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Analyse non linéaire

Isotropic singularities of solutions of nonlinear elliptic inequalities

by

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ABSTRACT. — If g is nondecreasing function satisfying the weak singularities existence condition then all the positive solutions of $\Delta u \leqslant g(u)+f$ in $B_1(0)\setminus\{0\}$ where f is radial and integrable in $B_1(0)$ are isotropic in measure near 0. We apply this result to solutions of $\Delta u \pm g(u)=0$ in particular when $g(r)\sim r|r|^{q-1}$, $g(r)\sim e^{\beta r}$, or $g(r)=r(L_n^+ r)^{\alpha}$.

Key words : Elliptic equations, fundamental solutions, singularities, convergence in measure.

RÉSUMÉ. — Si g est une fonction croissante sur $\mathbb{R}$ vérifiant la condition d'existence de singularités faibles et f une fonction intégrable radiale dans $B_1(0)$, alors toutes les solutions positives de $\Delta u \leqslant g(u)+f$ dans $B_1(0)\setminus\{0\}$ sont isotropes en mesure près de 0. Nous appliquons ce résultat aux solutions de $\Delta u \pm g(u)=0$, en particulier quand $g(r)\sim r|r|^{q-1}$, $g(r)\sim e^{\beta r}$ ou $g(r)=r(L_n^+ r)^{\alpha}$.

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0. INTRODUCTION

Let Ω be an open subset of $\mathbb{R}^N$ containing 0 and $\Omega'=\Omega\setminus\{0\}$. In the past few years many results about the behaviour near 0 of a positive function $u\in C^2(\Omega')$ satisfying

$$(0.1)\qquad \Delta u=u^q$$

or

$$(0.2)\qquad \Delta u=-u^q$$

($q>1$) in Ω' have been published ([1], [2], [7], [8], [11], [23]). Although those equations are very different (existence or nonexistence of a comparison principle between their solutions), there exists a great similarity between them in the case $N\geqq 3$ and $1<q<N/(N-2)$ in the sense that there always exist solutions satisfying

$$(0.3)\qquad \lim_{x\to 0}|x|^{N-2}u(x)=\gamma$$

with $\gamma>0$, which implies that

$$(0.4)\qquad \Delta u=u^q-C(N)\gamma\delta_0$$

or

$$(0.5)\qquad \Delta u=-u^q-C(N)\gamma\delta_0$$

holds in $D'(\Omega)$ ([23], [11]) where δ_0 is the Dirac measure at 0 and $C(N)=(N-2)\,|S^{N-1}|$ if $N\geqq 3$, $C(2)=2\pi$, but the two proofs of this phenomenon run very differently. In fact the main point to notice is that for a u satisfying (0.3) u^q is integrable near 0 and this leads us to a new type of isotropy which is the key-stone for the study of isolated singularities of positive solutions of nonlinear elliptic inequalities of the following type

$$(0.6)\qquad \Delta u\leqq g(u)+f.$$

Assume $N\geqq 3$, g *is a continuous nondecreasing function defined on* $[0,+\infty)$ *satisfying the weak singularities existence condition*

$$(0.7)\qquad \int_0^1 g(r^{2-N})r^{N-1}\,dr<+\infty,$$

$f\in L^1_{loc}(\Omega)$ *is radial near* 0 *and* $u\in C^2(\Omega')$ *is a positive solution of* (0.6) *in* Ω'. *Then*

(i) *either there exists* $\gamma\in[0,+\infty)$ *such that* $r^{N-2}u(r,.)$ *converges in measure on* S^{N-1} *to* γ *as* r *tends to* 0,

(ii) *or* $\lim_{x \to 0} |x|^{N-2} u(x) = +\infty$.

