Ann. Inst. Henri Poincaré,
Vol. 6, n° 1, 1989, p. 1-14. *Analyse non linéaire*

On a regularizing effect of Schrödinger type groups

by

Mikhaël BALABANE
Université de Reims
et École Normale Supérieure-C.M.A.

ABSTRACT. — We give estimates for the regularity in the space variables of solutions of Schrödinger type evolution equation with Cauchy Data in $L^1(R^n)$. We study the behaviour of these estimates with respect to the time variable, near the origin and at infinity. This is done under the assumption of non degeneracy of the curvature of the wave surfaces. A local decay result is also given. These results are first steps in the study of nonlinear Schrödinger type evolution equations.

Key words : Schrödinger equation, regularization, L^p estimates, regularity of mean values.

RÉSUMÉ. — Nous estimons la régularité dans les variables spatiales des solutions d'équations de type de Schrödinger non stationnaire pour des données initiales dans $L^1(R^n)$. Nous analysons le comportement de ces solutions pour les valeurs de la variable temporelle au voisinage de l'origine et de l'infini. L'hypothèse de base concerne le non-dégénérescence de la courbure des surfaces d'ondes. Puis nous déduisons un résultat de décroissance locale des solutions. Ceci fournit la première étape d'une analyse des opérateurs non linéaires du type de Schrödinger.

Classification A.M.S. : 35 B 45, 35 D 10, 35 H 05, 35 J 10, 42 A 45, 58 G 15.

Annales de l'Institut Henri Poincaré - Analyse non linéaire - 0294-1449
Vol. 6/89/01/01/14/$3,40/

I. INTRODUCTION

If one considers the one parameter group of operators $\exp(it\Delta)$, because of the group law, and because these operators map isomorphically any $H^s(R^n)$ onto itself, no regularizing effects can be expected in the $H^s(R^n)$ framework. Nevertheless results given by Strichartz [5], and Ginibre and Velo ([3], [4]) show that, for any f which belongs to $L^2(R^n)$, something more than being an $L^2(R^n)$ function can be asserted about $\exp(it\Delta)f$. More precisely they proved that for any fixed f in $L^2(R^n)$ and for almost all values of t in R, $\exp(it\Delta)f$ belongs to some $L^p(R^n)$. This does not contradict the group law because such p is greater than two, and the Schrödinger operators do not act on such a space.

The aim of this paper is to show that even regularity can be asserted. To prevent obstruction due to the group law, one considers Cauchy Data in $L^1(R^n)$ and what is proved is regularity of $W^{r,\infty}(R^n)$ type (with $r>2$). An interesting remark is that higher is the order of a pseudo-differential operator P(D) and more regularizing is $\exp(it\Delta)$. Furthermore this regularization is more effective in high space dimension. This translates the dispersion of the waves.

A weaker result was quoted in Balabane-Emami Rad [1] and here we make use of many of the same tools as in that article. These tools are direct computations and estimates using the stationnary phase lemma. In [1] we emphasized boundedness whereas here we give regularity results. These are essential in view of non linear Schrödinger type evolution equations.

II. NOTATIONS

We denote by t a real variable and by $x=(x_1, \ldots, x_n)$ a variable in R^n. Let F be the Fourier Transform in this x variable, the dual variable being denoted by $\xi=(\xi_1, \ldots, \xi_n)$. Let F^- be the Inverse Fourier Transform.

Let S be the unit sphere in R^n. We write, $x=\|x\|u\,(u\in S)$ and $\xi=\rho\omega\,(\rho>0,\ \omega\in S)$, in polar coordinates.

We recall that if Σ is a compact, d-dimensionnal, C^∞ manifold, the Symbol Class $S^k_{1,0}(\Sigma\times R)$ is the set of C^∞ functions $a(\omega, s)$ on $\Sigma\times R$

fulfilling the estimate:

$$\forall \alpha \in \mathrm{N}^k, \quad \forall \beta \in \mathrm{N}, \quad \exists \mathrm{C}_{\alpha,\beta}; \qquad |\partial_\omega^\alpha \partial_s^\beta a(\omega, s)| \leqq \mathrm{C}_{\alpha,\beta}(1+|s|)^{k-|\beta|}$$

In short, $\mathrm{S}^k_{1,0}(\Sigma \times \mathrm{R})$ will be denoted by S^k.

