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Infinite cup length in free loop spaces with an application to a problem of the N-body type

by

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ABSTRACT. — Cup lengths in the cohomology of the space of free loops (over fields of finite characteristic) are computed and results applied to prove the existence of infinitely many solutions of a Hamiltonian system of N-body type.

RÉSUMÉ. — « Cup » longueurs dans la cohomologie de l'espace des lacets libres (sur corps de caractéristique finie) sont obtenues et sont appliquées à démontrer, l'existence d'une infinité de solutions d'un système Hamiltonien à N-corps.

1. INTRODUCTION

Let E denote the Sobolev space $W_T^{1,2}(\mathbb{R}^n)$ of T-periodic functions from the reals $\mathbb{R}$ to Euclidean n-space $\mathbb{R}^n$. Then, if U is an open set in $\mathbb{R}^n$, set

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E(U) equal to those $q \in E$ whose range is in U. The topology of E(U) plays an important role in many problems in Analysis. For example, if U is connected, simply connected and not contractible, it is known that the (Lusternik-Schnirelmann) category of E(U) is infinite [FH1] and this fact is instrumental in establishing the existence of infinitely many solutions to singular Hamiltonian systems, where U is the domain of the corresponding potential function (*see* e. g., [AC], [BR], [C], [R1]). A special case of this situation is the so-called N-body problem in $\mathbb{R}^k$, where $E = W_T^{1,2}(\mathbb{R}^{kN})$ and for non-collision solutions the subset E(U), where $U = F_N(\mathbb{R}^k)$ is the N-th configuration space of $\mathbb{R}^k$, plays an essential role (*see* e. g. [BR]). The proof of the above result [FH1] that cat E(U) is infinite does not require knowledge of the cup length in the cohomology of E(U). In fact, the cup length of E(U) may very well be finite over certain coefficient fields. For example, when $n \geqq 3$ is odd, and $U = \mathbb{R} - \{0\}$, the cup length of E(U) over the field of rationals $\mathbb{Q}$ is 1 [VS].

The topological analogue of E(U) is the space of free loops ΛU, where for any topological space M, $\Lambda M = \mathscr{C}^0(S^1, M)$, the space of continuous maps from the circle S^1 to the space M. E(U) has the same homotopy type as ΛM [K] so that category and cup length of E(U) may be studied through ΛU. For example, the general result in [FH1] states that for simply connected space M with non-trivial finitely generated cohomology (over some field), ΛM has infinite category and this implies the corresponding result for E(U) mentioned above. (The case where M is finite dimensional but not necessarily simply connected or of finite type is contained in [FH2].)

Even though the cup length in the cohomology of ΛM does not play a role in the above general results in [FH1], nevertheless, it is still useful to know that over some fields, depending on M, the cup length is infinite. We will show in this note, that for spheres S^m, ΛS^m has infinite cup length over $\mathbb{Z}_2$ and for complex projective space $\mathbb{C}P^n$, $\Lambda \mathbb{C}P^n$ has infinite cup length over $\mathbb{Z}_r$, where r divides $n+1$. Incidentally, $\Lambda \mathbb{C}P^n$ has only finite cup length over the rational field [VS]. The first result will allow us to compute the relative category [F1] of the pair $(\Lambda M, \Lambda N)$, where M is a wedge of spheres and N a "subwedge". In this case both ΛM and ΛN have infinite category and the relative category cat $(\Lambda M, \Lambda N)$ is shown to be infinite using a cup length type argument. This result provides an alternative tool for the topological part of a theorem of Bahri-Rabinowitz [BR] of 3-body type and should be useful in the general case N>3 (*see* § 2).

For our N-body problem application, we will require the category of a certain subspace of ΛS^m described as follows. Let $\mathbb{Z}_2 = \{1, \zeta\}$ act on ΛS^m by the action $(\zeta q)(t) = -q\left(t + \frac{1}{2}\right)$, $0 \leqq t \leqq 1$, where q is considered

1 − periodic. Let $\Lambda_0 S^m$ denote the fixed point set under this action. $\Lambda_0 S^m$ fibers over S^m but for m even, this fibration does not admit a section, which is a requirement for the main tool in [FH1]. Nevertheless, we show that the cup length of $\Lambda_0 S^m$ over $\mathbb{Z}_2$ is infinite and hence the category of $\Lambda_0 S^m$ is infinite. As an application of these results, it follows that the subspace $E(F_N(\mathbb{R}^k))$ of the Sobolev space $W_T^{1,2}(\mathbb{R}^{kN})$ corresponding to the free loops $\Lambda F_N(\mathbb{R}^k)$ on the N-th configuration space of $\mathbb{R}^k$, also has infinite cup length over $\mathbb{Z}_2$. Furthermore, if we let $\Lambda_0(F_N(\mathbb{R}^k))$ denote the subspace of $\Lambda(F_N(\mathbb{R}^k))$ which is the fixed point set of the $\mathbb{Z}_2$-action defined above, then $\Lambda_0(F_N(\mathbb{R}^k))$, has infinite cup length over $\mathbb{Z}_2$ and hence cat $\Lambda_0(F_N(\mathbb{R}^k))=\infty$. This result if the key to proving a critical point theorem for the functional associated with a problem of N-body type, which improves a result fo Coti-Zelati [C] who minimizes the appropriate functional to obtain a critical point. In addition to yielding an unbounded sequence of critical values, the theorem (*see* Section 3) allows a non-autonomous potential $V(q, t)$ which is C^1 and T/2-periodic in t.

