}(x)=u_\varepsilon(x-a)$, and

$$\xi_a\in \mathrm{C}_0^\infty(\Omega) \qquad \text{with} \quad \xi_a\geqq 0 \quad \text{and} \quad \xi_a=1 \text{ near } a. \tag{1.7}$$

Set

$$\mathcal{F}=\left\{\begin{array}{ll} h:[0,1]\to \mathrm{H} \text{ continuous, } h(0)=u_0 & \\ & h(1)=\mathrm{R}_0\,\xi_a\,u_{\varepsilon,a}\end{array}\right\}$$

$\mathrm{R}_0>0$ fixed.

We have:

THEOREM 4. – For a suitable choice of $\mathrm{R}_0>0$, $a\in\Omega$ and $\varepsilon>0$ the value

$$c=\inf_{h\in\mathcal{F}}\max_{t\in[0,1]}\mathrm{I}(h,(t))$$

defines a critical value for I, and $c\geqq c_1$. □

It is not clear whether or not $c=c_1$. So no additional multiplicity can be claimed for (1.2). However, in case $c=c_1$ then it is possible to claim a critical point of mountain-pass type (*cf.* [H]) for I in Λ^-. This follows by a refined version of the mountain-pass lemma (*see* [A-R]) obtained by Ghoussoub-Preiss and the fact that Λ^- cannot contain local minima for I (*see* [G.P., theorem (ter) part *a*]).

The referee has brought to our attention a paper of O. Rey (*See* [R.]) where, by a different approach, a result similar to that of Theorem 3 is established when $f\neq 0$, $f\geqq 0$ and $\|f\|_{\mathrm{H}^{-1}}$ is sufficiently small.

2. THE PROOF OF THEOREM 1

To obtain the proof of Theorem 1 several preliminary results are in order.

We start with a lemma which clarifies the purpose of assumption (*).

LEMMA 2.1. – *Let $f\neq 0$ satisfy (*). For every $u\in\mathrm{H}$, $u\neq 0$ there exists a unique $t^+=t^+(u)>0$ such that $t^+u\in\Lambda^-$. In particular:*

$$t^+>\left[\frac{\|\nabla u\|_2^2}{(p-1)\|u\|_p^p}\right]^{1/(p-2)}:=t_{\max}$$

and $\mathrm{I}(t^+u)=\max_{t\geqq t_{\max}}\mathrm{I}(tu)$

Moreover, if $\int_\Omega fu>0$, then there exists a *unique* $t^-=t^-(u)>0$ such that $t^-u\in\Lambda^+$.

In particular,

$$t^-<\left[\frac{\|\nabla u\|_2^2}{(p-1)\|u\|_p^p}\right]^{1/(p-2)}$$

and $\mathrm{I}(t^-u)\leqq\mathrm{I}(tu)$, $\forall\, t\in[0,\, t^+]$.

Proof. — Set $\varphi(t)=t\|\nabla u\|_2^2-t^{p-1}\|u\|_p^p$. Easy computations show that φ is concave and achieves its maximum at

$$t_{\max}=\left[\frac{\|\nabla u\|_2^2}{(p-1)\|u\|_p^p}\right]^{1/(p-2)}.$$

Also

$$\varphi(t_{\max})=\left[\frac{1}{p-1}\right]^{(p-1)/(p-2)}(p-2)\left[\frac{\|\nabla u\|_2^{2(p-1)}}{\|u\|_p^p}\right]^{1/(p-2)},$$

that is

$$\varphi(t_{\max})=c_N\frac{\|\nabla u\|_2^{(N+2)/2}}{\|u\|_p^{N/2}}.$$

Therefore if $\int_\Omega fu\leqq 0$ then there exists a *unique* $t^+>t_{\max}$ such that: $\varphi(t^+)=\int_\Omega fu$ and $\varphi'(t^+)<0$. Equivalently $t^+u\in\Lambda^-$ and $I(t^+u)\geqq I(tu)\ \forall t\geqq t_{\max}$.

In case $\int_\Omega fu>0$, by assumption (*) we have that necessarily

$$\int_\Omega fu<c_N\frac{\|\nabla u\|_2^{(N+2)/2}}{\|u\|_p^{N/2}}=\varphi(t_{\max}).$$

Consequently, in this case, we have unique $0<t^-<t_{\max}<t^+$ such that

$$\varphi(t^+)=\int_\Omega fu=\varphi(t^-)$$

and

$$\varphi'(t^-)>0>\varphi'(t^+).$$

Equivalently $t^+u\in\Lambda^-$ and $t^-u\in\Lambda^+$.

Also $I(t^+u)\geqq I(tu)$, $\forall t\geqq t^-$ and $I(t^-u)\leqq I(tu)$, $\forall t\in[0, t^+]$.

LEMMA 2.2. — *For* $f\neq 0$

$$\inf_{\|u\|_p=1}\left(c_N\|\nabla u\|^{(N+2)/2}-\int_\Omega fu\right):=\mu_0 \tag{2.1}$$

is achieved. In particular if f *satisfies* (*), *then* $\mu_0>0$.

The proof of Lemma 2.2 is technical and a straightforward adaptation of that given in [B.N.1] for an analogous minimization problem.

It will be given in the appendix for the reader's convenience.

Next, for $u \neq 0$ set

$$\psi(u) = c_N \frac{\|\nabla u\|_2^{(N+2)/2}}{\|u\|_p^{N/2}} - \int_\Omega fu.$$

Since for $t>0$, $\|u\|_p = 1$ we have:

$$\psi(tu) = t\left[c_N \|\nabla u\|_2^{(N+2)/2} - \int_\Omega fu\right];$$

given $\gamma > 0$, from Lemma 2.2 we derive that

$$\inf_{\|u\| \geqq \gamma} \psi(u) \geqq \gamma \mu_0. \tag{2.2}$$

In particular if f satisfies (*) then the infinum (2.2) is bounded away from zero.
This remark is crucial for the following:

LEMMA 2.3. – *Let f satisfy (*). For every $u \in \Lambda$, $u \neq 0$ we have*

$$\|\nabla u\|_2^2 - (p-1)\|u\|_p^p \neq 0$$

(i. e. $\Lambda_0 = \{0\}$).

Proof. – Although the result also holds for $f=0$, we shall only be concerned with the case $f \neq 0$.
Arguing by contradiction assume that for some $u \in \Lambda$, $u \neq 0$ we have

$$\|\nabla u\|_2^2 - (p-1)\|u\|_p^p = 0 \tag{2.3}$$

Thus

$$0 = \|\nabla u\|_2^2 - \|u\|_p^p - \int_\Omega fu = (p-2)\|u\|_p^p - \int_\Omega fu. \tag{2.4}$$

Condition (2.3) implies

$$\|u\|_p \geqq \left(\frac{S}{p-1}\right)^{1/(p-2)} := \gamma,$$

and from (2.2) and (2.4) we obtain:

$$0<\mu_0\gamma\leqq\psi(u)=\left[\frac{1}{p-1}\right]^{(p-1)/(p-2)}(p-2)\left[\frac{\|\nabla u\|_2^{2(p-1)}}{\|u\|_p^p}\right]^{1/(p-2)}-\int_\Omega fu$$
$$=(p-2)\left(\left[\frac{1}{p-1}\right]^{(p-1)/(p-2)}\left[\frac{\|\nabla u\|_2^{2(p-1)}}{\|u\|_p^p}\right]^{1/(p-2)}-\|u\|_p^p\right)$$
$$=(p-2)\|u\|_p^p\left(\left[\frac{\|\nabla u\|_2^2}{(p-1)\|u\|_p^p}\right]^{(p-1)/(p-2)}-1\right)=0$$

which yields to a contradiction. □

As a consequence of Lemma 2.3 we have:

