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On the density of the range for some nonlinear operators

by

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ABSTRACT. — We study the existence of periodic solutions for a class of nonlinear Hamiltonian systems

$$\dot{z} - \mathcal{J}\,\mathrm{H}'(z) = f(t).$$

By using the Leray-Schauder Theorem to solve a modified problem and passing to a limit, we show that $\mathcal{J}\,d/dt - \mathcal{J}\,\mathrm{H}'(.)$ has a dense range in $\mathcal{L}^2$. We also obtain similar density results for nonlinear Schrödinger operators and other problems.

Key words : Hamiltonian systems, Schrödinger equations, Superlinear nonlinearity, Periodic solution, Dense range.

RÉSUMÉ. — Nous étudions l'existence de solutions périodiques pour une classe de systèmes Hamiltoniens non linéaires

$$\dot{z} - \mathcal{J}\,\mathrm{H}'(z) = f(t).$$

A l'aide du théorème de Leray-Schauder nous résolvons un problème modifié et passons à la limite. On montre que l'opérateur $\mathcal{J}\,d/dt - \mathcal{J}\,\mathrm{H}'(.)$ a une image dense dans $\mathcal{L}^2$. Nous obtenons aussi des résultats semblables pour d'autres opérateurs dont les opérateurs de Schrödinger non linéaires.

Classification A.M.S. : 34 C 25, 35 B 10, 35 J 10, 58 F 05.

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1. INTRODUCTION

During the past few years there has been a considerable amount of research on the existence of multiple solutions of "superlinear" differential equations. By superlinear we mean the equation possesses a nonlinearity which grows more rapidly than linearly at infinity. For example, in the setting of periodicity problems for Hamiltonian systems, Rabinowitz proved ([8], [9]).

THEOREM 1.1. — *Let* H *satisfy the following conditions:*

(H 1) $H \in C^1(\mathbf{R}^{2n}, \mathbf{R})$,

(H 2) *there are* $\mu > 2$ *and* $r > 0$ *such that* $0 < \mu H(z) \leqq H_z(z) \cdot z$, $\forall |z| \geqq r$.

Then for any $T > 0$, *the Hamiltonian system*

$$\dot{z} - \mathcal{J} H_z(z) = 0 \tag{1.2}$$

possesses infinitely many distinct T*-periodic solutions.*

The existence of multiple solutions is based on a variational formulation of (1.2) which is invariant under a group action of symmetries. When one considers perturbation problems, such symmetries break down.

Nevertheless the existence of multiple solutions for the perturbed problem

$$\dot{z} - \mathcal{J} H_z(z) = f(t) \tag{1.3}$$

has been proved under conditions (H 1) (H 2) and some further conditions on the growth of H (*cf.* [2], [6], [7]). A natural question to ask is whether (1.3) still possesses infinitely many solutions under conditions (H 1) and (H 2) only.

In this paper we will prove another kind of existence result for (1.3), namely,

THEOREM 1.4. — *Let* H *satisfy* (H 1) *and*

$$\lim_{|z| \to +\infty} \frac{H_z(z) \cdot z}{|z|^2} = +\infty. \tag{H 3}$$

Then for any $T > 0$, *there exists a dense set* A *in the space* E *of* T*-periodic functions in* $L^2([0, T], \mathbf{R}^{2n})$ *such that for any* $f \in A$, (1.3) *possesses a* T*-periodic solution, (i. e. the range of* $\mathcal{J}\, d/dt - \mathcal{J} H_z(.)$ *is dense in* E).

Note that we require much milder conditions on H than ([2], [6], [7]), but we also get a weaker existence statement. However Theorem 1.4 suggests that a stronger result than ([2], [6], [7]) may be true.

We also apply similar ideas to Schrödinger equations, beam equations, and other problems. Since these problems have some common feature and their proofs can be fitted into a common abstract frame work, we first prove this abstract result in paragraphe 2. Combining the abstract theorem with specific estimates we prove the density theorems for Hamiltonian systems in paragraph 3, and for Schrödinger equations and other problems in paragraph 4. This paper was motivated by an analogous kind of result for a semilinear wave equation by Tanaka [11].

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2. AN ABSTRACT DENSITY RESULT

Let E be a Hilbert space. We denote its norm and inner product by $\|.\|$ and $(\, , \,)$ respectively. Let $W_2 \subset W_1 \subset E$, where both are Banach spaces with norms $\|.\|_i$, $i=1, 2$. We assume that for $z \in W_1$, $\|z\| \leqq \|z\|_1$ and for $z \in W_2$, $\|z\|_1 \leqq \|z\|_2$.

