

AMERICAN PRICES EMBEDDED IN EUROPEAN PRICES

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ABSTRACT. – In this paper, we are interested in American option prices in the Black–Scholes model. For a large class of payoffs, we show that in the region where the European price increases with the time to maturity, this price is equal to the American price of another claim. We give examples in which we explicit the corresponding claims. The characterization of the American claims obtained in this way remains an open question.

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RÉSUMÉ. – Ce travail met en évidence un lien entre prix d'options européennes et prix d'options américaines dans le modèle de Black–Scholes. Nous montrons que pour une large classe de fonctions de payoff φ , dans la zone où il augmente avec la maturité, le prix de l'option européenne est égal au prix américain correspondant à une fonction de payoff modifiée $\widehat{\varphi}$. Nous donnons des exemples où il est possible d'expliciter $\widehat{\varphi}$. Mais la caractérisation de l'image de $\varphi \rightarrow \widehat{\varphi}$ reste un problème ouvert.

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Introduction

Consider the classical Black–Scholes model:

$$\begin{aligned}dX_t^x &= \rho X_t^x dt + \sigma X_t^x dB_t, \\X_0^x &= x > 0, \\ \rho &\in \mathbb{R}, \sigma > 0,\end{aligned}$$

where B is a standard Brownian motion, ρ the instantaneous interest rate and σ the volatility of X and denote by

$$\mathcal{A}f(x) = \frac{\sigma^2 x^2}{2} f''(x) + \rho x f'(x) - \rho f(x) \tag{0.1}$$

the Black–Scholes infinitesimal generator. Given a continuous function $\psi : \mathbb{R}_+^* \rightarrow \mathbb{R}_+$ satisfying some growth assumptions, the price of the so-called American option with payoff ψ , maturity $t > 0$ and spot x is given by the expression

$$v_\psi^{am}(t, x) = \sup_{\tau \in \mathcal{T}(0, t)} \mathbb{E}[e^{-\rho\tau} \psi(X_\tau^x)], \tag{0.2}$$

where τ runs across the set of stopping times of the Brownian filtration such that $\tau \leq t$ almost surely. Except for some very particular class of payoffs ψ (e.g. payoffs satisfying $\forall x > 0, \mathcal{A}\psi(x) \geq 0$ or $\forall x > 0, \mathcal{A}\psi(x) \leq 0$), in general, there is no closed-form expressions for $v_\psi^{am}(t, x)$. The computation of $v_\psi^{am}(t, x)$ usually relies either on finite-difference type methods or Markov-chain approximation methods to solve the corresponding optimal stopping problem in a discrete time-space framework. There is also a huge literature on special approximation methods designed for some particular payoffs, among which the case of the Put option, given by $\psi(x) = (K - x)^+$ where K is some positive constant (the strike of the option) has received much attention.

The purpose of this paper is to exhibit a new class of payoffs ψ for which a closed-form expression for $v_\psi^{am}(t, x)$ is available. The idea originates from the analytic properties of the function v_ψ^{am} : this function is greater than ψ by (0.2) and typically the space $]0, \infty[\times \mathbb{R}_+^*$ splits into two regions, the so-called Exercise region where by definition $v_\psi^{am} = \psi$ and its complement the Continuation region where $v_\psi^{am} > \psi$. It is known that v_ψ^{am} solves the evolution equation associated with (0.1)

$$\partial_t v_\psi^{am} = \mathcal{A}v_\psi^{am}$$

in the Continuation region (at least in the distribution sense). Moreover, as from (0.2) $t \mapsto v_\psi^{am}(t, x)$ is non-decreasing, $\partial_t v_\psi^{am} \geq 0$ holds. In fact, since v_ψ^{am} is a continuous function, it may be remarked that the knowledge of v_ψ^{am} in the Continuation region is enough to get v_ψ^{am} everywhere.

This leads to the natural idea to build American prices (i.e. functions v_ψ^{am} for some ψ) by picking up the classical solution $v_\varphi(t, x)$ of the evolution equation:

$$\begin{cases} \forall t, x > 0, \partial_t v_\varphi(t, x) = \mathcal{A}v_\varphi(t, x), \\ \forall x > 0, v_\varphi(0, x) = \varphi(x), \end{cases}$$

in the region where it increases with time. From a financial point of view, $v_\varphi(t, x)$ is the Black–Scholes price of the European option with payoff φ and maturity t i.e. $v_\varphi(t, x) = \mathbb{E}[e^{-\rho t} \varphi(X_t^x)]$. This embedding idea has been worked out in [2] in case $\rho = 0$. A similar approach has also been developed in the different context of the free boundary arising in a two-phases problem (see [1]). Trying to generalize things to the

case $\rho \neq 0$, we ran across a probabilistic proof which allows a very compact statement of the embedding result.

The first section of the paper is devoted to some basic properties of European and American prices within the Black–Scholes model. Next we state and prove our embedding theorem (Section 2). Then we give some examples (Section 3). Lastly, we discuss some properties of the map which takes a payoff φ to the payoff $\widehat{\varphi}$ the American price of which is embedded in its European price (Section 4). The characterization of the payoffs $\widehat{\varphi}$ obtained in this way remains an open question.

1. European and American prices in the Black–Scholes model

In this section we recall the very few properties of European and American Black–Scholes prices we shall need in the next section.

Let $\alpha = 2\rho/\sigma^2$. The invariant functions of the semigroup associated with (0.1) are easily seen to be the vector space generated by x and $x^{-\alpha}$. We shall consider payoffs φ such that

$$x \in \mathbb{R}_+^* \mapsto \varphi(x) \in \mathbb{R}_+ \text{ is continuous and } \sup_{x>0} \frac{\varphi(x)}{x + x^{-\alpha}} < \infty \tag{H0}$$

The growth assumption is only there to grant the existence of the various expectations involved. It seems that the continuity assumption could be removed, but the connection with American options would be more intricate, so we keep this hypothesis.

PROPOSITION 1 ([3]). – *Under (H0) the price of the European option with maturity $t \geq 0$ and payoff φ is given by*

$$v_\varphi(t, x) = \mathbb{E}[e^{-\rho t} \varphi(X_t^x)].$$

In particular $v_\varphi(0, x) = \varphi(x)$.

The function v_φ is continuous from $[0, \infty[\times \mathbb{R}_+^*$ into \mathbb{R} , and for any $t > 0$, the process $(e^{-\rho u} v_\varphi(t - u, X_u^x))_{0 \leq u \leq t}$ is a continuous square-integrable martingale.

Last, for $\phi(y) = \varphi(e^y)/(e^y + e^{-\alpha y})$,

$$v_\varphi(t, x) - x\mathbb{E}\left(\phi\left(\sigma B_t + \left(\rho + \frac{\sigma^2}{2}\right)t\right)\right) - x^{-\alpha}\mathbb{E}\left(\phi\left(\sigma B_t - \left(\rho + \frac{\sigma^2}{2}\right)t\right)\right)$$

converges to 0 as $t \rightarrow +\infty$ locally uniformly in $x > 0$.

Proof. – We only prove the last assertion which is quite unusual. By definition of ϕ ,

$$v_\varphi(t, x) = \mathbb{E}\left(e^{-\rho t} X_t^x \phi\left(\ln(x) + \sigma B_t + \left(\rho - \frac{\sigma^2}{2}\right)t\right)\right) + \mathbb{E}\left(e^{-\rho t} (X_t^x)^{-\alpha} \phi\left(\ln(x) + \sigma B_t + \left(\rho - \frac{\sigma^2}{2}\right)t\right)\right).$$

Since $e^{-\rho t} X_t^x = x e^{\sigma B_t - \frac{\sigma^2}{2}t}$ and $e^{-\rho t} (X_t^x)^{-\alpha} = x^{-\alpha} e^{-\frac{2\rho}{\sigma} B_t - \frac{2\rho^2}{\sigma^2}t}$, by Girsanov theorem, we deduce that

$$v_\varphi(t, x) = x \mathbb{E} \left(\phi \left(\ln(x) + \sigma B_t + \left(\rho + \frac{\sigma^2}{2} \right) t \right) \right) + x^{-\alpha} \mathbb{E} \left(\phi \left(\ln(x) + \sigma B_t - \left(\rho + \frac{\sigma^2}{2} \right) t \right) \right).$$