LEMMA 2.4. – *Let $f\neq 0$ satisfy* (∗). *Given $u\in\Lambda$, $u\neq 0$ there exist $\varepsilon>0$ and a differentiable function $t=t(w)>0$, $w\in H\,\|w\|<\varepsilon$ satisfying the following:*

$$t(0)=1,\qquad t(w)(u-w)\in\Lambda,\quad \text{for } \|w\|<\varepsilon,$$

and

$$\langle t'(0), w\rangle=\frac{2\int_\Omega\nabla u.\nabla w-p\int_\Omega|u|^{p-2}uw\int_\Omega fw}{\|\nabla u\|_2^2-(p-1)\|u\|_p^p}. \tag{2.5}$$

Proof. – Define $F:\mathbb{R}\times H\to\mathbb{R}$ as follows:

$$F(t, w)=t\|\nabla(u-w)\|_2^2-t^{p-1}\|u-w\|_p^p-\int_\Omega f(u-w).$$

Since $F(1, 0)=0$ and $F_t(1,0)=\|\nabla u\|_2^2-(p-1)\|u\|_p^p\neq 0$ (by Lemma 2.3), we can apply the implicit function theorem at the point $(1,0)$ and get the result. □

We are now ready to give:

The Proof of Theorem 1

We start by showing that I is bounded from below in Λ. Indeed for $u\in\Lambda$ we have:

$$\int_\Omega|\nabla u|^2-\int_\Omega|u|^p-\int_\Omega fu=0.$$

Thus:

$$I(u)=\frac{1}{2}\int_\Omega|\nabla u|^2-\frac{1}{p}\int_\Omega|u|^p-\int_\Omega fu=\frac{1}{N}\int_\Omega|\nabla u|^2-\left(1-\frac{1}{p}\right)\int_\Omega fu$$
$$\geqq\frac{1}{N}\|\nabla u\|_2^2-\frac{N+2}{2N}\|f\|_{H^{-1}}\|\nabla u\|_2\geqq-\frac{1}{16N}[(N+2)\|f\|_{H^{-1}}]^2.$$

In particular

$$c_0 \geqq -\frac{1}{16\,\mathrm{N}}[(\mathrm{N}+2)\,\|f\|_{\mathrm{H}^{-1}}]^2. \tag{2.6}$$

We first obtain our result for f satisfying $(*)$. The more general situation where f satisfies $(*)_0$ will be subsequently derived by a limiting argument.

So from now on we assume that f satisfy $(*)$.

In order to obtain an upper bound for c_0, let $v \in \mathrm{H}$ be the unique solutions for $-\Delta u = f$. So for $f \neq 0$

$$\int_\Omega f v = \|\nabla v\|_2^2 > 0.$$

Set $t_0 = t^-(v) > 0$ as defined by Lemma 2.1.

Hence $t_0 v \in \Lambda^+$ and consequently:

$$\mathrm{I}(t_0 v) = \frac{t_0^2}{2}\|\nabla v\|_2^2 - \frac{t_0^p}{p}\|v\|_p^p - t_0\|\nabla v\|_2^2$$
$$= -\frac{t_0^2}{2}\|\nabla v\|_2^2 + \frac{p-1}{p}t_0^p\|v\|_p^p < -\frac{t_0^2}{\mathrm{N}}\|\nabla v\|_2^2 = -\frac{t_0^2}{\mathrm{N}}\|f\|_{\mathrm{H}^{-1}}^2$$

This yields,

$$c_0 < -\frac{t_0^2}{\mathrm{N}}\|f\|_{\mathrm{H}^{-1}}^2 < 0. \tag{2.7}$$

Clearly Ekeland's variational principle (*see* [A.E.], Corollary 5.3.2) applies to the minimization problem (1.3). It gives a minimizing sequence $\{u_n\} \subset \Lambda$ with the following properties:

(i) $\mathrm{I}(u_n) < c_0 + \dfrac{1}{n}$.

(ii) $\mathrm{I}(w) \geqq \mathrm{I}(u_n) - \dfrac{1}{n}\|\nabla(w - u_n)\|_2,\ \forall w \in \Lambda$.

By taking n large, from (2.7) we have:

$$\mathrm{I}(u_n) = \frac{1}{\mathrm{N}}\int_\Omega |\nabla u_n|^2 - \frac{\mathrm{N}+2}{2\,\mathrm{N}}\int_\Omega f u_n < c_0 + \frac{1}{n} < -\frac{t_0^2}{\mathrm{N}}\|f\|_{\mathrm{H}^{-1}}^2 \tag{2.8}$$

This implies

$$\int_\Omega f u_n \geqq \frac{2}{\mathrm{N}+2}t_0^2\|f\|_{\mathrm{H}^{-1}}^2 > 0. \tag{2.9}$$

Consequently $u_n \neq 0$, and putting together (2.8) and (2.9) we derive:

$$\frac{2\,t_0^2}{\mathrm{N}+2}\|f\|_{\mathrm{H}^{-1}} \leqq \|\nabla u_n\|_2 \leqq \frac{\mathrm{N}+2}{2}\|f\|_{\mathrm{H}^{-1}}. \tag{2.10}$$

Our goal is to obtain $\|I'(u_n)\| \to 0$ as $n \to +\infty$.

Hence let us assume $\|I'(u_n)\| > 0$ for n large (otherwise we are done).

Applying Lemma 2.4 with $u = u_n$ and $w = \delta \dfrac{I'(u_n)}{\|I'(u_n)\|}$ $\delta > 0$ small, we find, $t_n(\delta) := t\left[\delta \dfrac{I'(u_n)}{\|I'(u_n)\|}\right]$ such that

$$w_\delta = t_n(\delta)\left[u_n - \delta \frac{I'(u_n)}{\|I'(u_n)\|}\right] \in \Lambda.$$

From condition (ii) we have:

$$\frac{1}{n}\|\nabla(w_\delta - u_n)\|_2 \geqq I(u_n) - I(w_\delta) = (1 - t_n(\delta))\langle I'(w_\delta), u_n\rangle$$
$$+ \delta\, t_n(\delta)\left\langle I'(w_\delta), \frac{I'(u_n)}{\|I'(u_n)\|}\right\rangle + o(\delta).$$

Dividing by $\delta > 0$ and passing to the limit as $\delta \to 0$ we derive:

$$\frac{1}{n}(1 + |t_n'(0)|\,\|\nabla u_n\|_2) \geqq -t_n'(0)\langle I'(u_n), u_n\rangle + \|I'(u_n)\| = \|I'(u_n)\|$$

where we have set $t_n'(0) = \left\langle t'(0), \dfrac{I'(u_n)}{\|I'(u_n)\|}\right\rangle$.

Thus from (2.10) we conclude:

$$\|I'(u_n)\| \leqq \frac{C}{n}(1 + |t_n'(0)|)$$

for a suitable positive constant C.

We are done once we show that $|t_n'(0)|$ is bounded uniformly on n.

From (2.5) and the estimate (2.10) we get:

$$|t_n'(0)) \leqq \frac{C_1}{|\|\nabla u_n\|_2^2 - (p-1)\|u_n\|_p^p|}$$

$C_1 > 0$ suitable constant.

Hence we need to show that $|\|\nabla u_n\|_2^2 - (p-1)\|u_n\|_p^p|$ is bounded away from zero.