2. CUP LENGTH IN SOME FREE LOOP SPACES

We employ the following notation. I is the unit interval [0, 1]; M^I is the space of maps from I to a space M; ΛM is the free loop space on M given by $\Lambda M=\{\alpha\in M^I:\alpha(0)=\alpha(1)\}$; and $\Omega(M)=(\Omega M, *)$, the space of based loops, *i.e.* loops $\alpha\in\Lambda M$ such that $\alpha(0)=\alpha(1)=*\in M$. If we consider the (Hurewicz) fibration.

$$\Omega(M)\to M^I\xrightarrow{q} M\times M \tag{1}$$

where $q(\alpha)=(\alpha(0), \alpha(1))$, then the diagonal map $\Delta: M\to M\times M$ induces the fibration.

$$\Omega M\to\Lambda M\xrightarrow{p} M \tag{2}$$

where $p(\alpha)=\alpha(0)=\alpha(1)$. We will also make use of the induced fibrations:

$$\begin{array}{ccccccc} P^1 M & \xrightarrow{\hat{i}_1} & M^I & \qquad & P^2 M & \xrightarrow{\hat{i}_2} & M^I \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle q} & & {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle q} \\ M & \xrightarrow{i_1} & M\times M & & M & \xrightarrow{i_2} & M\times M \end{array} \tag{3}$$

where $i_1(x)=(*, x), i_2(x)=(x, *)$ and $P^1 M$, $P^2 M$ are contractible.

The Leray-Serre spectral sequences of (1), (2), p_1 and p_2 will be denoted by $(E^{p,q}, d)$, $(\bar{E}^{p,q}, \bar{d})$, $('E^{p,q}, d')$, $(''E^{p,q}, d'')$.

We consider first the case $M=S^{m+1}$, $m+1$ even, $m\geqq 1$ and prove several lemmas under this assumption. If u is a generator of $'E_{m+1}^{m+1,0}=H^{m+1}(M)$, we define $x\in H^m(\Omega M)$ by $d'_{m+1}x=u$. Furthermore, we define $y\in H^{2m}(\Omega M)$ by $d'_{m+1}y=ux$. With these definitions, the $\mathbb{Z}$-cohomology of ΩS^{m+1} has the form $H^*(S^m)\otimes H^*(\Omega S^{2m+1})$ where the first factor has generator x in dimension m and the second factor is a divided polynomial algebra with generators $y_1, y_2, \ldots, y_k, \ldots$ in dimensions $2km$ and $y=y_1$ (*see* [1]).

2.1. LEMMA. – $d''_{m+1}x=-u$.

Proof. – This is a simple calculation using the reverse map.

$$\begin{array}{ccc} P^2M & \overset{\nu}{\to} & P^1M \\ & {\scriptstyle p_2}\searrow \quad \swarrow{\scriptstyle p_1} & \\ & M & \end{array} \tag{4}$$

where $(\nu\alpha)(t)=\alpha(1-t)$. Then, $\nu_0=\nu|\Omega M$, has the property that $\nu_0^*(x)=-x$ and hence $d''_{m+1}x=d''_{m+1}\nu_0^*(-x)=d'_{m+1}(-x)=u$.

Comparing, the spectral sequences (SS) of p_1 and p_2 with that of q we have:

2.2. LEMMA. – $d_{m+1}(x)=u\times 1-1\times u$ *in* $H^{m+1}(S^{m+1})=E_{m+1}^{m+1,0}$.

We consider next the differential operator d_{m+1}: $E_{m+1}^{m+1,m}\to E_{m+1}^{2m+2,0}$.

2.3. LEMMA. – (*a*) $d_{m+1}((1\times u)x)=-u\times u$

(*b*) $d_{m+1}((u\times 1)x)=u\times u$

and the kernel of

$$d_{m+1}:\quad E_{m+1}^{m+1,m}\to E_{m+1}^{2m+2,0}$$

is generated by $((u\times 1)x+(1\times u)x)$.

Proof. – To prove (*a*) consider

$$d_{m+1}((1\times u)x)=(-1)^{m+1}(1\times u)(1\times u-u\times 1)=1\times u^2-u\times u=-u\times u$$

(*b*) follows from a similar argument. Thus

$$d_{m+1}((1\times u)x+(u\times 1)x)=0$$

and an easy argument shows that the kernel of $d_{m+1}^{m+1,m}$ has $(u\times 1)x+(1\times u)x$ as generator.

We now consider the diagram:

$$\begin{array}{ccc} \Lambda M & \overset{\hat\Delta}{\to} & M^I \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ M & \overset{\Delta}{\to} & M\times M \end{array} \tag{5}$$

Recall that $\bar{d}$ is the differential in the SS for p, and $x \in \bar{E}^{0,m}_{m+1}$, $y \in \bar{E}^{0,2m}_{m+1}$.

2.4. LEMMA. – $\bar{d}_{m+1}(x)=0$, $\bar{d}_{m+1}(y)=2ux$

Proof. – First observe that from Lemma 2.2.

$$\bar{d}_{m+1}(x)=\bar{d}_{m+1}\hat{\Delta}^*(x)=\Delta^* d_{m+1}(x)=\Delta^*(u\times 1-1\times u)=u-u=0.$$

Since, $M^I \sim M$, in the SS for q, it follows that

$$d_{m+1}(y)=(1\times u)x+(u\times 1)x \in E^{m,m+1}_{m+1}$$

and comparison with the SS for p_1 forces the plus sign. Hence,

$$\bar{d}_{m+1}(y)=\bar{d}_{m+1}\hat{\Delta}^*(y)=\Delta^* d_{m+1}(y)=2ux.$$

2.5. LEMMA. – *Let* $i:\Omega M\to\Lambda M$, *denote the inclusion map, where* $M=S^{m+1}$ *as above. Then,*

$$i^*:\quad H^q(\Lambda M;\mathbb{Z}_2)\to H^q(\Omega M;\mathbb{Z}_2),\qquad q\geqq 0$$

is surjective.

Proof. – Consider the terms, $E^{0,*}_{m+1}=H^*(\Omega M)$ in the spectral sequence for p over $\mathbb{Z}$. Let y_k denote one of the generators of the divided polynomial algebra $H^*(\Omega S^{2m+1})$. Then, using induction,

$$\bar{d}_{m+1}(y_1 y_{k-1})=k\bar{d}_{m+1}(y_k)=(2ux)y_{k-1}+y,(2uxy_{k-2})=k\,2uxy_{k-1}$$

Therefore, $\bar{d}_{m+1}(y_k)=2uxy_{k-1}$. Hence, over $\mathbb{Z}_2$, $\bar{d}_{m+1}(y_k)=0$. This is sufficient to verify the lemma.

