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On a classical problem of the calculus of variations without convexity assumptions

by

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ABSTRACT. — We show that the functional

$$I(x)=\int_0^T g(t, x(t))\,dt+\int_0^T h(t, x'(t))\,dt$$

attains a minimum under the condition that g be concave in x.

Key words : Calculus of variations, normal integrals, convex functionals.

RÉSUMÉ. — Nous montrons que le fonctionnel

$$I(x)=\int_0^T g(t, x(t))\,dt+\int_0^T h(t, x'(t))\,dt$$

atteint le minimum sous la condition de concavité sur g.

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INTRODUCTION

We consider the problem of the existence of the minimum for the integral functional I (x):

$$I(x)=\int_0^T g(t, x(t))\,dt+\int_0^T h(t, x'(t))\,dt$$

on the set of functions x belonging to $W^{1,p}([0, T], \mathbb{R}^n)$, $p \geqq 1$ and satisfying: $x(0)=a$, $x(T)=b$; $x'(t) \in \Phi(t)$ a.e. on [0, T]. The set-valued map $\Phi: [0, T] \to 2^{\mathbb{R}^n}$ is measurable with non-empty, closed (not necessarily bounded nor convex) values, and each of the functions g and h satisfies Carathéodory conditions. Our purpose is to show that, for the existence of the minimum, Tonelli's assumption of convexity of h with respect to x' can be replaced by the condition of concavity of g with respect to x, all other requirements (e.g. growth conditions) being the same. In particular, we do not impose any regularity on g, h and h^{**}. Notice that the subset of $W^{1,p}$ on which the minimum is seeked is not weakly closed, due to the lack of convexity of the values of Φ.

The problem of avoiding convexity has been considered by: Aubert-Tahraoui [A-T1] and Marcellini [M1] with $g \equiv 0$ and $\Phi \equiv \mathbb{R}^1$; with g linear and on a control theory setting, by Olech [O] and Cesari [Ce1]; under different conditions on g and h and with $\Phi \equiv \mathbb{R}^1$, by Aubert-Tahraoui [A-T1] and Marcellini [M1] (*see* also the references in [M2] and in [Ce2]). In addition, necessary and sufficient conditions for the existence of minima were given by Ekeland [E] and Raymond [R], under regularity assumptions for the integrands. Our theorem neither contains nor is contained in either Theorem 2 of [M1] or in the results of [A-T1], which concern the case $n=1$, while it generalizes Theorem 16.7.i of Cesari [Ce2]. Our main tool is Liapunov's theorem on the range of vector measures as presented in the book of Cesari (§ 16).

ASSUMPTIONS AND PRELIMINARY RESULTS

We shall assume the following hypothesis.

HYPOTHESIS (H). – The set-valued map $\Phi: [0, T] \to 2^{\mathbb{R}^n}$ is measurable [C-V] with non-empty closed values. In addition we assume that there exists at least one $v \in L^p([0, T], \mathbb{R}^n)$ such that $v(t) \in \Phi(t)$ a.e. and $\int_0^T v(t)\,dt = b-a$.

The map $g: [0, T] \times \mathbb{R}^n \to \mathbb{R}$ is such that

(g_1) $t \to g(t, x)$ is measurable for each x;

(g_2) $x \to g(t, x)$ is continuous for a.e. t;
(g_3) $x \to g(t, x)$ is concave for a.e. t.
Moreover there exist a constant γ_1 and a function $\gamma_2 \in L^1$, such that
(g_4) $g(t, x) \geqq -\gamma_1 |x|^p - \gamma_2(t)$.
The map $h: [0, T] \times \mathbb{R}^n \to \mathbb{R}$ is such that
(h_1) $t \to h(t, x')$ is measurable for each x';
(h_2) $x' \to h(t, x')$ is continuous for a.e. t.
Moreover:
(h_3) if $p=1$, there exist: a convex lower semicontinuous monotonic function $\psi: \mathbb{R}^+ \to \mathbb{R}$ and a function $\xi_1(.)$ in L^1 such that

$$h(t, x') \geqq \psi(|x'|) - \xi_1(t)$$

and

$$\lim_{r \to \infty} \frac{\psi(r)}{r} = +\infty.$$

If $p > 1$, there exist: a positive constant ξ_2 and a function $\xi_3(.)$ in L^1 such that $h(t, x') \geqq \xi_2 |x'|^p - \xi_3(t)$ and γ_1/ξ_2 is strictly smaller than the best Sobolev constant in $W_0^{1,p}([0, T])$. ■