In the case $N=2$ it is necessary to introduce the exponential order of growth of g [20]

$$(0.8) \qquad a_g^+ = \inf\{a>0: \int_0^{+\infty} e^{-ar} g(r)\,dr < +\infty\},$$

and we prove that *under the same conditions on* f *and* u *satisfying* (0.6) *in* Ω'; *then*

– *if* $a_g^+ = 0$ *we have either* (i) *or* (ii) *with* $|x|^{2-N}$ *replaced by* $\mathrm{Ln}(1/|x|)$

– *if* $a_g^+ > 0$ *we have*

(iii) *either there exists* $\gamma \in [0, 2/a_g^+)$ *such that* $u(r,.)/\mathrm{Ln}(1/r)$ *converges in measure to* γ *on* S^1 *as* r *tends to* 0,

(iv) *or* $\varliminf_{x \to 0} u(x)/\mathrm{Ln}(1/|x|) \geqq 2/a_g^+$.

Those results play an important role for the description of isolated singularities of nonnegative solutions of

$$(0.9) \qquad \Delta u = g(u).$$

For example, when $N \geqq 3$ we prove that *if* g *is nondecreasing and satisfies the weak singularities existence condition, then any* $u \in C^2(\Omega')$ *nonnegative and satisfying* (0.9) *in* Ω' *is such that* $|x|^{N-2} u(x)$ *converges to some* $\gamma \in \mathbb{R}^+ \cup \{+\infty\}$ *as* x *tends to* 0. This result extends to the case $N=2$ with some minor modifications. An other important tool for proving this type of result is Serrin and Ni's symmetry theorem [12].

When g *has nonpositive values we prove that when* $N \geqq 3$ *any nonnegative solution* $u \in C^2(\Omega')$ *of* (0.9) *is such that* $r^{N-2} u(r,.)$ *converges in* $L^1(S^{N-1})$ *to some* $\gamma \in [0, +\infty)$ *as* r *tends to* 0. Under a moderate growth assumption on g we prove that $\lim_{x \to 0} |x|^{N-2} u(x) = \gamma$. When $N=2$ the situation is quite more complicated. Using a result due to John and Nirenberg we prove that *when* g *has nonpositive values and is of exponential or subexponential type any nonnegative solution* u *of* (0.9) *in* Ω' *satisfies*

$$(0.10) \qquad \lim_{x \to 0} u(x)/\mathrm{Ln}(1/|x|) = \gamma \in [0, 2/a_g^+).$$

The last section is devoted to the study of the behavior near 0 of positive solutions of

$$(0.11) \qquad \Delta u = u(\mathrm{Ln}^+ u)^\alpha$$

in $\Omega'(\alpha>0)$. This equation reduces to a Hamilton-Jacobi equation in setting $v=\mathrm{Ln}^+ u$ and v satisfies

$$\Delta v+|Dv|^2=v^\alpha \tag{0.12}$$

on $\{x\in\Omega' : u(x)\geqq 1\}$. If we set $g(r)=r(\mathrm{Ln}^+ r)^\alpha$, it is clear that (0.7) is always satisfied, hence for any $\gamma\geqq 0$ there always exist solutions satisfying (0.3); however Vazquez *a priori* estimate condition

$$\int_{r_0}^{+\infty}\frac{ds}{\sqrt{s\,g(s)}}<+\infty \tag{0.13}$$

for some $r_0>0$ is satisfied if and only if $\alpha>2$ and we prove the following:

Assume $N\geqq 3$ *and* $u\in C^2(\Omega')$ *is a nonnegative solution of* (0.11) *in* Ω'; *then*

– *if* $0<\alpha\leqq 2$

(i) *either u can be extended to* Ω *as a* C^2 *solution of* (0.11) *in* Ω

(ii) *or there exists* $\gamma>0$ *such that* $\lim_{x\to 0}|x|^{N-2}u(x)=\gamma$.

– *if* $\alpha>2$

(iii) *either u behaves as in (i) or (ii)*

(iv) *or* $u(x)=\gamma(\alpha,N)\ e^{\gamma(\alpha)|x|^{2/(2-\alpha)}}(1+O(|x|^{2/(\alpha-2)})$ *near* 0 *with* $\gamma(\alpha)=\left(\frac{2}{\alpha-2}\right)^{2/(\alpha-2)}$ *and* $\gamma(\alpha,N)=e^{(\alpha-(N-1)(\alpha-2))/2\alpha}$. This result extends in dimension 2.