Now,

$$\begin{aligned} & \left| \mathbb{E} \left(\phi \left(\ln(x) + \sigma B_t + \left(\rho + \frac{\sigma^2}{2} \right) t \right) \right) - \mathbb{E} \left(\phi \left(\sigma B_t + \left(\rho + \frac{\sigma^2}{2} \right) t \right) \right) \right| \\ &= \frac{1}{\sqrt{2\pi}} \left| \int_{\mathbb{R}} \phi \left(\sigma \sqrt{t} y + \left(\rho + \frac{\sigma^2}{2} \right) t \right) \left(e^{-\frac{(y - \ln(x)/(\sigma\sqrt{t}))^2}{2}} - e^{-\frac{y^2}{2}} \right) dy \right| \\ &\leq \frac{1}{\sqrt{2\pi}} \sup_{z>0} \frac{\varphi(z)}{z + z^{-\alpha}} \int_{\mathbb{R}} \left| e^{-\frac{(y - \ln(x)/(\sigma\sqrt{t}))^2}{2}} - e^{-\frac{y^2}{2}} \right| dy \\ &\leq \left(e^{\frac{\ln^2(x)}{2\sigma^2 t}} - 1 \right) \sup_{z>0} \frac{\varphi(z)}{z + z^{-\alpha}} \end{aligned}$$

converges to 0 as $t \rightarrow +\infty$ locally uniformly for $x > 0$.

We deal with $\mathbb{E}(\phi(\ln(x) + \sigma B_t - (\rho + \sigma^2/2)t))$ in the same way to conclude. \square

Let us now turn to American options:

PROPOSITION 2 ([3]). – *If ψ satisfies (H0), the price of the American option with maturity $t \geq 0$ and payoff ψ is given by*

$$v_\psi^{am}(t, x) = \sup_{\tau \in \mathcal{T}(0,t)} \mathbb{E}[e^{-\rho\tau} \psi(X_\tau^x)],$$

where τ runs across the set of stopping times of the Brownian filtration such that $\tau \leq t$ almost surely. In particular $v_\psi^{am}(0, x) = \psi(x)$.

The function v_ψ^{am} is continuous from $[0, \infty[\times \mathbb{R}_+^*$ into \mathbb{R} , and for any $x \in \mathbb{R}_+^*$ the map $t \mapsto v_\psi^{am}(t, x)$ is non-decreasing.

2. Embedding American prices in European prices

Our main result relates the price $v_\varphi(t, x)$ of the European option with payoff φ to the price $v_\varphi^{am}(t, x)$ of the American option with payoff $\widehat{\varphi}(x) = \inf_{t \geq 0} v_\varphi(t, x)$:

THEOREM 3. – *Under (H0) let*

$$\widehat{\varphi}(x) = \inf_{t \geq 0} v_\varphi(t, x).$$

Then

$$\forall (t, x) \in [0, +\infty) \times \mathbb{R}_+^*, \quad \sup_{\tau \in \mathcal{T}(0,t)} \mathbb{E}[e^{-\rho\tau} \widehat{\varphi}(X_\tau^x)] \leq v_\varphi(t, x), \tag{2.1}$$

where the supremum is taken over all the stopping times τ of the filtration of the Brownian motion smaller than t .

Moreover, if there exists a continuous function $\widehat{t} : \mathbb{R}_+^* \rightarrow [0, +\infty]$ such that

$$\forall x > 0, \quad \inf_{t \geq 0} v_\varphi(t, x) = v_\varphi(\widehat{t}(x), x)$$

(where $v_\varphi(\infty, x)$ is defined as $\liminf_{t \rightarrow +\infty} v_\varphi(t, x)$) then the converse inequality holds for $t \geq \widehat{t}(x)$ and

$$\forall (t, x) \in [0, +\infty) \times \mathbb{R}_+^*, \quad \sup_{\tau \in \mathcal{T}(0,t)} \mathbb{E}[e^{-\rho\tau} \widehat{\varphi}(X_\tau^x)] = v_\varphi(t \vee \widehat{t}(x), x). \quad (2.2)$$

If, lastly, either $\widehat{t}(x_0) < +\infty$ for some $x_0 > 0$, or

$$\exists C > 0, \forall x, y > 0, \quad |\varphi(x) - \varphi(y)| \leq C(|x - y| + |x^{-\alpha} - y^{-\alpha}|)$$

or the function $\frac{\varphi(x)}{x+x^{-\alpha}}$ admits limits both for $x \rightarrow 0$ and $x \rightarrow +\infty$, then the function $\widehat{\varphi}$ satisfies (H0) and

$$\forall (t, x) \in [0, +\infty) \times \mathbb{R}_+^*, \quad v_\varphi^{am}(t, x) = v_\varphi(t \vee \widehat{t}(x), x). \quad (2.3)$$

Proof. – Let $(t, x) \in \mathbb{R}_+ \times \mathbb{R}_+^*$. According to Proposition 1, the process $(e^{-\rho u} v_\varphi(t - u, X_u^x))_{u \in [0,t]}$ is a martingale. If $\tau \leq t$ is a stopping time, by Doob optional sampling theorem

$$v_\varphi(t, x) = \mathbb{E}[e^{-\rho\tau} v_\varphi(t - \tau, X_\tau^x)] \geq \mathbb{E}[e^{-\rho\tau} \widehat{\varphi}(X_\tau^x)].$$

Since τ is arbitrary, we deduce that (2.1) holds.

To prove (2.2), we suppose the existence of $\widehat{t} : \mathbb{R}_+^* \rightarrow [0, +\infty]$ continuous such that

$$\forall x > 0, \quad \widehat{\varphi}(x) = v_\varphi(\widehat{t}(x), x)$$

and we make a distinction between the two following situations:

- Case $\widehat{t}(x) = +\infty$: since $s \rightarrow \sup_{\tau \in \mathcal{T}(0,s)} \mathbb{E}[e^{-\rho\tau} \widehat{\varphi}(X_\tau^x)]$ is increasing, for $u \geq t$

$$\widehat{\varphi}(x) \leq \sup_{\tau \in \mathcal{T}(0,t)} \mathbb{E}[e^{-\rho\tau} \widehat{\varphi}(X_\tau^x)] \leq \sup_{\tau \in \mathcal{T}(0,u)} \mathbb{E}[e^{-\rho\tau} \widehat{\varphi}(X_\tau^x)] \leq v_\varphi(u, x) \quad \text{by (2.1).}$$

Letting $u \rightarrow +\infty$, we deduce that

$$\forall t \geq 0, \quad \sup_{\tau \in \mathcal{T}(0,t)} \mathbb{E}[e^{-\rho\tau} \widehat{\varphi}(X_\tau^x)] = \widehat{\varphi}(x) = v_\varphi(\infty, x) = v_\varphi(t \vee \widehat{t}(x), x).$$

- Case $\widehat{t}(x) < +\infty$: let $t \geq \widehat{t}(x)$ and $\tau_0 = \inf\{u : t - u - \widehat{t}(X_u^x) \leq 0\}$. Since $\widehat{t}(X_t^x) \geq 0$, the stopping time τ_0 is smaller than t . By continuity of $u \rightarrow t - u - \widehat{t}(X_u^x)$, $t - \tau_0 = \widehat{t}(X_{\tau_0}^x)$. Hence

$$\begin{aligned} v_\varphi(t, x) &= \mathbb{E}[e^{-\rho\tau_0} v_\varphi(t - \tau_0, X_{\tau_0}^x)] = \mathbb{E}[e^{-\rho\tau_0} v_\varphi(\widehat{t}(X_{\tau_0}^x), X_{\tau_0}^x)] \\ &= \mathbb{E}[e^{-\rho\tau_0} \widehat{\varphi}(X_{\tau_0}^x)] \leq \sup_{\tau \in \mathcal{T}(0,t)} \mathbb{E}[e^{-\rho\tau} \widehat{\varphi}(X_\tau^x)]. \end{aligned}$$

The converse inequality (2.1) is already proved. Hence $\forall t \geq \hat{t}(x)$, $\sup_{\tau \in \mathcal{T}(0,t)} \mathbb{E}[e^{-\rho\tau} \hat{\varphi}(X_\tau^x)] = v_\varphi(t, x)$ and τ_0 is an optimal stopping time. Since $t \rightarrow v_\varphi^{am}(t, x)$ is increasing, for $t \leq \hat{t}(x)$,

$$\hat{\varphi}(x) \leq \sup_{\tau \in \mathcal{T}(0,t)} \mathbb{E}[e^{-\rho\tau} \hat{\varphi}(X_\tau^x)] \leq \sup_{\tau \in \mathcal{T}(0,\hat{t}(x))} \mathbb{E}[e^{-\rho\tau} \hat{\varphi}(X_\tau^x)] = \hat{\varphi}(x),$$

which concludes the proof of (2.2).