We list some notations and preliminary results. The closed unit ball of $\mathbb{R}^n$ is $\bar{B} = \{x \in \mathbb{R}^n : |x| \leqq 1\}$. The characteristic function of a set E is $\chi_E(.)$. Let (X, d) be a metric space and $F: X \to 2^{\mathbb{R}^n}$ be a map from X into the nonempty compact subsets of $\mathbb{R}^n$: F is called upper semicontinuous on X if, for every $x \in X$ and for every $\varepsilon > 0$, there exists $\delta = \delta(x, \varepsilon)$ such that $d(x, y) < \delta \Rightarrow F(y) \subseteqq B(F(x), \varepsilon)$. A set-valued map F whose graph is closed and whose values are all contained in a compact set is upper semicontinuous. We also set $\|F(x)\| = \max\{|y| : y \in F(x)\}$.

Let $f^{**}(t, x)$ be the bipolar of the function $x \to f(t, x)$. We have the following

PROPOSITION 1 ([E-T] Prop. I.4.1; Lemma IX.3.3; Prop. IX.3.1). — (*a*) *$f^{**}(t, x)$ is the largest convex (in x) function not larger than $f(t, x)$.*

(*b*) *Under the growth assumption* (h_3) *on* f

$$f^{**}(t, x) = \min\left\{\sum_{i=1}^{n+1} \lambda_i f(t, \xi_i) : x = \sum_{i=1}^{n+1} \lambda_i \xi_i;\ \lambda_i \geqq 0;\ \sum_{i=1}^{n+1} \lambda_i = 1\right\}.$$

(*c*) *Let $x'(.)$ be measurable. Then there exist measurable $p_i: I \to [0, 1]$ and measurable $v_i: I \to \mathbb{R}^n$, $i = 1, \ldots, n+1$, such that:*

$$\sum_i p_i(t) = 1; \qquad x'(t) = \sum_i p_i(t) v_i(t); \qquad f^{**}(t, x'(t)) = \sum_i p_i(t) f(t, v_i(t)).$$

The following properties of the subdifferential of a convex function ([E-T], § I.5.1) will be used later.

LEMMA 1. — *Let* $f: [0, T] \times \mathbb{R}^n \to \mathbb{R}$ *satisfy*

(i) $f(t, x) \leqq k|x|^p + b(t)$ $(k > 0, b \in L^1)$;

(ii) $t \to f(t, x)$ *is measurable for every* x;

(iii) $x \to f(t, x)$ *is convex and continuous for almost every* t.

Then, for any continuous $x: [0, T] \to \mathbb{R}^n$, *the set valued map*

$$t \to \partial_x f(t, x(t))$$

admits a selection $\delta(.) \in L^1$.

Proof. — (*a*) We claim that the map $t \to \partial_x f(t, x(t))$ is measurable. In fact, fix $\Delta > 0$; then $|f(t, x)| \leqq k\Delta^p + b(t)$ in $[0, T] \times \Delta\bar{B}$. By the Corollary to Proposition 2.2.6 in [C] we have that

$$\|\partial_x f(t, x)\| \leqq \frac{2}{\Delta}(k(2\Delta)^p + b(t)) \quad \text{for a. e. } t \in [0, T], \quad \text{for all } x \in \Delta\bar{B}. \quad (1)$$

Fix $\varepsilon > 0$ and let, by Scorza Dragoni's theorem, $E_\varepsilon \subseteqq [0, T]$ be closed and such that: $m([0, T] \setminus E_\varepsilon) \leqq \varepsilon$; the restriction of f to $E_\varepsilon \times \Delta\bar{B}$ is continuous as well as the restriction of b to E_ε. We prove first that the map $(t, x) \to \partial_x f(t, x)$ is upper semicontinuous on $E_\varepsilon \times \Delta\bar{B}$. Let us show that it has closed graph. Let (t_n, x_n) be in $E_\varepsilon \times \Delta\bar{B}$, $(t_n, x_n) \to (t, x)$ and let v_n be in $\partial_x f(t_n, x_n)$, $v_n \to v$. From

$$f(t_n, x_n) - f(t_n, y) \geqq \langle v_n, x_n - y \rangle, \qquad y \in \mathbb{R}^n,$$

and the continuity of f, we have

$$\langle f(t, x) - f(t, y) \rangle \geqq \langle v, x - y \rangle, \qquad y \in \mathbb{R}^n,$$

so that $v \in \partial_x f(t, x)$. By (1) and the boundedness of b on E_ε, the upper semicontinuity follows.