2.6. THEOREM. – *If* $M=S^{m+1}$, $m+1$ *even, then as algebras,*

$$H^*(\Lambda S^{m+1};\mathbb{Z}_2)\simeq H^*(S^{m+1},\mathbb{Z}_2)\otimes_{\mathbb{Z}_2}H^*(\Omega S^{m+1};\mathbb{Z}_2).$$

Proof. – This is immediate from the Leray-Hirsch theorem and the fact that i^* is an isomorphism in dimensions which are multiplies of m.

The case when $M=S^{m+1}$ with $m+1$ odd is considerably easier. $H^*(\Omega S^{m+1})$ is the divided polynomial algebra on generators $y_1,\ldots,y_k,\ldots$ and in the SS for p, $\bar{d}_{m+1}y_k=0$, so that i^* is surjective over $\mathbb{Z}$ as well as over $\mathbb{Z}_2$.

2.7. THEOREM. – *If* $M=S^{m+1}$, $m+1$ *odd, then as algebras over* $\mathbb{Z}$,

$$H^*(\Lambda S^m)\simeq H^*(S^{m+1})\otimes H^*(\Omega S^{m+1}).$$

2.8. COROLLARY. – *If* $M=S^{m+1}$, $m\geqq 1$, *the cup length of* $H^*(\Lambda S^{m+1};\mathbb{Z}_2)$ *is infinite.*

Proof. – $H^*(\Lambda S^{m+1};\mathbb{Z}_2)$ contains the divided polynomial algebra over $\mathbb{Z}_2$ on generators $y_1, y_2,\ldots,y_k$ and calculating binomial coefficients mod 2 we find that the cup product

$$y_2\cdot y_4\cdot y_8\cdots y_{2^k}=y_r,\qquad r=2^{k+1}-2$$

is non-zero for all $k \geqq 1$.

2.9. *Remark.* – Corollary 2.8 implies that the category of ΛS^{m+1}, $m \geqq 1$, is infinite. However, the direct argument in [FH1] is simpler. Nevertheless, we will need Corollary 2.8 later on to compute a relative cup product. Our next example cannot be handled using [FH1].

Let f denote any map $f: M \to M$ and consider its graph $1 \times f: M \to M \times M$. Let $\Lambda_f M$ denote the total space of the induced fibration as in the following diagram:

$$\begin{array}{ccc} \Omega M & & \Omega M \\ \downarrow & & \downarrow \\ \Lambda_f M & \xrightarrow{\hat{f}} & M^I \\ {\scriptstyle p_f}\downarrow & & \downarrow{\scriptstyle q} \\ M & \xrightarrow{1 \times f} & M \times M \end{array}$$

An important case for us in the application to be given in Section 3, is $M = S^{m+1}$, $m+1$ even, and f the antipodal map. (If $m+1$ is odd, f is homotopic to the identity and $\Lambda_f M \sim \Lambda M$). Notice that in this case p_f does not admit a section which is why [FH1] does not apply.

Let $\tilde{d}$ denote the differential in the SS over $\mathbb{Z}$ for the fiber map p_f. The analogue of Lemma 2.4 is:

2.10. Lemma. – $\tilde{d}_{m+1}(x) = -2u$, $\tilde{d}_{m+1}(y) = 0$.

Proof:

$$\tilde{d}_{m+1}(x) = \tilde{d}_{m+1} \hat{f}^*(x) = (1 \times f)^* d_{m+1}(x) = (1 \times f)^*(u \times 1 - 1 \times u) = 2u.$$

Furthermore,

$$\begin{aligned} \tilde{d}_{m+1}(y) &= \tilde{d}_{m+1} \hat{f}^*(y) \\ &= (1 \times f)^* d_{m+1}(y) = (1 \times f)^*((u \times 1)x + (1 \times u)x) = ux - ux = 0. \end{aligned}$$

2.11. Lemma. – *Let* $i: \Omega M \to \Lambda_f M$, *where* $M = S^{m+1}$, $m+1$ *even, and* f *the antpodal map. Then,*

$$i^*: \quad H^q(\Lambda_f M; \mathbb{Z}_2) \to H^q(\Omega M; \mathbb{Z}_2), \qquad q \geqq 0$$

is surjective. Furthermore, over $\mathbb{Z}$, *the image of*

$$i^*: \quad H^*(\Lambda_f M) \to H^*(\Omega M)$$

contains the divided polynomial algebra in $H^*(\Omega M)$.

Proof. – It follows by induction that $\tilde{d}_{m+1} y_k = 0$ over $\mathbb{Z}$ where, as above, the y_k are generators of the polynomial algebra $H^*(\Omega S^{2m+1})$. This observation suffices to prove the lemma.

2.12. COROLLARY. — *$\Lambda_f M$, where $M=S^{m+1}$, $m\geqq 1$, has infinite cup length over $\mathbb{Z}$ and $\mathbb{Z}_2$ and hence $\Lambda_f M$ has infinite category.*

2.13. COROLLARY:

$$H^*(\Lambda_f M;\mathbb{Z}_2)\simeq H^*(S^{m+1},\mathbb{Z}_2)\otimes_{\mathbb{Z}_2} H^*(\Omega S^{m+1};\mathbb{Z}_2),$$

as algebras.

Our next example is the computation on the cup length of $\Lambda\mathbb{C}P^n$. We will make use of the fact that $\Omega\mathbb{C}P^n$ has the same homotopy type as $S^1\times\Omega S^{2n+1}$. Working over $\mathbb{Z}$ and employing the diagrams (3), (4) and (5), let x denote a generator of $H^1(\Omega\mathbb{C}P^n)$, u a generator of $H^2(\mathbb{C}P^n)$, and $y_1, y_2, \ldots y_k, \ldots$ the generators of the divided polynomial algebra in $H^*(\Omega\mathbb{C}P^n)$ corresponding to $H^*(\Omega S^{2n+1})$. Also let $y_1=y$, to conform to some previous notation. We may assume that $d_2'(x)=u$ in the SS for p_1. Then, Lemmas 2.1-2.2 obtain with only notational adjustments and $d_2''(x)=-u$, $d_2(x)=1\times u-u\times 1$ and $\bar{d}_2(x)=0$. $\bar{d}_2(u^k x)=0$, $k=1,\ldots,n$ and the differential operators $\bar{d}_j$ are all trivial for $2\leqq j\leqq 2n-1$. At the level $\bar{E}_{2n}^{**}$, we have

$$\bar{E}_{2n}^{i,0}=\langle u^i\rangle,\ \bar{E}_{2n}^{i,1}=\langle u^i x\rangle,\ \bar{E}_{2n}^{0,2n}=\langle y\rangle,$$

$i=0,\ldots,n$, where $\langle\quad\rangle$ indicates "generated by".