Let $\mathscr{L}(W, E)$ be the family of linear operators from W to E, and $C(W, E)$ be the family of continuous operators from W to E.

We consider the following conditions (DL), (F), (S), and (E).

(DL). There are operators $D, B \in \mathscr{L}(W_1, E)$, $L \in \mathscr{L}(W_2, E)$, such that

(1) $(Dz, z)=0$, $\forall z \in W_1$.

(2) The domain $\mathscr{D}(D^*)$ of D^*, the adjoint operator of D, is dense in E.

(3) $(Lz, Dz)=0$, $\forall z \in W_2$.

(4) There are constants $\alpha_1>0$, $\alpha_2 \geqq 0$ such that

$$(Lz, z) \geqq \alpha_1 \|Bz\|^2 - \alpha_2 (\|Dz\|^2 + \|Dz\| \, \|z\|), \qquad \forall z \in W_2.$$

(5) There is a constant $\alpha_3>0$ such that

$$\|z\|_2 \leqq \alpha_3 (\|Lz\| + \|z\|_1), \qquad \forall z \in W_2.$$

(F) There is a compact map $F \in C(W_1, E)$ such that:

(1) $(F(z), Dz)=0$, $\forall z \in W_1$.

(2) For any given $K>0$, there is $C_K>0$ such that

$$(F(z), z) \geqq K \|z\|^2 - C_K, \qquad \forall z \in W_1.$$

(S) (1) For any given $\varepsilon \in (0, 1)$ and $f \in E$, the equation

$$Lz + \varepsilon Dz + \varepsilon z = f \tag{2.1}$$

possesses a solution $z \in W_2$.

(2) There is a constant $\eta>0$ depending on ε such that if $z \in W_2$ is a solution of (2.1), then

$$\|z\|_2 \leqq \eta \|f\|.$$

(E) For any given $\varepsilon \in (0, 1)$ and $f \in E$, there is a constant $\beta>0$ depending on ε and f but independent of $\lambda \in [0, 1]$ such that if $z \in W_2$ is a solution of the equation

$$L z+\varepsilon D z+\varepsilon z+\lambda F(z)=\lambda f \tag{2.2}$$

then

$$\|z\|_1 \leqq \beta(\|Dz\|+\|Bz\|+\|z\|+1).$$

We have the following.

PROPOSITION 2.3. — *Assume conditions* (DL), (F), (S), *and* (E) *hold. Then the range of the operator* L+F *is dense in* E.

Proof. — We carry out the proof in several steps. In step 1, we give an *a priori* bound for $\|z\|_2$ for any solution z of (2.2). Then in step 2 using the Leray-Schauder Theorem we prove the existence of a solution of (2.2) with $\lambda=1$ in W_2. Finally in step 3 by letting ε tend to zero we prove that L+F has dense range in E.

Step 1. — Given $\varepsilon\in(0,1)$, $\lambda\in[0,1]$ and $f\in E$. Assuming $z\in W_2$ is a solution of (2.2), we shall obtain an *a priori* bound for $\|z\|_2$.

Multiplying (2.2) by Dz, using (1), (3) of (DL) and (1) of (F) we get that

$$\varepsilon\|Dz\|^2=\lambda(f, Dz)\leqq\|f\|\,\|Dz\|. \tag{2.4}$$

So

$$\|Dz\|\leqq\frac{1}{\varepsilon}\|f\|. \tag{2.5}$$

Multiplying (2.2) by z, using (1), (4) of (DL) and (2) of (F) we get

$$\alpha_1\|Bz\|^2-\alpha_2(\|Dz\|^2+\|Dz\|\,\|z\|)+\varepsilon\|z\|^2-C_1\leqq\|f\|\,\|z\|.$$

So

$$\alpha_1\|Bz\|^2+\varepsilon\|z\|^2\leqq C_1+\alpha_2\|Dz\|^2+\frac{\delta}{2}\alpha_2\|z\|^2 +\frac{1}{2\delta}\|Dz\|^2+\frac{\delta}{2}\|z\|^2+\frac{1}{2\delta}\|f\|^2.$$

Letting $\delta=\varepsilon/(1+\alpha_2)$ and transferring all $\|z\|^2$ terms to the left yields

$$\alpha_1\|Bz\|^2+\frac{\varepsilon}{2}\|z\|^2\leqq C_1+\left(\alpha_2+\frac{1}{2\delta}\right)\|Dz\|^2+\frac{1}{2\delta}\|f\|^2. \tag{2.6}$$

