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Exact controllability in short time for the wave equation

by

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ABSTRACT. — We present an elementary and constructive method to obtain in several cases the "optimal" estimates needed in the Hilbert uniqueness method of J. L. Lions for the exact controllability of linear evolution systems.

Key words : Exact controllability, wave equation.

1. INTRODUCTION

In [LIONS 2] and [LIONS 3] a general method was given for the exact control of evolution systems. It is based on the construction of new Hilbert spaces corresponding to different uniqueness theorems. In general the results involve a "sufficiently large" time. The proofs are based on *a priori* estimates of the type introduced in [HO]. The purpose of the present

(1) Some results of the present paper were obtained during the author's stay at the Université Paris-VI (Laboratoire Analyse numérique) as searcher associated to the C.N.R.S. and at the Université de Savoie, Chambéry, as invited professor.

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paper is to give a method to obtain in several cases the "optimal" estimates without using uniqueness theorems. This procedure provides more elementary and constructive proofs.

In this paper we consider the wave equation although the method applies for more general equations, too. In particular, we shall improve some results of L. F. Ho and J. L. Lions concerning the *boundary* control of Dirichlet resp. Neumann type (*cf.* [HO], [LIONS 3], [LIONS 4]). The optimality of our results will also be investigated. Some results of the present paper were stated without proof in [KOMORNIK].

For the general theory of exact controllability we refer to [LIONS 3], [LIONS 4] and [RUSSELL]. The connection between the exact controllability and the stabilizability is not considered here; for these questions we refer to [RUSSELL], [LIONS 3], [CHEN], [LAGNESE], [LASIECKA-TRIGGIANI] and to [KOMORNIK-ZUAZUA]. Optimal time estimates are obtained for other equations by different methods in [ZUAZUA]. The Hilbert uniqueness method applies also to the interior control, *cf.* [HARAUX 1] and [HARAUX 2].

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2. NEUMANN ACTION

Let Ω be a non-empty, bounded, open set of class C^∞ in $\mathbb{R}^N$ with boundary Γ, Ω being locally on one side of Γ, and denote $\nu(x)=(\nu_1(x), \ldots, \nu_N(x))$ the unit normal vector to Γ, directed towards the exterior of Ω.

Let $A=\partial_i(a_{ij}\partial_j)$ be a second-order elliptic differential operator with coefficients $a_{ij}\in W^{1,\infty}(\Omega)$ (throughout this paper we use the summation convention for repeated indices), then

$$a_{ij}=a_{ji} \quad \text{for all } i, j=1, \ldots, N \tag{1}$$

and there exist positive constants α, β such that

$$\alpha|\xi|^2 \leqslant a_{ij}(x)\xi_i\xi_j \leqslant \beta|\xi|^2 \text{ for all } x\in\Omega \text{ and } \xi=(\xi_1, \ldots, \xi_N)\in\mathbb{R}^N. \tag{2}$$

Fix a point $x^0\in\mathbb{R}^N$ such that, putting $h(x)=x-x^0$, the differential operator $\partial_i((2a_{ij}-h_k\partial_k a_{ij})\partial_j)$ be also elliptic. Then there is a positive constant γ such that

$$(2a_{ij}-h_k\partial_k a_{ij})(x)\xi_i\xi_j \geqq \gamma a_{ij}(x)\xi_i\xi_j$$

for all (3)

$$x\in\Omega \quad \text{and} \quad \xi=(\xi_1, \ldots, \xi_N)\in\mathbb{R}^N.$$

Let η denote the smallest positive number such that

$$(h_k(x)\xi_k)^2 \leqq \eta^2 a_{ij}(x)\xi_i\xi_j$$

for all (4)

$$x\in\Omega \quad \text{and} \quad \xi=(\xi_1, \ldots, \xi_N)\in\mathbb{R}^N$$

and set

$$T_0 = T_0(x^0) = 4\eta/\min\{\gamma, 2N\}. \tag{5}$$

Example. — If $\Omega=\{x\in\mathbb{R}^N: |x|\leqq 1\}$, $A=\Delta$ (i. e. $a_{ij}=\delta_{ij}$) and $x^0=0$, then (3), (4) are satisfied and $T_0=2$.

