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Minimal interface criterion for phase transitions in mixtures of Cahn-Hilliard fluids

by

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ABSTRACT. — In this paper we extend the Van der Waals-Cahn-Hilliard theory of phase transitions to the case of a mixture of n non-interacting fluids. By describing the state of the mixture as given by a vector density function $u=(u_1, \ldots, u_n)$, the problem consists in studying the asymptotic behaviour as $\varepsilon \to 0^+$ of minimizers of the energy functionals:

$$E_\varepsilon(u)=\int_\Omega |\varepsilon^2 |Du|^2 + W(u)| \, dx$$

under the volume constraint $\int_\Omega u(x)\,dx=m$, with $m \in \mathbf{R}^n$ fixed. The function W, which represents the Gibbs free energy, is non-negative and vanishes only in a finite number of points $\alpha_1, \ldots, \alpha_k \in \mathbf{R}^n$. The result is that the minimizers asymptotically approach a configuration which corresponds to a partition of the container Ω into k subsets whose boundaries satisfy a minimality condition.

Key words : Calculus of variations, Γ-convergence, relaxation, fluids, phase transitions.

RÉSUMÉ. — Dans cet article nous étendons la théorie des transitions de phase de Van der Waals-Cahn-Hilliard au cas d'un mélange de n fluides non intérageants. En supposant l'état du mélange décrit par une fonction

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vectorielle de densité $u=(u_1, \ldots, u_n)$, le problème consistera dans l'étude du comportement asymptotique par $\varepsilon \to 0^+$ des minimisants des énergies:

$$E_\varepsilon(u)=\int_\Omega |\varepsilon^2 |Du|^2+W(u)|\,dx$$

sous la contrainte de volume $\int_\Omega u(x)\,dx=m$, avec $m\in \mathbf{R}^n$ fixé. La fonction W représente l'énergie libre de Gibbs, à valeurs non négatives et qui est nulle sur un nombre fini de points $\alpha_1, \ldots, \alpha_k \in \mathbf{R}^n$. Nous obtenons alors que les minimisants approchent asymptotiquement une configuration qui corresponds à une partition du container Ω en k sous-ensembles dont les bords satisfont à une certaine condition de minimalité.

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1. INTRODUCTION AND STATEMENT OF THE MAIN RESULT

We briefly recall the theory for a single fluid (*see*, for example, [GM]). Suppose we have a fluid, contained in an open set $\Omega \subset \mathbf{R}^N$, of total mass m and density given by a function u defined on Ω.

Under isothermal conditions, the stable configurations of the fluid are given by the functions u which minimize the total energy:

$$E_\varepsilon(u)=\int_\Omega [\varepsilon^2 |Du|^2+W(u)|\,dx \to \min \qquad (1.1)$$

among all u such that $\int_\Omega u(x)\,dx=m$ and $u(x)\geqq 0$. Here W is a nonnegative function that represents the Gibbs free energy relative to an ideal fluid, and ε is a small parameter that takes into account the energy connected with the formation of interfaces.

Assume that W acts as in Figure 1 and that $\alpha|\Omega|<m<\beta|\Omega|$, where $|\Omega|$ denotes the volume of Ω.

Note that the solutions of the variational problem (1.1) have a certain regularity, at least $u\in H^1(\Omega)$, while it is physically reasonable that the density u should be a piecewise constant function admitting only the values α and β (*i.e.* the fluid splits into two phases). On the other hand, experience indicates that stable configurations must minimize the surface area of the interfaces between the phases, and so we cannot simply neglect the gradient term in (1.1).

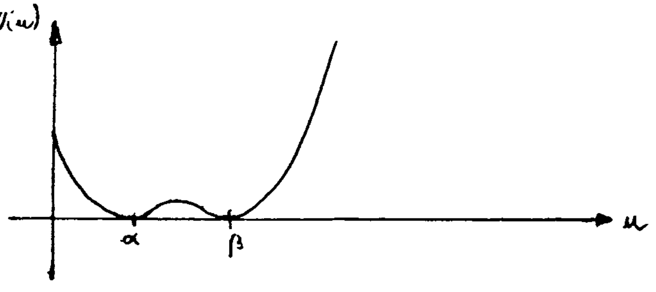

FIG.

The idea to overcome such difficulties is to study the asymptotic behaviour of the variational problem (1.1) as $\varepsilon \to 0^+$. The problem was recently solved in full generality by Modica [M1]. He proved that, if u_ε is a solution of the variational problem (1.1) and u_ε converges in $L^1(\Omega)$ as $\varepsilon \to 0^+$ to a function u_0, then u_0 takes only the values α and β and the set $E = \{x \in \Omega : u(x) = \alpha\}$ minimizes the surface area of its boundary among all subsets of Ω having the same volume.

It should be remarked that Modica's result does not work for more than two phases. For instance, if $W(\alpha) = W(\beta) = W(\gamma) = 0$ with $\alpha < \beta < \gamma$, the limit density u_0 takes again only two values: either α and β, or β and γ, depending on m. The multiphases systems, as well as the mixtures of non-interacting fluids, can be studied by passing from the scalar case $u \in \mathbf{R}$ to the vector case $u \in \mathbf{R}^n$. This is the aim of the present paper.

Let $\Omega \subset \mathbf{R}^N$, $N \geqq 2$ be open and bounded, with lipschitz-continuous boundary. Suppose we have n fluids that we can describe with a vector-valued density function $u = (u^1, \ldots, u^n) \in L^1(\Omega; \mathbf{R}^n)$: each scalar component of u is the density of an ingredient of the mixture. The appropriate constraint are again, with obvious meaning of the symbols,

$$\int_\Omega u(x)\,dx = m, \qquad u > 0$$

where $m = (m^1, \ldots, m^n) \in \mathbf{R}^n$ is given and satisfies the condition $\min\{\alpha_1^i, \ldots, \alpha_k^i\} \leqq \dfrac{m^i}{|\Omega|} \leqq \max\{\alpha_1^i, \ldots, \alpha_k^i\}$ for every $i = 1, \ldots, n$. By physical considerations, it seems to be reasonable that the free energy of the mixture is the sum of the free energies of the components. So we assume that the free energy of the fluid is a nonnegative, continuous function $W(u)$, defined for $u \in \mathbf{R}^n$, $u > 0$, which vanishes in a finite number of points $\alpha_1, \ldots, \alpha_k \in \mathbf{R}^n$.

For example, if $n=2$ and $W(u^1,u^2)=W_1(u^1)+W_2(u^2)$ with W_1 and W_2 acting as in Figure 1 with zeros α_1, β_1 and α_2, β_2, then $k=4$ and $\alpha^1=(\alpha_1,\alpha_2)$, $\alpha^2=(\alpha_1,\beta_2)$, $\alpha^3=(\beta_1,\alpha_2)$, $\alpha^4=(\beta_1,\beta_2)$.

We need also the following technical assumption on W: there exist K_1, $K_2 \in \mathbf{R}$ with $0<K_1<K_2$, such that:

$$W(u) \geqq \sup\{W(v): v\in[K_1,K_2]^n\} \tag{1.2}$$

for every $u \notin [K_1,K_2]^n$.

We can now state our main result. For every $A \subset \mathbf{R}^N$ denote by $\mathbf{1}_A$ its characteristic function, by $\partial^* A$ its reduced boundary (*cf.* Giusti [Gi]), by $|A|$ its Lebesgue measure, by $H_{N-1}(A)$ its $(N-1)$-dimensional Hausdorff measure. For every $i,j\in\{1,\dots,k\}$ define

$$d(\alpha_i,\alpha_j)=\inf\left\{\int_0^1 W^{1/2}(\gamma(t))\,|\gamma'(t)|\,dt : \gamma\in C^1([0,1];\mathbf{R}^n),\right.$$
$$\left.\gamma>0,\gamma(0)=\alpha_i,\gamma(1)=\alpha_j\right\}.$$

THEOREM. — *For every* $\varepsilon>0$, *let* u_ε *be a solution of the minimization problem:*

$$\min\left\{\int_\Omega[\varepsilon^2|Du|^2+W(u)]\,dx : u\in H^1(\Omega;\mathbf{R}^n), \int_\Omega u(x)\,dx=m, u>0\right\}.$$

Assume that the functions u_ε *converge in* $L^1(\Omega,\mathbf{R}^n)$, *as* $\varepsilon\to 0^+$, *to a function* u_0. *Then*

$$u_0(x)=\sum_{i=1}^k \alpha_i \mathbf{1}_{s_i}(x)$$

where $S_1,\dots,S_k$ *is a partition of* Ω *which minimizes the quantity*

$$\sum_{i,j=1}^k d(\alpha_i,\alpha_j)\,H_{N-1}(\partial^* S_i\cap\partial^* S_j)$$

among all other partitions of Ω *such that* $\sum_{i=1}^k |S_i|\alpha_i=m$.

We mention that results of this type in the vector case have been recently obtained by P. Sternberg [S] and I. Fonseca & L. Tartar [FT]. In [S] the problem is studied by taking $n=2$ and W which vanishes on two simple smooth curves in the plane and without assuming any constraint. In [FT] the problem is studied with n arbitrary, but $k=2$, and with the same volume constraint as in the present paper. Furthermore. Ambrosio [A] has announced general results in the case in which the set of zeros of W is essentially any closed subset of $\mathbf{R}^n$. All the mentioned papers, as well as the present paper, rely on the Γ-convergence theory.