We use the notations $\partial_t = \partial/\partial t$ and $\mathrm{D} = (-i\,\partial/\partial x_1, \ldots, -i\,\partial/\partial x_n)$ for first order differential operators on $\mathrm{R} \times \mathrm{R}^n$, and define a constant coefficients pseudo-differential operator on R^n by the usual formula:

$$\mathrm{P}(\mathrm{D})\,u = \mathrm{F}^-(\mathrm{P}(\xi)(\mathrm{F}(u)(\xi))) \qquad \textit{for} \quad u \in \mathrm{S}(\mathrm{R}).$$

when $\mathrm{P}(\xi)$ belongs to a symbol class $\mathrm{S}^k(\mathrm{R}^n \times R^n)$, and does not depend on the first n variables.

III. ASSUMPTIONS

Let P(D) be a pseudo-differential operator on R^n with constant coefficients. We assume that:

H1. $\mathrm{P}(\xi) = p(\xi) + \mathrm{R}(\xi)$ where P and R are real valued, P is elliptic of order $2\,m$ and $\mathrm{R}(\xi)$ belongs to $\mathrm{S}^{2m-1}_{1,0}$. p is homogeneous.

H2. For $u \in \mathrm{S}$ (the unit sphere in R^n), the restriction of $\Psi(\xi) = \langle u, \xi \rangle p^{-1/2m}(\xi)$ to S *has* only non-degenerate critical points [i. e. if $\omega_0 \in \mathrm{S}$ and $d_\omega \Psi(\omega_0) = 0$, then $d^2_{\omega\omega} \Psi(\omega_0)$ is a non degenerate quadratic form on $\mathrm{T}_{\omega_0}\mathrm{S}$].

Remark. — H1 and H2 are automatically fulfilled if $\mathrm{P}(\mathrm{D}) = (-\Delta)^m + \mathrm{R}(\mathrm{D})$ with $\mathrm{R}(\xi)$ a real symbol which belongs to $\mathrm{S}^{2m-1}_{1,0}$.

IV. THE MAIN STATEMENTS

We consider the Cauchy problem:

$$\text{(S)} \qquad \partial_t u - i\,\mathrm{P}(\mathrm{D})\,u = 0 \qquad \textit{with} \quad u(0, x) = u_0(x) \in \mathrm{S}(\mathrm{R}^n)$$

We denote its solution by $u(t, x) = e^{it\mathrm{P}(\mathrm{D})} u_0(x)$.

Then we have:

THEOREM I (Regularizing effect, L^∞ estimates). — *Assume* H1 *and* H2 *hold. Then the distribution in the t variable $e^{itP(D)}$ is for $t \neq 0$ a function of t with values in the space of bounded linear mappings from* $L^1(R^n)$ *to* $W^{r,\infty}(R^n)$ *and we have the estimate:*

$$\| e^{itP(D)} u_0 \|_{W^{r,\infty}(R^n)} < C(1 + |t|^{-1-c}) \| u_0 \|_{L^1(R^n)}$$

where C *is an absolute constant, provided that:*

(i) $$c = \left[\frac{r+n}{2m-1} \right];$$

(ii) *n is odd and* $0 \leqq r \leqq (m-1)(n-1)-2$, *or, n is even and* $0 \leqq r \leqq (m-1)(n-2)-2$.

THEOREM II (Regularizing effect, L^q estimates). — *Assume* H1 *and* H2 *hold. Then the distribution in the t variable $e^{itP(D)}$ is for $t \neq 0$ a function of t with values in the space of bounded linear mappings from* $L^1(R^n)$ to $W^{r,q}(R^n)$ *and we have the estimate:*

$$\| e^{itP(D)} u_0 \|_{W^{r,q}(R^n)} < C(|t|^d + |t|^{-1-c}) \| u_0 \|_{L^1(R^n)}$$

where C *is an absolute constant provided that:*

(i) *n is odd and* $0 \leqq r \leqq (m-1)(n-3)-3$, *or n is even and* $0 \leqq r \leqq (m-1)(n-2)-2$;

(ii) $$q > \frac{2n}{n-3-2l} \text{ with } l = \left[\frac{2r+n+1}{2(2m-1)} \right];$$

(iii) $$c = \left[\frac{r+n}{2m-1} \right] \text{ and } d = 2\left[\frac{n}{2q} \right] + 2.$$