If we check that $\hat{\varphi}$ is continuous under the various assumptions made in the last assertion of the theorem then (2.3) follows immediately since $v_\varphi^{am}(t, x) = \sup_{\tau \in \mathcal{T}(0,t)} \mathbb{E}[e^{-\rho\tau} \hat{\varphi}(X_\tau^x)]$.

- If $\hat{t}(x_0) < +\infty$ for some $x_0 > 0$ then for x in a neighbourhood of x_0 , $\hat{t}(x) < +\infty$ and since $t \rightarrow \sup_{\tau \in \mathcal{T}(0,t)} \mathbb{E}[e^{-\rho\tau} \hat{\varphi}(X_\tau^x)]$ is increasing, by (2.2) $v_\varphi(t, x)$ has a limit for $t \rightarrow +\infty$. By the last assertion of Proposition 1, we deduce that for $\phi(y) = \varphi(e^y)/(e^y + e^{-\alpha y})$, $\mathbb{E}(\phi(\sigma B_t + (\rho + \sigma^2/2)t))$ and $\mathbb{E}(\phi(\sigma B_t - (\rho + \sigma^2/2)t))$ admit limits as $t \rightarrow +\infty$. We denote the limits by a and b . Proposition 1 then yields that $v_\varphi(t, x)$ converges to the invariant function $ax + bx^{-\alpha}$ as $t \rightarrow +\infty$ locally uniformly for $x > 0$. The continuity of $\hat{\varphi}$ follows easily.
- If $\varphi(x)/(x + x^{-\alpha})$ admits limits for $x \rightarrow 0$ and $x \rightarrow +\infty$ then $\phi(y)$ admits limits for $y \rightarrow -\infty$ and $y \rightarrow +\infty$. We deduce that $\mathbb{E}(\phi(\sigma B_t + (\rho + \sigma^2/2)t))$ and $\mathbb{E}(\phi(\sigma B_t - (\rho + \sigma^2/2)t))$ admit limits as $t \rightarrow +\infty$ and we conclude like in the previous case.
- If $\forall x, y > 0, |\varphi(x) - \varphi(y)| \leq C(|x - y| + |x^{-\alpha} - y^{-\alpha}|)$, then $\forall t \geq 0$,

$$\begin{aligned} |v_\varphi(t, x) - v_\varphi(t, y)| &\leq \mathbb{E}[e^{-\rho t} |\varphi(X_t^x) - \varphi(X_t^y)|] \\ &\leq C(|x - y| \mathbb{E}[e^{-\rho t} X_t^1] + |x^{-\alpha} - y^{-\alpha}| \mathbb{E}[e^{-\rho t} (X_t^1)^{-\alpha}]) \\ &\leq C(|x - y| + |x^{-\alpha} - y^{-\alpha}|). \end{aligned}$$

Hence the functions $x \rightarrow v_\varphi(t, x)$ indexed by $t \geq 0$ are equicontinuous, which ensures the continuity of $x \rightarrow \inf_{t \geq 0} v_\varphi(t, x) = \hat{\varphi}(x)$. \square

Remark 4. – The continuity of the argument of the infimum is granted in the following uniqueness situation: suppose that $\forall x > 0, \exists \hat{t}(x) \leq T(x), \hat{\varphi}(x) = v_\varphi(\hat{t}(x), x)$ where $T: \mathbb{R}_+^* \rightarrow \mathbb{R}_+$ is continuous. Then by the continuity of T and v_φ , it is easy to see that $\hat{\varphi}(x) = \inf_{t \in [0, T(x)]} v_\varphi(t, x)$ is continuous. Moreover, since $\hat{t}(x) = \inf\{t \geq 0: \hat{\varphi}(x) = v_\varphi(t, x)\}$ (respectively $\hat{t}(x) = \sup\{t \leq T(x): \hat{\varphi}(x) = v_\varphi(t, x)\}$), \hat{t} is lower semi-continuous (respectively upper semi-continuous) i.e. \hat{t} is continuous and (2.3) holds.

In the above theorem it may happen that the function $\hat{\varphi}$ is nil: in case $\lim_{x \rightarrow 0} \frac{\varphi(x)}{x+x^{-\alpha}} = \lim_{x \rightarrow +\infty} \frac{\varphi(x)}{x+x^{-\alpha}} = 0$, we easily check that

$$\forall x > 0, \lim_{t \rightarrow +\infty} v_\varphi(t, x) = 0.$$

In such a situation, the following localized version of our main result is far more interesting than Theorem 3. It is proved by the same arguments, after noticing that the

continuity of $(t, x) \in [0, +\infty) \times \mathbb{R}_+^* \rightarrow v_\varphi(t, x)$ implies the continuity of $x \rightarrow \widehat{\varphi}^T(x) = \inf_{0 \leq t \leq T} v_\varphi(t, x)$ where $T > 0$.

THEOREM 5. – *Let $T > 0$. The function $\widehat{\varphi}^T(x) = \inf_{0 \leq t \leq T} v_\varphi(t, x)$ satisfies (H0) and*

$$\forall (t, x) \in [0, T] \times \mathbb{R}_+^*, \quad v_{\widehat{\varphi}^T}^{am}(t, x) \leq v_\varphi(t, x).$$

Moreover, if there exists a continuous function $\widehat{t} : \mathbb{R}_+^* \rightarrow [0, T]$ such that

$$\forall x > 0, \quad \inf_{0 \leq t \leq T} v_\varphi(t, x) = v_\varphi(\widehat{t}(x), x),$$

then

$$\forall (t, x) \in [0, T] \times \mathbb{R}_+^*, \quad v_{\widehat{\varphi}^T}^{am}(t, x) = v_\varphi(t \vee \widehat{t}(x), x).$$

Remark 6. – The only feature of the Black–Scholes model which is required in the above results is time-homogeneity. In fact, Propositions 1 and 2 and Theorems 3 and 5 can be adapted to the so-called generalized Black–Scholes model:

$$\begin{aligned} X_t^x &= x \exp\left(\sigma B_t + \left(\rho - \delta - \frac{\sigma^2}{2}\right)t\right), \\ v_\varphi(t, x) &= \mathbb{E}\left[e^{-\rho t} \varphi(X_t^x)\right] \quad \text{and} \\ v_\psi^{am}(t, x) &= \sup_{\tau \in \mathcal{T}(0, t)} \mathbb{E}\left[e^{-\rho \tau} \psi(X_\tau^x)\right], \end{aligned}$$

or to the more general time-homogeneous model:

$$\begin{aligned} X_0^x &= x, \quad dX_t^x = X_t^x (\sigma(X_t^x) dB_t + (\rho(X_t^x) - \delta(X_t^x)) dt) \\ v_\varphi(t, x) &= \mathbb{E}\left[e^{-\int_0^t \rho(X_s^x) ds} \varphi(X_t^x)\right] \quad \text{and} \\ v_\psi^{am}(t, x) &= \sup_{\tau \in \mathcal{T}(0, t)} \mathbb{E}\left[e^{-\int_0^\tau \rho(X_s^x) ds} \psi(X_\tau^x)\right] \end{aligned}$$

and also to the multidimensional versions of these models.

Of course it would be of great interest to give conditions on φ which ensure the existence of a continuous curve in the argument of the infimum. One way is to perform explicit computations, since the Black–Scholes semigroup is explicit. Nevertheless this is not very illuminating. We ran across the following statement, for the local embedding result, which is maybe the simplest in this direction:

PROPOSITION 7. – *Let φ be a C^4 function which satisfies (H0) and such that for some $x_c \in \mathbb{R}_+^*$,*

- (i) $\mathcal{A}\varphi(x_c) = 0$ and either $\forall x > 0, (x - x_c)\mathcal{A}\varphi(x) \geq 0$
or $\forall x > 0, (x - x_c)\mathcal{A}\varphi(x) \leq 0$.
- (ii) $\mathcal{A}^2\varphi(x_c) > 0$ and $\partial_x \mathcal{A}\varphi(x_c) \neq 0$.