I wish to thank Professor L. Modica for suggesting me this research as subject of my thesis for the "Laurea in Matematica" at the university of Pisa.

2. PRELIMINARY RESULTS

For any $v \in L^1(\Omega)$, it is usual to define

$$\int_A |Dv| = \sup\left\{\int_A v \operatorname{div} g\, dx : g \in C_0^1(\Omega; \mathbf{R}^N), |g| \leqq 1\right\}$$

for any open subset A of Ω. If $\int_\Omega |Dv| < +\infty$, then $u \in BV(\Omega)$; in this case the set function $A \mapsto \int_A |Dv|$ is the trace on the open subsets of Ω of a borel measure on Ω, which will be denoted by $\int_E |Dv|$ for any borel subset E of Ω. If v is the characteristic function $\mathbf{1}_S$ of a measurable subset S of Ω, then we let

$$\int_\Omega |Dv| = P_\Omega(S)$$

and $P_\Omega(S)$ is called perimeter of S in Ω, because $P_\Omega(S) = H_{N-1}(\partial S \cap \Omega)$ in the case that S has a smooth boundary. For every S, it is possible to construct a subset $\partial^* S \subset \partial S$, called reduced boundary of S, such that

$$P_\Omega(S) = H_{N-1}(\partial^* S \cap \Omega).$$

Denote by $\mathbf{R}_+^n$ the set of all $u \in \mathbf{R}^n$ such that $u \geqq 0$, that is $u_1 \geqq 0, \ldots, u_n \geqq 0$. On $\mathbf{R}_+^n$ we define the following metric:

$$d(\zeta_1, \zeta_2) = \inf\left\{\int_0^1 W^{1/2}(\gamma(t))\,|\gamma'(t)|\,dt : \gamma \in C^1([0,1]; \mathbf{R}_+^n), \gamma(0) = \zeta_1, \gamma(1) = \zeta_2\right\}$$

which is the riemannian metric derived from $W^{1/2}$.

For $\zeta \in \mathbf{R}_+^n$, let $\varphi_i(\zeta) = d(\zeta, \alpha_i)$, $i = 1, \ldots, k$, where α_i are the zeros of W. We begin with a simple result on the metric d.

PROPOSITION 2.1. — *The function φ_i is locally lipschitz-continuous. Moreover, if $u \in H^1(\Omega) \cap L^\infty(\Omega)$, then $\varphi_i \circ u \in W^{1,1}(\Omega)$ and the following inequality holds:*

$$\int_\Omega |D(\varphi_i \circ u)|\,dx \leqq \int_\Omega W^{1/2}(u(x))\,|Du(x)|\,dx. \tag{2.1}$$

Proof. – We omit the very easy proof that φ_i is locally lipschitz-continuous. If the following inequality holds:

$$\int_{\Omega'} |D(\varphi_i \circ u)| \leqq \int_{\Omega'} W^{1/2}(u(x)) |Du(x)| \, dx \tag{2.2}$$

for any Ω' open set in Ω, then, by the Radon-Nikodym Theorem, $D(\varphi_i \circ u)$ is a L^1 function and (2.1) immediately follows. Let us prove (2.2).

Consider the case $u \in C^1(\Omega)$. The function $\varphi_i \circ u$ is locally lipschitz-continuous, and then differentiable almost everywhere in Ω. Let x be a differentiability point for $\varphi_i \circ u$, $\{x_h\}$ be a sequence converging to x in Ω and σ_h be the segment $\sigma_h(t) = (1-t)x + tx_h$. We have, by definition of the metric:

$$\begin{aligned} |\varphi_i(u(x)) - \varphi_i(u(x_h))| &\leqq d(u(x), u(x_h)) \\ &\leqq \int_0^1 W^{1/2}(u(\sigma_h(t)) |Du(\sigma_h(t))| \, |x_h - x| \, dt. \end{aligned}$$

By applying the Mean Value Theorem to the right-hand side integral, dividing by $|x_h - x|$, and passing to the limit we obtain

$$|D(\varphi_i(u(x)))| \leqq W^{1/2}(u(x)) |Du(x)|,$$

and (2.2) follows.

Suppose now $u \in H^1(\Omega) \cap L^\infty(\Omega)$, and let $\{u_h\} \subset C^1(\Omega)$ be a sequence such that $u_h \to u$ in $H^1(\Omega)$. Suppose also that $u_h(x) \to u(x)$ and $Du_h(x) \to Du(x)$ almost everywhere in Ω.

By using the Dominated Convergence Theorem and the Fatou's Lemma we obtain for any $g \in C_0^1(\Omega'; \mathbf{R}^N)$, $|g| \leqq 1$ in Ω':

$$\begin{aligned} \int_{\Omega'} (\varphi_i \circ u) \operatorname{div} g \, dx &= \lim_{h \to +\infty} \int_{\Omega'} (\varphi_i \circ u_h) \operatorname{div} g \, dx \\ &- \lim_{h \to +\infty} \int_{\Omega'} < D(\varphi_i \circ u_h), g > dx \leqq \lim_{h \to +\infty} \int_{\Omega'} |D(\varphi_i \circ u_h)| \, dx \\ &\leqq \limsup_{h \to +\infty} \int_{\Omega'} W^{1/2}(u_h(x)) |Du_h(x)| \, dx \leqq \int_{\Omega'} W^{1/2}(u(x)) |Du(x)| \, dx \end{aligned}$$

and the proposition is proved. ■

Let μ and ν be two regular positive Borel measures on Ω. We define the supremum $\mu \vee \nu$ of μ and ν as the smallest regular positive measure which is greater than or equal to μ and ν on all borel subsets of Ω. We have:

$$(\mu \vee \nu)(A) = \sup \{ \mu(A') + \nu(A'') : A' \cap A'' = \varnothing, \\ A' \cup A'' \subset A, A' \textit{ and } A'' \textit{ are open sets in } \Omega \}$$

for any open subset A of Ω.

Let u be a function such that $W(u(x)) = 0$ almost everywhere and $\varphi_i \circ u \in BV(\Omega)$; hence

$$u(x) = \sum_{i=1}^{k} \alpha_i \mathbf{1}_{s_i}(x) \tag{2.3}$$

where $S_1, \ldots, S_k$ are pairwise disjoint sets in Ω such that $\left| \Omega \quad \bigcup_{i=1}^{k} S_i \right| = 0$.

PROPOSITION 2.2. – *Denote* μ_i *the Borel measure* $\mu_i : E \mapsto \int_E |D(\varphi_i \circ u)|$. *Then* $P_\Omega(S_i) < +\infty$ *for* $i = 1, \ldots, k$ *and*

$$\left(\bigvee_{i=1}^{k} \mu_i \right)(\Omega) = 1/2 \sum_{i,\, j=1}^{k} d(\alpha_i, \alpha_j) H_{N-1}(\partial^* S_i \cap \partial^* S_j \cap \Omega). \tag{2.4}$$

Proof. – Let us prove that the perimeters of the sets S_i in Ω are bounded. By applying the coarea-type Fleming-Rishel formula:

$$\int_\Omega |D(\varphi_i \circ u)| = \int_{-\infty}^{+\infty} P_\Omega(\{x : \varphi_i(u(x)) \leqq t\})\, dt \\ \geqq \int_0^{d_i} P_\Omega(\{x : \varphi_i(u(x)) \leqq t\})\, dt = d_i P_\Omega(S_i)$$

where $d_i = \min_{\substack{j=1,\ldots,k \\ j \neq i}} (d(\alpha_i, \alpha_j))$; hence $P_\Omega(S_i) < +\infty$.

Suppose we have proved that for every index $i = 1, \ldots, k$ and for every open set $\Omega' \subset \Omega$ we have:

$$2 \int_{\Omega'} |D(\varphi_i \circ u)| = \sum_{j,\, h=1}^{k} H_{N-1}(\partial^* S_h \cap \partial^* S_j \cap \Omega') |\varphi_i(\alpha_h) - \varphi_i(\alpha_j)|. \tag{2.5}$$

Since we have $|\varphi_l(\alpha_i) - \varphi_l(\alpha_j)| \leqq d(\alpha_i, \alpha_j)$ for any $i, j, l = 1, \ldots, k$, and the equality holds for $l = i$ or $l = j$, the result follows from the lemma below, by passing to the measure theoretical supremum in (2.5).