Let T be a positive number and set $Q=(0, T)\times\Omega$, $\Sigma=(0, T)\times\Gamma$. Consider the evolution system

$$y''-Ay=0 \quad \text{in } Q, \tag{6}$$

$$\partial y/\partial \nu_A = v \quad \text{on } \Sigma, \tag{7}$$

$$y(0)=y^0 \quad \text{and} \quad y'(0)=y^1 \quad \text{on } \Omega, \tag{8}$$

where, as usual, $\partial y/\partial \nu_A = \nu_i a_{ij}\partial_j y$ denotes the "A-normal" derivative of y.

We recall that the system (6), (7), (8) is called exactly controllable if, for *all* initial data y^0, y^1 from a suitable space, there exists a corresponding control v from a suitable space driving the system to rest at time T i. e. such that

$$y(T)=y'(T)=0 \quad \text{on } \Omega. \tag{9}$$

Due to the finite speed of propagation this is not the case unless T is sufficiently large.

Let us introduce the notations

$$\Gamma_+=\{x\in\Gamma: h_i(x)\nu_i(x)>0\}, \qquad \Sigma_+=(0, T)\times\Gamma_+$$

and

$$\Gamma_-=\{x\in\Gamma: h_i(x)\nu_i(x)\leqq 0\}, \qquad \Sigma_-=(0, T)\times\Gamma_-.$$

The purpose of this section is to give a more constructive proof of the following theorem due to J. L. Lions:

THEOREM 1. — *Assume that* $T > T_0$. *Then for any initial data* $y^0\in L^2(\Omega)$, $y^1\in(H^1(\Omega))'$ [*=the dual space of* $H^1(\Omega)$] *there exist control functions* v_0, $v_1\in L^2(0, T; L^2(\Gamma_+))$ *and* $v_2\in L^2(0, T; (H^1(\Gamma_-))')$ *such that, putting*

$$v=\begin{vmatrix} v_0+\partial v_1/\partial t & \text{on } \Sigma_+, \\ v_2 & \text{on } \Sigma_-, \end{vmatrix}$$

the solution of (6), (7), (8) *satisfies the final conditions* (9). ■

In [LIONS 3] this result was obtained by the Hilbert Uniqueness Method (HUM) as follows. Given φ^0, $\varphi^1\in\mathcal{D}(\Omega)$ arbitrarily, first one solves the problem

$$\varphi'' - A\varphi = 0 \quad \text{in } Q, \tag{10}$$

$$\partial\varphi/\partial\nu_A = 0 \quad \text{on } \Sigma, \tag{11}$$

$$\varphi(0) = \varphi^0 \qquad \text{and} \qquad \varphi'(0) = \varphi^1 \quad \text{on } \Omega \tag{12}$$

and the problem

$$y'' - A y = 0 \quad \text{in } Q,$$

$$\partial y/\partial\nu_A = \left| \begin{array}{ll} \varphi'' - \varphi & \text{on } \Sigma_+, \\ \Delta_\Gamma \varphi & \text{on } \Sigma_-, \end{array} \right.$$

$$y(T) = y'(T) = 0 \quad \text{on } \Omega$$

(in a suitable sense defined in [LIONS 3]). One obtains in this way a continuous linear map

$$\Lambda: \quad \mathscr{D}(\Omega) \times \mathscr{D}(\Omega) \to \mathscr{D}'(\Omega) \times \mathscr{D}'(\Omega), \qquad \Lambda(\varphi^0, \varphi^1) = (y'(0), -y(0)).$$

An easy calculation shows that

$$\langle \Lambda(\varphi^0, \varphi^1), (\psi^0, \psi^1) \rangle = \int_{\Sigma_+} \varphi\psi + \varphi'\psi' \, d\Gamma\, dt + \int_{\Sigma_-} \nabla_\sigma \varphi \nabla_\sigma \psi \, d\Gamma\, dt$$

for all $\varphi^0, \varphi^1, \psi^0, \psi^1 \in \mathscr{D}(\Omega)$ where $\nabla_\sigma \varphi$ denotes the tangential part of $\nabla\varphi$ and where ψ is defined by (10), (11), (12), replacing (φ^0, φ^1) by (ψ^0, ψ^1).