Combining (2.5), (2.6) with (E), we get a constant $\gamma_1=\gamma_1(\varepsilon, \|f\|)>0$ independent of λ and z such that

$$\|z\|_1\leqq\gamma_1. \tag{2.7}$$

By the condition (F), F is compact and hence bounded from W_1 to E. Consequently there is a constant $\gamma_2=\gamma_2(F, \gamma_1)>0$ independent of λ and z

such that

$$\|F(z)\| \leqq \gamma_2.$$

Therefore from (2.2) we get

$$\|Lz\| \leqq \varepsilon \|Dz\| + \varepsilon \|z\| + \gamma_2 + \|f\|.$$

Combining with (5) of (DL), (2.5), (2.6) and (2.7) yield a constant $\gamma_3 = \gamma_3(\varepsilon, \|f\|, F) > 0$ independent of $\lambda \in [0, 1]$ and z such that

$$\|z\|_2 \leqq \gamma_3. \tag{2.8}$$

Step 2. — The condition (S) implies the uniqueness of the solution z_ε, of (2.1) in W_2. So the solution operator $S_\varepsilon : E \to W_2$ of (2.1), $S_\varepsilon(f) = z_\varepsilon$, is well-defined and $S_\varepsilon \in C(E, W_2)$ by the condition (S).

For given $\varepsilon \in (0, 1)$, $f \in E$, define $Q(z) = f - F(z)$ for $z \in W_2$. By the condition (F), $Q \in C(W_2, E)$ is compact. Define $P_\varepsilon = S_\varepsilon \circ Q(z)$ for $z \in W_2$, then $P_\varepsilon \in C(W_2, W_2)$ is compact.

For any $\lambda \in [0, 1]$, if $z \in W_2$ is a solution of $z = \lambda P_\varepsilon(z)$, (2.8) gives a bound of $\|z\|_2$ uniformly in λ. So by the Leray-Schauder Theorem (*cf.* [3]) we get a fixed point z of P_ε in W_2, which is a solution of (2.2) with $\lambda = 1$. That is

$$Lz + \varepsilon Dz + \varepsilon z + F(z) = f. \tag{2.9}$$

Step 3. — Given any $f \in \mathscr{D}$, the domain of D^*, for every $\varepsilon \in (0, 1)$ (2.9) possesses a solution z_ε in W_2 by step 2. For convenience we omit the subindex ε of z_ε, in the following.

Multiplying (2.9) by Dz, as in (2.4) we get

$$\varepsilon \|Dz\|^2 \leqq (f, Dz) = (D^* f, z) \leqq \|D^* f\| \|z\|, \tag{2.10}$$

and (2.5).

Multiplying (2.9) by z and using (1), (4) of (DL), (2) of (F), for any $K > 0$ we get a constant $C_K > 0$ such that

$$\alpha_1 \|Bz\|^2 - \alpha_2 (\|Dz\|^2 + \|Dz\| \|z\|) + \varepsilon \|z\|^2 + K \|z\|^2 - C_K \leqq \|f\| \|z\|.$$

So

$$\begin{aligned}
K\|z\|^2 &\leqq C_K + \frac{1}{2}\|z\|^2 + \frac{1}{2}\|f\|^2 + \alpha_2 (\|Dz\|^2 + \|Dz\| \|z\|) \\
&\leqq C_K + \frac{1}{2}\|z\|^2 + \|f\|^2 + \alpha_2 \left(\frac{1}{\varepsilon^2}\|f\|^2 + \frac{1}{\varepsilon}\|f\| \|z\|\right) \quad \text{(by (2.5))} \\
&\leqq C_K + \frac{1}{2}\|z\|^2 + \frac{1}{2}\|f\|^2 + \frac{\alpha_2}{\varepsilon^2}\|f\|^2 + \frac{\alpha_2}{2\varepsilon^2}\|f\|^2 + \frac{\alpha_2}{2}\|z\|^2.
\end{aligned}$$

Thus

$$\left(K - \frac{1}{2}(1 + \alpha_2)\right) \|\varepsilon z\|^2 \leqq \varepsilon^2 \left(C_K + \frac{1}{2}\|f\|^2\right) + \frac{3}{2}\alpha_2 \|f\|^2.$$