Arguing by contradiction, assume that for a subsequence, which we still call u_n, we have:

$$\|\nabla u_n\|_2^2 - (p-1)\|u_n\|_p^p = o(1). \tag{2.11}$$

From the estimate (2.10) and (2.11) we derive:

$$\|u_n\|_p \geqq \gamma \qquad (\gamma > 0 \text{ suitable constant})$$

and

$$\left[\frac{\|\nabla u_n\|_2^2}{p-1}\right]^{(p-1)/(p-2)} - [\|u_n\|_p^p]^{(p-1)/(p-2)} = o(1).$$

In addition (2.11), and the fact that $u_n \in \Lambda$ also give:

$$\int_\Omega f u_n = (p-2)\|u_n\|_p^p + o(1).$$

This, together with (2.2) implies:

$$\begin{aligned}0<\mu_0\gamma^{(N+2)/2} &\leqq \|u_n\|_p^{p/(p-2)}\psi(u_n)\\ &= (p-2)\left[\left[\frac{\|\nabla u_n\|_2^2}{p-1}\right]^{(p-1)/(p-2)} - [\|u_n\|_p^p]^{(p-1)/(p-2)}\right] = o(1).\end{aligned}$$

which is clearly impossible.

In conclusion:

$$\|I'(u_n)\| \to 0 \quad \text{as } n \to +\infty. \tag{2.12}$$

Let $u_0 \in H$ be the weak limit in $H_0^1(\Omega)$ of (a subsequence of) u_n.
From (2.9) we derive that:

$$\int_\Omega f u_0 > 0$$

and from (2.12) that

$$\langle I'(u_0), w\rangle = 0, \qquad \forall w \in H,$$

i.e. u_0 is a weak solution for (1.2).

In particular, $u_0 \in \Lambda$.

Therefore:

$$c_0 \leqq I(u_0) = \frac{1}{N}\|\nabla u_0\|_2^2 - \int_\Omega f u_0 \leqq \lim_{n\to+\infty} I(u_n) = c_0.$$

Consequently $u_n \to u_0$ strongly in H and $I(u_0) = c_0 = \inf_\Lambda I$. Also from Lemma 2.1 and (2.12) follows that necessarily $u_0 \in \Lambda^+$.

To conclude that u_0 is a local minimum for I, notice that for every $u \in H$ with $\int_\Omega f u > 0$ we have:

$$I(su) \geqq I(t^- u)$$

$$\text{for every } 0 < s < \left[\frac{\|\nabla u\|_2^2}{(p-1)\|u\|_p^p}\right]^{1/(p-2)} \tag{2.13}$$

(*see* Lemma 2.1).

In particular for $u=u_0\in\Lambda^+$ we have:

$$t^- = 1 < \left[\frac{\|\nabla u_0\|_2^2}{(p-1)\|u\|_p^p}\right]^{1/(p-2)}. \tag{2.14}$$

Let $\varepsilon>0$ sufficiently small to have:

$$1 < \frac{\|\nabla (u_0-w)\|_2^2}{(p-1)\|u_0-w\|_p^p}$$

for $\|w\|<\varepsilon$.
From Lemma 2.4, let $t(w)>0$ satisfy $t(w)(u_0-w)\in\Lambda$ for every $\|w\|<\varepsilon$.

Since $t(w)\to 1$ as $\|w\|\to 0$, we can always assume that

$$t(w) < \left[\frac{\|\nabla (u_0-w)\|_2^2}{(p-1)\|u_0-w\|_p^p}\right]^{1/(p-2)}$$

for every $w:\|w\|<\varepsilon$.

Namely, $t(w)(u_0-w)\in\Lambda^+$ and for $0<s<\left[\frac{\|\nabla (u_0-w)\|_2^2}{(p-1)\|u_0-w\|_p^p}\right]^{1/(p-2)}$ we have,

$$\mathrm{I}(s(u_0-w)) \geqq \mathrm{I}(t(w)(u_0-w)) \geqq \mathrm{I}(u_0).$$

From (2.14) we can take $s=1$ and conclude:

$$\mathrm{I}(u_0-w) \geqq \mathrm{I}(w), \qquad \forall w\in\mathrm{H}, \quad \|w\|<\varepsilon.$$

Furthermore if $f\geqq 0$, take, $t_0=t^-(|u_0|)>0$ with $t_0|u_0|\in\Lambda^+$.

Necessarily $t_0\geqq 1$, and

$$\mathrm{I}(t_0|u_0|) \leqq \mathrm{I}(|u_0|) \leqq \mathrm{I}(u_0).$$

So we can always take $u_0\geqq 0$.

To obtain the proof when f satisfies $(*)_0$ we shall apply an approximation argument. To this purpose, notice that if f satisfies $(*)_0$ then $f_\varepsilon=(1-\varepsilon)f$ satisfies $(*)$ $\forall\varepsilon\in(0,1)$.

Set

$$\mathrm{I}_\varepsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{p}\int_\Omega |u|^p + (1-\varepsilon)\int_\Omega fu, \qquad u\in\mathrm{H}.$$

Let $u_\varepsilon\in\Lambda_\varepsilon^+=\{u\in\mathrm{H}: \langle \mathrm{I}'_\varepsilon(u), u\rangle=0, \|\nabla u\|_2^2-(p-1)\|u\|_p^p>0\}$ satisfy:

$$\mathrm{I}_\varepsilon(u_\varepsilon) = \inf_{\Lambda_\varepsilon} \mathrm{I}_\varepsilon := c_\varepsilon$$

and

$$\langle \mathrm{I}'_\varepsilon(u_\varepsilon), w\rangle = 0, \qquad \forall w\in\mathrm{H}. \tag{2.15}$$

Clearly $\|\nabla u_\varepsilon\|_2 \leqq \mathrm{C}_2$, for $0<\varepsilon<1$ and $\mathrm{C}_2>0$ a suitable constant.

Let $u\in\Lambda^{+}$, necessarily $\int_\Omega fu>0$ and consequently

$$(1-\varepsilon)\int_\Omega fu>0,\qquad 0<\varepsilon<1.$$

From Lemma 2.1 applied with $f=f_\varepsilon$ we find:

$$0<t_\varepsilon^-<\left[\frac{\|\nabla u\|_2^2}{(p-1)\|u\|_p^p}\right]^{1/(p-2)}$$

with $t_\varepsilon^- u\in\Lambda_\varepsilon^+$.

Since $1<\dfrac{\|\nabla u\|_2^2}{(p-1)\|u\|_p^p}$, from (2.13) it follows that

$$I_\varepsilon(t_\varepsilon^- u)\leqq I_\varepsilon(u)$$

and consequently:

$$c_\varepsilon\leqq I_\varepsilon(t_\varepsilon u)\leqq I_\varepsilon(u)\leqq I(u)+\varepsilon\|f\|_{H^{-1}}\|\nabla u\|_2\leqq I(u)+\varepsilon C_3$$

(with $C_3>0$ a suitable constant).

Estimate (2.6) with $f=f_\varepsilon$ and the above inequality imply:

$$-\frac{1}{16N}[(N+2)\|f\|_{H^{-1}}]^2\leqq-\frac{1}{16N}[(N+2)\|f_\varepsilon\|_{H^{-1}}]^2\leqq c_\varepsilon\leqq c_0+\varepsilon C_3.$$

Let $\varepsilon_n\to 0$, $n\to+\infty$ and $u_0\in H$ satisfy:

(*a*) $c_{\varepsilon_n}\to\bar c\leqq c_0$, $n\to+\infty$

(*b*) $u_{\varepsilon_n}\to u_0$, $n\to+\infty$ weakly in H.