The contents of this article is the following:

1. Isotropic solutions of elliptic inequalities
2. Singular solutions of $\Delta u=\pm g(u)$
3. Singularities of $\Delta u=u(\mathrm{Ln}^+ u)^\alpha$.

1. ISOTROPIC SOLUTIONS OF ELLIPTIC INEQUALITIES

Throughout this section Ω is an open subset of $\mathbb{R}^N$, $N\geqq 2$ containing 0, $\Omega'=\Omega\setminus\{0\}$ and g is a nondecreasing function. For the sake of simplicity we shall assume that g is continuous. If $N\geqq 3$ it is wellknown that the following condition

$$\int_0^1 g(r^{2-N})\,r^{N-1}\,dr<+\infty, \tag{1.1}$$

is a necessary and sufficient condition for the existence for any $\gamma \geqq 0$ of a solution ψ belonging to some appropriate Marcinkiewicz space of

$$-\Delta\psi + g(\psi) = C(N)\gamma\delta_0 \tag{1.2}$$

in $D'(\Omega)$ [3], or equivalently of a solution of

$$-\Delta\psi + g(\psi) = 0 \tag{1.3}$$

in Ω' with a weak singularity at 0, that is such that

$$\lim_{x \to 0} |x|^{N-2} u(x) = \gamma, \tag{1.4}$$

[22]. Moreover $g(\psi) \in L^1_{loc}(\Omega)$.

If $N=2$ the situation is more complicated and we define the exponential order of growth of g

$$a_g^+ = \inf\left\{ a > 0 : \int_0^{+\infty} e^{-ar} g(r)\, dr < +\infty \right\} \tag{1.5}$$

[20], and the condition $\gamma \in [0, 2/a_g^+]$ is a necessary and sufficient condition for the existence of a function $\psi \in C^2(\Omega')$ satisfying (1.3) in Ω' and

$$\lim_{x \to 0} \psi(x)/Ln(1/|x|) = \gamma. \tag{1.6}$$

Moreover for such a ψ, $g(\psi) \in L^1_{loc}(\Omega)$ and (1.2) holds in $D'(\Omega')$ [21]. Our first result is the following

PROPOSITION 1.1. — *Assume* $\bar{B}_R = \{x \in \mathbb{R}^N : |x| \leqq R\} \subset \Omega$, $g(0)=0$, $f \in L^1_{loc}(\Omega)$ *is nonnegative and* $u \in C^2(\Omega')$ *is a nonnegative solution of*

$$\Delta u \leqq g(u) + f \tag{1.7}$$

in Ω'. *If* $v \in C^2(\bar{B}_R \setminus \{0\})$ *is a radial nonnegative solution of*

$$\Delta v = g(v) \tag{1.8}$$

in $B_R \setminus \{0\}$ *such that* $g(v+\bar{\delta}) \in L^1(B_R)$ *for some* $\bar{\delta} > 0$, *then there exists* $\alpha \geqq 0$ *such that for any* $q \in [1, \infty)$

$$\lim_{x \to 0} |x|^{1-N} \int_{|y|=|x|} |\alpha - \omega(y)/\mu(y)|^q \, dS = 0 \tag{1.9}$$

where $\omega = \inf(u, v)$, $\mu(x) = |x|^{2-N}$ *if* $N \geqq 3$ *and* $\mu(x) = Ln(1/|x|)$ *if* $N=2$.

The main ingredient for proving this result is the following theorem due to Brezis and Lions [5].

LEMMA 1.1. — *Assume* $N \geqq 2$, $\omega \in L^1_{loc}(\Omega')$ *satisfies*

$\Delta\omega \in L^1_{loc}(\Omega')$ *in the sense of distributions in* Ω',

(1.10)
$$\omega \geqq 0 \quad a.e. \text{ in } \Omega',$$
$$\Delta\omega \leqq a\omega + F \quad a.e. \text{ in } \Omega',$$

where a is some nonnegative constant and $F \in L^1_{loc}(\Omega)$. *Then* $\omega \in L^1_{loc}(\Omega)$ *and there exist* $\alpha \geqq 0$ *and* $\Phi \in L^1_{loc}(\Omega)$ *such that*

(1.11)
$$-\Delta\omega = \Phi + \alpha C(N)\,\delta_0$$

in $D'(\Omega)$.