Let Δ be such that $|x(t)| \leqq \Delta$, $t \in I$: then the map $t \to \partial_x(t, x(t))$ is upper semicontinuous on E_ε. An application of Lusin's theorem for multivalued maps yields our claim.

(*b*) By the theorem of Kuratowski-Ryll Nardzewski (*see* Theorem III.6 in [C-V]) there exists a measurable selection $\delta(t) \in \partial_x(t, x(t))$. We have

$$\delta(t) \leqq \|\partial_x f(t, x(t))\| \leqq \frac{2}{\Delta}|k(2\Delta)^p + b(t)|,$$

so that $\delta \in L^1$. ■

MAIN RESULT

THEOREM 1. — *Let* Φ*;* f*;* g *satisfy hypothesis* (H). *Then the problem*

$$\text{(M)} \qquad \text{Minimize} \int_0^T g(t, x(t))\, dt + \int_0^T h(t, x'(t))\, dt$$

on the subset of $W^{1,p}$ *of those* $x(.)$ *satisfying:* $x(0)=a$, $x(T)=b$; $x'(t)\in\Phi(t)$ *a. e. in* $[0, T]$, *admits at least one solution.*

Proof. — The argument of the proof goes by showing first that the relaxed problem has a solution $\tilde{x}$; then by constructing from $\tilde{x}$ a different function, a solution to the original problem.

(*a*) Let us consider the function h_Φ defined as

$$h_\Phi(t, x)=\begin{cases} +\infty & \text{for } x\notin\Phi(t) \\ h(t, x) & \text{for } x\in\Phi(t). \end{cases}$$

Then Problem (M) is equivalent to minimizing the functional I, with h_Φ replacing h, on the functions of $W^{1,p}$ satisfying the boundary conditions.

Set h^c to be h_Φ^{**} and consider the problem

$$\text{(MR)} \qquad \text{Minimize} \int_0^T g(t, x(t))\,dt+\int_0^T h^c(t, x'(t))\,dt$$

for x in $W^{1,p}$, $x(0)=a$, $x(T)=b$. By Proposition 1 and the convexity (with respect to x') of the functions appearing in (h_3), h^c satisfies the same growth condition (h_3). Then it is known that problem (MR) has a solution $\tilde{x}$. On it, $h^c(t, \tilde{x}'(t))<+\infty$ a. e.; by (*b*) of Proposition 1, $\tilde{x}'(t)$ belongs to $\operatorname{co}\Phi(t)$ a. e. and, by (*c*), there exist measurable functions p_i and v_i such that

$$\begin{gathered} \sum_{1}^{n+1}{}_i\, p_i(t)\, v_i(t)=\tilde{x}'(t) \\ \sum_{1}^{n+1}{}_i\, p_i(t)\, h_\Phi(t, v_i(t))=h^c(t, \tilde{x}'(t)). \end{gathered} \tag{2}$$

Let us remark that any $v_i(t)$ can be in the complement of $\Phi(t)$ on a set E of positive measure only if $p_i\equiv 0$ on it. In this case, we can modify v_i on E by an arbitrary integrable selection from Φ without affecting (2). Hence we can as well assume that $v_i(t)\in\Phi(t)$ a. e., so that $h_\Phi(t, v_i(t))=h(t, v_i(t))$ a. e.

(*b*) We consider the integrability of a function that will be used in the remainder of the proof. By Lusin's theorem there exists a sequence $(K_j)_j$ of disjoint compact subsets of I, and a null set N, such that $I=N\cup(\bigcup_j K_j)$ and the restriction of each of the maps $t\to h(t, v_i(t))$ to each K_j is continuous. Set $S_m=\bigcup_{j\leq m} K_j$. We claim: Let $(E_j^i)_i$, $i=1, \ldots, n+1$, be a measurable partition of K_j with the property that, for every j,

$$\int_{K_j}\Big(\sum_i p_i(t)\, h(t, v_i(t))\Big)\,dt=\int_{K_j}\Big(\sum_i \chi_{E_j^i}(t)\, h(t, v_i(t))\Big)\,dt. \tag{3}$$