So

$$\overline{\lim_{\varepsilon\to 0}} \|\varepsilon z\|^2 \leq \frac{1}{K-(1/2)(1+\alpha_2)} \cdot \frac{3}{2}\alpha_2 \|f\|^2 \to 0 \quad \text{as } K\to +\infty.$$

Therefore

$$\lim_{\varepsilon\to 0} \|\varepsilon z\| = 0. \tag{2.11}$$

Combining with (2.10) we get

$$\lim_{\varepsilon\to 0} \|\varepsilon Dz\| = 0. \tag{2.12}$$

(2.11) and (2.12) imply that

$$Lz+F(z)=f-\varepsilon Dz-\varepsilon z\to f \quad \text{in E as } \varepsilon\to 0.$$

This proves that $\mathscr{D}$ is contained in the closure of the range of L+F in E. By (2) of (DL), $\mathscr{D}$ is dense in E. Therefore this completes the proof of Proposition 2.3.

Q.E.D.

3. DENSITY RESULTS FOR SUPERQUADRATIC HAMILTONIAN SYSTEMS

We consider the Hamiltonian system

$$\mathscr{J}\dot{z}+H'(z)=f(t), \tag{3.1}$$

where $\dot{z}=dz/dt$, $H:\mathbf{R}^{2n}\to\mathbf{R}$, H' is its gradient. $\mathscr{J}=\begin{pmatrix} 0 & -\mathscr{I} \\ \mathscr{I} & 0 \end{pmatrix}$, $\mathscr{I}$ is the identity matrix on $\mathbf{R}^n$. Given $T>0$, let E be the space of all T-periodic functions in $L^2([0, T], \mathbf{R}^{2n})$, $f\in E$. We now give the:

Proof of Theorem 1.4. – We shall apply Proposition 2.3 to get Theorem 1.4.

For $z\in E$, denote its norm and inner product by

$$\|z\| = \left(\int_0^T |z(t)|^2\,dt\right)^{1/2} \quad \text{and} \quad (z, w)=\int_0^T z.w\,dt.$$

Let $W=W_1=W_2=W^{1,2}([0, T], \mathbf{R}^{2n})\cap E$ with norm

$$\|z\|_1 = \left(\int_0^T (|\dot{z}(t)|^2+|z(t)|^2)\,dt\right)^{1/2}.$$

Define $Dz=dz/dt$, $Bz=0$, $Lz=\mathscr{J}\,dz/dt$. Then D, B, $L\in\mathscr{L}(W_1, E)$, and it is easy to verify that (DL) holds. Note that the domain of $D^*=-D$ is

W_1 which is dense in E. For $z \in W_1$, define $F(z) = H'(z)$. Since W_1 is compactly embedded into $C([0, T], \mathbf{R}^{2n})$, $F \in C(W_1, E)$ is compact. Since $\int_0^T H'(z) \cdot \dot{z}\, dt = \int_0^T \frac{d}{dt} H(z)\, dt = H(z(T)) - H(z(0)) = 0$, (1) of (F) holds. Given any $K > 0$, by (H 3) there is $C_K > 0$ such that $H'(z) . z \geqq K |z|^2 - C_K$ for all $z \in \mathbf{R}^{2n}$. So for $z \in W_1$

$$(F(z), z) = \int_0^T H'(z) \cdot z\, dt \geqq K \|z\|^2 - TC_K.$$

This shows that (2) of (F) holds. The condition (E) holds not only for solutions of (2.2) but also for all $z \in W_1$ with $\beta = 1$ by the definition of $\|.\|_1$. So we only left to verify the condition (S), i. e. for any given $\varepsilon > 0, f \in E$, there exists a solution $z \in W_1$ of

$$\mathcal{J} \dot{z} + \varepsilon \dot{z} + \varepsilon z = f, \tag{3.2}$$

and there exists *a priori* estimates for solutions of (3.2) in W_1 in terms of $\|f\|$.