LEMMA 1.2. — *Assume* $N \geqq 2$, $h \in L^1(B_R)$ *is radial and* φ *is a nonnegative radial solution of*

(1.12)
$$-\Delta\varphi = h$$

in $D'(B_R \setminus \{0\})$ [*resp. in* $D'(B_R)$]. *Then there exists* $\nu \in [0, +\infty)$ *such that* $\lim_{x \to 0} \varphi(x)/\mu(x) = \nu$ [*resp.* $\lim_{x \to 0} \varphi(x)/\mu(x) = 0$].

Proof. — From Lemma 1.1 there exists $\nu \geqq 0$ such that

(1.13)
$$-\Delta\varphi = h + \nu C(N)\,\delta_0$$

in $D'(B_R)$ and $\tilde{\varphi} = \varphi - \nu\mu$ satisfies (1.12) in $D'(B_R)$. Without any loss of generality we can assume that h is nonnegative in $B(0, R)$, hence $r \mapsto r^{N-1}\tilde{\varphi}_r(r)$ is nonincreasing and then keeps a constant sign near 0.

Case 1. — $r^{N-1}\tilde{\varphi}_r(r) > 0$ on $(0, \varepsilon]$. For n large enough define

(1.14)
$$\eta_n(r) = \begin{cases} 1 & \text{if } 0 \leqq r \leqq \dfrac{1}{n}, \\[2mm] \dfrac{1}{2}\left(1 + \cos\left(n\pi\left(r - \dfrac{1}{n}\right)\right)\right) & \text{if } \dfrac{1}{n} \leqq r \leqq \dfrac{2}{n}, \\[2mm] 0 & \text{if } \dfrac{2}{n} \leqq r \leqq \varepsilon. \end{cases}$$

$0 \leqq \eta_n \leqq 1$ on $[0, \varepsilon]$ and $\int_0^\varepsilon \eta_{nr}(r)\,dr = -1$. From (1.12) we get

$$\left| \int_0^\varepsilon \tilde{\varphi}_r(r)\,\eta_{nr}(r)\,r^{N-1}\,dr \right| = \int_0^\varepsilon h(r)\,\eta_n(r)\,r^{N-1}\,dr.$$

Using the monotonicity of $r^{N-1}\varphi_r(r)$ we deduce

(1.15)
$$0 \leqq \left(\frac{2}{n}\right)^{N-1} \tilde{\varphi}_r\left(\frac{2}{n}\right) \leqq \left| \int_{1/n}^{2/n} \tilde{\varphi}_r(r)\,\eta_{nr}(r)\,r^{N-1}\,dr \right| \leqq \int_0^{2/n} h(r)\,r^{N-1}\,dr$$

which implies $\lim_{n \to +\infty} \left(\frac{2}{n}\right)^{N-1} \tilde{\varphi}_r\left(\frac{2}{n}\right)=0$ and

$$\lim_{r \to 0} r^{N-1} \tilde{\varphi}_r(r)=0. \tag{1.16}$$

Case 2. — $r^{N-1} \tilde{\varphi}_r(r) \leqq 0$ on $(0, \varepsilon]$. Using the same method as above we get

$$0 \leqq -\left(\frac{1}{n}\right)^{N-1} \tilde{\varphi}_r\left(\frac{1}{n}\right) \leqq \int_0^{2/n} h(r) r^{N-1} dr \tag{1.17}$$

which again implies (1.16).

From (1.16) it is clear that $\lim_{x \to 0} \tilde{\varphi}(x)/\mu(x)=0$.