THEOREM III (Local decay). — *Assume* H1 *and* H2 *hold. Assume that* $P(\xi)$ *has only non-degenerate critical points in* R^n. *Let* u_0 *belong to* $L^1(R^n)$ *and be compactly supported by* K. *Then for any bounded open set* Ω *of* R^n *and any* $r > 0$, *there exists a function* $C_{K,\Omega,r}(t)$ *bounded in the neighbourhood* *of infinity such that:*

$$\| e^{itP(D)} u_0 \|_{W^{r,\infty}(\Omega)} \leqq C_{K,\Omega,r}(t) |t|^{-n/2} \| u_0 \|_{L^1(\mathbb{R}^n)}$$

V. TWO TECHNICAL LEMMAS

The heart of the proof of these theorems will be a foliation of a neighbourhood of infinity in the Fourier variables, fitting the wave surfaces of P(D) [i. e. the surfaces $P(\xi)=\text{const.}$]. The following lemmas analyze the behaviour of $\|\xi\|$ with respect to this foliation. Their proofs have been given in Balabane-Emami Rad [1].

We write $\xi=\rho\omega$, $\omega\in S$, to have:

LEMMA 1. — *There exists two positive constants a and b, and a function* $\rho(\omega, s)\in C^\infty(S\times]a, \infty[)$ *such that:*

(i) *The mapping* $(\omega, s)\to(\omega, \rho)$ *is a* C^∞ *diffeomorphism from* $S\times]a, \infty[$ *onto the complementary set of* $P(\xi)\leqq a$ *in* $\mathbb{R}^n$;

(ii) $P(\rho(\omega, s))=s$;

(iii) $\rho(\omega, s)>b$ *for* $s>a$.

LEMMA 2. — $\rho(\omega, s)=(s/p(\omega))^{1/2m}+\sigma(\omega, s)$ *where* $\sigma(\omega, s)$ *belongs to* $S^0(S\times]a, \infty[)$.

VI. PROOF OF THE THEOREM

The solution of the Cauchy Problem (S) is:

$$u(t, x)=F^{-}(e^{itP(\xi)}(F u_0)(\xi))$$

In order to estimate $\|u(t, x)\|_{W^{r,q}(\mathbb{R}^n)}$, we use:

$$\|u(t, x)\|_{W^{r,q}(\mathbb{R}^n)}=\|F^{-}(e^{itP(\xi)}) * u_0\|_{W^{r,q}(\mathbb{R}^n)}$$
$$\leqq \|F^{-}(e^{itP(\xi)})\|_{W^{r,q}(\mathbb{R}^n)} \|u_0\|_{L^1(\mathbb{R}^n)}$$

and so

$$\|u(t, x)\|_{W^{r,q}(\mathbb{R}^n)}\leqq \sup_{|\beta|\leqq r} \|F^{-}(\xi^\beta e^{itP(\xi)})\|_{L^q(\mathbb{R}^n)} \|u_0\|_{L^1(\mathbb{R}^n)} \tag{0}$$

We compute the inverse Fourier transform of $\xi^\beta e^{itP(\xi)}$ by writing it as an oscillatory integral and splitting it into two terms:

$$F^-(\xi^\beta e^{itP(\xi)}) = I_0(t, x) + I_\infty(t, x)$$

with (for $j=0$ or ∞)

$$I_j(t, x) = \int_{\mathbf{R}^n} e^{ix\cdot\xi} e^{itP(\xi)} \xi^\beta a_j(\xi)\, d\xi$$

Here $(a_0(\xi), a_\infty(\xi))$ is a C^∞ partition of unity in $\mathbf{R}^n$, fitting the open sets $P(\xi) < a+1$ and $P(\xi) > a$.

We assume that $t \neq 0$.

A. Estimating $I_0(t, x)$

$$I_0(t, x) = \int_{\mathbf{R}^n} e^{ix\cdot\xi} e^{itP(\xi)} \xi^\beta a_0(\xi)\, d\xi$$

appears as the Inverse Fourier Transform of an infinitely differentiable function with compact support, and so it is a C^∞ function, rapidly decreasing at infinity in the x variable. It is obvious to control its behaviour with respect to the t-variable:

For x in a bounded set of $\mathbf{R}^n$:

$$|I_0(t, x)| \leqq \int_{\mathbf{R}^n} \|\xi\|^{|\beta|} |a_0(\xi)|\, d\xi$$

Integrating by parts $2k$ times in order to control the decrease for x in a neighbourhood of infinity, we have:

$$|I_0(t, x)| \leqq \left| \int_{\mathbf{R}^n} \frac{e^{ix\cdot\xi}}{\|x\|^{2k}} \Delta(e^{itP(\xi)} \xi^\beta a_0(\xi))\, d\xi \right| \leqq c_{2k} \frac{(1+|t|^{2k})}{\|x\|^{2k}}$$

where c_{2k} is an absolute constant.