We now prove (2.5). Note that for any $J \subset \{1, \ldots, k\}$ we have

$$P_{\Omega'}(\bigcup_{j \in J} S_j) = \sum_{j \in J} \sum_{\substack{i=1 \\ i \notin J}}^{k} H_{N-1}(\partial^* S_i \cap \partial^* S_j \cap \Omega') \tag{2.6}$$

In fact, reasoning as in Vol'pert [Vo]:

$$P_{\Omega'}(S_i \cup S_j) = H_{N-1}(\partial^* S_i \Delta \partial^* S_j) \qquad (i \neq j)$$

$$\partial^* S_i = \bigcup_{\substack{j=1 \\ j\neq i}}^{k} (\partial^* S_i \cap \partial^* S_j) \cup N \qquad \textit{where} \quad H_{N-1}(N) = 0.$$

An inductive argument on the cardinality of J proves (2.6). We now return to the proof of (2.5). Suppose for simplicity that $i=1$ and that the indices are numbered in such a way that

$$0 = \varphi_1(\alpha_1) < \varphi_1(\alpha_2) < \ldots < \varphi_1(\alpha_k).$$

(It's possible that there are indices j such that $\varphi_1(\alpha_j) = \varphi_1(\alpha_{j+1})$, but the changes in the proof are trivial.) By the Fleming-Rishel formula we obtain:

$$\begin{aligned}\int_{\Omega'} |D(\varphi_1 \circ u)| &= \int_0^{\varphi_1(\alpha_1)} P_{\Omega'}(\{x \in \Omega' : \varphi_1(u(x)) \leq t\})\, dt \\ &= \sum_{j=1}^{k-1} [\varphi_1(\alpha_{j+1}) - \varphi_1(\alpha_j)]\, P_{\Omega'}\left(\bigcup_{l=1}^{j} S_l\right) \\ &= \sum_{j=1}^{k-1} \sum_{l=1}^{j} \sum_{m=j+1}^{k} [\varphi_1(\alpha_{j+1}) - \varphi_1(\alpha_j)]\, H_{N-1}(\partial^* S_l \cap \partial^* S_m).\end{aligned}$$

Reordering the terms in the sum, we finally obtain (2.5). ■

LEMMA 2.3. – *Let μ be a regular positive borel measure on Ω, $B_1, \ldots, B_m$ be disjoint borel subsets of Ω with finite μ-measure and c_i^h, $i=1, \ldots, m$, $h=1, \ldots, k$ be positive coefficients. Define*

$$\mu_h(A) = \sum_{i=1}^{m} c_i^h \mu(A \cap B_i); \qquad \nu(A) = \sum_{i=1}^{m} (\max_h c_i^h)\, \mu(A \cap B_i).$$

Then $\nu = \bigvee_{i=1}^{k} \mu^h$.

Proof. – We omit the easy proof. ■

On the basis of the previous proposition, we define, for any $u \in L^1(\Omega)$:

$$F_0(u) = \begin{cases} 2 \bigvee_{i=1}^{k} \mu_i = 2 \bigvee_{i=1}^{k} \int_{\Omega} |D(\varphi_i \circ u)| \\ \qquad \text{if} \quad \varphi_i \circ u \in BV(\Omega), \quad W(u(x)) = 0 \text{ a.e.}, \quad \int_{\Omega} u\, dx = m \\ +\infty \qquad \text{otherwise} \end{cases}$$

and we obtain that, if $F_0(u) < +\infty$, then

$$F_0(u) = \sum_{i,j=1}^{k} d(\alpha_i, \alpha_j)\, H_{N-1}(\partial^* S_i \cap \partial^* S_j \cap \Omega).$$

We now recall the definition of Γ-convergence [DGF]:

Let $\{G_h\}$ be a sequence of real-extended functionals on $L^1(\Omega)$, G_0 a functional of the same kind. We say that G_h Γ-converges to G_0 as $h \to +\infty$ at a point $u \in L^1(\Omega)$, and we write:

$$\Gamma - \lim_{h \to +\infty} G_h(u) = G_0(u)$$

if and only if the following relations hold:

$$\forall u_h \to u \text{ in } L^1(\Omega) \text{ we have } G_0(u) \leqq \liminf_{h \to +\infty} G_h(u_h) \tag{2.8}$$

$$\exists u_h \to u \text{ in } L^1(\Omega) \text{ such that } G_0(u) \geqq \limsup_{h \to +\infty} G_h(u_h) \tag{2.9}$$

As immediate consequence, we obtain the following result (*see* [DGF]):

PROPOSITION 2.4. – *Assume* $G_0(u) = \Gamma - \lim_{h \to +\infty} G_h(u)$ *for every* $u \in L^1(\Omega)$. *Let* u_h *be such that* $G_h(u_h) = \min\{G_h(u) : u \in L^1(\Omega)\}$ *for every* $h \in N$. *If* $u_h \to u_0$ *in* $L^1(\Omega)$, *then*:

$$\lim_{h \to +\infty} G_h(u_h) = G_0(u_0) = \min\{G_0(u) : u \in L^1(\Omega)\}.$$

The result we shall prove in the next sections is the following. Define, for any $u \in L^1(\Omega)$,

$$F_\varepsilon(u) = \begin{cases} \int_\Omega \left[\varepsilon |Du|^2 + \frac{1}{\varepsilon} W(u)\right] dx & \text{if } u \in H^1(\Omega; \mathbf{R}^n_+) \text{ and } \int_\Omega u(x)\,dx = m \\ +\infty & \text{otherwise.} \end{cases}$$

THEOREM 2.5. – *For every* $\varepsilon_h \downarrow 0$,

$$\Gamma - \lim_{h \to +\infty} F_{\varepsilon_h}(u) = F_0(u), \qquad \forall u \in L^1(\Omega).$$

Our main result is now a consequence of Theorem 2.5. Indeed, by the hypothesis $u_\varepsilon \to u_0$. Proposition 2.4 yields that F_0 admits a minimum value and u_0 is a minimum point of F_0. Furthermore, the fact that this minimum value is obviously finite gives the formula for the asymptotic value of the energies $F_\varepsilon(u_\varepsilon)$.

3. THE PROOF OF THE MAIN RESULT

To prove our main result in the form of Theorem 2.5 we only need to prove the statements (2.8) and (2.9) in the definition of Γ-convergence.

Proof of (2.8). — Let $\varepsilon_h \downarrow 0$ be a fixed sequence. It is not restrictive to assume $\lim_{h \to +\infty} F_{\varepsilon_h}(u_h)$ exists and it is finite, being trivial the other cases.

By choosing a subsequence u_{h_k} that converges to u pointwise almost everywhere in Ω, and by Fatou's Lemma, we obtain:

$$\int_\Omega W(u)\,dx \leqq \liminf_{k \to +\infty} \int_\Omega W(u_{h_k})\,dx \leqq \liminf_{k \to +\infty} \varepsilon_{h_k} F_{\varepsilon_{h_k}}(u_{h_k}) = 0;$$

hence $W(u(x)) = 0$ almost everywhere in Ω because W is a continuous and nonnegative function.

We now need to prove that

$$2 \bigvee_{i=1}^{k} \int_\Omega |D(\varphi_i \circ u)| \leqq \lim_{h \to +\infty} \int_\Omega \left[\varepsilon_h |D_{u_h}|^2 + \frac{1}{\varepsilon_h} W(u_h) \right] dx \qquad (3.1)$$

We further reduce the problem by making the sequence u_h to be equibounded. If this is not true, by using the technical assumption (1.2) on W, we can replace u_h by the $\tilde{u}_h$ obtained trough truncation of each scalar component by K_1 and K_2. Note that $\tilde{u}_h \to u$ in $L^1(\Omega)$ and the integrals in the right-hand side of (3.1) decrease. Then, $(\varphi_i \circ \tilde{u}_h) \to (\varphi_i \circ u)$ in $L^1(\Omega)$ for every $i = 1, \ldots, k$, because of the continuity of φ_i, and lower semicontinuity yields:

$$|D(\varphi_i \circ u)|(A) \leqq \liminf_{h \to +\infty} |D(\varphi_i \circ u_h)|(A)$$

for every $i = 1, \ldots, k$, A *open set in* Ω.

Thus:

$$\bigvee_{i=1}^{k} |D(\varphi_i \circ u)|(\Omega) = \sup \left\{ \sum_{i=1}^{k} |D(\varphi_i \circ u)|(A_i) : \bigcup_{i=1}^{k} A_i \subset \Omega, \right.$$
$$\left. A_i \cap A_j = \varnothing \textit{ if } i \neq j, A_i \textit{ open sets in } \Omega \right\}$$
$$\leqq \liminf_{h \to +\infty} \bigvee_{i=1}^{k} |D(\varphi_i \circ u_h)|(\Omega)$$

Recalling Proposition 2.1 we finally obtain:

$$\liminf_{h \to +\infty} \bigvee_{i=1}^{k} |D(\varphi_i \circ u_h)|(\Omega) \leqq \liminf_{h \to +\infty} \int_\Omega W^{1/2}(u_h) |Du_h(x)|\,dx$$
$$\leqq \frac{1}{2} \liminf_{h \to +\infty} \int_\Omega \left[\varepsilon_h |Du_h|^2 + \frac{1}{\varepsilon_h} W(u_h) \right] dx. \quad \blacksquare$$

Proof of (2.9). — Fix $\varepsilon_h \downarrow 0$. If $u \in L^1(\Omega)$ is such that $F_0(u) = +\infty$, the construction of the sequence u_{ε_h} is trivial. Therefore assume

$u(x)=\sum_{i=1}^{k}\alpha_i \mathbf{1} S_i(x)$, with $S_1, \ldots, S_k$ pairwise disjoint sets with finite perimeter in Ω such that $\left|\Omega\setminus\bigcup_{i=1}^{k} S_i\right|=0$.