Proof of Proposition 1.1. — Let p be the $C^{1,1}$ even convex function defined on $\mathbb{R}$ by

$$p(t)=\begin{cases} |t|-\delta/2 & \text{for } |t| \geqq \delta>0 \\ t^2/2\delta & \text{for } |t| \leqq \delta \end{cases}$$

and let ω_δ be $\frac{1}{2}(u+v-p(u-v))$. Then

$$\Delta\omega_\delta=\frac{1}{2}\Delta(u+v)-\frac{1}{2}p'(u-v)\Delta(u-v)-\frac{1}{2}p''(u-v)|\nabla(u-v)|^2 \tag{1.18}$$

It is clear that $\Delta\omega_\delta \in L^1_{loc}(B_R \setminus \{0\})$ and $0 \leqq \omega \leqq \omega_\delta \leqq \omega+\delta/4$. Moreover

$$\Delta\omega_\delta \leqq \frac{1}{2}\Delta(u+v)-\frac{1}{2}p'(u-v)\Delta(u-v)=F. \tag{1.19}$$

We now set $B_R \setminus \{0\}=G_1 \cup G_2 \cup G_3$ with

$$\begin{aligned} G_1&=\{x \in B_R \setminus \{0\} : (u-v)(x)>\delta\} \\ G_2&=\{x \in B_R \setminus \{0\} : (u-v)(x)<-\delta\} \\ G_3&=\{x \in B_R \setminus \{0\} : |(u-v)(x)| \leqq \delta\}. \end{aligned} \tag{1.20}$$

On G_1, $p'(u-v)=1$ and $F=\Delta v=g(v)=g\left(\omega_\delta-\frac{\delta}{4}\right)$. On G_2, $p'(u-v)=-1$ and

$$F=\Delta u \leqq g(u)+f=g\left(\omega_\delta-\frac{\delta}{4}\right)+f \leqq g(v)+f.$$

On G_3, $p'(u-v)=(u-v)/\delta$, hence

$$(1.21)\quad F=\frac{1}{2}\left(1-\frac{u-v}{\delta}\right)\Delta u+\frac{1}{2}\left(1+\frac{u-v}{\delta}\right)\Delta v$$
$$\leqq\frac{1}{2}\left(1-\frac{u-v}{\delta}\right)g(u)+\frac{1}{2}\left(1+\frac{u-v}{\delta}\right)g(v)+f$$

and by the continuity of g there exists $\theta=\theta(x)\in[0,1]$ such that $F\leqq g(\theta u+(1-\theta)v)+f$. If we assume for example that $v\leqq u\leqq v+\delta$, then $F\leqq g(u)+f$ and $0\leqq u-\omega_\delta\leqq\frac{3}{4}\delta$ which implies that

$$F\leqq g\left(\omega_\delta+\frac{3}{4}\delta\right)+f\leqq g(v+\delta)+f.$$

We do the same if $u\leqq v\leqq u+\delta$ and finally

$$(1.22)\qquad \Delta\omega_\delta\leqq g\left(\omega_\delta+\frac{3}{4}\delta\right)+f\leqq g(v+\delta)+f$$

holds in $B_R\setminus\{0\}$. We take now $\delta\leqq\bar{\delta}$, so the right-hand side of (1.22) is integrable in B_R and there exists $\alpha\geqq 0$ such that

$$(1.23)\qquad -\Delta\omega_\delta=\Phi+\alpha C(N)\delta_0$$

in $\mathbf{D}'(B_R)$ with $\Phi\in L^1_{loc}(B_R)$.

From Lemma 1.2. $\omega_\delta(x)/\mu(x)$ remains bounded near 0 and it is the same with $\varphi_\delta=\omega_\delta-\alpha\mu$. Moreover φ_δ satisfies

$$(1.24)\qquad -\Delta\varphi_\delta=\Phi$$

in $\mathbf{D}'(B_R)$. Let

$$\bar{\varphi}_\delta(r)=\frac{1}{|S^{N-1}|}\int_{S^{N-1}}\varphi_\delta(r,\sigma)\,d\sigma$$