From (2.15) it follows $\langle I'(u_0), w\rangle=0$, $\forall w\in H$ (*i.e.* u_0 is a critical point for I) and $I(u_0)\leqq c_0$.

In particular $u_0\in\Lambda$ and necessarily $I(u_0)=c_0$, (*i.e.* $u_{\varepsilon_n}\to u_0$ strongly in H).

This completes the proof. □

3. THE PROOF OF THEOREMS 2 AND 4

The functional I involves the limiting Sobolev exponent $p=\dfrac{2N}{N-2}$. This compromises its compactness properties, and a possible failure of the P.S. condition is to be expected.

Our first task is to locate the levels free from this noncompactness effect.

We refer to [B] and [S] for a survey on related problems where such an approach has been successfully used.

In this direction we have:

PROPOSITION 3.1. – *Every sequence* $\{u_n\} \subset H$ *satisfying*:

(*a*) $I(u_n) \to c$ *with* $c < c_0 + \frac{1}{N} S^{N/2}$

[c_0 *as defined in* (1.3)].

(*b*) $\| I'(u_n) \| \to 0$

as a convergent subsequence.

Namely the (P.S) condition holds for all level $c < c_0 + \frac{1}{N} S^{N/2}$.

Proof. – It is not difficult to see that (*a*) and (*b*) imply that $\|\nabla u_n\|_2$ is uniformly bounded.

Hence for a subsequence of u_n (which we still call u_n), we can find a $w_0 \in H$ such that

$$u_n \to w_0 \text{ weakly in H.}$$

Consequently from (*b*) we obtain:

$$\langle I'(w_0), w \rangle = 0, \qquad \forall w \in H. \tag{3.1}$$

That is w_0 is a solution in $H_0^1(\Omega)$ for (1.2). In particular $w_0 \neq 0$, $w_0 \in \Lambda$ and $I(w_0) \geqq c_0$.

Write $u_n = w_0 + v_n$ with $v_n \to 0$ weakly in H.

By a Lemma of Brezis-Lieb [B.L.] we have:

$$\|u_n\|_p^p = \|w_0 + v_n\|_p^p = \|w_0\|_p^p + \|v_n\|_p^p + o(1).$$

Hence, for n large, we conclude:

$$c_0 + \frac{1}{N} S^{N/2} > I(w_0 + v_n) = I(w_0) + \frac{1}{2}\|\nabla v_n\|_2^2 - \frac{1}{p}\|v_n\|_p^p + o(1)$$
$$\geqq c_0 + \frac{1}{2}\|\nabla v_n\|_2^2 - \frac{1}{p}\|v_n\|_p^p + o(1).$$

which gives:

$$\frac{1}{2}\|\nabla v_n\|_2^2 - \frac{1}{p}\|v_n\|_p^p < \frac{1}{N} S^{N/2} + o(1). \tag{3.2}$$

Also from (*b*) follows:

$$o(1) = \langle I'(u_n), u_n \rangle = \|\nabla w_0\|^2 - \|w_0\|_p^p - \int_\Omega f w_0 + \|\nabla v_n\|_2^2 - \|v_n\|_p^p + o(1)$$
$$= \langle I'(w_0), w_0 \rangle + \|\nabla v_n\|_2^2 - \|v_n\|_p^p + o(1):$$

and taking into account (3.1) we obtain:

$$\|\nabla v_n\|_2^2 - \|v_n\|_p^p = o(1). \tag{3.3}$$

We claim that conditions (3.2) and (3.3) can hold simultaneously only if $\{v_n\}$ admits a subsequence, $\{v_{n_k}\}$ say, which converges strongly to zero, *i.e.* $\|v_{n_k}\| \to 0$, $k \to +\infty$.

Arguing by contradiction assume that $\|v_n\|$ is bounded away from zero. That is for some constant $c_4 > 0$ we have $\|v_n\| \geqq c_4$, $\forall n \in \mathbb{N}$.

From (3.3) then it follows:

$$\|v_n\|_p^{p-2} \geqq \mathrm{S} + o(1),$$

and consequently

$$\|v_n\|_p^p \geqq \mathrm{S}^{\mathrm{N}/2} + o(1).$$

This yields a contradiction since from (3.2) and (3.3) we have:

$$\frac{1}{\mathrm{N}}\mathrm{S}^{\mathrm{N}/2} \leqq \frac{1}{\mathrm{N}}\|v_n\|_p^p + o(1) = \frac{1}{2}\|\nabla v_n\|_2^2 - \frac{1}{p}\|v_n\|_p^p + o(1) < \frac{1}{\mathrm{N}}\mathrm{S}^{\mathrm{N}/2}$$

for n large.

In conclusion, $u_{n_k} \to w_0$ strongly. □

At this point it would not be difficult to derive Theorem 2, if we had the inequality:

$$\inf_{\Lambda^-} \mathrm{I} = c_1 < c_0 + \frac{1}{\mathrm{N}}\mathrm{S}^{\mathrm{N}/2}. \tag{3.4}$$

However it appears difficult to derive (3.4) directly.

We shall obtain it by comparison with a mountain-pass value.

To this end, recall that $u_0 \neq 0$. Following [B.N.1] we set $\Sigma \subset \Omega$ to be a set of positive measure such that $u_0 > 0$ on Σ (replace u_0 with $-u_0$ and f with $-f$ if necessary).

$$\text{Set } \mathrm{U}_{\varepsilon,a}(x) = \xi_a(x)\, u_{\varepsilon,a}(x), \qquad x \in \mathbb{R}^{\mathrm{N}};$$

[$u_{\varepsilon,a}$ and ξ_a defined in (1.6) and (1.7)].

LEMMA 3.1. — *For every* $\mathrm{R} > 0$ *and a.e.* $a \in \Sigma$, *there exists* $\varepsilon_0 = \varepsilon_0(\mathrm{R}, a) > 0$ *such that*:

$$\mathrm{I}(u_0 + \mathrm{R}\mathrm{U}_{\varepsilon,a}) < c_0 + \frac{1}{\mathrm{N}}\mathrm{S}^{\mathrm{N}/2}$$

for every $0 < \varepsilon < \varepsilon_0$.

Proof. — We have:

$$\mathrm{I}(u_0 + \mathrm{R}\mathrm{U}_{\varepsilon,a}) = \int_\Omega \frac{|\nabla u_0|^2}{2} + \mathrm{R}\int_\Omega \nabla u_0 \nabla \mathrm{U}_{\varepsilon,a} + \frac{\mathrm{R}^2}{2}\int_\Omega |\nabla \mathrm{U}_{\varepsilon,a}|^2 - \frac{1}{p}\int_\Omega |u_0 + \mathrm{R}\mathrm{U}_{\varepsilon,a}|^p - \int_\Omega f u_0 - \mathrm{R}\int_\Omega f \mathrm{U}_{\varepsilon,a}. \tag{3.5}$$

A careful estimate obtained by Brezis-Nirenberg (*see* formulae (17) and (22) in [B.N.1]) shows that:

$$\| u_0 + \mathrm{R} \mathrm{U}_{\varepsilon, a} \|_p^p = \| u_0 \|_p^p + \mathrm{R}^p \| \mathrm{U}_{\varepsilon, a} \|_p^p + p \mathrm{R} \int_\Omega | u_0 |^{p-2} u_0 \mathrm{U}_{\varepsilon, a}$$

$$+ p \mathrm{R}^{p-1} \int_\Omega \mathrm{U}_{\varepsilon, a}^{p-1} u_0 + o [\varepsilon^{(\mathrm{N}-2)/2}] \quad \text{for a.e. } a \in \Sigma.$$