Then the map

$$t \to \sum_{j}^{\infty} \sum_{i}^{n+1} \chi_{E_j^i}(t)\, h(t, v_i(t)) \tag{4}$$

belongs to L^1. As a consequence, since, for $p > 1$,

$$\left|\sum_{i,j} \chi_{E_j^i}(t)\, v_i(t)\right|^p = \sum_{i,j} \chi_{E_j^i}(t)\, |v_i(t)|^p \leqq \frac{1}{\xi_2} \sum_{i,j} \chi_{E_j^i}(t)\,(h(t, v_i(t)) + \xi_3(t)),$$

the function $\sum_{ij} \chi_{E_j^i} v_i$ belongs to L^p. Analogously for the case $p=1$. To prove the claim, remark that on one hand the map $t \to \sum_i p_i(t)\, h(t, v_i(t))$ is integrable since it equals $t \to h^c(t, \tilde{x}'(t))$. On the other hand the sequence of maps

$$s_m(t) = \sum_{j \leqq m} \left(\sum_i \chi_{E_j^i}(t)\, v_i(t)\,(h(t, v_i(t)) + \xi_3(t))\right)$$

is monotone non decreasing and

$$\int_0^T s_m(t)\, dt = \sum_{j \leqq m} \int_{K_j} \sum_i \chi_{E_j^i}(t)\,(h(t, v_i(t)) + \xi_3(t))\, dt.$$

By (3) the right hand side equals

$$\sum_{j \leqq m} \int_{K_j} \sum_i p_i(t)\,(h(t, v_i(t)) + \xi_3(t))\, dt$$
$$= \int_0^T \chi_{S_m}(t)\,(h^c(t, \tilde{x}'(t)) + \xi_3(t))\, dt$$
$$\leqq \int_0^T (h^c(t, \tilde{x}'(t)) + \xi_3(t))\, dt < +\infty.$$

Hence

$$\int_0^T \left(\sum_{i,j} \chi_{E_j^i}(t)\,(h(t, v_i(t)) + \xi_3(t)\right) dt$$
$$= \int_0^T (\lim s_m(t))\, dt = \lim \int_0^T s_m(t)\, dt = \int_0^T (h^c(t, \tilde{x}'(t)) + \xi_3(t))\, dt. \tag{5}$$

(*c*) Set $\partial^x g(t, x)$ to be $-\partial_x(-g(t, x))$ and consider the map $t \to \partial^x g(t, \tilde{x}(t))$.

Lemma 1 shows that there exists an integrable function $\delta(.)$, a selection from $\partial^x g(t, \tilde{x}(t))$. Set $B(t)$ to be

$$B(t) = \int_0^t \delta(s)\, ds.$$

Consider the vector measures ν_i and the scalar measures η_i, β_i defined, for $i=1, \ldots, n+1$, by

$$\nu_i(E)=\int_E v_i(t)\,dt$$

$$\eta_i(E)=\int_E h(t, v_i(t))\,dt$$

$$\beta_i(E)=\int_E \langle v_i(t), B(T)-B(t)\rangle\,dt.$$

Each of them is a vector valued non-atomic measure on [0, T], hence on every K_j. By an extension of Liapunov's theorem on the range of vector measures ([Ce] 16.1.v) there exists a measurable partition of each K_j, $(E_j^i)_j$, $i=1, \ldots, n+1$, such that

$$\sum_i \int_{K_j} \chi_{E_j^i}(t)\,d\nu_i(t)=\sum_i \int_{K_j} p_i(t)\,d\nu_i(t); \qquad (6)$$

$$\sum_i \int_{K_j} \chi_{E_j^i}(t)\,d\eta_i(t)=\sum_i \int_{K_j} p_i(t)\,d\eta_i(t); \qquad (7)$$

$$\sum_i \int_{K_j} \chi_{E_j^i}(t)\,d\beta_i(t)=\sum_i \int_{K_j} p_i(t)\,d\beta_i(t). \qquad (8)$$

(*d*) We claim that the function $x: I \to \mathbb{R}^n$ defined as

$$x'(t)=\sum_j{}_{i}^{\infty} \sum_i{}_{1}^{n+1} \chi_{E_j^i}(t)\,v_i(t),\; x(0)=\tilde{x}(0)$$

is a solution to problem (M).

First, let us remark that almost every t in [0, T] belongs to exactly one of the E_j^i, so that, for almost every t, $x'(t)$ equals one of the $v_i(t)$ and hence belongs to $\Phi(t)$. Moreover

$$h(t, x'(t))=h(t, \sum_{i,j} \chi_{E_j^i}(t)\,v_i(t))=\sum_{i,j} \chi_{E_j^i}(t)\,h(t, v_i(t)).$$

Hence, by point (*b*), $h(t, x'(t))$ is integrable whenever (3) holds, and this follows from the definition of the measures η_i and equality (7). Again by point (*b*), we have that $x'(.)\in L^p$, hence x is in $W^{1,p}$. Moreover,

$$x(T)=\tilde{x}(0)+\int_0^T \sum_{j,i} \chi_{E_j^i}(t)\,v_i(t)\,dt=\tilde{x}(0)+\sum_j \int_{K_j} \sum_i \chi_{E_j^i}(t)\,v_i(t)\,dt$$

and, from (6), the last integral equals

$$\sum_j \int_{K_j} \sum_i p_i(t)\,v_i(t)\,dt=\int_0^T \sum_i p_i(t)\,v_i(t)\,dt,$$

so that $x(\mathrm{T})=\tilde{x}(\mathrm{T})$.