To get an estimate on $\|z\|_1$ in terms of $\|f\|$, we multiply (3.2) by $\dot{z}$, then integrate on [0, T]. This yields

$$\|\dot{z}\| \leqq \frac{1}{\varepsilon} \|f\|. \tag{3.3}$$

Multiplying (3.2) by z and integrating on [0, T] yield

$$\begin{aligned} \varepsilon \|z\|^2 &= (f, z) - (\mathcal{J} \dot{z}, z) \\ &\leqq \|f\| \|z\| + \|\dot{z}\| \|z\| \leqq \left(1 + \frac{1}{\varepsilon}\right) \|f\| \|z\| \quad (\text{by} (3.2)) \\ &\leqq \frac{\varepsilon}{2} \|z\|^2 + \frac{1}{2\varepsilon} \left(1 + \frac{1}{\varepsilon}\right)^2 \|f\|^2. \end{aligned}$$

So

$$\|z\| \leqq \frac{1+\varepsilon}{\sqrt{2}\varepsilon^2} \|f\|.$$

Combining with (3.2) we get

$$\|z\|_1 \leqq \|\dot{z}\| + \|z\| \leqq \left(\frac{1}{\varepsilon} + \frac{1+\varepsilon}{\sqrt{2}\varepsilon^2}\right) \|f\|. \tag{3.4}$$

Denote $\varphi_j = \cos(j \omega t)$, $\psi_j = \sin(j \omega t)$ for $j \in \{0\} \cup \mathbf{N}$, where $\omega = 2\pi/T$. Write f in its Fourier series form,

$$f = a_0 + \sum_{j=1}^{\infty} (a_j \varphi_j + b_j \psi_j),$$

where a_0, a_j, $b_j \in \mathbf{R}^{2n}$ for $j \in \mathbf{N}$. Using the estimate (3.4) and the linearity of (3.2), it suffices to solve (3.2) for $f = a\varphi_j + b\psi_j$, where $j \in \{0\} \cup \mathbf{N}$ and a, $b \in \mathbf{R}^{2n}$.

For $j=0$, i.e. $f=a$, take $z = a/\varepsilon$.

For $j \in \mathbf{N}$, let $z = \xi\varphi_j + \eta\psi_j$, then (3.2) yields

$$\left.\begin{aligned} \varepsilon\xi + j\omega(\mathcal{J} + \varepsilon\mathcal{I})\eta = a \\ -j\omega(\mathcal{J} + \varepsilon\mathcal{I})\xi + \varepsilon\eta = b \end{aligned}\right\} \tag{3.5}$$

where $\mathcal{I}$ is the identity matrix on $\mathbf{R}^{2n}$. Since

$$\det\begin{pmatrix} \varepsilon\mathcal{I} & j\omega(\mathcal{J} + \varepsilon\mathcal{I}) \\ -j\omega(\mathcal{J} + \varepsilon\mathcal{I}) & \varepsilon\mathcal{I} \end{pmatrix} = \left[\left(\varepsilon - j^2\omega^2\frac{1-\varepsilon^2}{\varepsilon}\right)^2 + 4\varepsilon^2\right]^{2n}\varepsilon^{2n} > 0.$$

(3.5) possesses a unique solution (ξ, η). Therefore (3.2) is solvable in W^2.

Now applying Proposition 2.3, we get Theorem 1.4.

Q.E.D.

Remark. — Results similar to Theorem 1.4 also hold for general Hamiltonian systems

$$\dot{z} - \mathcal{J} H_z(t, z) = f(t)$$

and second order Hamiltonian systems

$$\ddot{q} + V_q(t, q) = f(t)$$

under corresponding conditions. Since their proofs are close to those of Proposition 2.3 and Theorem 1.4, we omit them here. For such details we refer to [5] and [6].

4. DENSITY RESULTS FOR NONLINEAR SCHRÖDINGER EQUATIONS AND BEAM EQUATIONS

In this section, we first consider the nonlinear Schrödinger equation

$$i\varphi_t - \varphi_{xx} + |\varphi|^{p-1}\varphi = g \tag{4.1}$$

where $p > 1$ is a constant and $i = \sqrt{-1}$. It can be transformed to a very similar problem to (3.1). Indeed let $\varphi = u + iv$, $g = g_1 + ig_2$. We get

$$\begin{cases} -v_t - u_{xx} + (u^2+v^2)^{(p-1)/2} u = g_1 \\ u_t - v_{xx} + (u^2+v^2)^{(p-1)/2} v = g_2. \end{cases}$$

Thus we may consider the periodic boundary value problem of (4.1) in a more general setting,

$$\left.\begin{array}{c} \mathcal{J} z_t - z_{xx} + H'(z) = f, \quad \forall (x, t) \in (0, \pi) \times \mathbf{R} \\ z(x, t+T) = z(t), \quad \forall (x, t) \in [0, \pi] \times \mathbf{R} \\ z(0, t) = z(\pi, t) = 0, \quad \forall t \in \mathbf{R} \end{array}\right\} \tag{4.2}$$

where $z: [0, \pi] \times \mathbf{R} \to \mathbf{R}^{2n}$, $H: \mathbf{R}^{2n} \to \mathbf{R}$, H' is its gradient. $\mathcal{J} = \begin{pmatrix} 0 & -\mathcal{J} \\ \mathcal{J} & 0 \end{pmatrix}$, $\mathcal{J}$ is the identity matrix on $\mathbf{R}^n$, $z_t = \partial z/\partial t$, $z_{xx} = \partial^2 z/\partial x^2$.