We will need the following in the SS for q in (5).

2.14. LEMMA. — *Let $u_1=1\times u$ and $u_2=u\times 1$ in* $H^*(\mathbb{C}P^n\times\mathbb{C}P^n)$. *Then, in the* SS *for q the element*

$$w=(u_1^n+u_1^{n-1}u_2+\ldots+u_1u_2^{n-1}+u_2^n)\,x\in E_2^{2n,1}$$

is a d_2 cocycle, i.e., $d_2(w)=0$. *w is a generator of kernel d_2 chosen so that $d_{2n}(y)=w$. Therefore, in the* SS *for p in (5) we have $\bar{d}_{2n}(y)=\bar{d}_{2n}(y_1)=(n+1)\,u^n x$.*

Proof:

$$\begin{aligned} d_2 w &= (u_1^n+u_1^{n-1}u_2+\ldots+u_1u_2^{n-1}+u_2^n)\,d_2(x)\\ &= (u_1^n+u_1^{n-1}u_2+\ldots u_1u_2^{n-1}+u_2^n)(u_1-u_2)=u_1^{n+1}-u_2^{n+1}=0. \end{aligned}$$

On the other hand, if

$$(a_1u_1^n+a_2u_1^{n-1}u_2+\ldots+a_nu_1u_2^{n-1}+a_{n+1}u_2^n)(u_1-u_2)=0$$

we have, by equating coefficients, $a_1=a_2=\ldots=a_{n+1}$ and w generates $\ker d_2$. Thus, $d_2y=\pm w$ since $H^{2n+1}(M^I)=0$ and w cannot survive. There is no loss of generality if we stipulate that $d_2(y)=w$. Finally, in the SS for p, we have

$$\bar{d}_{2n}(y)=\bar{d}_{2n}\hat{\Delta}^*(y)=\hat{\Delta}^*\bar{d}_{2n}y=(n+1)\,u^n x.$$

2.15. LEMMA. — *On the* SS *for p we have*

$$d_{2n}(y_k)=(n+1)\,u^n x y_{k-1}.$$

Proof. — We use induction on k.

$$
\begin{aligned}
d_{2n}(y_1 \cdot y_{k-1}) &= (n+1)\, u^n x y_{k-1} + y_1 d_{2n} y_{k-1} \\
&= (n+1)\, u^n x y_{k-1} + y_1 (n+1)\, u^n x y_{k-2} \\
&= (n+1)\, u^n x [y_{k-1} + (k-1)\, y_{k-1}] = k(n+1)\, u^n x y_{k-1} = k d_{2n}(y_k).
\end{aligned}
$$

Hence, $d_{2n}(y_k) = (n+1)\, u^n x y_{k-1}$

2.16. COROLLARY. — *If r is a prime which divides $n+1$, then in the* SS *for p over $\mathbb{Z}_r$ we have $d_{2n}(x)=0$ and $d_{2n}(y_k)=0$ for all $k \geqq 1$. Hence, the inclusion map $i : \Omega \mathbb{C}\mathrm{P}^n \to \Lambda \mathbb{C}\mathrm{P}^n$ induces surjections*

$$i^*: \quad \mathrm{H}^q(\Lambda \mathbb{C}\mathrm{P}^n; \mathbb{Z}_r) \to \mathrm{H}^q(\Omega \mathbb{C}\mathrm{P}^n; \mathbb{Z}_r)$$

and hence, as algebras,

$$\mathrm{H}^*(\Lambda \mathbb{C}\mathrm{P}^n; \mathbb{Z}_r) \simeq \mathrm{H}^*(\mathbb{C}\mathrm{P}^n; \mathbb{Z}_r) \otimes_{\mathbb{Z}_r} \mathrm{H}^*(\Omega \mathbb{C}\mathrm{P}^n; \mathbb{Z}_r).$$

We need to extend some of these results to configuration spaces. First let $\mathrm{M} = \mathrm{F}_\mathrm{N}(\mathbb{R}^k)$, the N-th configuration space of Euclidean k-space $\mathbb{R}^k$. Recall, that

$$\mathrm{F}_\mathrm{N}(\mathbb{R}^k) = \{ (x_1, \ldots, x_\mathrm{N}) \in (\mathbb{R}^k)^\mathrm{N},\ x_i \neq x_j \text{ for } i \neq j \}.$$

Also, the projection $p_\mathrm{N} : \mathrm{F}_\mathrm{N}(\mathbb{R}^k) \to \mathrm{F}_{\mathrm{N}-1}(\mathbb{R}^k)$ given by

$$p_\mathrm{N}(x_1, \ldots, x_\mathrm{N}) = (x_1, \ldots, x_{\mathrm{N}-1})$$

is locally trivial with fiber $\mathbb{R}^k - \mathrm{Q}_\mathrm{N}$, where Q_N is a set of $(\mathrm{N}-1)$ points. In particular, p_2: $\mathrm{F}_2(\mathbb{R}^k) \to \mathbb{R}^k$, with fiber $\mathbb{R}^k - 0$. Hence $\mathrm{F}_2(\mathbb{R}^k) \sim \mathbb{R}^k - 0$ and we have for $k \geqq 3$

$$\mathrm{H}^*(\Lambda \mathrm{F}_2(\mathbb{R}^k); \mathbb{Z}_2) \simeq \mathrm{H}^*(\mathrm{S}^{k-1}; \mathbb{Z}_2) \otimes_{\mathbb{Z}_2} \mathrm{H}^*(\Omega \mathrm{S}^{k-1}; \mathbb{Z}_2)$$

so that $\Lambda \mathrm{F}_2 \mathbb{R}^k$ has infinite cup length over $\mathbb{Z}_2$. It is easy to see that p_N admits a section for $\mathrm{N} \geqq 3$. In fact we will produce an equivariant section which will be needed by the next example.