Then there exists a constant $T > 0$ such that the assumptions of Theorem 5 are satisfied.

Proof. – Since φ is C^4 , the function $v_\varphi(t, x)$ belongs to $C^{2,4}(\mathbb{R}_+ \times \mathbb{R}_+^*)$ (C^2 in t , C^4 in x) and satisfies the Black–Scholes partial differential equation $\partial_t v_\varphi = \mathcal{A}v_\varphi$ for $t \geq 0$ and not only $t > 0$. Consider the equation $\partial_t v_\varphi(t, x) = 0$ in a neighbourhood of $(0, x_c)$ in $\{(t, x), t \geq 0\}$. By derivation of the Black–Scholes evolution equation, $\partial_{t,x}^2 v_\varphi(0, x_c) = \partial_x \mathcal{A}\varphi(x_c) \neq 0$. Hence, by the implicit functions theorem, there is for some $\varepsilon > 0$ a curve \hat{x} :

$$\hat{x} : [0, \varepsilon] \rightarrow \mathbb{R}_+^*$$

continuous on $[0, \varepsilon]$, with $\hat{x}(0) = x_c$, such that $\partial_t v_\varphi(t, \hat{x}(t)) = 0$, and C^1 on $]0, \varepsilon[$ with

$$\partial_{t^2}^2 v_\varphi(t, \hat{x}(t)) + \partial_{t,x}^2 v_\varphi(t, \hat{x}(t))\hat{x}'(t) = 0.$$

Moreover by taking ε small enough we can assume that $\hat{x}'(t)$ does not vanish and keeps the same sign as $\hat{x}'(0^+) = -\frac{\mathcal{A}^2\varphi(x_c)}{\partial_x \mathcal{A}\varphi(x_c)}$. We deduce that there exists a continuous function $\hat{t} : [x_c, \hat{x}(\varepsilon)] \rightarrow [0, \varepsilon]$ such that $\hat{x}(\hat{t}(x)) = x$.

Assume $\hat{x}'(0^+) > 0$. Then the function \hat{t} is increasing. Moreover, $\partial_x \mathcal{A}\varphi(x) < 0$ which ensures $\forall x < x_c, \mathcal{A}\varphi(x) \geq 0$ and $\forall x > x_c, \mathcal{A}\varphi(x) \leq 0$. We set $T = \varepsilon$ and extend \hat{t} to \mathbb{R}_+^* by setting $\hat{t}(x) = T$ for $x > x_c + \varepsilon$ and $\hat{t}(x) = 0$ for $x < x_c$. The obtained function is continuous and the whole point is to show that for every x , the infimum of $t \mapsto v_\varphi(t, x)$ on $[0, T]$ is reached at $\hat{t}(x)$. This amounts to show that $\partial_t v_\varphi(t, x) = \mathcal{A}v_\varphi(t, x)$ is non-positive for (t, x) above \hat{x} (i.e. for $t \leq T$ and $x \geq \hat{x}(t)$) and non-negative below. If $(P_t)_{t \geq 0}$ denotes the semigroup associated with (0.1),

$$\mathcal{A}v_\varphi(t, x) = \mathcal{A}P_t\varphi(x) = P_t\mathcal{A}\varphi(x).$$

Let (t, x) belong to the above (respectively below) region. By the optimal stopping theorem, $\mathcal{A}v_\varphi(t, x)$ is equal to the expectation of the value of the martingale $(e^{-\rho u} P_{t-u}\mathcal{A}\varphi(X_u^x))_{0 \leq u \leq t}$ stopped at the border of the above (respectively below) region $\{(u, \hat{x}(u)), u \in [0, \varepsilon]\} \cup \{(0, x), x \geq x_c\}$ (respectively $\{(u, \hat{x}(u)), u \in [0, \varepsilon]\} \cup \{(0, x), x \leq x_c\}$) which is non-positive (respectively non-negative) since $\forall t \in]0, \varepsilon]$, $P_u\mathcal{A}\varphi(\hat{x}(u)) = \partial_u v_\varphi(u, \hat{x}(u)) = 0$ and $\forall x \geq x_c, \mathcal{A}\varphi(x) \leq 0$ (respectively $\forall x \geq x_c, \mathcal{A}\varphi(x) \geq 0$).

The case $\hat{x}'(0^+) < 0$ is handled in the same way. \square

Example 8. – As an application, consider the family of payoffs

$$\varphi_{a,b}(x) = x^{-\alpha} + x^a - x^b,$$

where $1 > a > b > -\alpha$. Then for $x \geq 1, x^a \geq x^b$, for $x < 1, x^{-\alpha} > x^b$ so that $\varphi_{a,b}$ is non-negative. Moreover

$$\lim_{x \rightarrow 0} \frac{\varphi_{a,b}(x)}{x + x^{-\alpha}} = 1, \quad \lim_{x \rightarrow +\infty} \frac{\varphi_{a,b}(x)}{x + x^{-\alpha}} = 0$$

and $\varphi_{a,b}$ satisfies (H0). Let $\lambda(y) = (\frac{\sigma^2}{2}y + \rho)(y - 1)$. Then

$$\mathcal{A}\varphi_{a,b}(x) = \lambda(a)x^a - \lambda(b)x^b,$$

which gives, since $\lambda(a) < 0$ and $\lambda(b) < 0$, $\mathcal{A}\varphi_{a,b}(x) < 0$ for $x > x_c$ and $\mathcal{A}\varphi_{a,b}(x) > 0$ for $x < x_c$ with $\lambda(a)x_c^a = \lambda(b)x_c^b$. Moreover

$$\mathcal{A}^2\varphi_{a,b}(x_c) = \lambda(a)^2 x_c^a - \lambda(b)^2 x_c^b = (\lambda(a) - \lambda(b))\lambda(b)x_c^b$$

and $\mathcal{A}^2\varphi_{a,b}(x_c) > 0$ as soon as $\lambda(b) > \lambda(a)$. Lastly,

$$\partial_x \mathcal{A}\varphi_{a,b}(x_c) = \lambda(a)ax_c^{a-1} - \lambda(b)bx_c^{b-1} = \left(\frac{a}{x_c} - \frac{b}{x_c}\right)\lambda(a)x_c^a \neq 0.$$

Of course, in this example, since $v_\varphi(t, x) = x^{-\alpha} + x^a e^{t\lambda(a)} - x^b e^{t\lambda(b)}$, everything can be computed explicitly and it is even possible to check the hypotheses of the global embedding result:

$$\left(\frac{x \vee x_c}{x_c}\right)^{a-b} = e^{\widehat{t}(x)(\lambda(b)-\lambda(a))}$$

and

$$\widehat{\varphi}_{a,b}(x) = x^{-\alpha} + x^a \left(\frac{x \vee x_c}{x_c}\right)^{\frac{\lambda(a)(a-b)}{\lambda(b)-\lambda(a)}} - x^b \left(\frac{x \vee x_c}{x_c}\right)^{\frac{\lambda(b)(a-b)}{\lambda(b)-\lambda(a)}}.$$

Similarly the hypothesis of Proposition 7 are satisfied by the payoff $x + x^b - x^a$ where $1 > a > b > -\alpha$ in case $\lambda(a) > \lambda(b)$.

In the global case, we could not find any simple condition on φ ensuring the existence of a continuous curve in the argument of $\inf_{t \geq 0} v_\varphi(t, x)$. Nevertheless, it is worth mentioning the following interesting class of European payoffs: if φ is a non-negative function equal to an invariant function $ax + bx^{-\alpha}$ with $a, b \geq 0, a + b > 0$, less a non-negative function ϕ satisfying $\lim_{x \rightarrow 0} \frac{\phi(x)}{x+x^{-\alpha}} = \lim_{x \rightarrow +\infty} \frac{\phi(x)}{x+x^{-\alpha}} = 0$, then

$$\forall x > 0, \forall t \geq 0, \quad v_\varphi(t, x) \leq ax + bx^{-\alpha} \quad \text{and} \\ \lim_{t \rightarrow +\infty} v_\varphi(t, x) = ax + bx^{-\alpha},$$

which implies that $\widehat{\varphi}(x) = \inf_{t \geq 0} v_\varphi(t, x)$ is not trivial and that $\forall x \in \mathbb{R}, \exists \widehat{t}(x) < +\infty, \widehat{\varphi}(x) = v_\varphi(\widehat{t}(x), x)$. The only assumption missing to apply Theorem 3 is the continuity of \widehat{t} .