Let

$$C^\infty = \{ z \in C^\infty([0, \pi] \times \mathbf{R}, \mathbf{R}^{2n}) \mid z(x, t+T) = z(x, t) \text{ for } (x, t) \in [0, \pi] \times \mathbf{R} \}$$

and E be the completion of C^∞ under the norm

$$\| z \| = (z, z)^{1/2} \quad \text{where} \quad (z, y) = \int_0^T \int_0^\pi z \cdot y \, dx \, dt \quad \text{for} \quad z, y \in C^\infty.$$

For the problem (4.2) we have

Theorem 4.3. — *Let* H *satisfy conditions* (H 1), (H 3) (*in Theorem* 1.4), *and*

(H 4) *There are constants a, b>0 such that*

$$H(z) \leq a H'(z) \cdot z + b, \qquad \forall z \in \mathbf{R}^{2n}.$$

Then for any $T>0$, *there exists a dense set* $A \subset E$ *such that for any* $f \in A$, (4.2) *possesses a* T-*periodic solution.*

We shall apply Proposition 2.3 to get Theorem 4.3.

Let $C_0^\infty = \{ z \in C^\infty \mid z(0, t) = z(\pi, t) = 0 \text{ for } t \in \mathbf{R} \}$. Let W_1 be the completion of C_0^∞ under the norm

$$\| z \|_1 = \| z_t \| + \sup_{0 \leq t \leq T} \left(\int_0^\pi | z_x |^2 dx \right)^{1/2}.$$

and W_2 be the completion of C_0^∞ under the norm

$$\| z \|_2 = \| z_{xx} \| + \| z \|_1.$$

Then W_1 and W_2 are Banach spaces and $W_2 \subset W_1 \subset E$. We have the following.

LEMMA 4.4. — (1) *W_1 is compactly embedded into* $C = C([0, \pi] \times [0, T], \mathbf{R}^{2n})$ *and*

$$\| z \|_C \equiv \max_{(x, t) \in [0, \pi] \times [0, T]} | z(x, t) | \leq \sqrt{\pi} \sup_{0 \leq t \leq T} \left(\int_0^\pi | z_x |^2 dx \right)^{1/2} \quad \text{for} \quad z \in W_1. \tag{4.5}$$

(2) *For* $z \in W_1$, $\| z \| \leq \pi \| z_x \|$.

(3) *For* $z \in W_2$, $(z_{xx}, z_t) = 0$ *and* $(z_{xx}, z) = -\| z_x \|^2$.

Proof. — (1) For $z \in C_0^\infty$ we have that

$$|z(x, t)| \leqq \int_0^\pi |z_x(x, t)|\, dx \leqq \sqrt{\pi}\left(\int_0^\pi |z_x|^2\, dx\right)^{1/2}. \tag{4.6}$$

Given $z \in W_1$. Choose $z_k \in C_0^\infty$ such that $z_k \to z$ in W_1 as $k \to \infty$. Replacing z in (4.6) by $z_k - z_m$, we get that $\{z_k\}$ converges uniformly in C. Therefore $z \in C$ and (4.6) therefore (4.5) holds for $z \in W_1$.

To prove the compactness of the embedding from W_1 to C, let

$$X_1 = W^{1,2}([0, \pi], \mathbf{R}^{2n}), \quad X_0 = C([0, \pi], \mathbf{R}^{2n}) \text{ and } Y = L^1([0, \pi], \mathbf{R}^{2n}).$$

Then $X_1 \subset X_0 \subset Y$ and the embedding from X_1 to X_0 is compact. If Q is a bounded set in W_1, Q is bounded in C ([0, T], X_1). For $z \in Q$,

$$\int_0^T \left(\int_0^\pi |z_t|\, dx\right)^2 dt \leqq \pi \int_0^T \int_0^\pi |z_t|^2\, dx\, dt = \pi \|z_t\|^2.$$

So Q is also bounded in $W^{1,2}([0, T], Y)$. By a theorem of Simon (Corollary 5, [10]) Q is precompact in $C([0, T], X_0)$. Therefore Q is precompact in C. This yields (1).