2.17. LEMMA. — *For $\mathrm{N} \geqq 3$, $p_\mathrm{N} : \mathrm{F}_\mathrm{N}(\mathbb{R}^k) \to \mathrm{F}_{\mathrm{N}-1}(\mathbb{R}^k)$ admits a section σ with the property that $\sigma(-x) = -\sigma(x)$.*

Proof. — Let

$$\alpha = \alpha(x_1, \ldots, x_{\mathrm{N}-1}) = \min_{i \neq j} |x_i - x_j|.$$

Define

$$x_\mathrm{N} = x_\mathrm{N}(x_1, \ldots, x_{\mathrm{N}-1}) = \left(1 - \frac{\alpha}{2|x_2 - x_1|}\right) x_1 + \left(\frac{\alpha}{2|x_2 - x_1|}\right) x_2$$

and set

$$\sigma(x_1, \ldots, x_{\mathrm{N}-1}) = (x_1, \ldots, x_{\mathrm{N}-1}, x_\mathrm{N}).$$

2.18. THEOREM. – *$\Lambda F_N(\mathbb{R}^k)$ has infinite cup length over $\mathbb{Z}_2$ for $k \geqq 3$, $N \geqq 2$.*

Proof. – By the previous lemma $\Lambda p_N : \Lambda F_N(\mathbb{R}^k) \to \Lambda F_{N-1}(\mathbb{R}^k)$ admits a section for $N \geqq 3$ and hence by induction the result follows.

We now consider the configuration space analogue of Corollary 2.12. We define $\Lambda_A F_N(\mathbb{R}^k)$ as a pull-back by the diagram

$$\begin{array}{ccc} \Lambda_A F_N(\mathbb{R}^k) & \longrightarrow & [F_N(\mathbb{R}^k)]^I \\ {}^{p_A}\downarrow & & \downarrow {}^{q} \\ F_N(\mathbb{R}^k) & \xrightarrow{1 \times A} & F_N(\mathbb{R}^k) \times F_N(\mathbb{R}^k) \end{array}$$

where $A(x_1, \ldots, x_N) = (-x_1, \ldots, -x_N)$.

Thus, $\Lambda_A F_N(\mathbb{R}^k)$ is the space of paths $q = (q_1, \ldots, q_N)$ in $F_N(\mathbb{R}^k)$ such that $q_i(1) = -q_i(0)$. Then, the fibration

$$p_N : \quad F_N(\mathbb{R}^k) \to F_{N-1}(\mathbb{R}^k), \qquad N \geqq 3$$

induces

$$\bar{p}_N : \quad \Lambda_A F_N(\mathbb{R}^k) \to \Lambda_A F_{N-1}(\mathbb{R}^{k-1}).$$

$\bar{p}_N$ admits a section using the section σ of Lemma 2.17. It is easy to identify $\Lambda_A F_2(\mathbb{R}^k)$, up to homotopy type, with $\Lambda_f S^{k-1}$ of Corollary 2.12. Combining the remarks we obtain:

2.19. THEOREM. – *For $N \geqq 2$, $k \geqq 3$, the cup length of $\Lambda_A F_N(\mathbb{R}^k)$ is infinite.*

Our final computation concerns the relative category of a certain pair which can be estimated by considering $H^*(X; A)$ as a module over $H^*(X)$. In [BR], Bahri and Rabinowitz exploited a purely topological result that the free loop spaces $\Lambda F_3(\mathbb{R}^k)$ and $\Lambda F_2(\mathbb{R}^k)$ were not of the same homotopy type to prove a theorem of the 3-body type concerning the existence of an unbounded sequence of critical values without a symmetry condition on the potential (*see* Section 3). This topological result is derived from a result of Vigué-Poirier and Sullivan [VS] to the effect that the rational Betti numbers of $\Lambda F_3(\mathbb{R}^k)$, $k \geqq 3$, were unbounded, while those of $\Lambda F_2(\mathbb{R}^k)$ were bounded. The following result (theorem) provides an alternative tool for the Bahri-Rabinowitz theorem and will, hopefully, play a role in the case $N > 3$.

First we recall one of the definitions of relative category introduced in [F1], [F2].

2.20. DEFINITION. – Let (X, A) be a topological pair. A categorical cover for (X, A) of length n is an $(n+1)$-tuple of open sets $(V_0, V_1, \ldots, V_n)$ such that $\bigcup V_j \supset X$, V_0 deforms in X to A relative to A, and V_i, $i \geqq 1$, deforms in X to a point. cat (X, A) is the minimum length of such categorical covers if such categorical covers exist. Otherwise, set cat $(X, A)=\infty$.

The next result is the analogue for cup length in this setting, using the fact that over any commutative ring of coefficients, $H^*(X, A; R)$ is a module over $H^*(X; R)$. Although, the result depends on coefficients we will not display it in the notation.

2.21. PROPOSITION [F1]. – *If there exist n elements* $u_1, \ldots, u_n$ *in* $H^*(X)$ *of positive dimension such that the product* $u_1 u_2 \ldots u_n$ *is not in the annihilator of* $H^*(X, A)$, *then* $\operatorname{cat}(X, A) > n$.

Now, let $M = S_1 \vee \ldots \vee S_m$ denote a wedge of spheres of dimension ≥ 2 and M′ a "subwedge" which we take to be $S_1 \vee \ldots \vee S_k$, $k < m$. We employ Proposition 2.21 and Corollary 2.8 to prove the following.

2.22. THEOREM. – $\operatorname{cat}(\Lambda M, \Lambda M') = \infty$.

Proof. – We work with $\mathbb{Z}_2$ coefficients.

Consider the diagram

$$\begin{array}{ccccccc} 0 \to H^q(\Lambda M, \Lambda M') & \xrightarrow{j^*} & H^q(\Lambda M) & \xrightarrow{i^*} & H^q(\Lambda M') \to 0 \\ & & {\scriptstyle r^*}\downarrow & \nearrow {\scriptstyle 0} & \\ & & H^q(\Lambda S_m) & & \end{array}$$

where i^* surjects because $\Lambda M'$ is a retract of ΛM and r^* injects, where r: $\Lambda M \to \Lambda S_m$ is a retraction which takes $\Lambda M'$ to a point. Since $H^*(\Lambda S_m)$ has infinite cup length the result follows.