and

$$\bar{\Phi}(r)=\frac{1}{|S^{N-1}|}\int_{S^{N-1}}\Phi(r,\sigma)\,d\sigma$$

be the spherical averages of φ_δ and Φ respectively, (r,σ) being the spherical coordinates in $\mathbb{R}^N\setminus\{0\}$, then

$$(1.25)\qquad -\Delta\bar{\varphi}_\delta=\bar{\Phi}\leqq|\bar{\Phi}|.$$

Applying Lemma 1.2 we deduce that $\lim_{r \to 0} \bar{\varphi}(r)/\mu(r)=0$. As a consequence

$$\lim_{r \to 0} \int_{S^{N-1}} |\omega_\delta(r,.)/\mu(r)-\alpha|\, d\sigma=0,$$

which implies (with the uniform boundedness)

$$\lim_{r \to 0} \int_{S^{N-1}} |\omega_\delta(r,.)/\mu(r)-\alpha|^q\, d\sigma=0 \tag{1.26}$$

for any $q \in [1, +\infty)$. As $0 \leqq \omega \leqq \omega_\delta \leqq \omega+\delta/4$ we deduce

$$\lim_{r \to 0} \int_{S^{N-1}} |\omega(r,.)/\mu(r)-\alpha|^q\, d\sigma=0, \tag{1.27}$$

which is (1.9).

Remark 1.1. — As $\{\Delta\omega_\delta\}=\Phi$ is integrable in B_R and $\Phi=\Delta\omega_\delta=F-\frac{1}{2}p''(u-v)|\nabla(u-v)|^2$ we get

$$\frac{1}{2}p''(u-v)|\nabla(u-v)|^2 \leqq \Phi+g(v+\delta)+f \tag{1.28}$$

and then $p''(u-v)|\nabla(u-v)|^2 \in L^1(B_R)$.

DEFINITION 1.1. — Assume (E, Σ, μ) is an abstract measure space where Σ is a σ-algebra of subsets of E and μ a positive σ-additive and complete measure such that $\mu(E) < +\infty$, and $\{\psi_r\}_{r \in (0, R)}$ a subset of measurable functions (for the measure μ) with value in $\mathbb{R}$. We say that $\{\psi_r\}$ *converges in measure* to some measurable function ψ as r tends to 0 if for any $\varepsilon>0$ we have

$$\lim_{r \to 0} \mu(\{x \in E : |\psi_r(x)-\psi(x)|>\varepsilon\})=0. \tag{1.29}$$

It is equivalent to say that from any sequence $\{r_n\}$ converging to 0 we can extract a subsequence $\{r_{n_k}\}$ such that $\{\psi_{r_{n_k}}\}$ converges to ψ $\mu-a.e.$ on E as n_k goes to $+\infty$.

The generic isotropy result is the following

THEOREM 1.1. — *Assume* $N \geqq 3$, g *satisfies* (1.1), $f \in L^1_{loc}(\Omega')$ *is radial near* 0 *and* $u \in C^2(\Omega')$ *is nonnegative and satisfies*

$$\Delta u \leqq g(u)+f \tag{1.30}$$

in Ω'. *Then we have the following*

(i) *either* $r^{N-2}u(r,.)$ *converges in measure on* S^{N-1} *to some nonnegative real number* γ *as* r *tends to* 0,

(ii) *or*

$$\lim_{x \to 0} |x|^{N-2} u(x) = +\infty. \tag{1.31}$$

Proof. — We recall that $(r, \sigma) \in (0, +\infty) \times S^{N-1}$ are the spherical coordinates in $\mathbb{R}^N \setminus \{0\}$. For $\lambda > 0$ let v_λ be the solution of

$$\begin{aligned} \Delta v_\lambda &= g(v_\lambda) + |f| \quad \text{in } B_R \setminus \{0\} \subset \Omega' \\ v_\lambda &= 0 \quad \text{on } \partial B_R \\ \lim_{x \to 0} & |x|^{N-2} v_\lambda(x) = \lambda. \end{aligned} \tag{1.32}$$

Such a v_λ exists, is radial and positive near 0. As $|f|$ is radial it does not affect the behaviour of v_λ near 0 (*see* Lemma 1.2).