In short, for any integer k, there exists an absolute constant C_{2k} with:

$$|I_0(t, x)| \leqq C_{2k} \frac{(1+|t|^{2k})}{(1+\|x\|^{2k})} \tag{1}$$

Taking $k=0$ shows that I_0 belongs to $L^\infty(\mathbf{R}^n)$ uniformly in t.

B. Estimating $I_\infty(t, x)$

$$I_\infty(t, x)=\int_{\mathbf{R}^n} e^{ix\cdot\xi}\, e^{it\mathrm{P}(\xi)}\, \xi^\beta\, a_\infty(\xi)\, d\xi$$

Because $a_\infty(\xi)$ is zero on $\mathrm{P}(\xi)<a$, we can apply lemmas 1 and 2 in order to write I_∞ as an oscillatory integral in the (ω, s) variables. Let us use, in polar coordinates, the notation:

$$\xi=\rho\omega, \qquad x=\|x\|u$$
$$\lambda=\|x\|s^{1/2m}$$
$$\Phi(\omega, s)=\langle u.\omega\rangle\rho s^{-1/2m}$$
$$A_0(\omega, s)=(\rho\omega)^\beta a_\infty(\rho\omega)\rho^{n-1}\partial_s\rho$$

By lemma 2, $A_0(\omega, s)$ is a symbol which belongs to $S^{(|\beta|+n-2m)/2m}$, and $\Phi(\omega, s)$ is a symbol which belongs to S^0. We have:

$$I_\infty(t, x)=\int_0^\infty e^{its}\left(\int_S e^{i\lambda\Phi} A_0\, d\omega\right) ds$$

Using l times the classical formula for Fourier transform of derivatives in the s variables gives, for $t\neq 0$,

$$I_\infty(t, x)=\int_0^\infty \frac{e^{its}}{(-it)^l}\left(\int_S \partial_s^l[e^{i\lambda\Phi} A_0]\, d\omega\right) ds$$

Differentiating $e^{i\lambda\Phi}A_0$ one time in s gives:

$$\partial_s[e^{i\lambda\Phi}A_0]=e^{i\lambda\Phi}[i\lambda A_0\partial_s\Phi+i\Phi A_0\partial_s\lambda+\partial_s A_0]=(1+\|x\|)\,e^{i\lambda\Phi}A_1$$

Because $\lambda\in S^{1/2m}$, $\Phi\in S^0$, $A_0\in S^{(|\beta|+n-2m)/2m}$ we have the following estimate for the three terms which appear in this derivative,

(i) $|\partial_s^\alpha\partial_\omega^\gamma\lambda A_0\partial_s\Phi|$

$$\leqq \sum_{h+k+l=\alpha}\ \sum_{|d|+|e|+|f|=|\gamma|} C(h, k, l, d, e, f)\,|[\partial_s^h\partial_\omega^d\lambda].[\partial_s^k\partial_\omega^e A_0].[\partial_s^f\partial_\omega^l\partial_s\Phi]|$$

$$\leqq \sum_{h+k+l=\alpha}\ \sum_{|d|+|e|+|f|=|\gamma|} C'(h, \ldots, f)\,\|x\|$$

$$(1+s)^{(1-2mh)/2m}(1+s)^{(|\beta|+n-2m(k+1))/2m}(1+s)^{-1-l}$$

$$\leqq C''\,\|x\|\,(1+s)^{(|\beta|+n+1-2m(2+\alpha))/2m};$$

(ii) *The same computation works for* $\partial_s^\alpha \partial_\omega^\gamma A_0 \Phi \partial_s \lambda$;

(iii) $|\partial_s^\alpha \partial_\omega^\gamma \partial_s A_0| \leqq C'''(1+s)^{|\beta|+n-2m(2+\alpha)/2m}$

where C'' and C''' are absolute constants, not depending on $\|x\|$. This proves that, uniformly in the $\|x\|$ parameter, A_1 belongs to $S^{(|\beta|+n-4m+1)/2m}$.