One then shows that for sufficiently large T the right side of this identity defines a scalar product on the vector space $\mathscr{D}(\Omega) \times \mathscr{D}(\Omega)$ of the pairs (φ^0, φ^1) (this corresponds to a uniqueness theorem). Then by the Lax-Milgram theorem Λ extends to an isomorphism $\Lambda: F \to F'$ of a Hilbert space onto its dual.

Hence the exact controllability follows. Indeed, given $(y^1, -y^0) \in F'$ arbitrarily, one takes

$$v = \left| \begin{array}{ll} \varphi'' - \varphi & \text{on } \Sigma_+, \\ \Delta_\Gamma \varphi & \text{on } \Sigma_-, \end{array} \right.$$

where φ is the solution of (10), (11), (12) corresponding to $(\varphi^0, \varphi^1) := \Lambda^{-1}(y^1, -y^0)$.

Originally, the above crucial uniqueness theorem was proved by a compactness argument. Now we shall prove directly the following more precise inequality:

Proposition 1. — *Let*

$$\varphi \in C([0, T]; H^2(\Omega)) \cap C^1([0, T]; H^1(\Omega)) \cap C^2([0, T]; L^2(\Omega))$$

be an arbitrary function satisfying (10) *and* (11). *Define the "energy" of* φ

by the formula

$$E=E(t)=2^{-1}\int_{\Omega} a_{ij}(\partial_i \varphi(t))(\partial_j \varphi(t))+|\varphi'(t)|^2\,dx,\ t\in[0,T]. \quad (13)$$

Then E *does not depend on* t *and for any positive number* ε *we have*

$$\int_{\Sigma_+} h_k \nu_k((\varphi')^2+\varphi^2)\,d\Gamma\,dt+\int_{\Sigma_-} |h_k\nu_k|\,a_{ij}(\partial_i\varphi)(\partial_j\varphi)\,d\Gamma\,dt$$

$$\geqq C_1(\varepsilon)(T-T_0-\varepsilon)E+C_2(\varepsilon)\int_{\Omega}\varphi(0)^2\,dx \quad (14)$$

where C_1 *and* C_2 *are suitable positive constants independent of the choice of* φ. ■

(The constants C_1 and C_2 may be given explicitely by the proof below.)

The proof of this proposition is based on the following identity, implicitely proved in [HO]:

LEMMA. — *Let*

$$\varphi\in C([0,T];H^2(\Omega))\cap C^1([0,T];H^1(\Omega))\cap C^2([0,T];L^2(\Omega))$$

be an arbitrary function satisfying (10) *and let* C *be an arbitrary real number. Then*

$$\int_{\Sigma}(\partial\varphi/\partial\nu_A)(2h_k\partial_k\varphi+C\varphi)+h_k\nu_k((\varphi')^2-a_{ij}(\partial_i\varphi)(\partial_j\varphi))\,d\Gamma\,dt$$

$$=\int_Q (N-C)(\varphi')^2+((C-N+2)a_{ij}-h_k\partial_k a_{ij})(\partial_i\varphi)(\partial_j\varphi)\,dx\,dt$$

$$+\int_{\Omega}\varphi'(2h_k\partial_k\varphi+C\varphi)\,dx\,\Big|_0^T. \quad (15)$$

Proof. — Integrate by parts in the identity

$$\int_Q (2h_k\partial_k\varphi)(\varphi''-\partial_i(a_{ij}\partial_j\varphi))\,dx\,dt=0.$$

It follows that

$$\int_Q (2 h_k \partial_k \varphi) \varphi'' \, dx \, dt = \int_\Omega 2 \varphi' h_k \partial_k \varphi \, dx \Big|_0^T$$

$$- \int_Q (2 h_k \partial_k \varphi') \varphi' \, dx \, dt = \int_\Omega 2 \varphi' h_k \partial_k \varphi \, dx \Big|_0^T$$

$$- \int_Q h_k \partial_k (\varphi')^2 \, dx \, dt = \int_\Omega 2 \varphi' h_k \partial_k \varphi \, dx \Big|_0^T$$