The next section is dedicated to a family of payoffs φ included in the above class. In these examples, we explicit some American prices with a non-trivial Exercise region thanks to Theorem 3. We also check that the above mentioned continuity of \widehat{t} is not always satisfied.

3. Case study: $\alpha > -1$ and $\varphi(x) = x(\mathbf{1}_{\{x < K_1\}} + \mathbf{1}_{\{x > K_2\}})$

To be able to compare the invariant functions x and $x^{-\alpha}$, we need to compare $-\alpha$ and 1. We choose the case $\alpha > -1$ which is the more interesting from a financial point of view since $\rho \geq 0 \Leftrightarrow \alpha \geq 0$. The payoff φ is equal to the invariant function x less the

function $\phi(x) = x 1_{\{K_1 \leq x \leq K_2\}}$. Since

$$\forall x > 0, \forall t > 0, \quad 0 < v_\phi(t, x) < x \quad \text{and} \quad \lim_{t \rightarrow +\infty} v_\phi(t, x) = x$$

the function $t \rightarrow v_\phi(t, x)$ is likely to be increasing for $K_1 < x < K_2$ and decreasing then increasing otherwise. This remark together with the easiness of computations motivate the choice of this example. The function ϕ satisfies the growth assumption in (H0) but is not continuous. Therefore, even if we make the computations for ϕ , we shall after all apply our results to a suitable regularization of ϕ .

3.1. The case $K_1 = 0$

To simplify notations, we replace K_2 by K and write $\phi(x) = x 1_{\{x > K\}}$. This payoff corresponds to the sum of one Call and K Digit options with common strike K . Its simplicity allows to compute explicitly $\hat{\phi}$ and \hat{t} .

PROPOSITION 9. – Let $\phi(x) = x 1_{\{x > K\}}$. Then

$$v_\phi(t, x) = xN(d_1(t, x)),$$

where

$$d_1(t, x) = \frac{\ln(\frac{x}{K}) + (\rho + \frac{\sigma^2}{2})t}{\sigma\sqrt{t}}$$

and $N(d) = \int_{-\infty}^d e^{-\frac{y^2}{2}} \frac{dy}{\sqrt{2\pi}}$ is the cumulative distribution function of the normal law. Moreover,

$$\hat{\phi}(x) = x 1_{\{x > K\}} N\left(\frac{2}{\sigma} \sqrt{\left(\rho + \frac{\sigma^2}{2}\right) \ln\left(\frac{x}{K}\right)}\right) = v_\phi(\hat{t}(x), x)$$

and

$$\hat{t}(x) = \frac{\ln(x/K) 1_{\{x \geq K\}}}{\rho + \frac{\sigma^2}{2}},$$

and $\forall x > 0, t \rightarrow v_\phi(t, x)$ is strictly decreasing on $[0, \hat{t}(x)]$ and strictly increasing on $[\hat{t}(x), +\infty)$.

Proof. – Using Girsanov theorem, we get

$$\begin{aligned} v_\phi(t, x) &= \mathbb{E}\left[x e^{\sigma B_t - \frac{\sigma^2}{2}t} 1_{\{x e^{\sigma B_t + (\rho - \sigma^2/2)t} \geq K\}}\right] \\ &= x P(x e^{\sigma B_t + (\rho + \frac{\sigma^2}{2})t} \geq K) = xN(d_1(t, x)). \end{aligned}$$

By the chain rule, $\partial_t v_\phi(t, x) = xN'(d_1(t, x))\partial_t d_1(t, x)$. Since $\forall x, t > 0, xN'(d_1(t, x)) > 0$ and

$$\partial_t d_1(t, x) = \frac{(\rho + \frac{\sigma^2}{2})t - \ln(\frac{x}{K})}{2\sigma t^{3/2}},$$

we obtain that

$$\forall x > 0, \quad \begin{cases} \forall t \in]0, \hat{t}(x)[, \partial_t v_\varphi(t, x) < 0, \\ \forall t > \hat{t}(x), \partial_t v_\varphi(t, x) > 0. \end{cases}$$

Hence $\inf_{t \geq 0} v_\varphi(t, x) = v_\varphi(\hat{t}(x), x)$ and the explicit expression of this function is easily computed. \square

Let us now regularize things in order to apply our theorem. Let $u > 0$. The function $x \rightarrow v_\varphi(u, x)$ is continuous. Let $(P_t)_{t \geq 0}$ denote the semigroup associated with (0.1). By the semigroup property, the price of the European option with payoff $v_\varphi(u, x)$ is $P_t(P_u \varphi) = P_{t+u} \varphi$. If we set $\hat{\varphi}_u = \inf_{t \geq 0} P_t(P_u \varphi)$, then by the previous proposition, $\hat{\varphi}_u(x) = v_\varphi(u \vee \hat{t}(x), x) = P_{0 \vee (\hat{t}(x) - u)}(P_u \varphi)(x)$. Since \hat{t} is a continuous function with values in $[0, +\infty)$, so is $\hat{t}_u(x) = 0 \vee (\hat{t}(x) - u)$. Applying Theorem 3, we obtain the price of the American option with payoff $\hat{\varphi}_u$:

COROLLARY 10. – *Let $u > 0$. The price of the American option with payoff $\hat{\varphi}_u(x) = v_\varphi(u \vee \hat{t}(x), x)$ is*

$$\begin{aligned} v_{\hat{\varphi}_u}^{am}(t, x) &= v_\varphi((t + u) \vee \hat{t}(x), x) \\ &= x \left(N \left(\frac{2}{\sigma} \sqrt{\left(\rho + \frac{\sigma^2}{2} \right) \ln \left(\frac{x}{K} \right)} \right) 1_{\{t+u \leq \ln(x/K)/(\rho+\sigma^2/2)\}} \right. \\ &\quad \left. + N \left(\frac{\ln(\frac{x}{K}) + (\rho + \frac{\sigma^2}{2})(t + u)}{\sigma \sqrt{t + u}} \right) 1_{\{t+u > \ln(x/K)/(\rho+\sigma^2/2)\}} \right) \end{aligned}$$

and the Exercise region is given by $\{(t, x) : t + u \leq \ln(x/K)/(\rho + \sigma^2/2)\}$.

Remark 11. – Although the payoff $\hat{\varphi}_u$ has no financial meaning, this example provides a very interesting benchmark for numerical procedures devoted to American options since the price and the Exercise boundary are explicit. Let us also notice that this is a two-parameter (K and u) family of closed-formula. The payoff is of course obtained by setting t to zero in $v_{\hat{\varphi}_u}^{am}(t, x)$.

3.2. The case $K_1 > 0$

The main purpose of this subsection is to design an example where there is no continuous curve in the argument of the infimum (Proposition 13). By a slight modification of the computations made in the proof of Proposition 9, we get

$$\begin{aligned} v_\varphi(t, x) &= x(N(-d_1(t, x)) + N(d_2(t, x))), \\ \text{where for } i = 1, 2 \quad d_i(t, x) &= \frac{\ln(\frac{x}{K_i}) + (\rho + \frac{\sigma^2}{2})t}{\sigma \sqrt{t}}. \end{aligned}$$

It is not possible to compute $\hat{\varphi}$ explicitly but using the implicit functions theorem, we can study the sign of $\partial_t v_\varphi(t, x)$ to obtain:

LEMMA 12. – *There exist two differentiable functions $t \in \mathbb{R}_+^* \rightarrow \xi_1(t) < \xi_2(t)$ satisfying*