The following lemma, whose proof is given in appendix, and a diagonal argument allows us to consider partitions such that the sets $S_1, \ldots, S_k$ are polygonal domains in $\mathbf{R}^N$, with $H_{N-1}(\partial S_i \cap \partial\Omega)=0$:

LEMMA 3.1. — *Let $\{S_1, \ldots, S_k\}$ as before. Then there exists a sequence $\{S_1^h, \ldots, S_k^h\}_{h\in\mathbf{N}}$ of partitions of Ω such that:*

(i) *S_i^h is a polygonal domain and $H_{N-1}(\partial S_i^h \cap \partial\Omega)=0$ for any $i=1, \ldots, k$, $h\in\mathbf{N}$;*

(ii) *if $u_h(x)=\sum_{i=1}^{k}\alpha_i \mathbf{1}_{S_i^h}(x), u(x)=\sum_{i=1}^{k}\alpha_i \mathbf{1}_{S_i}(x)$, then $u_h\to u$ in $L^1(\Omega)$ as $h\to+\infty$;*

(iii) $\int_\Omega u_h(x)\,dx=\int_\Omega u(x)\,dx=m$ *for any $h\in\mathbf{N}$;*

(iv) $\lim_{h\to+\infty}\bigvee_{i=1}^{k}\int_\Omega |D(\varphi_i\circ u_h)|=\bigvee_{i=1}^{k}\int_\Omega |D(\varphi_i\circ u)|$.

The following lemma, that will be essential in the construction of our sequence u_h, generalizes an idea of Modica [M2].

We suppose for simplicity that for any $i, j=1, \ldots, k$, $i\neq j$, there exists the distance-minimizing geodesic connecting α_i and α_j, *i.e.* we suppose that there exists a C^1-path γ_{ij} such that $\gamma_{ij}(0)=\alpha_i$, $\gamma_{ij}(1)=\alpha_j$ and $d(\alpha_i,\alpha_j)=\int_0^1 W^{1/2}(\gamma_{ij}(t))\,|\gamma'_{ij}(t)|\,dt$. We shall see later how to change the proof if such geodesics does not exist. Note that it is not restrictive to assume $|\gamma'_{ij}(t)|\neq 0$, $\forall t\in(0,1)$.

LEMMA 3.2. — *Consider the following ordinary differential equations:*

$$(y_\varepsilon^{ij})'^2=\frac{\delta+W(\gamma_{ij}(y_\varepsilon^{ij}))}{\varepsilon^2\,|\gamma'_{ij}(y_\varepsilon^{ij})|^2}\tag{3.2}$$

where $i, j=1, \ldots, k, i\neq j$ and $\delta>0$ is a fixed constant.

Thus, for every $\varepsilon>0$, there exist a lipschitz-continuous function $\chi_\varepsilon\colon\mathbf{R}^{k-1}\to\mathbf{R}^n$ and three constants C_1, C_2, C_3 (depending only on δ) such that:

(i) $\chi_\varepsilon(t_1, \ldots, t_{k-1})=\alpha_1$ *if* $t_1<0$.

$\chi_\varepsilon(t)=\alpha_i$ *if* $t_1>C_1\varepsilon, \ldots, t_{i-1}>C_1\varepsilon, t_i<0$ *for any* $i=2, \ldots, k-1$.

$\chi_\varepsilon(t)=\alpha_k$ *if* $t_1>C_1\varepsilon, \ldots, t_k>C_1\varepsilon$.

(ii) $0<|\chi_\varepsilon|<C_2$, $|D_{\chi_\varepsilon}|<C_3/\varepsilon$ *a.e. in* $\mathbf{R}^{k-1}$

(iii) *If* $j>i$, *on the set* $\{t\in\mathbf{R}^{k-1}: 0<t_i<C_1\,\varepsilon;\ t_j<0;\ t_h>C_1\,\varepsilon$ *for any* $h\neq i,j\}$ χ_ε *depends only on* t_i, *and we can write*:
$\chi_\varepsilon(t_i)=\gamma_{ij}(y_\varepsilon^{ij}(t_i))$ [*where* y_ε^{ij} *solves* (3.2)] *for any* t_i *such that* $\chi_\varepsilon(t_i)\neq\alpha_i$)
(*Note: if* $j=k$, *ignore the condition* $t_j<0$, *that makes no sense.*)

Proof. — To prove the lemma, we only need to find the constants C_1, C_2, C_3 and to define χ_ε at the points different from those considered in (i). First we search the solutions of (3.1), for fixed i, j. The function

$$\psi_\varepsilon(t)=\int_0^t \frac{\varepsilon\,|\gamma'_{ij}(s)|}{[\delta+W(\gamma_{ij}(s))]^{1/2}}\,ds \qquad (t\in[0,1])$$

is obviously increasing and, if $\eta_\varepsilon=\psi_\varepsilon(1)$, we immediately obtain: $\eta_\varepsilon\leqq\varepsilon\delta^{-1/2}$ *length* (γ_{ij}). Now, the inverse function $\tilde{y}_\varepsilon:[0,\eta_\varepsilon]\to[0,1]$ of ψ_ε satisfies the differential equation (3.1). We extend the function to the whole $\mathbf{R}$ by putting:

$$y_\varepsilon(t)=\begin{cases} 0 & t\leqq 0\\ \tilde{y}_\varepsilon & 0\leqq t\leqq n_\varepsilon\\ 1 & t\geqq\eta_\varepsilon.\end{cases}$$

Now y_ε is a lipschitz-continuous function satisfying (3.1) in all the points where $y_\varepsilon\neq 1$. Putting:

$$C_1=\max_{i,j=1,\ldots,k}\{\delta^{-1/2}\,\mathrm{length}(\gamma_{ij})\}$$

we can define χ_ε on the strips as in (iii). We choose

$$C_2>\sup\left\{|y|: y\in\bigcup_{i,j=1}^{k}\gamma_{ij}([0,1])\right\}$$

If

$$K=\sup_{\substack{i,j=1,\ldots,k\\ i\neq j,\ t\in[0,1]}}\left[\frac{[\delta+W(\gamma_{ij}(t))]^{1/2}}{|\gamma'_{ij}(t)|}\right]$$

we have $|D_{\chi_\varepsilon}|\leqq K/\varepsilon$. Standard extension results for lipschitz-continuous functions allow one to define χ_ε on the whole $\mathbf{R}^{k-1}$, with $C_3>K$ suitably chosen. ■

The last lemma has to do with tubular neighborhood of polygonal domains, and we omit the standard proof.

LEMMA 3.3. — *Let* Ω *be an open set in* $\mathbf{R}^N$, A *a polygonal domain in* $\mathbf{R}^N$ *with* ∂A *compact and such that* $H_{N-1}(\partial A\cap\partial\Omega)=0$. *Put*:

$$h(x)=\begin{cases} \operatorname{dist}(x,\partial A) & \textit{if}\quad x\notin A\\ -\operatorname{dist}(x,\partial A) & \textit{if}\quad x\in A.\end{cases}$$

Then there exists a constant $\eta>0$ *such that* h *is lipschitz-continuous on* $H_\eta=\{x\in\mathbf{R}^N:|h(x)|<\eta\}$, *and* $|Dh(\alpha)|=1$ *for almost all* $x\in H_\eta$.

Finally, if S_t *denotes the set* $\{x\in\mathbf{R}^N:h(x)=t\}$ *we have*:

$$\lim_{t\to 0^\pm} H_{N-1}(S_t\cap\Omega)=H_{N-1}(\partial A\cap\Omega).$$

Let us return to the proof of (2.9) for polygonal domains. We define, for $i\in\{1,\ldots,k\}$:

$$h_i(x)=\begin{cases}\operatorname{dist}(x,\partial S_i) & \text{if } x\notin S_i\\ -\operatorname{dist}(x,\partial S_i) & \text{if } x\in S_i\end{cases}$$

Fix $\delta>0$. For ε small enough we have $|Dh_i(x)|=1$ a.e. on the set $\{x\in\Omega:|h_i(x)|<C_1\varepsilon\}$, for all $i=1,\ldots,k$.