$$- \int_\Sigma h_k \nu_k (\varphi')^2 \, d\Gamma \, dt + \int_Q \mathrm{N} (\varphi')^2 \, dx \, dt$$

and (using also the symmetry of a_{ij}) that

$$\int_Q (2 h_k \partial_k \varphi) \partial_i (a_{ij} \partial_j \varphi) \, dx \, dt$$

$$= \int_\Sigma (\partial\varphi/\partial\nu_A) 2 h_k \partial_k \varphi \, d\Gamma \, dt - \int_Q (a_{ij} \partial_j \varphi) \partial_i (2 h_k \partial_k \varphi) \, dx \, dt$$

$$= \int_\Sigma (\partial\varphi/\partial\nu_A) 2 h_k \partial_k \varphi \, d\Gamma \, dt - \int_Q 2 (\partial_i h_k) a_{ij} (\partial_j \varphi) (\partial_k \varphi) \, dx \, dt$$

$$- \int_Q h_k \partial_k (a_{ij} (\partial_i \varphi) (\partial_j \varphi)) \, dx \, dt + \int_Q h_k (\partial_k a_{ij}) (\partial_i \varphi) (\partial_j \varphi)) \, dx \, dt$$

$$= \int_\Sigma (\partial\varphi/\partial\nu_A) 2 h_k \partial_k \varphi \, d\Gamma \, dt - \int_Q 2 a_{ij} (\partial_i \varphi) (\partial_j \varphi) \, dx \, dt$$

$$- \int_\Sigma h_k \nu_k a_{ij} (\partial_i \varphi) (\partial_j \varphi) \, d\Gamma \, dt + \int_Q \mathrm{N} a_{ij} (\partial_i \varphi) (\partial_j \varphi)) \, dx \, dt$$

$$+ \int_Q h_k (\partial_k a_{ij}) (\partial_i \varphi) (\partial_j \varphi)) \, dx \, dt.$$

Hence the case $C=0$ of the lemma follows. To conclude in the general case, it is sufficient to prove that

$$\int_\Sigma \varphi \nu_i a_{ij} \partial_j \varphi \, d\Gamma \, dt = \int_Q a_{ij} (\partial_i \varphi) (\partial_j \varphi) - (\varphi')^2 \, dx \, dt + \int_\Omega \varphi \varphi' \, dx \Big|_0^T.$$

This follows by integrating by parts in the identity

$$\int_Q \varphi (\varphi'' - \partial_i (a_{ij} \partial_j \varphi)) \, dx \, dt = 0;$$

indeed,

$$\int_Q \varphi\varphi''\,dx\,dt = \int_\Omega \varphi\varphi'\,dx\,\Big|_0^T - \int_Q (\varphi')^2\,dx\,dt$$

and

$$\int_Q \varphi\,\partial_i(a_{ij}\partial_j\varphi)\,dx\,dt = \int_\Sigma \varphi\nu_i a_{ij}\partial_j\varphi\,d\Gamma\,dt - \int_Q a_{ij}(\partial_i\varphi)(\partial_j\varphi)\,dx\,dt. \quad \blacksquare$$

Proof of Proposition 1. — The time-independence of the energy is well-known. Applying the identity (15) of the above lemma and taking into account (3) and (11), we obtain the inequality

$$\int_{\Sigma_+} h_k\nu_k(\varphi')^2\,d\Gamma\,dt + \int_{\Sigma_-} |h_k\nu_k|\,a_{ij}(\partial_i\varphi)(\partial_j\varphi)\,d\Gamma\,dt$$
$$\geqq \int_Q (\mathrm{N}-\mathrm{C})(\varphi')^2 + (\mathrm{C}-\mathrm{N}+\gamma)\,a_{ij}(\partial_i\varphi)(\partial_j\varphi)\,dx\,dt$$
$$+ \int_\Omega \varphi'(2h_k\partial_k\varphi + \mathrm{C}\varphi)\,dx\,\Big|_0^T. \quad (16)$$

