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Symmetry breaking for a class of semilinear elliptic problems

by

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ABSTRACT. — In this article we consider the problem

$$\begin{aligned} -\Delta u &= u^p + \lambda u \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \qquad (P_\lambda) \\ u &> 0 \end{aligned}$$

where Ω is an annulus and $p > 1$. We prove there is always a break of symmetry along the branch of radial solutions if the thickness of Ω is sufficiently small.

Key words : Symmetry Breaking, Critical Exponents, Annulus, Morse Index.

RÉSUMÉ. — Dans cet article, nous considérons le problème

$$\begin{aligned} -\Delta u &= u^p + \lambda u, \quad \text{dans } \Omega, \\ u &= 0 \quad \text{sur } \partial\Omega \qquad (P_\lambda) \\ u &> 0, \end{aligned}$$

ou Ω est un anneau et $p > 1$. Nous montrons qu'il y a toujours une casse de symétrie dans la branche des solutions radiales, si l'épaisseur de Ω est suffisamment petite.

Classification A.M.S. : Primary 35 J 25; Secondary 35 P 30.

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1. INTRODUCTION

In this article we consider the parameter dependent problem

$$\begin{aligned} -\Delta u &= u^p + \lambda u \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \qquad (\mathrm{P}_\lambda) \\ u &> 0 \end{aligned}$$

where $\Omega \subset \mathbb{R}^n$ is an annulus and prove that along the branch of radial solutions (u_λ, λ) of (P_λ) the symmetry breaks. That is, we show there exists a critical λ_0 such that every neighbourhood of $(u_{\lambda_0}, \lambda_0)$ contains nonradial solutions of (P_λ).

In [1] using variational arguments Brezis and Nirenberg proved that for p close to $\frac{n+2}{n-2}$ including the case $p=\frac{n+2}{n-2}$ there exists both radial and nonradial solutions for (P_λ) when λ is close to zero. Further they also posed the question whether these nonradial solutions come from a break of symmetry along the branch of radial solutions of (P_λ). The main result of this article answers this question partially, in the sense that we show symmetry always breaks if the annulus is thin. Moreover our arguments only ask $p > 1$ $\left(p \text{ can be greater than } \frac{n+2}{n-2}\right)$, but how thin the annulus should be depends on p. We refer the reader to [4] and to the references quoted there for other results on symmetry breaking.

2. MAIN RESULT

Before we state and prove the main result we will specify the following. Throughout this section we shall assume $\Omega = \{x \in \mathbb{R}^n : 1 < |x| < 1+\varepsilon\}$ and consider the problem (P_λ) as posed on this Ω. Also we will use a and b in place of 1 and $1+\varepsilon$ respectively. We will use $C_0^{1+\alpha}(\Omega)$ to denote the set of continuously differentiable functions on Ω which vanish on $\partial\Omega$ and whose first order derivatives are Holder continuous with exponent α. We will use $C_{0,R}^{1+\alpha}(\Omega)$ to denote the subspace of radial functions. We shall also use X to denote another closed subspace of $C_0^{1,\alpha}(\Omega)$ which we define below.

$$X = \{u \in C_0^{1,\alpha}(\Omega) : u(x_1, x_2, \ldots, x_{n-1}, x_n) = u(-x_1, -x_2, \ldots, -x_{n-1}, x_n)\}.$$

It is well known (*see* [5]) that (P_λ) admits an unique continuous branch of radial solutions (u_λ, λ) in $C_0^{1,\alpha}(\Omega) \times \mathrm{R}$ and the projection on the λ axis is $(-\infty, \lambda_1)$ where λ_1 denotes the first eigenvalue of $(-\Delta)$ on the domain Ω. Moreover the solutions u_λ are radially nondegenerate if ε is small, *see*

Theorem 1.7 of [5]. Throughout what follows we shall assume our ε is such that radial nondegeneracy holds.

THEOREM. — *Along the branch of radial solutions* (u_λ, λ) *the symmetry breaks at some point* $(u_{\lambda_0}, \lambda_0)$ *with* $0 < \lambda_0 < \lambda_1$, *if* ε *is small enough.*

We break up the proof of this Theorem into several steps.

Step 1. — The Morse-Index of the solution $(u_0, 0)$ (*i.e.* the point on the branch (u_λ, λ) corresponding to $\lambda=0$) is bigger or equal to $(n+1)$ in the space $C_0^{1+\alpha}(\Omega)$.