- (1) $\lim_{t \rightarrow 0} \xi_i(t) = K_i$ ($i = 1, 2$),
- (2) $\forall t > 0, \xi'_2(t) > 0$ and $\exists(\beta, T), 0 < \beta < T \leq (1 + \ln \sqrt{K_2/K_1})/(2\rho + \sigma^2)$,
 $\forall t < \beta, \xi'_1(t) > 0$ and $\forall t > T, \xi'_1(t) < 0$,
- (3) $\forall t > 0$,

$$\xi_2(t) > K_2 e^{(\rho + \frac{\sigma^2}{2})t} \quad \text{and} \quad \xi_1(t) < K_1 e^{(\rho + \frac{\sigma^2}{2})t} \wedge \sqrt{K_1 K_2} e^{-(\rho + \frac{\sigma^2}{2})t}$$

and such that

$$\forall t > 0, \forall x \in]\xi_1(t), \xi_2(t)[, \quad \partial_t v_\varphi(t, x) > 0 \quad \text{and}$$

$$\forall x \notin [\xi_1(t), \xi_2(t)], \quad \partial_t v_\varphi(t, x) < 0.$$

Proof. – An easy computation yields that $\partial_t v_\varphi(t, x)$ is equal to the product of a strictly positive function with $f(t, \ln x)$ where

$$f(t, y) = (y - a_1)e^{b_1 y + c_1} + (a_2 - y)e^{b_2 y + c_2} \quad \text{where for } i = 1, 2,$$

$$a_i(t) = \ln K_i + \left(\rho + \frac{\sigma^2}{2}\right)t, \quad b_i(t) = \frac{\ln K_i}{\sigma^2 t},$$

$$c_i(t) = \left(\frac{\rho}{\sigma^2} + \frac{1}{2}\right) \ln K_i - \frac{\ln^2 K_i}{2\sigma^2 t}.$$

Since $a_1 < a_2, f(t, a_2) = (a_2 - a_1)e^{b_1 a_2 + c_1} > 0$. Hence the function $y \rightarrow f(t, y)$ vanishes at the same points as

$$y \rightarrow g(t, y) = e^{(b_2 - b_1)y + (c_2 - c_1)} - \frac{y - a_1}{y - a_2}.$$

As $a_1 < a_2$ and $b_1 < b_2$, the function $y \rightarrow g(t, y)$ is strictly increasing from -1 to $+\infty$ on $] -\infty, a_2[$ and from $-\infty$ to $+\infty$ on $]a_2, +\infty[$, so it vanishes exactly twice. Let $y_1 < a_2 < y_2$ denote the corresponding points. Since $e^{(b_2 - b_1)y_1 + (c_2 - c_1)} > 0$ and $\frac{y_1 - a_1}{y_1 - a_2} < 1$, we obtain respectively $y_1 < a_1$ and $(b_2 - b_1)y_1 < c_1 - c_2$. We combine these upper-bounds to get

$$y_1(t) < a_1(t) \wedge \left(\ln \sqrt{K_1 K_2} - \left(\rho + \frac{\sigma^2}{2}\right)t\right). \tag{3.1}$$

We deduce that $x \rightarrow \partial_t v_\varphi(t, x)$ vanishes exactly twice, at the points $\xi_1(t) = e^{y_1(t)}$ and $\xi_2(t) = e^{y_2(t)}$ which satisfy statement (3). As $f(t, a_2) > 0, \partial_t v_\varphi(t, x)$ is strictly positive for $x \in (\xi_1(t), \xi_2(t))$. Moreover as $b_1 < b_2, f(t, y) < 0$ for $|y|$ large and $\partial_t v_\varphi(t, x)$ is strictly negative for $0 < x < \xi_1(t)$ and for $x > \xi_2(t)$.

Let us study more precisely the functions $y_1(t)$ and $y_2(t)$. Since $\forall t > 0, \forall y \neq a_2(t), \partial_y g(t, y) > 0$, by the implicit function theorem, for $i = 1, 2, y_i(t)$ is continuously differentiable and $y'_i(t)$ has the same sign as $-\partial_t g(t, y_i(t))$. Expliciting the dependence of g on the time variable, we get

$$g(t, y) = \exp\left(\frac{\ln(K_2/K_1)}{\sigma^2 t} \left(y + \left(\rho + \frac{\sigma^2}{2}\right)t - \ln \sqrt{K_1 K_2}\right)\right) - 1$$

$$\begin{aligned}
 & + \frac{\ln(K_1/K_2)}{y - \ln(K_2) - (\rho + \frac{\sigma^2}{2})t}, \\
 \partial_t g(t, y) = & \frac{\ln(K_1/K_2)}{\sigma^2 t^2} (y - \ln \sqrt{K_1 K_2}) \exp\left(\frac{\ln(K_2/K_1)}{\sigma^2 t} \left(y + \left(\rho + \frac{\sigma^2}{2}\right)t - \ln \sqrt{K_1 K_2}\right)\right) + \frac{(\rho + \frac{\sigma^2}{2}) \ln(K_1/K_2)}{(y - \ln K_2 - (\rho + \frac{\sigma^2}{2})t)^2}.
 \end{aligned}$$

Since $y_2(t) > a_2(t) > \ln \sqrt{K_1 K_2}$, $\partial_t g(t, y_2(t))$ is strictly negative and $\forall t > 0, y_2'(t) > 0$. Moreover, when $t \rightarrow 0$ the first term in $g(t, y_2(t))$ has a limit equal to $+\infty$ and the equation $g(t, y_2(t)) = 0$ implies that the second term goes also to ∞ which gives $\lim_{t \rightarrow 0} y_2(t) = \ln K_2$.

By (3.1), $y_1(t) < a_1(t) = \ln K_1 + (\rho + \sigma^2/2)t$. Hence when $t \rightarrow 0$ the first term in $g(t, y_1(t))$ has a limit equal to 0. By considering the other terms we deduce that $\lim_{t \rightarrow 0} y_1(t) = \ln K_1$. Hence the first term in $\partial_t g(t, y_1(t))$ goes to 0 and $\lim_{t \rightarrow 0} \partial_t g(t, y_1(t)) < 0$. Therefore $\exists \beta > 0, \forall t \in]0, \beta[, y_1'(t) > 0$.

Using the equality $g(t, y_1(t)) = 0$ to replace the exponential in $\partial_t g(t, y_1(t))$ and multiplying by $(y_1(t) - \ln K_2 - (\rho + \sigma^2/2)t)^2 / \ln(K_2/K_1)$, we obtain that $\partial_t g(t, y_1(t))$ has the same sign as

$$\begin{aligned}
 & \frac{-1}{\sigma^2 t^2} (y_1(t) - \ln \sqrt{K_1 K_2}) \left(y_1(t) - \ln K_1 - \left(\rho + \frac{\sigma^2}{2} \right) t \right) \\
 & \times \left(y_1(t) - \ln K_2 - \left(\rho + \frac{\sigma^2}{2} \right) t \right) - \left(\rho + \frac{\sigma^2}{2} \right).
 \end{aligned}$$

As by (3.1), $y_1(t) < \ln \sqrt{K_1 K_2} - (\rho + \sigma^2/2)t$, we conclude that for some $T \leq (1 + \ln \sqrt{K_2/K_1}) / (2\rho + \sigma^2), \forall t > T, \partial_t g(t, y_1(t)) > 0$ and $y_1'(t) < 0$. \square

So the situation looks like in Fig. 1.

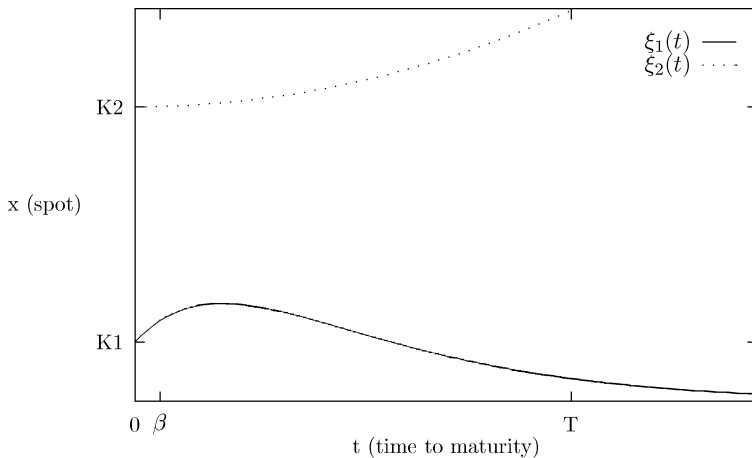


Fig. 1.