Now we show the inequality

$$\left|\int_\Omega \varphi'(2h_k\partial_k\varphi + \mathrm{C}\varphi)\,dx\right| \leqq 2\eta\mathrm{E} - \mathrm{C}(2\mathrm{N}-\mathrm{C})(4\eta)^{-1}\int_\Omega \varphi^2\,dx$$
$$+ \mathrm{C}(2\eta)^{-1}\int_{\Gamma^+} h_k\nu_k\varphi^2\,d\Gamma \quad (17)$$

for all $t\in[0, \mathrm{T}]$. We apply the Cauchy-Schwarz inequality, the Green formula and the definition of (4) of η as follows:

$$\left|\int_\Omega \varphi'(2h_k\partial_k\varphi + \mathrm{C}\varphi)\,dx\right| \leqq 2^{-1}\int_\Omega 2\eta(\varphi')^2 + (2\eta)^{-1}(2h_k\partial_k\varphi + \mathrm{C}\varphi)^2\,dx$$
$$= 2^{-1}\int_\Omega 2\eta(\varphi')^2 + (2\eta)^{-1}(2h_k\partial_k\varphi)^2 + (2\eta)^{-1}\mathrm{C}^2\varphi^2 + \mathrm{C}\eta^{-1}h_k\partial_k(\varphi^2)\,dx$$
$$\leqq 2\eta\mathrm{E} - \mathrm{C}(2\mathrm{N}-\mathrm{C})(4\eta)^{-1}\int_\Omega \varphi^2\,dx + \mathrm{C}(2\eta)^{-1}\int_\Gamma h_k\nu_k\varphi^2\,d\Gamma$$

which implies (17).

Finally we prove that

$$\int_{\Gamma_+} h_k\nu_k\varphi^2\,d\Gamma \leqq (1+\mathrm{T}^{-1})\int_{\Sigma_+} h_k\nu_k\varphi^2\,d\Gamma\,ds + \int_{\Sigma_+} h_k\nu_k(\varphi')^2\,d\Gamma\,ds \quad (18)$$

for all $t \in [0, T]$. Introducing the function $f(t) = \int_{\Gamma_+} h_k \nu_k \varphi^2 \, d\Gamma$, for any t, $t' \in [0, T]$ we have

$$f(t) = f(t') + \int_{t'}^{t} f'(s)\, ds \leqq f(t') + \int_0^T |f'(s)|\, ds$$

$$= f(t') + \int_0^T \left| \int_{\Gamma_+} 2 h_k \nu_k \varphi\varphi' \, d\Gamma \right| ds$$

$$\leqq f(t') + \int_{\Sigma_+} h_k \nu_k \varphi^2 \, d\Gamma \, ds + \int_{\Sigma_+} h_k \nu_k (\varphi')^2 \, d\Gamma \, ds.$$

Integrating in t' on $[0, T]$ and dividing by T we obtain (18).

From (16), (17) and (18) we deduce the inequality

$$(1 + C\eta^{-1}(1 + T^{-1})) \left(\int_{\Sigma_+} h_k \nu_k ((\varphi')^2 + (\varphi)^2) \, d\Gamma \, dt \right.$$
$$\left. + \int_{\Sigma_-} |h_k \nu_k| \, a_{ij} (\partial_i \varphi)(\partial_j \varphi) \, d\Gamma \, dt \right)$$
$$\geqq 2 \min\{N - C, C - N + \gamma\} TE - 4\eta E$$
$$+ C(2N - C)(4\eta)^{-1} \int_\Omega \varphi(0)^2 \, dx. \quad (19)$$

If $\gamma < 2N$, then (14) follows (also for $\varepsilon = 0$) by choosing $C = N - \gamma/2$ in (19). If $\gamma \geqq 2N$, then (14) follows by choosing a sufficiently small positive number C in (19). ■

Remark 1. — Proposition 1 yields the following uniqueness result (*cf.* [LIONS 3; Remark 5.3]): if

$$\varphi \in C([0, T]; H^2(\Omega)) \cap C^1([0, T]; H^1(\Omega)) \cap C^2([0, T]; L^2(\Omega))$$

is such that $\varphi'' - \Delta\varphi = 0$ in Q, $\partial\varphi/\partial\nu = 0$ on Σ, $\varphi = 0$ on Σ_+ and $\nabla\varphi = 0$ on Σ_- where $T > T_0$, then $\varphi \equiv 0$. This result may also be obtained as a consequence of Holmgren's theorem.