(2) Squaring both sides of (4.6) and integrating over $[0, \pi] \times [0, T]$ yield (2).

(3) For $z \in W_2$, choose $z_k \in C_0^\infty$ such that $z_k \to z$ in W_2 as $k \to \infty$. Then

$$0 = (z_{kxx}, z_{kt}) \to (z_{xx}, z_t) \quad \text{as } k \to \infty.$$

Thus $(z_{xx}, z_t) = 0$. Similarly, $(z_{xx}, z) = -\|z_x\|^2$.

Q.E.D.

With the aid of Lemma 4.4, we can give the:

Proof of Theorem 4.3. — For $z \in W_1$ define $Dz = z_t$, $Bz = z_x$. For $z \in W_2$ define $Lz = \mathcal{J} z_t - z_{xx}$. Then D, $B \in \mathscr{L}(W_1, E)$, $L \in \mathscr{L}(W_2, E)$. Using (3) of Lemma 4.4 it is easy to verify that (DL) holds with $\alpha_1 = 1$, $\alpha_2 = 0$, $\alpha_3 = 2$. For $z \in W_1$ define $F(z) = H'(z)$. By (1) of Lemma 4.4 $F \in C(W_1, E)$ is compact. From (H 1) and (H 3) we get (1) and (2) of (F). The verification of the condition (S) is very similar to that in the proof of Theorem 1.4, we omit it here (for detail *cf.* [6]).

To get the condition (E), we suppose $z \in W_2$ is a solution of

$$\mathcal{J} z_t - z_{xx} + \varepsilon z_t + \varepsilon z + \lambda H'(z) = \lambda f, \tag{4.7}$$

where $\varepsilon \in (0, 1]$ and $\lambda \in [0, 1]$. Multiplying (4.7) by z and integrating over $[0, \pi] \times [0, T]$ yields

$$(\mathcal{J} z_t, z) - (z_{xx}, z) + \varepsilon (z_t, z) + \varepsilon (z, z) + \lambda (H'(z), z) = \lambda (f, z).$$

By (3) of Lemma 4.4 we get

$$\varepsilon \|z\|^2 + \|z_x\|^2 + \lambda (H'(z), z) \leqq \lambda \|f\| \|z\| + \|z_t\| \|z\|. \tag{4.8}$$

Choose $z_k \in C_0^\infty$ such that $z_k \to z$ in W_2 as $k \to \infty$. Thus $z_{kxx} \to z_{xx} = \mathscr{J} z_t + \varepsilon z_t + \varepsilon z + \lambda H'(z) - \lambda f$ in E as $k \to \infty$. By (1) of Lemma 4.4, $z_k \to z$ in C as $k \to \infty$. Therefore $H(z_k) \to H(z)$ and $H'(z_k) \to H'(z)$ in C as $k \to \infty$.

Let

$$I(t) = \int_0^\pi \left(\frac{1}{2}|z_x|^2 + \lambda H(z)\right) dx,$$

$$I_k(t) = \int_0^\pi \left(\frac{1}{2}|z_{kx}|^2 + \lambda H(z_k)\right) dx,$$

and

$$f_k = \frac{1}{\lambda}[\mathscr{J} z_{kt} + \varepsilon z_{kt} + \varepsilon z_k - z_{kxx} + \lambda H'(z_k)].$$

Then $f_k \to f$ in E as $k \to \infty$, $I_k \in C^1(\mathbf{R}, \mathbf{R})$ and $I_k \to I$ in $C(\mathbf{R}, \mathbf{R})$ as $k \to \infty$. We have

$$\begin{aligned} I_k'(t) &= \int_0^\pi (z_{kx} \cdot z_{kxt} + \lambda H'(z_k) \cdot z_{kt})\, dx \\ &= \int_0^\pi (-z_{kxx} + \lambda H'(z_k)) \cdot z_{kt}\, dx = \int_0^\pi (\lambda f_k - \mathscr{J} z_{kt} - \varepsilon z_{kt} - \varepsilon z_k) \cdot z_{kt}\, dx \\ &= \int_0^\pi (\lambda f_k \cdot z_{kt} - \varepsilon |z_{kt}|^2 - \varepsilon z_k \cdot z_{kt})\, dx \to \int_0^\pi (\lambda f \cdot z_t - \varepsilon |z_t|^2 - \varepsilon z \cdot z_t)\, dx \end{aligned}$$

in $L^1([0, T], \mathbf{R})$ as $k \to \infty$. So $I \in W^{1,1}([0, T], \mathbf{R})$ and

$$I'(t) = \int_0^\pi (\lambda f \cdot z_t - \varepsilon |z_t|^2 - \varepsilon z \cdot z_t)\, dx.$$