Also from [B.N.2] we have:

$$\| \nabla \mathrm{U}_{\varepsilon, a} \|_2^2 = \mathrm{B} + O(\varepsilon^{\mathrm{N}-2}) \qquad \text{and} \qquad \| \mathrm{U}_{\varepsilon, a} \|_p^p = \mathrm{A} + O(\varepsilon^{\mathrm{N}})$$

where

$$\mathrm{B} = \int_{\mathbb{R}^\mathrm{N}} | \nabla u_1 (x) |^2 dx, \ \mathrm{A} = \int_{\mathbb{R}^\mathrm{N}} \frac{dx}{(1 + |x|^2)^\mathrm{N}}$$

and

$$\mathrm{S} = \frac{\mathrm{B}}{\mathrm{A}^{2/p}}. \tag{3.6}$$

Substituting in (3.5) and using the fact that u_0 satisfies (1.2) we obtain:

$$\mathrm{I}(u_0 + \mathrm{R} \mathrm{U}_{\varepsilon, a}) = \frac{1}{2} \int_\Omega | \nabla u_0 |^2 + \mathrm{R} \int_\Omega \nabla u_0 . \nabla \mathrm{U}_{\varepsilon, a} + \frac{\mathrm{R}^2}{2} \mathrm{B} - \frac{1}{p} \int_\Omega | u_0 |^p - \frac{\mathrm{R}^p}{p} \mathrm{A}$$

$$- \mathrm{R} \int_\Omega | u_0 | u_0^{p-2} \mathrm{U}_{\varepsilon, a} - \mathrm{R}^{p-1} \int_\Omega \mathrm{U}_{\varepsilon, a}^{p-1} u_0 - \int_\Omega f u_0 - \mathrm{R} \int_\Omega f \mathrm{U}_{\varepsilon, a} + o [\varepsilon^{(\mathrm{N}-2)/2}]$$

$$= \mathrm{I}(u_0) + \frac{\mathrm{R}^2}{2} \mathrm{B} - \frac{\mathrm{R}^p}{p} \mathrm{A} - \mathrm{R}^{p-1} \int_\Omega \mathrm{U}_{\varepsilon, a}^{p-1} u_0 + o [\varepsilon^{(\mathrm{N}-2)/2}]$$

for a.e. $a \in \Sigma$.

Set $u_0 = 0$ outside Ω, it follows:

$$\int_\Omega \mathrm{U}_{\varepsilon, a}^{p-1} u_0 = \int_{\mathbb{R}^\mathrm{N}} u_0 (x) \xi_a (x) \frac{\varepsilon^{(\mathrm{N}+2)/2}}{(\varepsilon^2 + |x - a|^2)^{(\mathrm{N}+2)/2}} dx$$

$$= \varepsilon^{(\mathrm{N}-2)/2} \int_{\mathbb{R}^\mathrm{N}} u_0 (x) \xi_a (x) \frac{1}{\varepsilon^\mathrm{N}} \psi_1 \left(\frac{x}{\varepsilon} \right) dx,$$

where $\psi_1 (x) = \dfrac{1}{(1 + |x|^2)^{(\mathrm{N}+2)/2}} \in \mathrm{L}^1 (\mathbb{R}^\mathrm{N})$.

Therefore, setting $\mathrm{D} = \displaystyle\int_{\mathbb{R}^\mathrm{N}} \frac{dx}{(1 + |x|^2)^{(\mathrm{N}+2)/2}}$ we derive:

$$\int_{\mathbb{R}^\mathrm{N}} u_0 (x) \xi_a (x) \frac{1}{\varepsilon^\mathrm{N}} \psi_1 \left(\frac{x}{\varepsilon} \right) dx \to u_0 (a) \mathrm{D}$$

for a.e. $a \in \Sigma$ (*see* [F]).

In other words,

$$\int_{\Omega} U_{\varepsilon, a}^{p-1}(x)\, u_0(x)\, dx = \varepsilon^{(N-2)/2} u_0(a)\, D + o(\varepsilon^{(N-2)/2}).$$

Consequently:

$$I(u_0 + RU_{\varepsilon, a}) = c_0 + \frac{R^2}{2} B - \frac{R^p}{p} A - R^{p-1} u_0(a)\, D\, \varepsilon^{(N-2)/2} + o[\varepsilon^{(N-2)/2}].$$

Define:

$$q(s) = \frac{s^2}{2} B - \frac{s^p}{P} A - s^{p-1} u_0(a)\, D\, \varepsilon^{(N-2)/2}, \qquad s > 0$$

and assume that $q(s)$ achieves its maximum at $s_\varepsilon > 0$.

Set

$$S_0 = \left(\frac{B}{A}\right)^{1/(p-2)}.$$

Since s_ε satisfies:

$$s_\varepsilon B - s_\varepsilon^{p-1} A = (p-1)\, u_0(a)\, D\, \varepsilon^{(N-2)/2} s_\varepsilon^{p-2} \tag{3.7}$$

necessarily $0 < s_\varepsilon < S_0$ and $s_\varepsilon \to S_0$ as $\varepsilon \to 0$.

Write $s_\varepsilon = S_0(1-\delta_\varepsilon)$. We study the rate at which $\delta_\varepsilon \to 0$ as $\varepsilon \to 0$.

From (3.7) we obtain:

$$\left(\frac{B^{p-1}}{A}\right)^{1/(p-2)} (1 - \delta_\varepsilon - (1-\delta_\varepsilon)^{p-1}) = (p-1)\frac{B}{A}(1-\delta_\varepsilon)^{p-2} \varepsilon^{(N-2)/2} u_0(a)\, D;$$

and expanding for δ_ε we derive:

$$(p-2)\left(\frac{B^{p-1}}{A}\right)^{1/(p-2)} \delta_\varepsilon = (p-1)\frac{B}{A} u_0(a)\, D\, \varepsilon^{(N-2)/2} + o(\varepsilon^{(N-2)/2}).$$

This implies:

$$\begin{aligned} I(u_0 + RU_{\varepsilon, a}) &\leq c_0 + \frac{s_\varepsilon^2}{2} B - \frac{s_\varepsilon^p}{p} B - s_\varepsilon^{p-1} u_0(a)\, D\, \varepsilon^{(N-2)/2} + o(\varepsilon^{(N-2)/2}) \\ &= c_0 + \frac{S_0^2}{2} B - \frac{S_0^p}{2} A - S_0^2 B \delta_\varepsilon + S_0^p A \delta_\varepsilon - S_0^{p-1} u_0(a)\, D\, \varepsilon^{(N-2)/2} + o(\varepsilon^{(N-2)/2}) \\ &= c_0 + \frac{1}{N} S^{N/2} - S_0^{p-1} u_0(a)\, D\, \varepsilon^{(N-2)/2} + o(\varepsilon^{(N-2)/2}). \end{aligned}$$

Therefore for $\varepsilon_0=\varepsilon_0(\mathrm{R}, a)>0$ sufficiently small we conclude

$$\mathrm{I}(u_0+\mathrm{RU}_{\varepsilon, a})<c_0+\frac{1}{\mathrm{N}}\mathrm{S}^{\mathrm{N}/2} \tag{3.8}$$

$\forall\, 0<\varepsilon<\varepsilon_0$. □

Our aim is to state a mountain pass principle that produces a value which is below the threshold $c_0+\frac{1}{\mathrm{N}}\mathrm{S}^{\mathrm{N}/2}$ but also compares with the value $c_1=\inf_{\Lambda^-}\mathrm{I}$.