Remark 2. — If $\Omega = \{x \in \mathbb{R}^N : |x| \leqq 1\}$ and $A = \Delta$, then the condition $T > T_0$ of Theorem 2 is optimal. Indeed, it was proved earlier in [GRAHAM-RUSSELL] that for $T < T_0$ our system is not exactly controllable. We mention that optimal time estimates for exact controllability are known for other domains, too, *cf.* [FATTORINI] and [LAGNESE]. We note also, that in a recent work of [BARDOS-LEBEAU-RAUCH] very precise estimates (of different type) are obtained on T by microlocal analysis.

Remark 3. – The condition $T > T_0$ of Theorem 1 is not always optimal. In such cases we can try to transform the equation and then to apply our results to the new equation. For example, in the one-dimensional case the more general equation

$$\rho u_{tt} - (p u_x)_x + q u = 0$$

with sufficiently smooth positive functions ρ, p and with a suitable function q, may be brought by a standard transformation of the independent and dependent variables to the form

$$u_{tt} - u_{xx} = 0$$

(*cf.* [LAGNESE 1]) and then our previous remark applies. This yields the optimal time estimates for the original equation, too.

3. DIRICHLET ACTION

We shall now study the exact controllability of the system

$$y'' - A y = 0 \quad \text{in } Q, \tag{20}$$

$$y = v \quad \text{on } \Sigma, \tag{21}$$

$$y(0) = y^0 \quad \text{and} \quad y'(0) = y^1 \quad \text{on } \Omega. \tag{22}$$

The purpose of this section is to improve an earlier result of [HO] by giving a better condition on T; *see* also [LIONS 3; Remark 1.7]. We adopt the notations and hypothesis (1)-(5) of the preceeding section.

THEOREM 2. – *Assume that* $T > T_0$. *Then for any initial data* $y^0 \in L^2(\Omega)$, $y^1 \in H^{-1}(\Omega)$, *there exists a corresponding control function* $v \in L^2(\Sigma)$ *such that the solution of the system* (20), (21), (22) *satisfies the final conditions*

$$y(T) = y'(T) = 0 \quad \text{on } \Omega. \quad \blacksquare$$

Applying the Hilbert Uniqueness Method (*cf.* [LIONS 3] for the details) it is sufficient to prove the following inequality:

PROPOSITION 2. – *Let*

$$\varphi \in C([0, T]; H^2(\Omega)) \cap C^1([0, T]; H_0^1(\Omega)) \cap C^2([0, T]; L^2(\Omega))$$

be an arbitrary function satisfying

$$\varphi'' - A\varphi = 0 \quad \textit{in } Q \tag{23}$$

and

$$\varphi = 0 \quad on\ \Sigma. \tag{24}$$

Define the "energy" of φ *by the formula*

$$E = E(t) = 2^{-1} \int_\Omega a_{ij}(\partial_i \varphi(t))(\partial_j \varphi(t)) + |\varphi'(t)|^2 dx, \ t \in [0, T]. \tag{25}$$

Then E *does not depend on t and*

$$\int_{\Sigma+} (a_{ij} \nu_i \nu_j)(h_k \nu_k)(\partial\varphi/\partial\nu)^2 \, d\Gamma \, dt \geqq \min\{\gamma, 2N\}(T - T_0) E. \tag{26}$$

Proof. The time-independence of E is well-known. We apply again the identity (15) of the Lemma. Now (24) implies that

$$\varphi' = 0 \qquad \text{and} \qquad \partial_i \varphi = \nu_i \partial\varphi/\partial\nu \quad \text{on } \Sigma. \tag{27}$$

Choosing

$$C = \max\{N - \gamma/2, 0\} \tag{28}$$

and using (3), (25) and the definition of Σ_+, we obtain that

$$\int_{\Sigma+} (a_{ij} \nu_i \nu_j)(h_k \nu_k)(\partial\varphi/\partial\nu)^2 \, d\Gamma \, dt \geqq 2(N - C) TE$$
$$+ \int_\Omega \varphi'(2 h_k \partial_k \varphi + C\varphi) \, dx \Big|_0^T.$$

Since $\gamma > 0$ implies that $N - C > 0$, to prove (26) it is sufficient to show that

$$\left| \int_\Omega \varphi'(2 h_k \partial_k \varphi + C\varphi) \, dx \Big|_0^T \right| \leqq 2\eta E \tag{29}$$

for all $t \in [0, T]$.