Thus

$$\int_0^T |I'(t)|\, dt \leqq \|f\|\, \|z_t\| + \varepsilon \|z_t\|^2 + \varepsilon \|z\|\, \|z_t\|. \tag{4.9}$$

By (H 4),

$$\begin{aligned} \int_0^T I(t)\, dt &= \frac{1}{2}\|z_x\|^2 + \lambda \int_0^T \int_0^\pi H(z)\, dx\, dt \leqq \frac{1}{2}\|z_x\|^2 + \lambda a (H'(z), z) + \lambda b T \pi \\ &\leqq a(\|z_x\|^2 + \lambda (H'(z), z)) + \left|\frac{1}{2} - a\right| \|z_x\|^2 + \lambda b T \pi \\ &\leqq \lambda a \|f\|\, \|z\| + a \|z_t\|\, \|z\| + \left|\frac{1}{2} - a\right| \|z_x\|^2 + \lambda b T \pi \quad (\text{by } (4.8)). \end{aligned}$$

Since $I \in C([0, T], \mathbf{R})$, there exists $\tau \in [0, T]$ such that $I(\tau) = \frac{1}{T}\int_0^T I(t)\,dt$. So

$$I(t) \leqq I(\tau) + \int_\tau^t |I'(s)|\,ds \leqq \frac{1}{T}\int_0^T I(s)\,ds + \int_0^T |I'(s)|\,ds.$$

Combining with (4.9) and (4.10) we get

$$I(t) \equiv \frac{1}{2}\int_0^\pi |z_x|^2\,dx + \lambda\int_0^\pi H(z)\,dx \leqq C_1(\|z_x\|^2 + \|z_t\|^2 + \|z\|^2 + 1). \quad (4.11)$$

By (H 3), we get

$$H(z) \geqq \frac{1}{2}|z|^2 - C_2, \qquad \forall z \in \mathbf{R}^{2n}$$

for some constant $C_2 \geqq 1$. Therefore from (4.11) we get

$$\int_0^\pi |z_x|^2\,dx \leqq 2(C_1 + \pi C_2)(\|z_x\|^2 + \|z_t\|^2 + \|z\|^2 + \lambda).$$

This yields the condition (E).

Now applying Proposition 2.3, we get Theorem 4.3.

Q.E.D.

Next we consider a nonlinear beam equation,

$$\left.\begin{array}{ll} u_{tt} + u_{xxxx} + g(u) = f(x, t), & (x, t) \in (0, \pi) \times \mathbf{R} \\ u(x, t+T) = u(x, t), & (x, t) \in [0, \pi] \times \mathbf{R} \\ u(0, t) = u(\pi, t) = u_{xx}(0, t) = u_{xx}(\pi, t) = 0, & t \in \mathbf{R} \end{array}\right\} \quad (4.14)$$

Let

$$C^\infty = \{u \in C^\infty([0, \pi] \times \mathbf{R}, \mathbf{R}) \mid u(x, t+T) = u(x, t) \text{ for } (x, t) \in [0, \pi] \times \mathbf{R}\}$$

and E be the completion of C^∞ under the norm

$$\|u\| = (u, u)^{1/2} \qquad \text{where} \quad (u, v) = \int_0^T \int_0^\pi u(x, t)\,v(x, t)\,dx\,dt.$$

We have:

THEOREM 4.13. — *Let g satisfy following conditions,*

(G 1) $g \in C(\mathbf{R}, \mathbf{R})$.

(G 2)

$$\lim_{|s| \to \infty} \frac{g(s)}{s} = +\infty.$$

(G 3) *Let* $G(s) = \int_0^s g(\tau)\,d\tau$. *There is a constant* $C > 0$ *such that*

$$G(s) \leqq C(sg(s) + 1), \qquad \forall s \in \mathbf{R}.$$

Then there exists a dense set $A \subset E$ *such that for any* $f \in A$, *problem* (4.12) *possesses a solution.*

Remark. — Since the proof of Theorem 4.13 is very similar to that of Theorem 4.3, we omit it here. For details we refer to [6].

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