Then by induction, we have:

$$\partial_s^l [e^{i\lambda\Phi} A_0] = (1 + \|x\|^l) e^{i\lambda\Phi} A_l$$

where A_l is a symbol which belongs to $S^{(|\beta|+n-2m-l(2m-1))/2m}$ uniformly in the $\|x\|$ parameter:

The formula becomes:

$$I_\infty(t, x) = \frac{1+\|x\|^l}{t^l} \int_0^\infty e^{its} \left(\int_S e^{i\lambda\Phi} A_l \, d\omega \right) ds \tag{2}$$

a. Estimating $I_\infty(t, x)$ *for* x *in any bounded region*

For bounded values of t, taking in (2)

$$l = \left[\frac{r+n}{2m-1} \right] + 1$$

implies that the order of the symbol A_l is less than -1, and the integral involved in (2) is absolutely convergent. So we have:

$$|I_\infty(t, x)| \leqq C' |t|^{-1-[r+n/2m-1]}$$

where C' does not depend on x if $\|x\|$ is bounded.

For t in a neighbourhood of infinity, (2) shows that $I_\infty(t, x)$ is rapidly decreasing.

The global (in the t variable) estimate for $I_\infty(t, x)$ is then:

$$\textit{For any } l \geqq 0, \quad |I_\infty(t, x)| \leqq C_l |t|^{-1-[r+n/2m-1]} (1+|t|)^{-l} \tag{3}$$

wheres C_l does not depend on x if $\|x\|$ is bounded.

b. Estimating $I_\infty(t, x)$ *for* x *in a neighbourhood of infinity*

Integration by parts in the s variable in formula (2) for $I_\infty(t, x)$ gave rise to a factor $\|x\|^l$. But when $\|x\|$ goes to infinity, the integral on S goes to zero because of oscillations due to the phase $\lambda\Phi$. We will use this fact to balance the increase of the term $\|x\|^l$. The asymptotic behaviour with respect to λ of the integral on S is given by the Stationnary Phase Lemma, which we can apply because assumption H2 implies:

LEMMA 3 (Balabane Emami Rad [1]). — *There exists a finite number of open sets Ω_i of S ($i=1, \ldots, N$) and a constant $d>a$ such that, for $s>d$:*

(1) *On the complementary set of $\bigcup_i \Omega_i$ in S, $\Phi(\omega, s)$ has no critical points in the ω variables and we have $\|d_\omega \Phi\| \geqq C>0$.*

(2) *On each Ω_i, $\Phi(\omega, s)$ has only one critical point $\omega^i(s) \in \Omega_i'$, where Ω_i' is an open set, $\Omega_i \supset \supset \Omega_i'$. At that point, $\Phi(\omega, s)$ is non-degenerate: the eigenvalues of the Hessian matrix of Φ in the ω variables at $(\omega^i(s), s)$ have their modulus bounded from below [i. e. let*

$$H(s)=\mathrm{Hess}_\omega(\Phi)(\omega^i(s), s)=d^2_{\omega\omega}\Phi(\omega^i(s), s),$$

then $\|H^{-1}(s)\| \leqq C'$].

(3) $\omega^i(s) \in C^\infty(]d, \infty[; S)$ *and the estimates are uniform in s (i. e. Ω_i, Ω_i', C, C' and the bounds on ω_i and its derivatives do not depend on $s>d$).*

We denote by Ω_0 the complementary set of $\bigcup_i \Omega_i'$, *and* we take a partition of unity on S related to the covering $\bigcup_i \Omega_i \cup \Omega_0$.

On Ω_0 there exists a first order differential operator L in the ω variables with coefficients uniformly bounded in the s variable such that ${}^t L\,\Phi=1$. We use k times the operator L to integrate by parts in the ω variables, to have:

$$\int_{\Omega_0} e^{i\lambda\Phi} A_l \, d\omega = \lambda^{-k} \int_{\Omega_0} e^{i\lambda\Phi} L^k(A_l)\, d\omega$$