Consider the sequence of functions:

$$\tilde{u}_\varepsilon(x)=\chi_\varepsilon(h_1(x),\ldots,h_{k-1}(x)).$$

Putting $\Sigma_i^t=\{x\in\Omega:h_i(x)=t\}$, $t>0$, $i=1,\ldots,k-1$, we obtain:

$$\int_\Omega|\tilde{u}_\varepsilon-u|\leqq\sum_{i=1}^{k-1}\int_{\{x\in\Omega:0<h_i(x)<C_2\varepsilon\}}|\tilde{u}_\varepsilon-u|\,dx$$
$$\leqq 2C_2\sum_{i=1}^{k-1}|\{x\in\Omega:0\leqq h_i(x)\leqq C_1\varepsilon\}|$$
$$=2C_2\sum_{i=1}^{k-1}\int_{\{x\in\Omega:0\leqq h_i(x)\leqq C_1\varepsilon\}}|Dh_i(x)|\,dx=2C_2\sum_{i=1}^{k-1}\int_0^{C_1\varepsilon}H_{N-1}(\Sigma_i^t)\,dt$$

where the last equality follows from the coarea formula. By Lemma 3.3 we have $\int_\Omega|\tilde{u}_\varepsilon-u|\,dx\leqq C_1\varepsilon$ and we conclude that $\tilde{u}_\varepsilon\to u$ in $L^1(\Omega)$. If $\int_\Omega\tilde{u}_\varepsilon=m$ we define u_ε. Otherwise we put:

$$\eta_\varepsilon=\int_\Omega\tilde{u}_\varepsilon(x)\,dx-\int_\Omega u(x)\,dx$$

Each scalar component of the vector η_ε is (in absolute value) less than or equal to $C_1\varepsilon$. Let x_0 be a point in the interior of the set $\Omega\cap S_1$ (if it is empty, the changes in the proof are trivial). If ε is small enough, the open ball $B_\varepsilon=B(x_0,\varepsilon^{1/N})$ is contained in $\{x:\tilde{u}_\varepsilon(x)=\alpha_1\}$, by definition of $\tilde{u}_\varepsilon$. Define:

$$u_\varepsilon(x)=\begin{cases}\tilde{u}_\varepsilon(x) & \text{if } x\in\Omega\setminus B_\varepsilon\\ \alpha_1+h_\varepsilon(1-\varepsilon^{-1/N}|x-x_0|) & \text{if } x\in B_\varepsilon\end{cases}$$

where $h_\varepsilon = -N\,\omega_{N-1}^{-1}\,\eta_\varepsilon\,\varepsilon^{(1-N)/N}$ and ω_{N-1} is the volume of the $(N-1)$-dimensional unit ball. The functions u_ε satisfy the constraint $\int u_\varepsilon = m$ for ε small enough, and converge in $L^1(\Omega)$ to u. It remains to give a sharp estimate of $\limsup_{\varepsilon\to 0^+} F_\varepsilon(u_\varepsilon)$. Let us consider the following partition of Ω:

(i) $\Omega_1^\varepsilon = S_1 \setminus B_\varepsilon$.

(ii) $\Omega_i^\varepsilon = \{x \in S_i : h_j(x) > C_1\,\varepsilon, j = 1, \ldots, i-1\}$ for $i = 2, \ldots, k$.

(iii) $\Omega_{ij}^\varepsilon = \{x \in \Omega : 0 < h_i(x) < C_1\,\varepsilon, h_j(x) < 0, h_l(x) > C_1\,\varepsilon$ for

$$l \in \{1, \ldots, k-1\} \setminus \{i, j\}\}$$

for $i, j \in \{1, \ldots, k\}$ and $i \neq j$.

(iv) $\Omega_0^\varepsilon = \Omega \quad B_\varepsilon \cup \left(\bigcup_{i=1}^{k} \Omega_i^\varepsilon\right) \cup \bigcup_{\substack{i,j=1\\ i\neq j}}^{k} \Omega_{ij}^\varepsilon$.

If we put:

$$F_\varepsilon(u, \Omega') = \int_{\Omega'} [\varepsilon\,|Du|^2 + \frac{1}{\varepsilon} W(u)]\,dx$$

we obviously have:

$$\limsup_{\varepsilon\to 0^+} F_\varepsilon(u_\varepsilon) \leqq \sum_{i=1}^{k} \limsup_{\varepsilon\to 0^+} F_\varepsilon(u_\varepsilon, \Omega_i^\varepsilon)$$
$$+ \sum_{i,j} \limsup_{\varepsilon\to 0^+} F_\varepsilon(u_\varepsilon, \Omega_{ij}^\varepsilon) + \limsup_{\varepsilon\to 0^+} F_\varepsilon(u_\varepsilon, B_\varepsilon) + \limsup_{\varepsilon\to 0^+} F_\varepsilon(u_\varepsilon, \Omega_0^\varepsilon). \quad (3.3)$$

The first sum of the right-hand side of (3.3) vanishes on Ω_i^ε because $u_\varepsilon \equiv \alpha_i$.

Since:

$$F_\varepsilon(u_\varepsilon, B_\varepsilon) = \varepsilon\,|h_\varepsilon|^2\,\varepsilon^{-2/N}\,|B_\varepsilon| + \varepsilon^{-1} \int_{B_\varepsilon} W(\alpha_1 + h_\varepsilon(1 - \varepsilon^{-1/N}|x - x_0|))\,dx$$
$$\leqq K\,\varepsilon^2 + \int_0^1 W(\alpha_1 + h_\varepsilon(1-r))\,r^{N-1}\,dr$$

and $h_\varepsilon \to 0$ for $\varepsilon \to 0$, we have

$$\limsup_{\varepsilon\to 0^+} F_\varepsilon(u_\varepsilon, B_\varepsilon) = 0.$$

Now define: $K_{ij}^\varepsilon = \{x \in \Omega : 0 < h_i(x) < C_1\,\varepsilon,\ 0 < h_j(x) < C_1\,\varepsilon\}$, $i > j$. Since $\Omega_0^\varepsilon \subset \bigcup_{ij} K_{ij}^\varepsilon$, by using Lemma 3.2 we obtain:

$$F_\varepsilon(u_\varepsilon, K_{ij}^\varepsilon) = \int_{K_{ij}^\varepsilon} [\varepsilon\,|Du_\varepsilon|^2 + \frac{1}{\varepsilon} W(u_\varepsilon)]\,dx \leqq C_5\,\varepsilon^{-1}\,|K_{ij}^\varepsilon|.$$

Putting now $S_j^t = \{ x \in \Omega : h_j(x) > t \text{ or } h_j(x) > C_1 \varepsilon \}$ and employing again the coarea formula we have:

$$|K_{ij}^\varepsilon| = \int_0^{C_1 \varepsilon} H_{N-1}(\{ x \in \Omega : h_i(x) = s, 0 < h_j(x) < C_1 \varepsilon \})\, ds$$
$$\leqq C_1 \varepsilon \sup_{0 \leqq s \leqq C_1 \varepsilon} H_{N-1}(\{ x \in \Omega : h_i(x) = s, 0 < h_j(x) < C_1 \varepsilon \})$$
$$= C_1 \varepsilon \sup_{0 \leqq s \leqq C_1 \varepsilon} H_{N-1}\left(\sum_i^s \; S_j^{C_1 \varepsilon} \right).$$

Remark that, for almost all $\rho > 0$, we have $H_{N-1}(\partial S_i \cap \partial S_j^\rho) = 0$. Thus, recalling Lemma 3.3, we have:

$$\limsup_{\varepsilon \to 0^+} \sup_{0 \leqq s \leqq C_1 \varepsilon} H_{N-1}\left(\sum_i^s \; S^{C_1 \varepsilon} \right)$$
$$\leqq \limsup_{\varepsilon \to 0^+} \sup_{0 \leqq s \leqq C_1 \varepsilon} H_{N-1}\left(\sum_i^s \; S_j^\rho \right) = H_{N-1}((\partial S_i \cap (\Omega \setminus S_j)) \setminus S_j^\rho).$$

Passing to the infimum for $\rho > 0$, we finally have $\limsup_{\varepsilon \to 0^+} F_\varepsilon(u_\varepsilon, K_{ij}^\varepsilon) = 0$, and then

$$\limsup_{\varepsilon \to 0^+} F_\varepsilon(u_\varepsilon, \Omega_0^\varepsilon) = 0.$$

It remains to estimate the terms of (3.3) of the form $\limsup_{\varepsilon \to 0^+} F_\varepsilon(u_\varepsilon, \Omega_{ij}^\varepsilon)$.

We have:

$$F_\varepsilon(u_\varepsilon; \Omega_{ij}^\varepsilon) = \int_{\Omega_{ij}^\varepsilon} \left[\varepsilon |Du_\varepsilon|^2 + \frac{1}{\varepsilon} W(u_\varepsilon) \right] dx$$
$$= \int_{\Omega_{ij}^\varepsilon} \left[\varepsilon |D(\gamma_{ij} \circ y_\varepsilon^{ij} \circ h_i)(x)|^2 + \frac{1}{\varepsilon} W(\gamma_{ij} \circ y_\varepsilon^{ij} \circ h_i)(x) \right] |D h_i(x)|\, dx$$

(by using the coarea formula)

$$\int_0^{C_1 \varepsilon} \left[\varepsilon \gamma_{ij}'^2 (y_\varepsilon^{ij}(t)) (y_\varepsilon^{ij})'^2 (t) + \frac{1}{\varepsilon} W(\gamma_{ij} \circ y_\varepsilon^{ij})(t) \right] H_{N-1}\left(\sum_i^t \cap S_j \right) dt$$

(by remarking that $\sigma_\varepsilon = \sup_{0 \leqq t \leqq C_1 \varepsilon} H_{N-1}\left(\sum_i^t \cap S_j \right) \to H_{N-1}(\partial S_i \cap \partial S_j)$ as $\varepsilon \to 0$)

$$\leqq \sigma_\varepsilon \int_0^{C_1 \varepsilon} \left[\varepsilon \gamma_{ij}'^2 (y_\varepsilon^{ij}(t)) ((y_\varepsilon^{ij})'(t))^2 + \frac{1}{\varepsilon} (\delta + W(\gamma_{ij} \circ y_\varepsilon^{ij})(t)) \right] dt$$