We claim now that

$$\int_0^{\mathrm{T}} h^c(t, \tilde{x}'(t))\,dt=\int_0^{\mathrm{T}} h(t, x'(t))\,dt \tag{9}$$

and

$$\int_0^{\mathrm{T}} g(t, \tilde{x}(t))\,dt=\int_0^{\mathrm{T}} g(t, x(t))\,dt. \tag{10}$$

Ad (9). Again from the definition of the η_i, (5) and (7),

$$\begin{aligned}\int_0^{\mathrm{T}} h^c(t, \tilde{x}'(t))\,dt &=\sum_j \int_{\mathrm{K}_j} \sum_i p_i(t)\, h(t, v_i(t))\,dt \\ &=\sum_j \sum_i \int_{\mathrm{K}_j} \chi_{\mathrm{E}_j^i}(t)\, h(t, v_i(t))\,dt \\ &=\sum_j \int_{\mathrm{K}_j} h(t, \sum_i \chi_{\mathrm{E}_j^i}(t)\, v_i(t))\,dt=\int_0^{\mathrm{T}} h(t, x'(t))\,dt.\end{aligned}$$

Ad (10). By the definition of $\delta(.)$ in (*c*), for every t and y, we have (*see* [E-T] § I.5.1)

$$g(t, y) \leqq g(t, \tilde{x}(t))+\langle \delta(t), y-\tilde{x}(t) \rangle. \tag{11}$$

We claim that

$$\int_0^{\mathrm{T}} \langle \delta(t), x(t)-\tilde{x}(t) \rangle\,dt=0. \tag{12}$$

Recalling the definition of B and denoting by u_l the l-th component of a vector u, the above integral can be written as

$$\begin{aligned}&\int_0^{\mathrm{T}} \sum_{l=1}^{n} \delta_l(t)(x_l(t)-\tilde{x}_l(t))\,dt \\ &=\sum_l \int_0^{\mathrm{T}} \delta_l(t) \int_0^t (x_l'(s)-\tilde{x}_l'(s))\,ds\,dt \\ &=\sum_l \int_0^{\mathrm{T}} (x_l'(s)-\tilde{x}_l'(s))(\mathrm{B}_l(\mathrm{T})-\mathrm{B}_l(s))\,ds \\ &=\int_0^{\mathrm{T}} \langle \sum_{j,i} \chi_{\mathrm{E}_j^i}(s)\, v_i(s)-\sum_i p_i(s)\, v_i(s), \mathrm{B}(\mathrm{T})-\mathrm{B}(s) \rangle\,ds \\ &=\sum_j \int_{\mathrm{K}_j} \sum_i (\chi_{\mathrm{E}_j^i}(s)-p_i(s)) \langle v_i(s), \mathrm{B}(\mathrm{T})-\mathrm{B}(s) \rangle\,ds \\ &=0, \quad \text{by (8).}\end{aligned}$$

By (11) this proves that

$$\int_0^T g(t, x(t))\, dt \leqq \int_0^T g(t, \tilde{x}(t))\, dt. \tag{12}$$

Since $\tilde{x}$ is a solution of the problem (MR), by (*a*) in Proposition 1,

$$\int_0^T g(t, x(t))\, dt + \int_0^T h(t, x'(t))\, dt \geqq \int_0^T g(t, \tilde{x}(t))\, dt + \int_0^T h^c(t, \tilde{x}'(t))\, dt.$$

On the other hand, by (9) and (12), (10) follows. This proves that x is a solution to the problem (M). ■

Remark. — In case $g(t, .)$ is strictly concave for almost every t, i.e. if there exists a selection δ from $\partial^x g(t, \tilde{x}(t))$ such that the inequality sign in (11) is strict for $y \neq \tilde{x}(t)$, the functions $x(t)$ and $\tilde{x}(t)$ have to coincide, otherwise the integral functional would assume on x a value strictly less than its infimum. Therefore, in this case, every x which is a minimizer for the relaxed problem (MR) is also a minimizer for the original problem (M).

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