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Remarks on critical exponents for Hessian operators

by

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ABSTRACT. — The critical exponent for the k-th Hessian operator ($k=2, \ldots, n$) is determined and the solvability of the associated Dirichlet problem with sub-critical nonhomogeneous term is discussed.

Key words : Hessian operators, Critical exponent, Associated Dirichlet problem.

RÉSUMÉ. — On détermine l'exposant critique pour les opérateurs de Hess, et l'on étudie le problème de Dirichlet associé, avec un terme non homogène sous-critique.

INTRODUCTION

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geqq 2$. It is well-known that the Sobolev exponent $\dfrac{n+2}{n-2}$ plays a critical role concerning the solvability of the Dirichlet problem

$$\Delta u + u^p = 0 \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega.$$

Namely, this problem admits no positive solutions in a star-shaped Ω when $p \geqq \dfrac{n+2}{n-2}$ and it has a positive solution in any Ω when $\dfrac{n+2}{n-2} > p > 0$

and p is not equal to one. In a previous work [6] we have studied a corresponding problem for the Monge-Ampère operator on a convex Ω:

$$\det \nabla^2 u = (-u)^p \quad \text{in } \Omega, \qquad u=0 \quad \text{on } \partial\Omega.$$

We found that in contrast to the semilinear case this problem admits non-zero convex solutions provided p is positive and is not equal to n. Since the Laplace operator and the Monge-Ampère operator are respectively the first and the last Hessian operators, we are led to the question of determining the "critical exponents" for the remaining Hessian operators.

Let $S_k(\nabla^2 u)$, $k=1, \ldots, n$, be the k-th Hessian operator, *i.e.*, it is the k-th elementary symmetric function of the Hessian matrix of u. Consider

$$\left.\begin{aligned} S_k(\nabla^2 u) &= (-u)^p \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned}\right\} \tag{1}$$

In this note we shall establish the following result:

THEOREM. – *Let* Ω *be a ball and* let $\gamma(k) = \begin{cases} \dfrac{(n+2)k}{n-2k} & 1 \leqq k < \dfrac{n}{2} \\ \infty & \dfrac{n}{2} \leqq k \leqq n \end{cases}$. *Then*

(i) (1) *has no negative solution in* $C^1(\bar{\Omega}) \cap C^4(\Omega)$ *when* $p \geqq \gamma(k)$; (ii) *It admits a negative solution which is radially symmetric and is in* $C^2(\bar{\Omega})$ *when* $0 < p < \gamma(k)$, *p is not equal to k.*

The non-existence result actually holds in more general situation. *See* Proposition 1 in the next section. Part (ii) of this theorem is contained in Propositions 2 ($0 < p < k$) and 3 ($k < p < \gamma(k)$) in Section 2. When $p=k$ one should study an eigenvalue problem. However, we shall not consider it here.

NOTATION. – Subscripts like those in u_i, u_{ij}, F_z, ..., stand for partial differentiations. Also, summation convention is always in effect.

SECTION 1. NON-EXISTENCE

Consider the following Dirichlet problem

$$\left.\begin{aligned} S_k(\nabla^2 u) &= f(x, -u) > 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}\right\} \tag{2}$$