2.23. *Remark.* – In the Lusternik-Schnirelmann method it is useful to know that when the category of a space X is infinite, there are compact subsets of arbitrarily high category in X. When the cup length of X, using singular cohomology, is infinite over some coefficient field, this is automatic [FH1]. We are indebted to Luis Montejano for suggesting the use of "infinite dimensional topology" to verify that when X is an ANR (sep.met.) and has infinite category, then X has compact subsets of arbitrarily high category. For example, if X is a Hilbert manifold modelled on a Hilbert space H, then by a result of D. Henderson [H], $X = P \times H$, where P is a locally finite polyhedron. If X has infinite category, then so does P. Since P is σ-compact, it is now an exercise to show that P has subpolyhedra of arbitrarily high category in P.

In the next section it will be necessary to apply some of the computations of this section to the corresponding Sobolev spaces. Let $W_T^{1,2}(\mathbb{R}^{kN})$ denote the Sobolev space of T periodic functions $q: \mathbb{R} \to \mathbb{R}^{kN}$ which are absolutely continuous and have square summable first derivatives. q can be represented by $q = ((q_1, \ldots, q_N)$ where each $q_i: \mathbb{R} \to \mathbb{R}^k$. The inner product for

$W_T^{1,2}(\mathbb{R}^{kN})$ is given by

$$\langle f, g\rangle = \int_0^T \langle \dot{f}(t), \dot{g}(t)\rangle dt + \int_0^T \langle f(t), g(t)\rangle dt.$$

Let $C_T^0(\mathbb{R}^{kN})$ denote the Banach space of continuous T-periodic functions with the uniform norm. We may readily identify $C_T^0(\mathbb{R}^{kN})$ with $\Lambda\mathbb{R}^{kN}$. It is a well-known fact [K] that the inclusion $i: W_T^{1,2}(\mathbb{R}^{kN}) \to C_T^0(\mathbb{R}^{kN})$ is a continuous injection, whose image is dense in $C_T^0(\mathbb{R}^{kN})$. Define an open set $\Lambda(N) \subset W_T^{1,2}(\mathbb{R}^{kN})$ as follows:

$$\Lambda(N) = \{(q_1, \ldots, q_N) \in W_T^{1,2}(\mathbb{R}^{kN}) : q_i(t) \neq q_j(t) \text{ for } 1 \leqq t \leqq T\}$$

Then,

$$\bar{i} = i \,|\, \Lambda(N) : \Lambda(N) \to \Lambda F_N(\mathbb{R}^k) \subset \Lambda\mathbb{R}^{kN}$$

where $\Lambda F_N(\mathbb{R}^k)$ is an open subset of $\Lambda\mathbb{R}^{kN}$. Then, by a theorem of Palais [P], $\bar{i}$ is a homotopy equivalence. Thus $\Lambda(N)$ has both infinite cup length over $\mathbb{Z}_2$ and infinite category. Now, introduce an action of $\mathbb{Z}_2 = \{1, \zeta\}$ on $W_T^{1,2}(\mathbb{R}^{kN})$ by $(\zeta q)(t) = -q\left(t + \frac{T}{2}\right)$, $0 \leqq t \leqq 1$. Let E_0 denote the fixed point set of this action, namely those q such that $\zeta q = q$. Then E_0 is a closed Hilbert subspace of $W_T^{1,2}(\mathbb{R}^{kN})$. Let C_0 denote the corresponding subspace of $C_T^0(\mathbb{R}^{kN})$. Then, again E_0 continuously injects into C_0, with image dense in C_0. Let $\Lambda_0(N) = \Lambda(N) \cap E_0$ and $\Lambda_0 F_N(\mathbb{R}^{kN}) = C_0 \cap \wedge F_N(\mathbb{R}^k)$. Then, by the same argument as above, $\Lambda_0(N)$ and $\Lambda_0 F_N(\mathbb{R}^k)$ have the same homotopy type. But $\Lambda_0 F_N(\mathbb{R}^k)$ may be identified with $\Lambda_A F_N(\mathbb{R}^k)$ of the theorem 2.19. Thus, $\Lambda_0(N)$ has infinite cup length over $\mathbb{Z}_2$ and infinite category. Summarizing:

2.25. THEOREM. – *Let*

$$\Lambda(N) = \{(q_1, \ldots, q_N) \in W_T^{1,2}(\mathbb{R}^{kN}) : q_i(t) \neq q_j(t),\ 0 \leqq t \leqq T\}$$

$$\Lambda_0(N) = \left\{(q_1, \ldots, q_N) \in \Lambda(N) : q_i(t) = -q_i\left(t + \frac{T}{2}\right),\ 0 \leqq t \leqq T,\ 1 \leqq i \leqq N\right\}.$$

Then, both $\Lambda(N)$ and $\Lambda_0(N)$ have infinite cup length over $\mathbb{Z}_2$ and hence infinite category (with compact subsets of arbitrarily high category).

3. A HAMILTONIAN SYSTEM OF THE N-BODY TYPE

Consider a potential function $V: F_N(\mathbb{R}^k) \times \mathbb{R} \to \mathbb{R}$ of the following form:

$$V(q_1, \ldots, q_N) = \frac{1}{2} \sum_{1 \leqq i \neq j \leqq N} V_{ij}((q_i - q_j), t)$$

and the following properties for $1 \leqq i \neq j \leqq N$, $q \in \mathbb{R}^k - 0$, $0 \leqq t \leqq T$.

(V_1) $V_{ij} \in C^1(\mathbb{R}^k - \{0\} \times \mathbb{R}, \mathbb{R})$, $V_{ij}(q, t) \leqq 0$.

(V_2) $V_{ij}(q, t) = V_{ji}(q, t)$ and $V_{ij}(q, t) = V_{ij}\left(q, t + \frac{T}{2}\right)$.

(V_3) $V_{ij}(q, t) \to -\infty$ as $q \to 0$ uniformly in t.