Because A_l belongs to $S^{(|\beta|+n-2m-l(2m-1))/2m}$ this implies:

$$\left| \int_{\Omega_0} e^{i\lambda\Phi} A_l \, d\omega \right| \leqq D_{k,l}\, \lambda^{-k} (1+s)^{(r+n-2m-l(2m-1))/2m}$$

where $D_{k,l}$ is an absolute constant. So

$$\frac{1+\|x\|^l}{|t|^l} \left| \int_0^\infty e^{its} \int_{\Omega_0} e^{i\lambda\Phi} A_l \, d\omega \, ds \right| \leqq D_{k,l} \frac{1+\|x\|^{l-k}}{|t|^l} \tag{4}$$

as soon as

$$(2m-1)l+k>r+n$$

On the other hand, on any open set Ω_i, we apply the Stationnary Phase Lemma with parameters (Duistermaat [2]) to have:

$$\left|\int_{\Omega_i} e^{i\lambda\Phi} A_l \, d\omega\right| \leqq D_l' \lambda^{-(n-1)/2} (1+s)^{(r+n-2m-l(2m-1))/2m}$$

where D′ is an absolute constant. So

$$\frac{1+\|x\|^l}{|t|^l}\left|\int_0^\infty e^{its}\int_{\Omega_i} e^{i\lambda\Phi} A_l \, d\omega \, ds\right| \leqq \frac{D_l'}{|t|^l}(1+\|x\|)^{l-((n-1)/2)} \tag{5}$$

as soon as

$$2l(2m-1)>2r+n+1$$

For t in a neighbourhood of infinity, taking

$$l=k=\left[\frac{r+n}{2m}\right]+1$$

fulfills the assumption of (4) and shows that

$$\left|\int_0^\infty e^{its}\int_{\Omega_0} e^{i\lambda\Phi} A_0 \, d\omega \, ds\right| \leqq D|t|^{-[(r+n)/2m]-1}$$

where D is an absolute constant.

For t in a neighbourhood of infinity, taking

$$l=\frac{n-1}{2} \textit{ if } n \textit{ is odd or } l=\frac{n-2}{2} \textit{ if } n \textit{ is even}$$

fulfills the assumptions of (5) because of the assumption on r in Theorem 1, and shows that

$$\left|\int_0^\infty e^{its}\int_{\Omega_i} e^{i\lambda\Phi} A_0 \, d\omega \, ds\right| \leqq D'|t|^{-[(n-1)/2]}$$

These two estimates show that, for x in a neighbourhood of infinity and t in a neighbourhood of infinity, we have:

$$|I_\infty(t, x)| \leqq D'' |t|^{-min([(n-1)/2], [(r+n)/2m]+1)} \leqq D'' |t|^{-[(n-1)/2]} \qquad (6)$$

because of the assumption on r. D'' is an absolute constant.

For t in a neighbourhood of zero, taking $l=0$, $k=r+n+1$ fulfills the assumptions of (4) and show that

$$\left| \int_0^\infty e^{its} \int_{\Omega_0} e^{i\lambda\Theta} A_0 \, d\omega \, ds \right| \leqq E$$

where E is an absolute constant.

For t in a neighbourhood of zero, taking

$$l = \left[\frac{2r+n+1}{2(2m-1)} \right] + 1$$

fulfills the assumptions of (5) because of the choice of r in Theorem 1 and shows that

$$\left| \int_0^\infty e^{its} \int_{\Omega_i} e^{i\lambda\Phi} A_0 \, d\omega \, ds \right| \leqq E' |t|^{-1-[(2r+n+1)/2(2m-1)]}$$

These two estimates show that, for x in the neighbourhood of infinity and t in a neighbourhood of zero, we have the estimate:

$$|I_\infty(t, x)| \leqq E'' |t|^{-1-[(2r+n+1)/2(2m-1)]} \qquad (7)$$

where E'' is an absolute constant.

The global estimate (in the t variable) for I_∞, when x is in a neighbourhood of infinity, is then given by (6) and (7) and turns to be:

$$|I_\infty(t, x)| \leqq G(1+|t|^{-1-[(2r+n+1)/2(2m-1)]})(1+|t|)^{-[(n-1)/2]} \qquad (8)$$

c. *Estimating* $I_\infty(t, x)$

Recollecting (3) and (8) gives, under the assumptions of Theorem 1 on r:

$$|I_\infty(t, x)| \leqq K(1+|t|^{-1-[(r+n)/(2m-1)]})(1+|t|)^{-[(n-1)/2]} \qquad (9)$$

C. Final estimate

Recollecting (1) and (9) gives, under the assumptions of Theorem 1 on r:

$$\| F^{-}(\xi^{\beta} e^{itP(\xi)}) \|_{W^{r,\infty}(\mathbb{R}^n)} \leqq L(1+|t|^{-1-[(r+n)/(2m-1)]})$$

where L is an absolute constant.