Many authors have studied the existence of negative solutions for this problem in the case $k=1$. Negative solutions are sought because they are important in many applications. However, when k is greater than one, negative solutions will be sought not only as an analogue of the semilinear case but mainly because they are precisely the "admissible ones". When k is greater than one a Hessian operator is nonlinear and so its type depends on the Hessian of u. According to Caffarelli-Nirenberg-Spruck [1] a C^2-function u is called *admissible* (with respect to S_k, $k \geqq 2$) if the eigenvalues of $\nabla^2 u(x)$, $x \in \Omega$, lie in the connected component Γ_k of the set $\{\lambda = (\lambda_1, \ldots, \lambda_n)$: The k-th elementary symmetric function of λ is positive$\}$ which contains the cone $\{\lambda : \lambda_i > 0$ for all $i\}$. It can be shown that for an admissible u $S_k(\nabla^2 u)$ is elliptic, *i.e.*, $\dfrac{\partial S_k(\nabla^2 u)}{\partial r_{ij}} \xi_i \xi_j > 0$, for all ξ, $|\xi| > 0$, and $x \in \Omega$. We claim that any solution of (2) which belongs to $C(\bar{\Omega}) \cap C^2(\Omega)$ is admissible if and only if it is negative. For, at a non-negative maximum of u, $S_k(\nabla^2 u)$ is non-positive. But since f is positive this is impossible. Conversely, if u is negative, it attains a minimum in Ω. At this minimum the eigenvalues of the Hessian of u lie in Γ_k. Hence by the continuity of $\nabla^2 u$ and the positivity of f u is also admissible.

The equation in (2) has a variational structure. To describe it we need to express S_k in terms of the Newtonian tensor. Recall that for $k=0, \ldots, n-1$, the k-th Newtonian tensor is given by

$$T_k(\nabla^2 u)_{ij} = \frac{1}{k!} \delta_{i_1 \ldots i_k i}^{j_1 \ldots j_k j} u_{i_1 j_1} \ldots u_{i_k j_k}.$$

Here $\delta_{i_1 \ldots i_k i}^{j_1 \ldots j_k j}$ is the generalised Kronecker delta: It is equal to 1 (resp. -1) if $i_1, \ldots, i_k, i$ are distinct and $\binom{j_1 \ldots j}{i_1 \ldots i}$ is an even (resp. odd) permutation; Otherwise it is equal to zero. A fundamental property of the Newtonian tensor is that it is of divergence free, *i.e.*,

$$\frac{\partial T_k(\nabla^2 u)_{ij}}{\partial x_j} = 0, \qquad i = 1, \ldots, n. \tag{3}$$

S_k is related to T_{k-1} via

$$\frac{\partial S_k(\nabla^2 u)}{\partial r_{ij}} = T_{k-1}(\nabla^2 u)_{ij} \tag{4}$$

and

$$S_k(\nabla^2 u) = \frac{1}{k} T_{k-1}(\nabla^2 u)_{ij} u_{ij}. \tag{5}$$

(5) follows from Euler's identity. For proofs of (3) and (4) see Reilly [5].

Using (3)-(5) one readily verifies that the equation in (2) is the Euler-Lagrange equation for the functional

$$J_k(u) = \frac{-1}{k+1}\int_\Omega u\, S_k(\nabla^2 u) + \int_\Omega F(x, u)$$

where $F(x, z) = \int_0^z f(x, t)\,dt$.

Now, consider a general integrand $\mathcal{F}(x, z, r_{ij})$, $r_{ij} = r_{ji}$. Let u be a C^4-solution of the Euler-Lagrange equation for $\mathcal{F}$. Then for any constant a the following identity holds:

$$\frac{\partial}{\partial x_i}\left[x_i \mathcal{F} + (x_k u_k + au)\frac{\partial \mathcal{F}_{r_{ij}}}{\partial x_j} - \frac{\partial}{\partial x_j}(x_k u_k + au)\mathcal{F}_{r_{ij}}\right]$$
$$= n\mathcal{F} + x_i \mathcal{F}_{x_i} - au\mathcal{F}_z - (a+2)\,u_{ij}\mathcal{F}_{r_{ij}}.$$

This identity, which can be verified directly, was first obtained in Pucci-Serrin [3]. Applying this identity to $\mathcal{F} = \dfrac{-z\, S_k(r_{ij})}{k+1} + F(x, z)$, we have, by (3)-(5),

$$\frac{\partial}{\partial x_i}\left[x_i\left(\frac{-u\, S_k(\nabla^2 u)}{k+1} + F(x, u)\right) + (x_k u_k + au)\left(\frac{-u_j T_{k-1}(\nabla^2 u)_{ij}}{k+1}\right)\right.$$
$$\left. + \frac{\partial}{\partial x_j}(x_k u_k + au)\frac{u\, T_{k-1}(\nabla^2 u)_{ij}}{k+1}\right]$$
$$= [k(a+2) + a - n]\frac{u\, S_k(\nabla^2 u)}{k+1} + n F - auf + x_i F_{x_i}. \tag{6}$$