(V_4) There exists $U_{ij} \in C^1(W - 0; \mathbb{R})$ on a neighborhood W of 0 in $\mathbb{R}^k$ such that:

(*a*) $U_{ij}(q) \to +\infty$ as $q \to 0$,

(*b*) $-V_{ij}(q, t) \geqq |U'_{ij}(q)|^2$, $q \in W - 0$, $t \in [0, T]$.

(V_4) was introduced by Gordon [G] and called the Strong Force Condition. Consider the following Hamiltonian system

$$m\ddot{q} + V_q(q, t) = 0, \qquad q = (q_1, \ldots, q_N) \tag{HS}$$

where $m = (m_1, \ldots, m_N)$ is the mass vector with $m_i > 0$. The functional corresponding to (HS) has the form

$$I(q) = \sum_{i=1}^{N} \frac{m_i}{2} \int_0^T |\dot{q}_i(t)|^2 \, dt - \int_0^T V(q_1(t), \ldots, q_N(t), t) \, dt \tag{$*$}$$

The arguments employed do not depend on the values m_i so we assume the masses $m_i = 1$ and write

$$I(q) = \frac{1}{2} \int_0^T |\dot{q}|^2 \, dt - \int_0^T V(q, t) \, dt \tag{$*$}$$

If we let E denote the Sobolev space $W_T^{1,2}(\mathbb{R}^{kN})$ of T-periodic, absolutely continuous functions with L^2 derivatives, then if (V_1) holds, $I(q)$ is C^1 and bounded from below by 0 on the open subset

$$\Lambda(N) = \{ q \in W_T^{1,2}(\mathbb{R}^{kN}) : q_i(t) \neq q_j(t),\ 1 \leqq i \neq j \leqq N,\ 0 \leqq t \leqq T \}$$

which corresponds to $W_T^{1,2}(F_N(\mathbb{R}^k))$. We also define a closed subspace E_0 of E as follows: Let $\mathbb{Z}_2 = \{1, \zeta\}$ denote the group of order 2 with non-trivial element ζ. Define the action of $\mathbb{Z}_2$ on E by $(\zeta q)(t) = -q\left(t + \frac{1}{2}\right)$.

Then,

$$E_0 = \{ q \in E, \zeta q = q \}.$$

We also set $\Lambda_0(N) = E_0 \cap \Lambda(N)$.

3.1. THEOREM. — *If the potential V satisfies* (V_1)−(V_4), *then* ($*$) *possesses an unbounded sequence of critical values.*

3.2. *Remark.* — The Coti-Zelati result [C] proves that when V is autonomous and T-periodic, that the minimum of ($*$) is a critical value. The Bahri-Rabinowitz result [BR] for $N = 3$, assumes no symmetry such as $V_{ij} = V_{ji}$, but imposes conditions on behaviour of V and V′ at infinity.

Before proceeding with the proof of Theorem 3.1 we observe that

(1) $$V(-q, t)=V(q, t), \qquad q\in\Lambda(N), \quad t\in[0, T].$$

and

(2) $$I(\zeta q)=I(q), \qquad q\in\Lambda(N).$$

3.3. PROPOSITION. – *Let* I_0 *denote* $I\,|\,\Lambda_0(N)$. *Then critical points of* I_0 *are critical points of* I.

Proof. – This is a general phenomenon. Namely if a functional I is invariant under the action of a finite group G, then critical points of the restriction I_0 to the fixed point set of the action are always critical points of I. In our case, if $u\in E$, then $u+\zeta u\in E_0$ and

$$I'(q)(u+\zeta u)=2\,I'(q)(u), \qquad q\in\Lambda_0(N)$$

and hence if q is a critical point for I_0, $I'(q)$ vanishes on E.

Theorem 3.1 Now follows from the following theorem.

3.4. THEOREM. – *If* V *satisfies* $(V_1)-(V_4)$, *then* $I_0=I\,|\,\Lambda_0(N)$ *possesses an unbounded sequence of critical values.*

The proof will be broken down into a series of lemmas.

3.5. LEMMA (Gordon's Lemma [G]). – *If* V *satisfies* (V_1), (V_3), (V_4) *and if a sequence* q^n *in* $\Lambda_0(N)$ *converges weakly to* $q\in E_0$, *then if* $q\in\partial\Lambda_0(N)$, *then* $I_0(q^n)\to+\infty$, *where* $\partial\Lambda_0(N)$, *is the boundary of* $\Lambda_0(N)$ *in* E_0.

3.6. LEMMA. – *If* V *satisfies* $(V_1)-(V_4)$, I_0 *satisfies the Palais-Smale condition* (PS) *on* $\Lambda_0(N)$.

Proof. – Let q^n denote a sequence in $\Lambda_0(N)$ such that $I_0(q^n)\to s\geqq 0$ and $I_0'(q^n)\to 0$. Then, we may assume $I_0(q^n)\leqq s+1$ and hence

$$\int_0^T |\dot{q}_n(s)|^2\,ds\leqq 2(s+1).$$

Then, since $q_n\in\Lambda_0(N)$

$$\int_t^{t+(1/2)} \dot{q}_n(s)\,ds=q_n\left(t+\frac{1}{2}\right)-q_n(t)=-2\,q_n(t)$$

it follows easily that the sequence q_n is bounded in the $W_T^{1,2}$ norm. Again, by standard arguments [R1], there is a subsequence, also denoted by q_n, such that q_n converges weakly to $q\in E_0$, and Gordon's lemma implies that $q_0\in\Lambda_0(N)$. Furthermore, I′ has the form $I'(q)=q-\mathscr{P}_q$, where $\mathscr{P}q_n$ has a (strongly) convergent subsequence in E_0. Since $I'(q_n)\to 0$, it follows that a subsequence of q_n converges strongly to q and I_0 is (PS) on $\Lambda_0(N)$.

The next lemma merely isolates the deformation theorem we employ (*see* [R2]).