To this end observe that under assumption (*), the manifold Λ^- disconnects H in exactly two connected components U_1 and U_2.

To see this, notice that for every $u\in\mathrm{H}$, $\|u\|=\|\nabla u\|_2=1$ by Lemma 2.1 we can find a unique $t^+(u)>0$ such that

$$t^+(u)\,u\in\Lambda^- \qquad \text{and} \qquad \mathrm{I}(t^+(u)\,u)=\max_{t\geq t_{\max}}\mathrm{I}(tu).$$

The uniqueness of $t^+(u)$ and its extremal property give that $t^+(u)$ is a continuous function of u.

Set

$$\mathrm{U}_1=\left\{u=0 \text{ or } u: \|u\|<t^+\left(\frac{u}{\|u\|}\right)\right\}$$

and

$$\mathrm{U}_2=\left\{u: \|u\|>t^+\left(\frac{u}{\|u\|}\right)\right\}.$$

Clearly $\mathrm{H}-\Lambda^-=\mathrm{U}_1\cup\mathrm{U}_2$ and $\Lambda^+\subset\mathrm{U}_1$.

In particular $u_0\in\mathrm{U}_1$.

The Proof of Theorem 4

Easy computations show that, for suitable constant $\mathrm{C}_5>0$ we have:

$$0<t^+(u)<\mathrm{C}_5, \qquad \forall\, u: \|u\|=1.$$

Set $\mathrm{R}_0=\left(\frac{1}{\mathrm{B}}|\mathrm{C}_5^2-\|u_0\|^2|\right)^{1/2}+1$ and fix $a\in\Sigma$ such that Lemma 3.2 applies, and the estimate (3.8) holds for all $0<\varepsilon<\varepsilon_0$.

We claim that

$$w_\varepsilon:=u_0+\mathrm{R}_0\,\xi_a\,u_{\varepsilon, a}\in\mathrm{U}_2 \tag{3.9}$$

for $\varepsilon>0$ small.

Indeed

$$\|\nabla w_\varepsilon\|_2^2 = \|\nabla(u_0 + R_0 \xi_a U_{\varepsilon, a})\|_2^2$$

$$= \|u_0\|_2^2 + R_0^2 B + o(1) > C_5^2 \geqq \left[t^+\left(\frac{w_\varepsilon}{\|w_\varepsilon\|}\right)\right]^2,$$

for $\varepsilon > 0$ small enough.

For such a choice of R_0 and $a \in \Sigma$, fix $\varepsilon > 0$ such that both (3.8) and (3.9) hold.

Set

$$\mathcal{F} = \left\{ \begin{array}{l} h : [0, 1] \to H \text{ continuous}, h(0) = u_0 \\ \qquad\qquad\qquad\qquad h(1) = R_0 \xi_a u_{\varepsilon, a} \end{array} \right\}$$

Clearly $h : [0, 1] \to H$ given by $h(t) = u_0 + t R_0 \xi_a u_{\varepsilon, a}$ belongs to $\mathcal{F}$. So by Lemma 2.3 we conclude:

$$c = \inf_{h \in \mathcal{F}} \max_{t \in [0, 1]} I(h(t)) < c_0 + \frac{1}{N} S^{N/2} \tag{3.10}$$

Also, since the range of any $h \in \mathcal{F}$ intersect Λ^-, we have

$$c \geqq c_1 = \inf_{\Lambda^-} I. \tag{3.11}$$

At this point the conclusion of Theorem 4 follows by Lemma 3.1 and a straightforward application of the mountain-pass lemma (*cf.* [A.R.]). □

The Proof of Theorem 2

Analogously to the proof of Theorem 1, one can show that the Ekeland's variational principle gives a sequence $\{u_n\} \subset \Lambda^-$ satisfying:

$$I'(u_n) \to c_1$$
$$\|I'(u_n)\| \to 0$$

But from (3.10) and (3.11), we have:

$$c_1 < c_0 + \frac{1}{N} S^{N/2}.$$

Thus, by Lemma 3.1, we obtain a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ and $u_1 \in H$ such that:

$$u_{n_k} \to u_1 \text{ strongly in } H.$$

Consequently u_1 is a critical point for I, $u_1 \in \Lambda^-$ (since Λ^- is closed) and $I(u_1) = c_1$.

Finally to see that $f \geqq 0$ yields $u_1 \geqq 0$, let $t^+ > 0$ satisfy

$$t^+ |u_1| \in \Lambda^-.$$

From Lemma 2.1 we conclude:

$$I(u_1) = \max_{t \geqq t_{\max}} I(tu_1) \geqq I(t^+ u_1) \geqq I(t^+ |u_1|).$$

So we can always take $u_1 \geqq 0$. □

4. APPENDIX

The Proof of Lemma 2.2

Let $\{u_n\}$ be a minimizing sequence for (2.1) such that for $u_0 \in H$ we have $u_n \to u_0$ weakly in H and $u_n \to u_0$ pointwise a.e. in Ω.

In general $\| u_0 \|_p \leqq 1$. We are done once we show $\| u_0 \|_p = 1$.

To obtain this, we shall argue by contradiction and assume

$$\| u_0 \|_p < 1.$$

Hence write $u_n = u_0 + w_n$ where $w_n \to 0$ weakly in H.

We have

$$\mu_0 + o(1) = c_n \| \nabla u_n \|^{(N+2)/2} - \int_\Omega f u_n = c_N (\| \nabla u_0 \|_2^2 + \| \nabla w_n \|_2^2)^{(N+2)/4} - \int_\Omega f u_0 + o(1) \quad (4.1)$$

On the other hand,

$$1 = \| u_0 + w_n \|_p^p = \| u_0 \|_p^p + \| w_n \|_p^p + o(1)$$

(*see* [B.L.]), which gives:

$$\| w_n \|_p^2 = (1 - \| u_0 \|_p^p)^{2/p} + o(1).$$

So from (4.1) we conclude:

$$\mu_0 + o(1) = c_N (\| \nabla u_0 \|_2^2 + \| \nabla w_n \|_2^2)^{(N+2)/4} - \int_\Omega f u_0$$

$$\geqq c_N [\| \nabla u_0 \|_2^2 + S(1 - \| u_0 \|_p^p)^{2/p} + o(1)]^{(N+2)/4} - \int_\Omega f u_0,$$

That is,

$$c_N [\| \nabla u_0 \|_2^2 + S(1 - \| u_0 \|_p^p)^{2/p}]^{(N+2)/4} - \int_\Omega f u_0 \leqq \mu_0. \quad (4.2)$$

Following [B.N.1] for every $u \in H$, $\| u \|_p < 1$ and $a \in \Omega$ let $c_\varepsilon = c_\varepsilon(a) > 0$ satisfy the following:

$$\| u + c_\varepsilon U_{\varepsilon, a} \|_p = 1$$

[recall $U_{\varepsilon,a}(x)=\xi_a(x)\,u_{\varepsilon,a}(x)$ with ξ_a and $u_{\varepsilon,a}$ given in (1.6) and (1.7)].

We have:

$$\|\nabla(u+c_\varepsilon U_{\varepsilon,a})\|_2^2=\|\nabla u\|_2^2+c_\varepsilon^2\|\nabla U_{\varepsilon,a}\|_2^2+o(1)$$
$$=\|\nabla u\|_2^2+c_\varepsilon^2 B+o(1) \quad (4.3)$$

and

$$1=\|u+c_\varepsilon U_{\varepsilon,a}\|_p^p=\|u\|_p^p+c_\varepsilon^p\|U_{\varepsilon,a}\|_p^p+o(1)=\|u\|_p^p+c_\varepsilon^p A+o(1)$$

[A, B as given in (3.6)].