[by using eq. (3.1)]

$$\sigma_\varepsilon \int_0^{C_1\varepsilon} \{ | \gamma'_{ij}(y^{ij}_\varepsilon(t)) | | y^{ij'}_\varepsilon(t) | [\delta + W(\gamma_{ij} \circ y^{ij}_\varepsilon)(t)]^{1/2} \} \, dt$$

$$\leqq 2\sigma_\varepsilon \int_0^1 [\delta + W(\gamma_{ij}(s))]^{1/2} | \gamma'_{ij}(s) | \, ds.$$

By collecting all previous inequalities, we conclude that

$$\limsup_{\varepsilon \to 0^+} F_\varepsilon(u_\varepsilon) \leqq \sum_{i,j=1}^{k} H_{N-1}(\partial S_i \cap \partial S_j) \int_0^1 [\delta + W(\gamma_{ij}(s))]^{1/2} | \gamma'_{ij}(s) | \, ds.$$

Recall that the functions $u_\varepsilon = u_\varepsilon^\delta$ depend on δ. Passing now to the infimum for $\delta > 0$ we obtain:

$$\inf_{\delta > 0} \limsup_{\varepsilon \to 0^+} F_\varepsilon(u_\varepsilon^\delta) \leqq F_0(u_0) \tag{3.4}$$

and, by a simple diagonal argument, (2.9) is proved. The proof is now almost completed, modulo the existence of geodesics connecting any two zeros of W. If such geodesics do not exist, we could choose approximate geodesics γ^h_{ij} such that

$$\int_0^1 W^{1/2}(\gamma^h_{ij}(s)) \, | \gamma^{h'}_{ij}(s) | \, ds \leqq d(\alpha_i, \alpha_j) + \frac{1}{h};$$

reasoning as above, we construct a sequence u^h_ε such that

$$\limsup_{\varepsilon \to 0^+} F_\varepsilon(u^h_\varepsilon) \leqq \sum_{ij} H_{N-1}(\partial S_i \cap \partial S_j)(d(\alpha_i, \alpha_j) + 1/h)$$

and again a diagonal argument completes the proof. ■

4. FINAL REMARKS

In this Section we prove a compactness result in $L^1(\Omega)$ for the family of minima.

PROPOSITION 4.1. – *Let* $\{u_\varepsilon\}_{\varepsilon > 0}$ *be a sequence such that, for each* $\varepsilon > 0$, u_ε *is a minimum point of* F_ε. *Let us suppose that there exists a constant* $M > 0$ *such that* $u_\varepsilon(x) \leqq M$ *for almost all* $x \in \Omega$ *and for all* $\varepsilon > 0$. *Then there exists a sequence* $\varepsilon_h \downarrow 0$ *such that* $\{u_{\varepsilon_h}\}_{h \in N}$ *converges in* $L^1(\Omega)$.

Proof. – Since the sequences $\{\varphi_i \circ u_\varepsilon\}_{\varepsilon > 0}$ are bounded in $L^1(\Omega)$ for every $i = 1, \ldots, k$, using the results obtained in Section 3 to prove the inequality (2.8), we have:

$$\int_{\Omega} |D(\varphi_i \circ u_\varepsilon)| \leqq \frac{1}{2} F_\varepsilon(u_\varepsilon), \qquad \forall \varepsilon > 0, \quad i = 1, \ldots, k \tag{4.1}$$

Furthermore, reasoning as for the proof of (2.9), we obtain that $\{F_\varepsilon(u_\varepsilon)\}$ is equibounded. Thus, by using Rellich's Theorem in $BV(\Omega)$, there exist k functions $f_1, \ldots, f_k$ and a sequence $\varepsilon_h \downarrow 0$ such that $\varphi_i \circ u_{\varepsilon_h} \to f_i$ in $L^1(\Omega)$ and pointwise a. e., for every $i = 1, \ldots, k$. We now define:

$$S_i = \{ x \in \Omega : f_i(x) = 0 \},$$

$$u_0(x) = \sum_{i=1}^{k} \alpha_i \mathbf{1}_{S_i}(x).$$

Let us turn to the convergence of $\{u_{\varepsilon_h}\}$: it is of course sufficient to prove pointwise a. e. convergence. Let $A = \{ x \in \Omega : \limsup_{\varepsilon \to 0^+} W(u_\varepsilon(x)) > 0 \}$; the equiboundedness of $F_\varepsilon(u_\varepsilon)$ implies that $|A| = 0$; hence, for a. a. $x \in \Omega$, the limit points of $u_{\varepsilon_h}(x)$ are in the set $\{\alpha_1, \ldots, \alpha_k\}$. On the other hand, for a. a. $x \in S_i$, if $u_{\varepsilon_{h_k}}(x) \to \alpha_j$, we have that $\varphi_i(u_{\varepsilon_{h_k}}(x)) \to \varphi_i(\alpha_j)$, $\varphi_i(u_{\varepsilon_{h_k}}(x)) \to f_i(x) = 0$; therefore $\varphi_i(\alpha_j) = 0$ and $i = j$. It follows that u_{ε_h} converges pointwise to u_0 and Proposition (4.1) is proved. ■

In order to check the equiboundedness-hypothesis of the previous proposition, we recall the following result, which is a straightforward generalization of a theorem by Gurtin and Matano (*see* [GM] and [LM]).

PROPOSITION 4.2. – *Suppose* $W \in C^1(\mathbf{R}^n_+)$,

$$W(t) = W_1(t_1) + \ldots + W_n(t_n)$$

(*see the introduction*). *Denote* $W'_i(t_i) = \dfrac{\partial W(t)}{\partial t_i}$, *and suppose that for an index* $i \in \{1, \ldots, k\}$ *the following properties hold:*

(i) $W'_i(0) < W'_i(\tau)$, $\forall \tau \in \mathbf{R}^+$,

(ii) $\lim_{\tau \to +\infty} W'_i(\tau) = +\infty$.

Then the family of minima $\{u_\varepsilon\}$ *is bounded in* $L^\infty(\Omega)$.

APPENDIX: PROOF OF LEMMA 3.1

For the moment, we ignore part (iii) of the Lemma. Let us recall the following result by T. Quentin de Gromard [QG]:

THEOREM A.1. – *Let* $\Omega \subset \mathbf{R}^N$ *be a bounded and open set, and let* $E \subset \mathbf{R}^N$ *be a set such that* $\mathbf{1}_E \in BV(\Omega)$. *Then there exists a sequence* E_h *of sets with* *bounded perimeter such that:*

(i) $|D(1_{E_h}-1_E)|(\Omega)\to 0$ *as* $n\to+\infty$.

(ii) $\Omega\cap\partial E_h$ *is contained in a finite union of* C^1*-hypersurfaces, for every* $h\in\mathbf{N}$.

(iii) $|(E_h\Delta E)\cap\Omega|\to 0$, $E_h\subset E+B(0,1/h)$, $\Omega\setminus E_h\subset(\Omega\setminus E)+B(0,1/h)$.

(iv) *If* E *contains a ball* $B(x,r)$, *then there exists* $r_0>0$ *such that* E_h *contains* $B(x,r_0)$ *for every* $h\in\mathbf{N}$. *The same holds for* $\Omega\setminus E$.

[(iv) follows from the proof given by T. Quentin de Gromard, even if it is not explicitly stated.]

The idea of the proof of Lemma 3.1 is as follows: we approximate the partition $S_1,\ldots,S_k$ with sets as in Theorem A.1, and then we approximate these sets with polygonal domains.

More precisely, we begin with the proof of the following result:

PROPOSITION A.2. – *Let* $S_1,\ldots,S_k$ *be a partition as in Lemma* 3.1. *Then there exists a sequence of partitions* $\{S_1^h,\ldots,S_k^h\}_{h\in\mathbf{N}}$ *such that:*

(i) S_i^h *is a closed set, and* ∂S_i^h *is contained in a finite union of* C^1*-hypersurfaces.*

(ii) $1_{S_i^h}\to 1_{S_i}$ *in* $BV(\Omega)$ *with the strong topology as* $h\to+\infty$.

(iii) *If* S_i *contains* $B(x,r)$, *then there exists* $r_0>0$ *such that* S_i^h *contains* $B(x,r_0)$ *for every* $h\in\mathbf{N}$.

Proof. – For any fixed $i=1,\ldots,k$ we pick a sequence $\tilde{S}_i^h$ as in Theorem A.1, and we define:

$$S_1^h=\tilde{S}_1^h$$
$$S_2^h=\tilde{S}_2^h\setminus S_1^h$$
$$\cdots$$
$$S_k^h=\tilde{S}_k^h\setminus\left(\bigcup_{i=1}^{k-1}S_i^k\right).$$

This sequence of partitions verifies (i) and (ii) by virtue of the following propositon, while (iii) is trivial to prove. ■

PROPOSITION A.3. – *Let* A, B *be disjoint sets with bounded perimeter in* Ω, *let* $\{A_h\}$ *and* $\{B_h\}$ *be sequences such that* $1_{B_k}\to 1_B$ *and* $1_{A_k}\to 1_A$ *in* $BV(\Omega)$ *with the strong topology. Then we have:*

(i) $1_{A_k\cup B_h}\to 1_{A\cup B}$ *in* $BV(\Omega)$.