Next we formulate the non-existence result. Recall that a bounded domain Ω is called star-shaped if for some x_0, $\langle x - x_0, \nu\rangle \geqq 0$ on $\partial\Omega$. (Here ν is the unit outer normal and $\langle\ .,\ .\ \rangle$ is the Euclidean inner product.)

PROPOSITION 1. — *Let Ω be a star-shaped C^2-domain with x_0 being the origin. Suppose that f belongs to $C(\bar\Omega \times (-\infty, 0]) \cap C^1(\Omega \times (-\infty, 0))$, positive in $\Omega \times (-\infty, 0)$ and equal to zero on $\Omega \times \{0\}$. There are no negative solutions to* (2) *which belong to $C^1(\bar\Omega) \cap C^4(\Omega)$ if*

$$n F(x, z) - \frac{n-2k}{k+1} z f(x, z) + x_i F_{x_i}(x, z) > 0 \tag{7}$$

in $\Omega\times(-\infty, 0)$. *The same conclusion holds under*

$$n\,F(x, z)-\frac{n-2k}{k+1}zf(x, z)+x_i F_{x_i}(x, z)\geqq 0 \tag{7'}$$

if $\langle x, \nu\rangle > 0$ *on* $\partial\Omega$.

Proof. – Let $\Omega_\varepsilon=\{x\in\Omega: u(x)<-\varepsilon\}$. By a strong maximum principle (*see* Lemma below) Ω_ε has C^4-boundary for small $\varepsilon>0$. Choose $a=\dfrac{n-2k}{k+1}$ so that $k(a+2)+a-n=0$ in (6) and then integrate both side of (6) over Ω_ε. By divergence theorem we have

$$\begin{aligned}\int_{\partial\Omega_\varepsilon}\langle x, \nu\rangle\left(\frac{\varepsilon S_k(\nabla^2 u)}{k+1}+F(x, -\varepsilon)\right)-\frac{1}{k+1}\int_{\partial\Omega_\varepsilon} x_k u_k u_j T_{k-1}(\nabla^2 u)_{ij}\nu_i\\ -\varepsilon\int_{\partial\Omega_\varepsilon}\frac{u_j T_{k-1}(\nabla^2 u)_{ij}\nu_i}{k+1}-\varepsilon\int_{\partial\Omega_\varepsilon}\frac{x_k u_{kj} T_{k-1}(\nabla^2 u)_{ij}\nu_i}{k+1}\\ =\int_{\Omega_\varepsilon}\left(n\,F(x, u)-\frac{n-2k}{k+1}uf(x, u)+x_i F_{x_i}(x, u)\right).\end{aligned}$$

A repeated application of (3)-(5) gives

$$\int_{\partial\Omega_\varepsilon} u_j T_{k-1}(\nabla^2 u)_{ij}\nu_i=k\int_{\Omega_\varepsilon} f(x, u) \tag{8}$$

and

$$\begin{aligned}\int_{\partial\Omega_\varepsilon} x_k u_{kj} T_{k-1}(\nabla^2 u)_{ij}\nu_i=k\int_{\Omega_\varepsilon} f(x, u)+\int_{\Omega_\varepsilon} x_k\frac{\partial}{\partial x_k}S_k(\nabla^2 u)\\ =(k-n)\int_\Omega f(x, u).\end{aligned}$$