Proof of Step 1. — Notice that by the definition of Morse Index it is enough to produce orthogonal functions which satisfy

$$\langle -\Delta v - pu^{p-1} v, v \rangle < 0 \tag{1}$$

to conclude that the Morse Index of u_0 is greater or equal to $(n+1)$. It is easy to see $v \equiv u_0$ satisfies (1) because u_0 is a solution of (P_0).

Now we construct $w \in C_0^{1,\alpha}(\Omega)$ with w orthogonal to u_0 and satisfying (1).

Define $w(x) = u_0(x) . w_0(x)$ where $w_0(x) = \sum_{i=1}^{n} x_i$. Clearly $w(x) \in C_0^{1,\alpha}(\Omega)$ and is orthogonal to u_0 since u_0 is radial. A simple computation yields

$$-\Delta w - pu_0^{p-1} w(x) = -(p-1) u_0^{p-1} w(x) - 2 \sum_{i=1}^{n} u_{0_{x_i}} \tag{2}$$

where $u_{0_{x_i}} = \dfrac{\partial u_0}{\partial x_i}$. Clearly (1) is satisfied by w if λ_1 the first eigenvalue of $(-\Delta)$ on Ω is bigger than $\dfrac{n}{(p-1)}$ as can be seen by the following.

Multiply (2) by w on both sides and integrate keeping in mind u_0 is radial and x_i's are orthogonal directions, the integration of terms on right of (2) yield

$$\begin{aligned}
\langle -\Delta w - pu_0^{p-1} w, w \rangle &= -(p-1) \int_\Omega u_0^{p+1} \left(\sum_{i=1}^{n} x_i^2 \right) + n \int_\Omega u_0^2 \\
&\leqq -(p-1) \int_\Omega u_0^{p+1} + n \int_\Omega u_0^2 \ldots \quad (\text{since } \textstyle\sum x_i^2 \geqq 1) \\
&= -(p-1) \int_\Omega |\nabla u_0|^2 + n \int_\Omega u_0^2 \ldots \quad (\text{since } u_0 \text{ is a solution of } P_0) \\
&\leqq -\lambda_1 (p-1) \int_\Omega u_0^2 + n \int_\Omega u_0^2 = (p-1) \left(\int_\Omega u_0^2 \right) \left[\frac{n}{(p-1)} - \lambda_1 \right] \\
&< 0 \quad \text{if} \quad \left(\lambda_1 > \frac{n}{p-1} \right).
\end{aligned}$$

Hence Step 1 follows.

Step 2. — In this step we draw some conclusions based on Step 1. From Step 1 it follows that as we move along the continuous branch (u_λ, λ) as λ varies over $(0, \lambda_1)$ there must be a point λ_0 such that $(u_{\lambda_0}, \lambda_0)$ lies on the branch and u_{λ_0} is a degenerate solution of (P_{λ_0}), *i.e.*

$$\left.\begin{aligned} -\Delta v &= (pu_{\lambda_0}^{p-1}+\lambda_0)\, v \quad \text{in } \Omega \\ v &= 0 \quad \text{on } \partial\Omega \end{aligned}\right\} \tag{3}$$

has a nontrivial solution v. It is classical that we can write v in the form

$$v(r, \theta) = \sum_{k=0}^{\infty} a_k(r)\, \varphi_k(\theta)\; a \leqq r \leqq b,\; \Theta \in S^{n-1}$$

where φ_0 is a constant and for $k \geqq 1$, φ_k is an eigenfunction of the Laplacian on the $(n-1)$ sphere S^{n-1} corresponding to the k-th nonradial eigenvalue. Also a_k's are solutions of the equation

$$\begin{gathered} -r^{1-n} \frac{d}{dr}(r^{n-1}\, w_r) + r^{-2}\, \mu_k\, w(r) = (pu_{\lambda_0}^{p-1}+\lambda_0)\, w(r) \\ w(a) = w(b) = 0 \end{gathered}$$

where $\mu_k = k(k+n-2)$ for $k \geqq 1$. Also from the remark on radial nondegeneracy made at the beginning of this section $a_0(r) = 0$. We now make another assertation in Step 3 keeping in mind the arguments above and also using the same notations.