Q.E.D.

VII. PROOF OF THEOREM II

Following (0) we must estimate $\| F^{-}(\xi^{\beta} e^{itP(\xi)}) \|_{L^q(\mathbb{R}^n)}$ for $|\beta| \leqq r$.

First we note that the assumptions of Theorem II are more restrictive than those of Theorem I. This implies that for any $q \geqq 1$,

$$F^{-}(e^{itP(\xi)}) \in W^{r,q}_{\mathrm{loc}}(\mathbb{R}^n)$$

It remains to estimate the behaviour of $F^{-}(e^{itP(\xi)})$ in a neighbourhood of infinity in the x variables. It follows from (1) that, for any integer k,

$$|I_0(t,x)| \leqq C_{2k} \frac{1+|t|^{2k}}{\|x\|^{2k}}$$

This implies that $I_0(t,x)$ belongs to $L^q(\mathbb{R}^n)$ for any $q \geqq 1$, and

$$\| I_0(t,x) \|_{L^q(\mathbb{R}^n)} \leqq C(1+|t|^{2(1+[n/2q])}) \tag{10}$$

In order to estimate $I_\infty(t,x)$, we first use the estimate (4) which gives:

$$\left| \int_0^\infty e^{its} \int_{\Omega_0} e^{i\lambda\Phi} A_0 \, d\omega \, ds \right| \leqq D_{l,k} \frac{\|x\|^{l-k}}{|t|^l}$$

provided that $(2m-1)l+k>r+n$, and this belongs to L^q if $(k-l)q>n$. Taking

$$l = 1 + \left[\frac{2r+n+1}{2(2m-1)} \right]$$

and k big enough solves these two inequalities.

We then use estimate (5) which gives:

$$\left| \int_0^\infty e^{its} \int_{\Omega_i} e^{i\lambda\Phi} A_0 \, d\omega \, ds \right| \leqq \frac{D'_l}{|t|^l} \| x \|^{l-((n-1)/2)}$$

provided that $2r+n+1<2l(2m-1)$. This belongs to $L^q(R^n)$ if $(n-1-2l)q>2n$. Taking

$$l=1+\left[\frac{2r+n+1}{2(2m-1)}\right]$$

fulfills the first inequality. The assumption of Theorem II on r are equivalent to $n-1>2l$. The assumption of Theorem II on q solves the second inequality.

The behaviour of these estimates when t goes to zero or to infinity implies the behaviour quoted in Theorem II.

Q.E.D.

VIII. PROOF OF THEOREM III

We have to prove that on any bounded open set Ω of R^n, $F^-(e^{itP(\xi)})$ is an infinitly differentiable function of the variables x when $t \neq 0$, and any of its local Sobolev norms decreases at least as $|t|^{-n/2}$ when t goes to infinity.

The estimate (3) shows that, for any $r \geqq 0$, and any integer l,

$$\| I_\infty(t, x) \|_{W^{r,\infty}(\Omega)} \leqq C_l |t|^{-l}$$

The assumption of non degeneracy of $P(\xi)$ at its critical points implies that we can estimate $I_0(t, x)$ using the Stationnary Phase Lemma for the integral in the ξ variables, on $P(\xi)<a+1$, the parameter being t.

Q.E.D.

REFERENCES

[1] M. BALABANE and H. A. EMAMI RAD, L^p Estimates for Schrödinger Evolution Equations, *Trans. Amer. Math. Soc.*, Vol. **292**, n° 1, 1985.
[2] J. J. DUISTERMAAT, *Fourier Integral Operators*, Courant Inst. Math. Sc. N.Y.U., 1973.
[3] J. GINIBRE and G. VELO, On the Global Cauchy Problem for Some Non Linear Schrödinger Equations, *Ann. Inst. Henri Poincaré*, Vol. **1**, n° 4, 1984.
[4] J. GINIBRE and G. VELO, The Global Cauchy Problem for the Non Linear Schrödinger Equation Revisited, *Ann. Inst. Henri Poincaré*, Vol. **2**, n° 4, 1985.
[5] R. STRICHARTZ, *Duke Math. Journal*, t. **44**, 1977.

(*Manuscrit reçu le* 2 *octobre* 1987.)