Let $u > 0$. The payoff $v_\varphi(u, x)$ satisfies (H0). Let

$$\widehat{\varphi}_u(x) = \inf_{t \geq 0} v_\varphi(t + u, x).$$

Since $(t, x) \rightarrow v_\varphi(t + u, x)$ is continuous and $t \rightarrow v_\varphi(t, x)$ is increasing for $t \geq t(x)$ where $t(x)$ is locally bounded (see Lemma 12), the function $\widehat{\varphi}_u(x)$ is continuous. According to Lemma 12, there exist $0 < \beta < T < +\infty$ such that $t \rightarrow \xi_1(t)$ is strictly increasing on $[0, \beta]$ and strictly decreasing on $[T, +\infty)$. Concerning the existence of a continuous function \widehat{t}_u such that $\widehat{\varphi}_u(x) = v_\varphi(\widehat{t}_u(x) + u, x)$ the situation depends on whether $u < \beta$ or $u \geq T$.

PROPOSITION 13. –

- If $u \geq T$, then $\widehat{\varphi}_u(x) = v_\varphi(\widehat{t}_u(x) + u, x)$ for the continuous function

$$\widehat{t}_u(x) = 1_{\{x \leq \xi_1(u)\}}(\xi_1^{-1}(x) - u) + 1_{\{x \geq \xi_2(u)\}}(\xi_2^{-1}(x) - u),$$

where ξ_1^{-1} denotes the inverse of the restriction of ξ_1 to $[T, +\infty)$ and the price of the American option with payoff $\widehat{\varphi}_u$ is

$$\begin{aligned} v_{\widehat{\varphi}_u}^{am}(t, x) &= v_\varphi((t \vee \widehat{t}_u(x)) + u, x) \\ &= v_\varphi((t + u) \vee \xi_1^{-1}(x), x) 1_{\{x \leq \xi_1(u)\}} + v_\varphi(t + u, x) 1_{\{\xi_1(u) < x < \xi_2(u)\}} \\ &\quad + v_\varphi((t + u) \vee \xi_2^{-1}(x), x) 1_{\{x \geq \xi_2(u)\}}. \end{aligned}$$

- If $u < \beta$, there is no continuous function \widehat{t}_u such that $\widehat{\varphi}_u(x) = v_\varphi(\widehat{t}_u(x) + u, x)$.
Moreover;

$$\forall t > 0, \forall x \in]\xi_1(t + u), \xi_2(t + u)[, \quad v_\varphi(t + u, x) > v_{\widehat{\varphi}_u}^{am}(t, x).$$

Proof. – We first suppose that $u \geq T$. According to Lemma 12, $t \in [0, +\infty) \rightarrow \xi_1(t + u)$ (respectively $t \in [0, +\infty) \rightarrow \xi_2(t + u)$) is decreasing (respectively increasing), and $\forall x \in]0, \xi_1^{-1}(u)[$ (respectively $\forall x \in]\xi_2^{-1}(u), +\infty[$) $t \rightarrow v_\varphi(t + u, x)$ is decreasing on $[0, \xi_1^{-1}(x) - u]$ (respectively $[0, \xi_2^{-1}(x) - u]$) and increasing on $[\xi_1^{-1}(x) - u, +\infty[$ (respectively $[\xi_2^{-1}(x) - u, +\infty[$). Moreover $\forall x \in [\xi_1(u), \xi_2(u)]$, $t \rightarrow v_\varphi(t + u, x)$ is increasing. Hence $\widehat{\varphi}_u(x) = v_\varphi(\widehat{t}_u(x) + u, x)$ for the continuous function

$$\widehat{t}_u(x) = 1_{\{x \leq \xi_1(u)\}}(\xi_1^{-1}(x) - u) + 1_{\{x \geq \xi_2(u)\}}(\xi_2^{-1}(x) - u),$$

and we deduce the price of the American option with payoff $\widehat{\varphi}_u$ by Theorem 3.

We turn to the case $u < \beta$. Let $F = \{(t, x) : v_\varphi(t + u, x) = \widehat{\varphi}_u(x)\}$. According to Lemma 12, $t \rightarrow \xi_1(t)$ is increasing on $[0, \beta]$. We deduce that $\forall t \in]u, \beta[$, $v_\varphi(t, \xi_1(t)) > v_\varphi(u, \xi_1(t))$ and $(t - u, \xi_1(t)) \notin F$. Hence

$$\begin{aligned} F &\subset F_1 \cup F_2 \quad \text{where } F_1 = \{(t - u, \xi_1(t)), t \geq \beta\} \quad \text{and} \\ F_2 &= \{(t - u, \xi_2(t)), t \geq u\} \cup \{(0, x), x \in [\xi_1(u), \xi_2(u)]\}. \end{aligned}$$

Let \hat{t}_u be such that $\forall x > 0$, $\hat{\varphi}_u(x) = v_\varphi(\hat{t}_u(x) + u, x)$ i.e. $(\hat{t}_u(x), x) \in F$. For x small enough $(\hat{t}_u(x), x) \in F_1$ and for x big enough $(\hat{t}_u(x), x) \in F_2$. Since F_1 and F_2 are not connected, the function \hat{t}_u is discontinuous.

Let $t > 0$ and $x \in (\xi_1(t + u), \xi_2(t + u))$. The positive continuous function

$$w \in W \rightarrow \Phi(w) = \inf_{s \in [0, t]} e^{-\rho s} \left(v_\varphi(t + u - s, x e^{\sigma w_s + (\rho - \frac{\sigma^2}{2})s}) - \hat{\varphi}_u(x e^{\sigma w_s + (\rho - \frac{\sigma^2}{2})s}) \right),$$

where $W = \{w \in C([0, t], \mathbb{R}), w(0) = 0\}$, is not constantly equal to 0. Indeed, when

$$\begin{aligned} \forall s \leq 0 \vee (t + u - \beta), \quad \xi_1(t + u - s) < x e^{\sigma w_s + (\rho - \frac{\sigma^2}{2})s}, \\ x e^{\sigma w_t + (\rho - \frac{\sigma^2}{2})t} < \xi_1(u), \\ \text{and } \forall s \leq t, \quad x e^{\sigma w_s + (\rho - \frac{\sigma^2}{2})s} < \xi_2(t + u - s), \end{aligned}$$

then $\forall s \in [0, t]$, $(t - s, x e^{\sigma w_s + (\rho - \frac{\sigma^2}{2})s}) \notin F$ and $\Phi(w) > 0$. As the support of the Wiener measure is W , $\mathbb{E}[\Phi((B_s)_{s \leq t})] > 0$. Let τ be a stopping time smaller than t . Then

$$\begin{aligned} v_\varphi(t + u, x) &= \mathbb{E}[e^{-\rho \tau} v_\varphi(t + u - \tau, X_\tau^x)] \\ &\geq \mathbb{E}[e^{-\rho \tau} \hat{\varphi}_u(X_\tau^x)] + \mathbb{E}[\Phi((B_s)_{s \leq t})]. \end{aligned}$$

Since τ is arbitrary, we conclude that $v_\varphi(t + u, x) - v_{\hat{\varphi}_u}^{am}(t, x) \geq \mathbb{E}[\Phi((B_s)_{s \leq t})] > 0$. \square

Remark 14. –

- (1) For any $x > 0$, $t \rightarrow v_\varphi(t + u, x)$ is continuous and increasing for t big enough. Hence $\tilde{t}_u(x) = \sup\{t: v_\varphi(t + u, x) = \hat{\varphi}_u(x)\}$ is finite. When $u < \beta$, $\forall t \leq \tilde{t}_u(x)$,

$$\begin{aligned} \hat{\varphi}_u(x) &\leq v_{\hat{\varphi}_u}^{am}(t, x) \leq v_{\hat{\varphi}_u}^{am}(\tilde{t}_u(x), x) \leq v_\varphi(\tilde{t}_u(x) + u, x), \\ \text{i.e. } v_{\hat{\varphi}_u}^{am}(t, x) &= \hat{\varphi}_u(x), \end{aligned}$$

but $\exists T(x)$ such that for $t \geq T(x)$, $x \in]\xi_1(t + u), \xi_2(t + u)[$ and we cannot deduce $v_{\hat{\varphi}_u}^{am}(t, x)$ from the price of the European option with payoff $\hat{\varphi}_u$.