Consequently,

$$\begin{aligned}\lim_{\varepsilon\downarrow 0}\int_{\partial\Omega_\varepsilon}\langle x, \nu\rangle\,|\nabla u|^2\,T_{k-1}(\nabla^2 u)_{ij}\nu_i\nu_j\\ =-(k+1)\int_\Omega\left(n\,F-\frac{n-2k}{k+1}uf+x_i F_{x_i}\right).\end{aligned}$$

If u is a negative solution of (2), $T_{k-1}(\nabla^2 u)_{ij}\nu_i\nu_j>0$ by ellipticity. Hence the left hand side of the above identity is non-negative. However, this contradicts with (7). On the other hand, if $\langle x, \nu\rangle\geqq\delta$ for some positive δ on $\partial\Omega$, by (8) and the lemma below the left hand side of the above identity is positive. But then this contradicts with (7)'. Proposition 1 is thus proved.

LEMMA. $-\dfrac{\partial u}{\partial \nu} > 0$ *on* $\partial\Omega$.

Proof. – Let $x \in \partial\Omega$ and let B be a ball which is inscribed inside Ω and touches $\partial\Omega$ at x. For a non-negative radially symmetric function g with $g(|x|) \leqq f(x, u(x))$ let w be the solution for

$$S_k(\nabla^2 w) = g \quad \text{in B}, \qquad w = 0 \quad \text{on } \partial B,$$

w can be solved explicitly in terms of g; *see* (10) in the next section. By the maximum principle $w \geqq u$ in B. Hence $\dfrac{\partial u}{\partial \nu}(x) \geqq \dfrac{\partial w}{\partial \nu}(x) > 0$. ∎

SECTION 3. EXISTENCE

Throughout this section Ω denotes a ball of radius R centered at the origin and $f(x, z)$ in (2) is equal to $g(r, z)$, $r = |x|$, where

$$g \in C([0, R] \times (-\infty, 0]), \qquad g > 0 \text{ in } [0, R) \times (-\infty, 0]. \tag{9}$$

We shall look for radially symmetric solutions for (2). Denote such a solution by $y(r)$. y satisfies

$$\begin{gathered} C_{k-1}^{n-1}\left(\frac{y'}{r}\right)^{k-1} y'' + C_k^{n-1}\left(\frac{y'}{r}\right)^k = g(r, y), \\ y(R) = 0, \qquad y'(0) = 0. \end{gathered} \tag{10}$$

Here C_{k-1}^{n-1} and C_k^{n-1} are combinatorial constants. It is easy to see that whenever y solves (10), $u(x) = y(|x|)$ is a solution for (3).

For a negative radially symmetric u, the functional $J_k(u)$ is equal to

$$\alpha \int_0^R (y')^{k+1} r^{n-k} + \tau \int_0^R G(r, y) r^{n-1}$$

where $G(r, z) = \displaystyle\int_0^z g(r, t)\, dt$, $\alpha = \dfrac{\tau}{k(k+1)} C_{k-1}^{n-1}$ and τ is the volume of the unit sphere in $\mathbf{R}^n$. It is more convenient to let J_k act on the whole space rather than on negative functions. To this end we extend g to be an even function in z (maintaining the same notation) and consider the functional

$$j_k(y)=\alpha\int_0^R |y'|^{k+1}\, r^{n-k}+\tau\int_0^R G(r, y)\, r^{n-1}.$$

LEMMA 1. – *Let* $E=\{y\in C^1([0, R]): y(R)=0\}$ (i) *For* $1\leqq k\leqq\frac{n}{2}$ *and* $0<\delta<\gamma(k)$, *there exists a constant* $C=C(\delta, k, R, n)$ *such that*

$$\left(\int_0^R |y|^{\delta+1}\, r^{n-1}\right)^{1/(\delta+1)}\leqq C\left(\int_0^R |y'|^{k+1}\, r^{n-k}\right)^{1/(k+1)}$$

for all $y\in E$; (ii) *For* $n\geqq k>\frac{n}{2}$, *there exists a constant* $C=C(k, R, n)$ *such that*

$$\|y\|_{C^{0,\sigma}([0, R])}\leqq C\left(\int_0^R |y'|^{k+1}\, r^{n-k}\right)^{1/(k+1)},\qquad \sigma=\frac{2k-n}{k+1}.$$