Observe that in the proof of the inequality (17) in the preceeding section no boundary condition on φ was used. Hence it remains valid in the present case, too:

$$\left| \int_\Omega \varphi'(2 h_k \partial_k \varphi + C\varphi) \, dx \right| \leqq 2\eta E - C(2N - C)(4\eta)^{-1} \int_\Omega \varphi^2 dx$$
$$+ C(2\eta)^{-1} \int_{\Gamma+} h_k \nu_k \varphi^2 \, d\Gamma.$$

However, the boundary integral on the right side of this inequality vanishes by (24), and the second term of the right side is always $\leqq 0$ by the choice (28) of C. Hence (29) follows. ■

Remark 4. — We note that the condition $T > T_0$ in Theorem 2 is optimal if $\Omega = \{x \in \mathbb{R}^N : |x| < 1\}$ and $A = \Delta$. This follows from the results of

[LAGNESE 1] or [BARDOS-LEBEAU-RAUCH]. Also, it is easy to prove it directly if $N=1$. Indeed, let $\Omega=(0,1)$ and let $0<T<1$ $(=T_0)$. We claim that, given $y^0 \in L^2(\Omega)$ and $y^1 \in H^{-1}(\Omega)$ arbitrarily, in general there do not exist control functions v_0 and $v_1 \in L^2(0, T)$ such that the solution $y(t, x)$ of the problem

$$y_{tt}-y_{xx}=0 \quad \text{in } (0, T)\times\Omega,$$
$$y(.,0)=v_0 \quad \text{and} \quad y(.,1)=v_1 \quad \text{in } (0, T),$$
$$y(0)=y^0 \quad \text{and} \quad y_t(0)=y^1 \quad \text{in } \Omega$$

satisfies the final conditions

$$y(T)=y_t(T)=0 \quad \text{in } \Omega. \tag{30}$$

To prove this, fix y^0 and $y^1 \in C^\infty(\mathrm{cl}(\Omega))$ such that

$$(y^0)'(x)+(y^1)(x) \not\equiv 0 \quad \text{near } x=1. \tag{31}$$

If there existed v_0 and $v_1 \in L^2(0, T)$ such that the solution of the above problem satisfies (30), then, extending v_0 and v_1 by zero to $T<t<1$, we would obtain a function $y(t, x)$ such

$$y_{tt}-y_{xx}=0 \quad \text{in } (0, 1)\times\Omega, \tag{32}$$
$$y(.,0)=v_0 \quad \text{and} \quad y(.,1)=v_1 \quad \text{in } (0, 1), \tag{33}$$
$$y(0)=y^0 \quad \text{and} \quad y_t(0)=y^1 \quad \text{in } \Omega,$$
$$y(1)=y_t(1)=0 \quad \text{in } \Omega. \tag{35}$$

Applying the method of d'Alembert it is easy to show that the solution of (32), (33), (34) is given by the formula

$$y(t, x)=-\frac{1}{2}y^0(2-t-x)-\frac{1}{2}y^0(t-x)$$
$$+\frac{1}{2}\int_0^{2-t-x} y^1-\frac{1}{2}\int_0^{t-x} y^1+v_1(t+x-1)+v_0(t-x)$$

for all $0 \leqq x \leqq 1$ and $\max\{x, 1-x\} \leqq t \leqq 1$. Hence (35) is equivalent to the existence of a constant C such that

$$v_0(x)=C+\frac{1}{2}y^0(x)-\frac{1}{2}\int_x^1 y^1$$

and

$$v_1(x)=-C+\frac{1}{2}y^0(1-x)+\frac{1}{2}\int_{1-x}^1 y^1$$

for all $0 \leqq x \leqq 1$. Differentiating the first equation and taking into account that $v_0(x)$ is constant near $x=1$ by definition, we obtain

$$(y^0)'(x)+(y^1)(x) \equiv 0 \quad \text{near } x=1$$

which contradicts to (31).

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