Thus

$$c_\varepsilon^2=\frac{1}{A^{2/p}}(1-\|u\|_p^p)^{2/p}+o(1). \quad (4.4)$$

Substituting in (4.3) we obtain:

$$\|\nabla(u+c_\varepsilon U_{\varepsilon,a})\|_2^2=\|\nabla u\|_2^2+\frac{B}{A^{2/p}}(1-\|u\|_p^p)^{2/p}+o(1)$$
$$=\|\nabla u\|_2^2+S(1-\|u\|_p^p)^{2/p}+o(1).$$

This yields:

$$\mu_0\leqq c_N\|\nabla(u+c_\varepsilon U_{\varepsilon,a})\|_2^{(N+2)/2}-\int_\Omega f(u+c_\varepsilon U_{\varepsilon,a})$$
$$=c_N(\|\nabla u\|_2^2+S(1-\|u\|_p^p)^{2/p})^{(N+2)/4}-\int_\Omega fu+o(1),$$

and passing to the limit as $\varepsilon\to 0$, we derive:

$$\mu_0\leqq c_N[\|\nabla u\|_2^2+S(1-\|u\|_p^p)^{2/p}]^{(N+2)/4}-\int_\Omega fu, \qquad \forall u\in H, \quad \|u\|_p<1.$$

Therefore from (4.2) we conclude:

$$c_N[\|\nabla u_0\|_2^2-S(1-\|u_0\|_p^p)^{2/p}]^{(N+2)/4}-\int_\Omega fu=\mu_0 \quad (4.5)$$

and that for every $w\in H$ necessarily:

$$\frac{d}{dt}\left[c_N[\|\nabla(u_0+tw)\|_2^2+S(1-\|u_0+tw\|_p^p)^{2/p}]^{(N+2)/4}-\int_\Omega f(u_0+tw)\right]_{t=0}=0.$$

That is:

$$\frac{N+2}{2}c_N\left[\|\nabla u_0\|_2^2+S(1-\|u_0\|_p^p)^{2/p}\right]^{(N-2)/4}$$
$$\times\left[\int_\Omega \nabla u_0\,.\,\nabla w-S(1-\|u_0\|_p^p)^{(2-p)/p}\int_\Omega |u_0|\,u_0^{p-2}\,w\right]$$
$$-\int_\Omega fw=0, \qquad \forall w\in H.$$

So setting $\sigma_0 = \frac{N+2}{2} c_N \left[\| \nabla u_0 \|_2^2 + S(1 - \| u_0 \|_p^p)^{2/p} \right]^{(N-2)/4} > 0$

and

$$\lambda_0 = \frac{S}{(1 - \| u_0 \|_p^p)^{(p-2)/p}}$$

we obtain that u_0 weakly satisfies:

$$-\Delta u_0 = \lambda_0 | u_0 |^{p-2} u_0 + \frac{1}{\sigma_0} f. \tag{4.5}$$

Since $f \neq 0$, in particular, we have that $u_0 \neq 0$.

Hence for a set of positive measure $\Sigma \subset \Omega$ we have:

$$u_0(a) > 0, \qquad \forall a \in \Sigma,$$

(replace u_0 with $-u_0$ and f with $-f$ if necessarily).

Let $a \in \Sigma$ and $c_\varepsilon = c_\varepsilon(a)$ satisfy:

$$\| u_0 + c_\varepsilon U_{\varepsilon, a} \|_p = 1.$$

We will reach a contradiction by showing that

$$I(u_0 + c_\varepsilon U_{\varepsilon, a}) < \mu_0$$

for a suitable choice of $a \in \Sigma$ and $\varepsilon > 0$ small enough.

To this end, let $c_0^p = \frac{1 - \| u_0 \|_p^p}{A}$. From (4.4) it follows that $c_\varepsilon \nearrow c_0$ as $\varepsilon \to 0$. Set $c_\varepsilon = c_0(1 - \delta_\varepsilon)$, $\delta_\varepsilon \to 0$ as $\varepsilon \to 0$. In [B.N.1], Brezis-Nirenberg have obtained a precise rate at which $\delta_\varepsilon \to 0$, by showing that, for a.e. $a \in \Sigma$, one has:

$$\delta_\varepsilon A c_0^p = \varepsilon^{(N-2)/2} \left[c_0 \int_\Omega | u_0(x) | u_0^{p-2}(x) \xi_a(x) \frac{dx}{| x - a |^{N-2}} + c_0^{P-1} u_0(a) D \right] + o(\varepsilon^{(N-2)/2}) \tag{4.7}$$

with

$$D = \int_{\mathbb{R}^N} \frac{dx}{(\varepsilon^2 + | x |^2)^{(N+2)/2}}.$$ (*See* formula (2.9) in [B.N.1].)

Now fix $a \in \Sigma$ for which (4.7) holds and

$$\int_\Omega \frac{| u_0 |^{p-2} u_0 \xi_a}{(\varepsilon^2 + | x - a |^2)^{(N-2)/2}} \to \int_\Omega \frac{| u_0 |^{p-2} u_0 \xi_a}{| x - a |^{N-2}} \quad \text{as } \varepsilon \to 0. \tag{4.8}$$

Using (4.5), (4.7) and the definition of c_0 we obtain:

$$\begin{aligned}
\mathrm{I}(u_0+c_0\,\mathrm{U}_{\varepsilon,a}) &= c_{\mathrm{N}}\left[\|\nabla u_0\|_2^2+2c_\varepsilon\int_\Omega \nabla u_0.\nabla \mathrm{U}_{\varepsilon,a}+c_\varepsilon^2\|\nabla \mathrm{U}_{\varepsilon,a}\|_2^2\right]^{(\mathrm{N}+2)/4} \\
&\qquad -\int_\Omega f u_0 - c_\varepsilon\int_\Omega f\mathrm{U}_{\varepsilon,a} \\
&= c_{\mathrm{N}}\left[\|\nabla u_0\|_2^2+2c_0\int_\Omega \nabla u_0.\nabla \mathrm{U}_{\varepsilon,a}+c_0^2(1-2\,\delta_\varepsilon)\,\mathrm{B}+o\,[\varepsilon^{(\mathrm{N}-2)/2}]\right]^{(\mathrm{N}+2)/4} \\
&\qquad -\int_\Omega f u_0 - c_\varepsilon\int_\Omega f\mathrm{U}_{\varepsilon,a} = c_{\mathrm{N}}\,[\|\nabla u_0\|_2^2+c_0^2\,\mathrm{B}]^{(\mathrm{N}+2)/4}-\int_\Omega f u_0 \\
&\qquad +\frac{\mathrm{N}+2}{4}c_{\mathrm{N}}\,[\|\nabla u_0\|_2^2+c_0^2\,\mathrm{B}]^{(\mathrm{N}-2)/4}\left[2c_0\int_\Omega \nabla u_0.\nabla \mathrm{U}_{\varepsilon,a}\right. \\
&\qquad \left.-2c_0^2\,\delta_\varepsilon\,\mathrm{B}\right]-c_0\int_\Omega f\mathrm{U}_{\varepsilon,a} \\
&\qquad +o\,[\varepsilon^{(\mathrm{N}-2)/2}] = \mu_0+c_0\left[\sigma_0\int_\Omega \nabla u_0.\nabla \mathrm{U}_{\varepsilon,a}\right. \\
&\qquad \left.-\int_\Omega f\mathrm{U}_{\varepsilon,a}\right]-\sigma_0\,c_0^2\,\mathrm{B}\,\delta_\varepsilon+o\,[\varepsilon^{(\mathrm{N}-2)/2}].
\end{aligned}$$