Proof. – Applying Hölder inequality to $y(r)=\int_R^r y'(s)\,ds$ we have

$$|y(r)|\leqq C\, r^{-(n-2k)/(k+1)}\left(\int_0^R |y'|^{k+1}\, r^{n-k}\right)^{1/(k+1)}.$$

Raising both side to their $(\delta+1)$-th power, multiplying by r^{n-1} and then integrading from 0 to R we obtain (i). (ii) can be proved in a similar way. ■

Remark. – In fact, a sharper result holds: There exists a constant $C=C(k, n)$ such that

$$C\left(\int_0^R |y'|^{k+1}\, r^{n-k}\right)^{1/(k+1)} \geqq \begin{cases} \left(\int_0^R |y|^{\gamma(k)+1}\, r^{n-1}\right)^{1/(\gamma(k)+1)} & \text{if } k<\frac{n}{2} \\ \|y\|_{C([0, R])} & \text{if } k=\frac{n}{2} \\ \|y\|_{C^{0,\sigma}([0, R])}, \quad \sigma=\frac{2k-n}{k+1}, & \text{if } k>\frac{n}{2} \end{cases}$$

for all y in E. For a proof see Lin [2].

Let W_k be the Banach space obtained by completing E under the norm $\left(\int_0^R |y'|^{k+1} r^{n-k}\right)^{1/(k+1)}$. Using Lemma 1 one can easily show that the following lemma holds:

LEMMA 2. — (i) W_k *is continuously embedded in* $\{y\in C^{0,\,1-(1/(k+1))}([r, R]) : y(R)=0.\}$ *for* $r>0$; (ii) W_k *is compactly embedded in* $L^{p+1}(r^{n-1}\,dr)$ *for* $p<\gamma(k)$ *when* $k \leqq \frac{n}{2}$ *and in* $C([0, R])$ *when* $k>\frac{n}{2}$.

We also have

LEMMA 3. — *Suppose g satisfies*

$$|g(r, z)| \leqq C(1+|z|^p), \qquad 0 \leqq r \leqq R \tag{11}$$

for some p, $0<p<\gamma(k)$. *Then* j_k *belongs to* $C^1(W_k; R)$. *Moreover,* $y \to \int_0^R G(r, y)\, r^{n-1}$ *is weakly continuous on* W_k.

The proof of this lemma is parallel to the proof for the corresponding result in the semilinear case; *See* Rabinowitz [4], Appendix B, for details. Any critical point of j_k will be called a *generalised solution* of (10). In other words, y is a generalised solution of (10) if and only if

$$(k+1)\,\alpha \int_0^R |y'|^{k-1} y'\, \varphi'\, r^{n-k} + \tau \int_0^R g(r, y)\, \varphi\, r^{n-1} = 0 \tag{12}$$

for all φ in W_k. The regularity of such solutions is given in

LEMMA 4. — *Under* (9) *and* (11), *any generalised solution of* (10) *is in* $C^2([0, R])$, *and solves* (10) *in classical sense. Moreover, it is negative in* $[0, R)$ *unless it vanishes identically.*

Proof. — We shall first show that any generalised solution is bounded. When k is greater than $\frac{n}{2}$ this follows from Lemma 1. So we assume k is less than or equal to $\frac{n}{2}$.