- (2) Let $u < \beta$ and $x^* = \sup\{x: (t, x) \in F_1 \cap F\}$ where F, F_1 are defined in the previous proof. Since F_1 and F are closed and $\lim_{t \rightarrow +\infty} \xi_1(t) = 0$, $\exists t^* \geq \beta - u$ such that $(t^*, x^*) \in F_1 \cap F$ i.e. $v_\varphi(t^* + u, x^*) = \hat{\varphi}_u(x^*)$. Since $x^* = \sup\{x: (t, x) \in F_1 \cap F\}$, $\forall x \in]x^*, \xi_2(u)[$, $v_\varphi(u, x) = \hat{\varphi}_u(x)$ and by continuity, $v_\varphi(u, x^*) = \hat{\varphi}_u(x^*)$. Hence $\{t \geq 0, v_\varphi(t + u, x^*) = \hat{\varphi}_u(x^*)\}$ contains at least two elements which is not surprising with regard to Remark 4.

4. Analyticity and some consequences

In this section we give some properties of the map $\varphi \rightarrow \hat{\varphi}$ which are consequences of the following analyticity of v_φ in the pair (t, x) :

PROPOSITION 15. – *The function $v_\varphi(t, x)$ is analytic in $]0, \infty[\times \mathbb{R}_+^*$.*

This is a consequence of the same property for the solution of the standard heat equation (which does not seem to be universally known in fact but can be shown by a direct estimation of the derivatives of the solution).

4.1. One-to-one property

Let us study now the injectivity of $\varphi \rightarrow \widehat{\varphi}$. Obviously if φ_1 is an invariant function (e.g. the nil function), then $\widehat{\varphi}_1 = \varphi_1$ and there are plenty of other functions φ such that $\widehat{\varphi} = \widehat{\varphi}_1$, for instance $\varphi = \varphi_1 + \phi$ where ϕ is a non-negative continuous function satisfying $\lim_{x \rightarrow 0} \frac{\phi(x)}{x+x^{-\alpha}} = \lim_{x \rightarrow +\infty} \frac{\phi(x)}{x+x^{-\alpha}} = 0$. The same phenomenon occurs if $\widehat{t}(x) = \infty$ everywhere. Therefore the following one-to-one statement is optimal:

PROPOSITION 16. – *Let φ_1 and φ_2 satisfy the assumptions of Theorem 3 and assume that φ_1 is not an invariant function and that there is a value x_0 such that $\widehat{t}_1(x_0) < \infty$. Then*

$$\widehat{\varphi}_1 = \widehat{\varphi}_2 \Rightarrow \varphi_1 = \varphi_2.$$

Proof. – For any $t_0 > \widehat{t}_1(x_0)$ there is an $\varepsilon > 0$ small enough such that the ball B centered at (t_0, x_0) with radius ε lies in the Continuation region of $\widehat{\varphi}_1$. In particular $v_{\widehat{\varphi}_1}^{am}(t, x) = v_{\varphi_1}(t, x)$ on B and $(t, x) \mapsto v_{\widehat{\varphi}_1}^{am}(t, x)$ is analytic on B . Now $v_{\widehat{\varphi}_1}^{am}(t, x) = v_{\widehat{\varphi}_2}^{am}(t, x)$, therefore $v_{\widehat{\varphi}_2}^{am}(t, x)$ is analytic on B . If $\widehat{t}_2(x_0) = +\infty$ then by taking ε small enough we can assume $v_{\widehat{\varphi}_2}^{am}(t, x) = \widehat{\varphi}_2(x)$ on B . Therefore $\partial_t v_{\varphi_1}(t, x) = 0$ on B and by the analyticity of v_{φ_1} , $\partial_t v_{\varphi_1}(t, x) = 0$ everywhere, which gives that φ_1 is an invariant function. Since this case is ruled out by assumption, $\widehat{t}_2(x_0) < \infty$. Now on the right of $\widehat{t}_1(x_0) \vee \widehat{t}_2(x_0)$, v_{φ_1} and v_{φ_2} match on some small enough ball, therefore everywhere, which gives $\varphi_1 = \varphi_2$ by the continuity of $t \mapsto v_{\varphi_i}(t, x)$ at $t = 0$. \square

4.2. On the range of $\varphi \rightarrow \widehat{\varphi}$

PROPOSITION 17. – *Let φ satisfy the assumptions of Theorem 3 and assume that φ is not an invariant function and that $0 < \widehat{t}(x_0) < \infty$ for some point x_0 . Then there is an open dense subset of $\widehat{t}^{-1}(\mathbb{R}_+^*)$ on which \widehat{t} , and therefore $\widehat{\varphi}$, is analytic.*

Proof. – First note that by composition $\widehat{\varphi}(x) = v_{\varphi}(\widehat{t}(x), x)$ is analytic as soon as \widehat{t} is. Pick up some point x_1 such that $0 < \widehat{t}(x_1) < \infty$. Since v_{φ} is analytic, by the implicit function theorem the equation $\partial_t v_{\varphi}(t, x) = 0$ on a small enough neighborhood V of $(\widehat{t}(x_1), x_1)$ defines an analytic curve $x \mapsto a(x)$ as soon as $\partial_t^2 v_{\varphi}(\widehat{t}(x_1), x_1) \neq 0$. Now by the continuity of \widehat{t} , $\partial_t v_{\varphi}(\widehat{t}(x), x) = 0$ in V so that $a \equiv \widehat{t}$ on a neighborhood of x_1 .

Unfortunately it is not granted that there exists x_1 such that $\partial_t^2 v_{\varphi}(\widehat{t}(x_1), x_1) \neq 0$.

Let us first remark that there is some point x_2 in $\widehat{t}^{-1}(\mathbb{R}_+^*)$ such that $t \mapsto v_{\varphi}(t, x_2)$ is not constant: otherwise $\partial_t v_{\varphi}(t, x) = 0$ would hold on some non-empty open set and therefore everywhere, thus φ would be an invariant function. Moreover the set of such points is dense in $\widehat{t}^{-1}(\mathbb{R}_+^*)$. Let $q = \inf\{n > 0, \partial_t^n v_{\varphi}(\widehat{t}(x_2), x_2) \neq 0\}$. Then q is finite. If $q = 2$ we are over. Otherwise notice first that the equation $\partial_{t^{q-1}} v_{\varphi}(t, x) = 0$ on a small enough neighborhood V of $(\widehat{t}(x_2), x_2)$ defines an analytic curve $x \mapsto b(x)$. Consider then the quantity $q(x) = \inf\{n > 0, \partial_t^n v_{\varphi}(\widehat{t}(x), x) \neq 0\}$ on a neighborhood of x_2 . By the analyticity of v_{φ} , $q(x) \leq q(x_2)$ on a sufficiently small neighborhood W of x_2 . Either $q(x) \equiv q(x_2)$, in which case $\partial_{t^{q-1}} v_{\varphi}(\widehat{t}(x), x) = 0$ on W , therefore $\widehat{t} \equiv b$ is analytic on W , otherwise there is some point x_3 in W such that $0 < \widehat{t}(x_3) < \infty$ and $q(x_3) < q(x_2)$.

By induction we thus either stop and get an analytic curve or reach the level $q = 2$ at some point. The proof is complete. \square

This proposition gives a first characterization statement about the functions $\widehat{\varphi}$:

COROLLARY 18. – *Let φ satisfy the assumptions of Theorem 3. Then $\widehat{\varphi}$ is either \mathcal{A} -superharmonic (i.e. $\mathcal{A}\widehat{\varphi} \geq 0$ in a weak sense) or \mathcal{A} -subharmonic (i.e. $\mathcal{A}\widehat{\varphi} \leq 0$ in a weak sense) or analytic on a non-empty open subset of \mathbb{R}_+^* . In particular $\varphi \rightarrow \widehat{\varphi}$ is not onto on the space of functions satisfying (H0).*

Proof. – The first case corresponds to $\widehat{t} = 0$ everywhere, the second one to the case $\widehat{t} = +\infty$ everywhere ($\widehat{\varphi}(x) = \liminf_{t \rightarrow +\infty} v_\varphi(t, x)$ is then an \mathcal{A} -subharmonic function by Fatou's lemma) and the last one to the previous proposition. \square

5. Conclusion

In this paper, for a fairly general class of payoffs φ , we deduce from the European price $v_\varphi(t, x)$ the American price of the claim with payoff $\widehat{\varphi}(x) = \inf_{t \geq 0} v_\varphi(t, x)$. We give examples of explicit computations. The characterization of the payoffs $\widehat{\varphi}$ obtained in this way remains an open question. A work devoted to design new approximations of the American Put price relying on our approach is in progress.

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