(ii) $1_{B_h\ A_h}\to 1_B$ *in* $BV(\Omega)$.

To prove this proposition, we need another lemma. We recall the definition of the $(N-1)$-dimensional Gross measure. If S, B are borel sets, we define:

$$\delta(S)=\sup\{L_{N-1}(\pi_a(S)):a\in S^{N-1}\}$$

$$G_{N-1}(B)=\sup_{\varepsilon>0}\inf\left\{\sum_{i=1}^{\infty}\delta(S_i):B\subset\bigcup_{i=1}^{\infty}S_i,\ S_i\ \textit{are borel sets},\ \operatorname{diam}S_i<\varepsilon\right\}$$

where L_{N-1} is the Lebesgue measure on the plane $\Pi_a = \{ \langle x, a \rangle = 0 \}$ and π_a denotes the orthogonal projection on Π_a. Furthermore, define $B_* = \{ x \in \Omega : B \textit{ has density } 1 \textit{ at } x \}$ and $B^* = \{ x \in \Omega : \Omega \setminus B \textit{ has density zero at } x \}$.

LEMMA A.4. – *Define* A, B, $\{A_h\}$, $\{B_h\}$ *as above. Then we have:*

$$\lim_{h \to +\infty} H_{N-1}(\partial^* A \cap B_{h*}) = 0.$$

Proof. – This proof has been communicated to me by L. Ambrosio. The set $\partial^* A \cap B_{h*}$ is rectifiable and then, by Theorem 3.2.26 of Federer [F], we have:

$$H_{N-1}(\partial^* A \cap B_{h*}) = G_{N-1}(\partial^* A \cap B_{h*}).$$

Define:

$$S = \{ x \in B_* : x \notin B_{h*} \textit{freq.} \} \cup \{ x \in B^* : x \notin B_h^* \textit{ freq.} \} \cup \{ x \in \partial^* B : x \notin \partial^* B_h \textit{freq.} \},$$

where freq. means for infinitely many h. Assume that $\delta(S) = 0$. Then $G_{N-1}(S) = 0$ and

$$G_{N-1}(\partial^* A \cap B_{h*}) = \int_\Omega \mathbf{1}_{\partial^* A \cap B_{h*}}(x)\, dG_{N-1}(x) \leq \int_{B^* \cup \partial^* B} \mathbf{1}_{B_{h*}}(x)\, dG_{N-1}(x) \to 0:$$

the inequality comes from the fact that $A \cap B = \varnothing$ and $\partial^* A \subset B^* \cup \partial^* B$ up to a set of null Gross-measure.

We now prove that $\delta(S) = 0$. This is obvious if $N = 1$ because the sets with bounded perimeter in **R** are finite unions of pairwise disjoint intervals, and then the strong convergence in $BV(\Omega)$ entails that the endpoints of the intervals definitively coincide. If $N > 1$ we choose $a \in S^{N-1}$ and we write $\mathbf{R}^N = \Pi_a \times \mathbf{R}$.

By the Vol'pert's decomposition of sets with bounded perimeter (*see* [Vo], th. 1.6) we obtain that, for L_{N-1} almost every $x \in \Pi_a$, the set

$$S_h(x) = \{ t \in \mathbf{R} : (x, t) \in B_* \Delta B_{h*} \textit{ or } (x, t) \in B^* \Delta B_h^* \textit{ or } (x, t) \in \partial^* B \Delta \partial^* B_h \}$$

is definitively empty. Thus:

$$L_{N-1}(\pi_a(S)) = L_{N-1}(\{ x \in \Pi_a(B) : \exists\, t \in \mathbf{R} : t \in S_h(x) \textit{ definitively} \}) \leq L_{N-1}(\{ x \in \Pi_a \textit{ such that } S_h(x) \neq \varnothing \textit{ freq.} \}) = 0$$

As this relation holds for every $a \in S^{N-1}$, we conclude that $\delta(S) = 0$. ■

The following result is well known (*see*, for example, Giusti [Gi]).

PROPOSITION A.5. – *Let* A, B *sets with bounded perimeter. Then we have:*

$$H_{N-1}(\{x \in \partial^* A \cap \partial^* B : \nu_A(x) \neq \pm \nu_B(x)\}) = 0.$$

Proof of Proposition A.3. – (i) We first prove that $\mathbf{1}_{A_h \cup B_h} \to \mathbf{1}_{A \cup B}$ in $BV(\Omega)$. It suffices to prove that $P_\Omega(A_h \cap B_h) \to 0$. Let G be a borel set; by using Lemma 2 of [QG] we obtain the following formula (*see* also [Vo]):

$$D\mathbf{1}_{E \cap F}(C) = D\mathbf{1}_F(C \cap E_*) + D\mathbf{1}_E(C \cap F_*) + 1/2 \int_{\partial^* A_k \cap \partial^* B_h \cap C} |\nu_{A_h}(x) + \nu_{B_h}(x)|\, dH_{N-1}(x).$$

Then:

$$\text{(A.1)} \quad P_\Omega(A_h \cap B_h) \leqq H_{N-1}(\partial^* A_h \cap B_{h*}) + H_{N-1}(\partial^* B_h \cap A_{h*}) + 1/2 \int_{\partial^* A_h \cap \partial^* B_h} |\nu_{A_h}(x) + \nu_{B_h}(x)|\, dH_{N-1}(x).$$

By using again Lemma 2 of [QG], we obtain:

$$\begin{aligned} H_{N-1}(\partial^* A_h \Delta \partial^* A) &= H_{N-1}(\partial^* A \cap (\Omega \setminus \partial^* A_h)) \\ &\quad + H_{N-1}(\partial^* A_h \cap (\Omega \setminus \partial^* A)) \\ &\leqq |D\mathbf{1}_A|(\Omega \setminus \partial^* A_h) + |D\mathbf{1}_A|(\Omega \setminus \partial^* A) \\ &= |D\mathbf{1}_{A_h \Delta A}|(\Omega) \leqq \int_\Omega |D(\mathbf{1}_{A_h} - \mathbf{1}_A)|; \end{aligned}$$

hence

$$\lim_{h \to +\infty} H_{N-1}(\partial^* A_h \Delta \partial^* A) = 0.$$

Observing that

$$\partial^* A_h \cap B_{h*} = ((\partial^* A_h \setminus \partial^* A) \cap B_{h*}) \cup ((\partial^* A_h \cap \partial^* A) \cap B_{h*})$$

we infer by Lemma A.4 that

$$\lim_{h \to +\infty} H_{N-1}(\partial^* A_h \cap B_{h*}) = 0$$

and analogously:

$$\lim_{h \to +\infty} H_{N-1}(\partial^* B_h \cap A_{h*}) = 0.$$

Finally, we have by Proposition A.5

$$\lim_{h \to +\infty} \int_{\partial^* A_h \cap \partial^* B_h \cap \partial^* A \cap \partial^* B} |\nu_{A_h}(x) - \nu_{B_h}(x)|\, dH_{N-1}(x) = 0$$

and also

$$\lim_{h \to +\infty} H_{N-1}((\partial^* A_h \cap \partial^* B_h) \Delta (\partial^* A \cap \partial^* B)) = 0$$

because of

$$(\partial^* A_h \cap \partial^* B_h) \setminus (\partial^* A \cap \partial^* B) = ((\partial^* A_h \setminus \partial^* A) \cap \partial^* B_h) \cup ((\partial^* B_h \setminus \partial^* B) \cap \partial^* A_h)$$
$$(\partial^* A \cap \partial^* B) \setminus (\partial^* A_h \cap \partial^* B_h) = ((\partial^* A \setminus \partial^* B_h) \cap \partial^* B) \cup ((\partial^* B \setminus \partial^* B_h) \cap \partial^* A)$$

(ii) It follows from

$$\mathbf{1}_{B_k \ A_h} = \mathbf{1}_{B_h} - \mathbf{1}_{B_h \cap A_h}. \quad \blacksquare$$

We have now to construct an approximation of a C^1 partition (C^1-regular up to a set with null H_{N-1} measure) by polygonal domains. We have a partition of Ω, say $S_1, \ldots, S_k$, made by sets with boundaries contained in a finite union of C^1-hypersurfaces, and we look for a sequence $\{S_1^h, \ldots, S_k^h\}_{h \in N}$ of partitions such that the S_i^h's are polygonal domains and such that the property (iv) of Lemma 3.1 holds. By the Fleming-Rishel formula, this will be proved if we verify the following property:

$$\forall J \subset \{1, \ldots, k\}, \lim_{h \to +\infty} P_\Omega(\bigcup_{i \in J} S_i^h) = P_\Omega(\bigcup_{i \in J} S_i). \qquad (A.2)$$

The first step consists in eliminating the singularities of the boundaries of the S_i's by including them in a family of cubes with little volume and perimeter, and by adding all these cubes to the set S_1.