Let y be a generalised solution. For $s \geqq 1$ and $N \geqq 1$ we define a function Φ in $C^1[1, \infty)$ by setting $\Phi(z)=z^s-1$ for $z\in[1, N]$ and taking it to be linear for $z \geqq N$. Select $\varphi = \int_1^{y^+ +1} |\Phi'(t)|^{k+1}\, dt$ where $y^+ = \max(y, 0)$

as a test function. Substituting φ into (12) we have

$$\int \left| \frac{d}{dr} \Phi(y^+ + 1) \right|^{k+1} r^{n-k} \leqq C \int g(r, y) \, |\Phi'(y^+ + 1)|^{k+1} y^+ r^{n-1}$$

after using $\varphi \leqq |\Phi'(y^+ + 1)|^{k+1} y^+$. It follows from Lemma 1, Hölder inequality and (11) that for a fixed β, $\max(p, k) < \beta < \gamma(k)$,

$$\left(\int \Phi(y^+ + 1)^{\beta+1} r^{n-1} \right)^{1/(\beta+1)} \leqq C \left(\int (\Phi'(y^+ + 1)(y^+ + 1))^{q+1} r^{n-1} \right)^{1/(q+1)}$$

where $q = p$ if $p > k$ and $q = k$ if $p \leqq k$ and C is a constant depending on $\|y\|_{W_k}$. Letting $N \to \infty$, we see that $y^+ \in L^{s(q+1)}(r^{n-1} dr)$ implies the stronger inclusion $y^+ \in L^{s(\beta+1)}(r^{n-1} dr)$. Also, we have

$$\|y^+ + 1\|_{L^{s\kappa(q+1)}(r^{n-1} dr)} \leqq (Cs)^{1/s} \|y^+ + 1\|_{L^{s(q+1)}(r^{n-1} dr)}$$

where $\kappa = \dfrac{1+\beta}{1+q} > 1$. Setting $s = \kappa^m$, $m \geqq 1$, an iteration yields

$$\sup y^+ \leqq C(1 + \|y\|_{L^{q+1}(r^{n-1} dr)}).$$

Similarly one can show that the above estimate holds for $y^- = -\min(y, 0)$ in place of y^+. Thus y is bounded.

Take $\varphi = \int_r^R \eta(t) \, dt$ where η is smooth as a test function in (12). After an integration by parts, we have

$$(k+1)\alpha \int_0^R |y'|^{k-1} y' \eta r^{n-k} dr = \tau \int_0^R \int_0^r g(t, y) t^{n-1} dt \, \eta \, dr.$$

Henceforth

$$(k+1)\alpha r^{n-k} |y'|^{k-1} y' = \tau \int_0^r g(t, -y) t^{n-1} dt \quad \text{a. e.}$$

Since g is non-negative, y is increasing (hence is negative) unless it vanishes identically. From this the desired conclusion of Lemma 4 can be drawn. ■

Now we can state the existence results.

PROPOSITION 2. — *Under* (9) *and*

$$\lim_{z \to \infty} \frac{g(r, z)}{z^p} = 0, \quad p < \gamma(k), \qquad \lim_{z \to 0} \frac{g(r, z)}{z^k} = \infty \tag{13}$$

uniformly on [0, R], (2) *admits a negative radially symmetric solution in* $C^2(\bar{\Omega})$.

This solution is obtained by minimizing j_k over W_k. (13) ensures that j_k is bounded from below and is negative somewhere. It follows from the direct method, which works in view of Lemma 3, that a non-zero absolute minimum exists.

PROPOSITION 3. — *In addition to* (9) *and* (11), *suppose g satisfies* $\lim_{z \to 0} \frac{g(r, z)}{z^k} = 0$ *uniformly in* [0, R], *and, further, there exist* $\theta \in (0, 1)$ *and a positive constant* M *such that*

$$G(r, z) \leqq \frac{\theta}{k+1} z g(r, z), \qquad 0 \leqq r \leqq R, \quad |z| \geqq M. \tag{14}$$

Then (2) *admits a negative radially symmetric solution which belongs to* $C^2(\bar{\Omega})$.

(14) implies that j_k fulfills the Palais-Smale condition. Proposition 3 follows from the mountain pass lemma. For details see the corresponding proof for the semilinear case in [4], Chapter 2.

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