Step 3. — We claim there exists $\lambda \in (0, \lambda_1)$ which we shall still denote by λ_0 (which may be different from λ_0 of Step 2) such that

$$\left.\begin{aligned} -\Delta v &= (pu_{\lambda_0}^{p-1}+\lambda_0)\, v \quad \text{in } \Omega \\ v &= 0 \quad \text{on } \partial\Omega \end{aligned}\right\} \tag{5}$$

has a nontrivial solution and this v has exactly the form

$$v(r, \Theta) = a_1(r)\, \varphi_1(\Theta), \qquad a_1 > 0 \text{ on } (a, b). \tag{6}$$

Proof of Step 3. — To prove our assertion we make a careful study of the set of equations we have in (4). We begin by considering the weighted eigenvalue problem

$$\begin{gathered} -r^{(1-n)} \frac{d}{dr}(r^{n-1}\, w_r) + r^{-2}\, \mu_1\, w(r) = \gamma_1(\lambda)\,(pu_\lambda^{p-1}+\lambda)\, w,\; \lambda \in (0, \lambda_1) \\ w(a) = w(b) = 0 \end{gathered} \tag{7}$$

where $\gamma_1(\lambda)$ is the first eigenvalue associated with the $(pu_\lambda^{p-1}+\lambda)$. It is clear $\gamma_1(\lambda)$ is a continuous function of λ as (u_λ, λ) lie on a continuous branch. Now as $\lambda \to \lambda_1$ along the branch it is clear $\gamma_1(\lambda) \to \gamma_1(\lambda_1) > 1$, since the operator on the left side of (7) is more positive than $(-\Delta)$.

Hence it is clear that if we have to meet any situation as described in Step 2 then $\gamma_1(\lambda)$ must be exactly 1 for some λ_0. This is because for all $k > 1$, the operator on the left hand side in (4) is more positive than when $k=1$. Hence it is clear that there exists a λ_0 such that (5) has a solution v with

$$v(r, \theta)=\sum_{k=1}^{\infty} a_k(r)\varphi_k(\theta)$$

with $a_1(r) > 0$ [since it corresponds to first eigenvalue $\gamma_1(\lambda_0)$]. We claim that in this situation, that is when $a_1 > 0$, all the a_k's $\equiv 0$ for $k \geqq 2$. This follows from (4) since $a_k(r)$ satisfy

$$\begin{gathered} -r^{1-n}\frac{d}{dr}(r^{n-1}w_r)+r^{-2}\mu_k w(r)=(pu_{\lambda_0}^{p-1}+\lambda_0)w(r) \\ w(a)=w(b)=0 \end{gathered} \tag{8}$$

with $\mu_k > \mu_1$, $\forall k > 1$, then a_k's should change sign for all $k \geqq 2$. But then Strum's comparison theorem would force a_1 to change sign. Hence since $a_1 > 0$ all a_k's $\equiv 0$ for $k \geqq 2$ (we have given this argument only because it is very general, however, in the situation we are in it follows in a straight forward way since the operator in the left in (8) is more positive for any $k \geqq 2$ than when $k=1$). This completes the proof of Step 3.

Proof of the Theorem. – Now we restrict our consideration of problem (P_λ) to the space X defined at the beginning of this section. Due to this restriction the n-dimensional degeneracy proved in Step 3 reduces to the case of an 1-dimensional degeneracy. From all the previous arguments it now follows that as we move along the branch (u_λ, λ) the topological degree has to change at the point $(u_{\lambda_0}, \lambda_0)$. This change in degree leads to a secondary bifurcation. By the uniqueness of radial solutions which we have assumed, it now follows that the the new solutions arising from the secondary bifurcation are nonradial and hence our claim of break of symmetry.

Remark 1. – In the case when $1 < p < \frac{n+2}{n-2}$ a combination of Step 1 and a result due to H. Hofer [2] leads to the conclusion that the equation (P_0) (*i.e.* when $\lambda=0$) has atleast one nonradial solution, since the radial solution cannot be the Mountain Pass Solution, for such a solution must have Morse Index 1.

Remark 2. – Using completely different arguments Li Yan-Yan has shown in [3], that the number of solutions of (P_0) tends to infinity as the thickness of the annulus goes to zero. We believe that this increase in the number of solutions occurs because as the thickness of the annulus goes to zero the Morse Index of the radial solution of (P_0) goes to infinity as was pointed out to the author by Prof. A. Bahri (To see this replace the function w_0 of Step 1 by the functions φ_k referred to in Step 2). We believe this produces more and more secondary bifurcation points on our branch (u_λ, λ) and the branches due to these secondary bifurcation lead to more and more solutions. However so far we are not able to prove this. Infact it seems very hard to use the results of Crandall-Rabinowitz on bifurcation from simple eigenvalues due to the difficulties involved in producing a good transvers direction.

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