Let B be the singular set of the given partition. It is possible to construct, for every $h \in \mathbf{N}$, a finite family of cubes $\{Q_1^h, \ldots, Q_{\nu(h)}^h\}$ such that:

(i) $B \subset \bigcup_{i=1}^{\nu(h)} Q_i^h$.

(ii) $P_\Omega\left(\bigcup_{i=1}^{\nu(h)} Q_i^h\right) < 2N/h, \left|\bigcup_{i=1}^{\nu(h)} Q_i^h\right| < 1/h$.

(iii) ∂Q_i *is transverse to* ∂S_j, $\forall i = 1, \ldots, \nu(h); j = 1, \ldots, k$.

For the sequence of partitions given by the sets $S_1 \cup \bigcup Q_j^h$; $S_2 \setminus \bigcup Q_j^h; \ldots; S_k \setminus \bigcup Q_j^h$, (A.2) holds. The boundaries of these sets are piecewise-C^1 hypersurfaces, whose singular sets are transverse intersections of regular surfaces with cubes. We now approximate the portions of the interfaces not belonging to the cubes by polygons, in such a way that (A.2) holds. Let Σ be one of these hypersurfaces, closed in the open set $\Omega' = \Omega \setminus \bigcup_i \bar{Q}_i$. Σ is an oriented hypersurface transverse to each one of the Q_i's. Putting

$$\Omega_\rho = ((\Sigma + B(0, \rho)) \setminus \bigcup_i Q_i) \cap \Omega$$

we have that, if ρ is small enough, Σ gives a partition of Ω_ρ in two parts, each of them belonging to one of the sets S_j. Let E be one of these portions: E is an open set with C^1 boundary in Ω_ρ.

LEMMA A.7. — *There exists a sequence* E_h, *with* ∂E_h *a piecewise-linear hypersurface, such that:*

(i) $\lim_{h \to +\infty} H_{N-1}(\partial E_h \cap \Omega_{\rho/2}) = H_{N-1}(\Sigma)$.

(ii) $\int_{\Omega_{\rho/2}} |\mathbf{1}_{E_h} - \mathbf{1}_E| \, dx \to 0$.

(iii) *The set* $\partial\Omega_{\rho/2} \setminus \partial\Omega'$ *has a neighborhood in* $\Omega_{\rho/2}$ *on which* $\mathbf{1}_E \equiv \mathbf{1}_{E_h}$, $\forall h \in \mathbf{N}$.

(iv) $H_{N-1}(\partial E_h \cap \partial\Omega') = 0$, $\forall h \in \mathbf{N}$.

Proof. — Let us call $\tilde{f}$ an extension of $\mathbf{1}_{E\,|\,\Omega_{\rho/2}}$ to an open neighborhood $\tilde{\Omega} \supset \bar{\Omega}_{\rho/2}$, such that $\int_{\partial\Omega_{\rho/2}} |D\tilde{f}| = 0$. This extension is possible because $\partial\Omega_{\rho/2} \cap \partial\Omega'$ is lipschitz continuous. Let $\tilde{f}_\varepsilon$ be a sequence converging to $\tilde{f}$ in $L^1(\tilde{\Omega})$ such that $\lim_{\varepsilon \to 0^+} \int_{\Omega_{\rho/2}} |D\tilde{f}_\varepsilon| = \int_{\Omega_{\rho/2}} |D\,\mathbf{1}_E| = H_{N-1}(\Sigma)$. It is not restrictive to assume that $f_\varepsilon \equiv \mathbf{1}_E$ on a neighborhood of $\partial\Omega_{\rho/2} \setminus \partial\Omega'$ in $\Omega_{\rho/2}$ and that $0 \leqq \tilde{f}_\varepsilon \leqq 1$ for every $x \in \tilde{\Omega}$. Let $f_h = \tilde{f}_{1/h}$.

There exists a sequence of piecewise linear function $g_h : \tilde{\Omega} \to \mathbf{R}$ such that:

$$\int_{\Omega_{\rho/2}} |g_h(x) - f_h(x)| < 1/h \tag{A.3}$$

$$\left| \int_{\Omega_{\rho/2}} |D g_h(x)| \, dx - \int_{\Omega_{\rho/2}} |D f_h(x)| \, dx \right| < 1/h$$

and such that $g_h \equiv \mathbf{1}_E$ on a neighborhood of $\partial\Omega_{\rho/2} \setminus \partial\Omega'$.

It follows that:

$$\left| \int_{\Omega_{\rho/2}} |D g_h| - H_{N-1}(\Sigma) \right| < 2/h. \tag{A.4}$$

Let $t \in [0, 1]$, and put: $E_{th} = \{ x \in \tilde{\Omega} : g_h(x) > t \}$. We remark that E_{th} is a polygonal domain for almost all $t \in [0, 1]$. We have:

$$g_h(x) - \mathbf{1}_E(x) > t \quad \textit{on } E_{th} \setminus E$$
$$\mathbf{1}_E - g_h \geqq 1 - t \quad \text{of } E \setminus E_{th}$$

Thus:

$$\int_{\Omega_{\rho/2}} |g_h - \mathbf{1}_E| \geqq \int_{E_{th}\setminus E} |g_h - \mathbf{1}_E| + \int_{E\setminus E_{th}} |g_h - \mathbf{1}_E|$$

$$\geqq t\,|E_{th}\setminus E| + (1-t)\,|E\setminus E_{th}| \geqq \min(t, 1-t) \int_{\Omega_{\rho/2}} |\mathbf{1}_{E_{th}} - \mathbf{1}_E|$$

and so $\mathbf{1}_{E_{th}} \to \mathbf{1}_E$ in $L^1(\Omega_{\rho/2})$. By the coarea formula:

$$\int_{\Omega_{\rho/2}} |Dg_h| = \int_0^1 P_{\Omega_{\rho/2}}(E_{th})\,dt. \tag{A.5}$$

By the lower semicontinuity of the total variation:

$$\liminf_{h\to+\infty} \int_{\Omega_{\rho/2}} |D\,\mathbf{1}_{E_{th}}| \geqq \int_{\Omega_{\rho/2}} |D\,\mathbf{1}_E|, \qquad \forall\, t\in[0,1] \tag{A.6}$$

By the Fatou's Lemma and (A.6) we have:

$$\int_{\Omega_{\rho/2}} |D\,\mathbf{1}_E| = \lim_{h\to+\infty} \int_{\Omega_{\rho/2}} |D g_h| \geqq \int_0^1 \liminf_{h\to+\infty} P_{\Omega_{\rho/2}}(E_{th})\,dt \geqq \int_{\Omega_{\rho/2}} |D\,\mathbf{1}_E|$$

Then, by using (A.5), we have that for almost all $t\in[0,1]$ the following formula holds:

$$\liminf_{h\to+\infty} \int_{\Omega_{\rho/2}} |D\,\mathbf{1}_{E_{th}}| = \int_{\Omega_{\rho/2}} |D\,\mathbf{1}_E| = H_{N-1}(\Sigma) \tag{A.7}$$

By using that $H_{N-1}(\partial\Omega') < +\infty$, we have that, for almost all $t\in[0,1]$, $H_{N-1}(\partial E_{ht} \cap \partial\Omega') = 0$. Thus, by choosing the appropriate t and passing to a subsequence in (A.7), we conclude the proof. ■

It remains to fulfil the condition (iii) of Lemma 3.1. By repeating the argument used by Modica [M1], and by remarking that the property of having non-empty interior is preserved by our construction of approximating partitions, we have only to indicate how to modify the original partition in such a way that each non-empty set S_i of the partition has non-empty interior. Take a point $x\in\Omega\cap\partial^* S_1 \cap \partial^* S_2$: the density of both S_1 and S_2 at x is exactly 1/2, and then, if $\rho_h \downarrow 0$, we have that $0 < |B_{\rho_h}(x) \cap S_2| < |B_{\rho h}(x)|$ for every $h\in N$. Then, for each $h\in N$, there exists $\tilde\rho_h$ such that $|B_{\tilde\rho h}(x)| = |B_{\rho h}(x)\setminus S_1|$. We can now divide the ball $B_{\tilde\rho h}(x)$ into $(k-1)$ sectors $\tilde B_{hl}$ with area $|B_{\tilde\rho h} \cap S_l|$, $l\neq 1$. Since obviously $H_{N-1}(\partial\tilde B_{hl}) \leqq K\,\tilde\rho_h$, where the constant K depends only on N, it follows that the perimeter of the (nonempty) sets $\tilde B_{hl}$ tends to zero. For the following sequence of partitions:

$$S_1^h = S_1 \cup (B_{\rho h}(x) \setminus B_{\tilde\rho h}(x))$$

$$S_l^h = (S_l \setminus B_{\rho h}(x)) \cup \tilde B_{hl}, \qquad l\neq 1$$

one has $|S_m^h| = |S_m|$ for each $m = 1, \ldots, k$, and for each $J \subset \{1, \ldots, k\}$,

$$\lim_{h \to +\infty} P_\Omega\Big(\bigcup_{l \in J} S_l^h\Big) = P_\Omega\Big(\bigcup_{l \in J} S_l\Big).$$

Moreover the sets S_1^h and S_2^h have nonempty interior. Repeating the same construction for the other elements of the partition, the proof is complete. ■

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