Thus from equation (4.6) we derive:

$$\mathrm{I}(u_0+c_\varepsilon\,\mathrm{U}_{\varepsilon,a})=\mu_0+\sigma_0\,\lambda_0\,c_0\int_\Omega |u_0|^{\mathrm{P}-2}u_0\,\mathrm{U}_{\varepsilon,a}-\delta_0\,c_0^2\,\mathrm{B}\,\delta_\varepsilon+o\,[\varepsilon^{(\mathrm{N}-2)/2}].$$

On the other hand from (4.8) we have:

$$\int_\Omega |u_0|^{p-2}u_0\,\mathrm{U}_{\varepsilon,a}=\varepsilon^{(\mathrm{N}-2)/2}\int_\Omega \frac{|u_0(x)|^{p-2}u_0(x)}{|x-a|^{\mathrm{N}-2}}\xi_a(x)\,dx+o\,[\varepsilon^{(\mathrm{N}-2)/2}].$$

Therefore:

$$\begin{aligned}
&\mathrm{I}(u_0+c_\varepsilon\,\mathrm{U}_{\varepsilon,a}) \\
&\quad=\mu_0+\sigma_0\left[\varepsilon^{(\mathrm{N}-2)/2}\lambda_0\int_\Omega \frac{|u_0(x)|^{p-2}u_0(x)}{|x-a|^{\mathrm{N}-2}}\xi_a-c_0^2\,\mathrm{B}\,\delta_\varepsilon\right]+o\,[\varepsilon^{(\mathrm{N}-2)/2}] \\
&\quad=\mu_0+\sigma_0\left[\frac{\mathrm{S}\,\varepsilon^{(\mathrm{N}-2)/2}}{(1-\|u_0\|_p^p)^{(p-2)/2}}c_0\int_\Omega \frac{|u_0|^{p-2}u_0}{|x-a|^{\mathrm{N}-2}}\xi_a-\mathrm{B}\,c_0^2\,\delta_\varepsilon\right]+o\,(\varepsilon^{(\mathrm{N}-2)/2}) \\
&\quad=\mu_0+\sigma_0\left[\frac{\mathrm{S}}{\mathrm{A}^{(p-2)/p}\,c_0^{p-2}}\varepsilon^{(\mathrm{N}-2)/2}\,c_0\right. \\
&\qquad\left.\times\int_\Omega \frac{|u_0|^{p-2}u_0}{|x-a|^{\mathrm{N}-2}}\xi_a-\mathrm{B}\,c_0^2\,\mathrm{A}\,\delta_\varepsilon\right]+o\,[\varepsilon^{(\mathrm{N}-2)/2}] \\
&\quad=\mu_0+\sigma_0\frac{\mathrm{B}}{\mathrm{A}\,c_0^{p-2}}\left[\varepsilon^{(\mathrm{N}-2)/2}\,c_0\int_\Omega \frac{|u_0|^{p-2}u_0}{|x-a|^{\mathrm{N}-2}}\xi_a-c_0^p\,\mathrm{A}\,\delta_\varepsilon\right]+o\,[\varepsilon^{(\mathrm{N}-2)/2}].
\end{aligned}$$

Finally, from (4.7) we conclude:

$$\mathrm{I}(u_0+c_\varepsilon \mathrm{U}_{\varepsilon,a})=\mu_0-\sigma_0\frac{\mathrm{B}}{\mathrm{A}}c_0\,u_0(a)\,\mathrm{D}\,\varepsilon^{(\mathrm{N}-2)/2}+o\,(\varepsilon^{(\mathrm{N}-2)/2}]<\mu_0$$

for $\varepsilon>0$ sufficiently small.

REFERENCES

[A.R.] A. AMBROSETTI and P. RABINOWITZ, Dual Variational Methods in Critical Point Theory and Applications, *J. Funct. Anal.*, Vol. **11**, 1973, pp. 349-381.

[A.E.] J. P. AUBIN and I. EKELAND, *Applied Nonlinear Analysis, Pure and Applied Mathematics*, Wiley Interscience Publications, 1984.

[B] H. BREZIS, Some Variational Problems with Lack of Compactness, *Proc. Symp. Pure Math.*, Vol. **45**, part 1, F. BROWER Ed., Amer. Math. Soc., 1986, pp. 165-201.

[B.L.] H. BREZIS and T. KATO, Remarks on the Schrodinger Operator with Singular Complex Potential, *J. Math. Pure Appl.*, **58**, 1979, pp. 137-151.

[B.K.] H. BREZIS et E. LIEB, A Relations Between Pointwise Convergence of Functions and Convergence of Integrals, *Proc. Amer. Math. Soc.*, Vol. **88**, 1983, pp. 486-490.

[B.N.1] H. BREZIS et L. NIRENBERG, A Minimization Problem with Critical Exponent and Non Zero Data, in "*Symmetry in Nature*", *Scuola Norm. Sup. Pisa*, 1989, pp. 129-140.

[B.N.2] H. BREZIS et L. NIRENBERG, Positive Solutions of Nonlinear Elliptic Equations Involving Critical Sobolev Exponents, *Comm. Pure Appl. Math.*, Vol. **36**, 1983, pp. 437-477.

[C.S.] L. CAFFARELLI et J. SPRUCK, Variational Problems with Critical Growth and Positive Dirichlet Data (to appear).

[C.R.] M. CRANDALL et P. RABINOWITZ, Some Continuation and Variational Method for Positive Solutions of Nonlinear Elliptic Eigenvalue Problems, *Arch. Rational Mech. Anal.*, Vol. **58**, 1975, pp. 207-218.

[F] G. FOLLAND, *Real Analysis*, Wiley Interscience, N.Y., 1984.

[G.P.] N. GHOUSSOUB et D. PREISS, A General Mountain Pass Principle for Locating and Classifying Critical Point, *Ann. I.H.P. Analyse non linéaire*, Vol. **6**, n° 5, 1989, pp. 321-330.

[H.] H. HOFER, A Geometric Description of the Neighbourhood of a Critical Point Given by the Mountain Pass Theorem, *J. London Math. Soc.*, Vol. **31**, 1985, pp. 566-570.

[M.] F. MERLE, Sur la non-existence de solutions positives d'équations elliptiques surlinéaires, *C. R. Acad. Sci. Paris*, T. **306**, Serie I, 1988, pp. 313-316.

[P.] S. POHOZAEV, Eigenfunctions of the Equation $\Delta u+\lambda f(u)=0$, *Soviet Math. Dokl.*, Vol. **6**, 1965, pp. 1408-1411.

[R.] O. REY, Concentration of Solutions to Elliptic Equations with Critical Nonlinearity (submitted).

[S.] M. STRUWE, Variational Methods and their Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems, *Lecture notes E.T.H.*, Zurich, 1989.

[T.] G. TALENTI, Best Constant in Sobolev Inequality, *Ann. Mat. Pure Appl.*, Vol. **110**, 1976, pp. 353-372.

[Z.] X. ZHENG, A Nonexistence Result of Positive Solutions for an Elliptic Equation, *Ann. I.H.P. Analyse nonlinéaire*, Vol. **7**, n° 2, 1990, pp. 91-96.

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