3.7. LEMMA. — *Let* Ω *denote an open set in a Hilbert space* E *and* $I:\Omega\to R$ *a* C^1 *functional which is bounded from below. Suppose* I *satisfies* (PS) *on* Ω *and we have the condition that when* $q_n\to q\in\partial\Omega$, $q_n\in\Omega$, *then* $I(q_n)\to+\infty$, i.e. I *is unbounded at the boundary* $\partial\Omega$. *Then, for* $c\in\mathbb{R}$, $\bar{\varepsilon}>0$ *and* U *a neighborhood of* $K_c=\{q\in\Omega:I(q)=c \text{ and } I'(q)=0\}$, *there is an* $\varepsilon>0$, $\varepsilon<\bar{\varepsilon}$ *and a deformation* $\varphi:\Omega\times I\to\Omega$ *such that*

(1) $\varphi_0=$ *identity*, $\varphi_t:\Omega\to\Omega$ *is a homeomorphism*, $t\in I$.

(2) $\varphi(q,t)=q$ *if* $|I(q)-c|\geqq\bar{\varepsilon}$, $t\in I$.

(3) $\varphi(q,1)\in I^{c-\varepsilon}$ *if* $q\in I^{c+\varepsilon}-U$, *where*

$$I^a=\{q\in\Omega:I(q)<a\}.$$

(4) *If* $K_c=\varnothing$ *we may take* $U=\varnothing$.

3.8. *Remark.* — The proof of Lemma 3.7 requires only a minor modification of the proof of theorem A.4 in [R2]. Following the notation in [R2], let

$$A\equiv\{u\in\Omega\,|\,I(u)\leqq c-\hat{\varepsilon}\}\cup\{u\in E\,|\,I(u)\geqq c+\hat{\varepsilon}\}$$

and

$$B\equiv\{u\in\Omega\,|\,c-\varepsilon\leqq I(u)\leqq c+\varepsilon\}$$

Then, A is closed in Ω but $A\cup(E-\Omega)=A_1$ is closed in E. B is closed in Ω but also closed in E because I is unbounded at $\partial\Omega$. Furthermore, $A_1\cap B=\varnothing$ because $\varepsilon<\hat{\varepsilon}$. After requiring the cutoff function to be 0 on A_1 and I on B, the proof proceeds verbatim and the Ω is forced to be invariant under the flow.

Next, the corresponding abstract critical point theorem.

3.9. LEMMA. — *Let* Ω *denote an open set in a Hilbert space* E *and* $f:\Omega\to\mathbb{R}$ *a* C^1 *functional which is bounded from below. Suppose* f *satisfies* (PS) *on* Ω *and* f *is unbounded at the boundary* $\partial\Omega$. *If* $\operatorname{cat}\Omega=+\infty$ *then* f *possesses an unbounded sequence of critical values.*

Proof. — The proof is quite standard after making a few remarks. First, since Ω is a Hilbert manifold, Ω possesses compact subsets of arbitrarily high category (*see* Section 2.) Thus, if we set $\Sigma_j=\{X\subset\Omega:\operatorname{cat}X\geqq j\}$ and

$$c_j=\inf_{\Sigma_j}\sup_{X}f(x),\qquad j\geqq1$$

we see that the c_j are finite and $c_j\leqq c_{j+1}$ with $c_1=\inf_{\Omega}f$. The usual arguments apply to show that the c_j are all critical values, and $\lim c_j=+\infty$ (*see* [R2]).

Proof of Theorem 3.3. — The proof is an immediate application of Lemma 3.2 to Lemma 3.9.

REFERENCES

[AC] A. AMBROSETTI and V. COTI-ZELATI, Periodic Solutions of Singular dynamical Systems, Periodic Solutions of Hamiltonian Systems and Related Topics, P. H. RABINOWITZ *et al.* Eds., 209, *NATO ASI series*, Reidel, 1987, pp. 1-10.

[BR] ABBAS BAHRI and P. H. RABINOWITZ, Periodic Solutions of Hamiltonian Systems of the 3-Body Type, *Annals Inst. H. Poincaré Analyse non linéaire*, Vol. **8**, 1991, pp. 561-649.

[C] V. COTI-ZELATI, *Periodic Solutions for N-body Type Problems*, Preprint 1989.

[F1] E. FADELL, Cohomological Methods in Non-Free G-Spaces, with Applications to General Borsuk-Ulam Theorems and Critical Point Theorems for Invariant Functionals, *Nonlinear Functional Analysis and its Applications*, Reidel 1986, pp. 1-47.

[F2] E. FADELL, *Lectures in Cohomological Index Theories of G-Spaces with Applications to Critical Point Theory*, Raccoltat Di Università della Calabria, 1987, Cosenza, Italy.

[FH1] E. FADELL and S. HUSSEINI, A Note on the Category of the Free Loop Space, *Proc. Am. Math. Soc.*, Vol. **107**, 1989, pp. 527-536.

[FH2] E. FADELL and S. HUSSEINI, Category of Loop Spaces of Open Subsets in Euclidean Space, *Nonlinear Analysis*, Vol. **17**, 1991, pp. 1153-1161.

[G] W. GORDON, Conservative Dynamical Systems Involving Strong Forces, *Trans. Am. Math. Soc.*, Vol. **204**, 1975, pp. 113-135.

[H] D. HENDERSON, Infinite Dimensional Manifolds Are Open Subsets of Hilbert Space, *Topology*, Vol. **9**, 1970, pp. 25-33.

[K] W. KLINGENBERG, *Lectures on Closed Geodesics*, Springer-Verlag, GMW No. 230, 1978.

[P] R. S. PALAIS, Homotopy Theory of Infinite Dimensional Manifolds, *Topology*, Vol. **5**, 1966, pp. 1-16.

[R1] P. H. RABINOWITZ, *Periodic Solutions for Some Forced Singular Hamiltonian Systems*, Festschrift in honor of Jürgen Moser, Analysis Et Cetera, Academic Press, 1990, pp. 521-544.

[R2] P. H. RABINOWITZ, Minimax Methods in Critical Point Theory with Applications to Differential Equations, *C.B.M.S. Ser. Math.*, Vol. **65**, 1986, A.M.S., Providence, R.I.

[VS] M. VIGUÉ-POIRIER and D. SULLIVAN, The Homology Theory of the Closed Geodesic Problem, *J. Diff. Geometry*, II, 1976, pp. 633-644.

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