From Proposition 1.1 there exists $\nu(\lambda) \geqq 0$ such that

$$\lim_{r \to 0} r^{N-2} \inf(u(r,.), v_\lambda(r)) = \nu(\lambda) \tag{1.33}$$

in $L^q(S^{N-1})$, $1 \leqq q < +\infty$, and $\nu(\lambda) \leqq \lambda$ from convexity. Moreover the function $\lambda \mapsto \nu(\lambda)$ is nondecreasing.

Case 1. — Assume $\lim_{\lambda \to +\infty} \nu(\lambda) = \gamma < +\infty$. For $\lambda > \gamma$ we have (1.33). Assume $\{r_n\}$ is some sequence converging to 0, then there exists a subsequence $\{r_{n_k}\}$ such that

$$\lim_{n_k \to +\infty} r_{n_k}^{N-2} \inf(u(r_{n_k}, \sigma), v_\lambda(r_{n_k})) = \nu(\lambda) \quad a.e. \text{ on } S^{N-1}. \tag{1.34}$$

As $\nu(\lambda) < \gamma$ and $\lim_{n_k \to +\infty} r_{n_k}^{N-2} v_\lambda(r_{n_k}) = \gamma$ we deduce that

$$\inf(u(r_{n_k}, \sigma), v_\lambda(r_{n_k})) = u(r_{n_k}, \sigma) \quad a.e. \text{ on } S^{N-1}$$

for n_k large enough and

$$\lim_{n_k \to +\infty} r_{n_k}^{N-2} u(r_{n_k}, \sigma) = \nu(\lambda) \quad a.e. \text{ on } S^{N-1}. \tag{1.35}$$

For $\lambda' > \lambda$ we repeat this operation with $\{r_n\}$ replaced by $\{r_{n_k}\}$ and there exists a subsequence $\{r_{n_{k_i}}\}$ such that

$$\lim_{n_{k_i} \to +\infty} r_{n_{k_i}}^{N-2} u(r_{n_{k_i}}, \sigma) = \nu(\lambda') \quad a.e. \text{ on } S^{N-1}. \tag{1.36}$$

From (1.35) and (1.36) we deduce that $\nu(\lambda') = \nu(\lambda) = \gamma$ for $\lambda > \gamma$ which implies (i).

Case 2. — Assume $\lim_{\lambda \to +\infty} \nu(\lambda) = +\infty$. For $\delta > 0$ we call p the function introduced in the proof of Proposition 1.1 and for $\lambda > 0$, $\tilde{\omega}_\delta = \frac{1}{2}(u + v_\lambda - p(u - v_\lambda)) + \frac{3}{4}\delta$. From (1.22) we have

$$(1.37) \qquad \Delta \tilde{\omega}_\delta \leqq g(\tilde{\omega}_\delta) + |f|.$$

Moreover $r^{N-2} \tilde{\omega}_\delta(r, .)$ converges to $\nu(\lambda)$ in $L^q(S^{N-1})$ $(1 \leqq q < +\infty)$ as r tends to 0. We consider now $w = v_{\nu(\lambda)}$ the solution of (1.32) and we set

$$s = \frac{r^{N-2}}{N-2},$$

$$w'(s) = r^{N-2} w(r), \quad \tilde{\omega}'_\delta(s, \sigma) = r^{N-2} \tilde{\omega}_\delta(r, \sigma), \varphi(s) = f(r).$$

Then (1.32) and (1.37) become

$$(1.38) \qquad \begin{aligned} & s^2 (\omega'_\delta)_{ss} + \frac{1}{(N-2)^2} \Delta_{S^{N-1}} \tilde{\omega}'_\delta \leqq k s^{N/(N-2)} \left(g\left(\frac{\tilde{\omega}'_\delta}{s(N-2)} \right) + \varphi \right), \\ & s^2 w'_{ss} = k s^{N/(N-2)} \left(g\left(\frac{w'}{s(N-2)} \right) + |\varphi| \right), \end{aligned}$$

where $k = k(N) = (N-2)^{(4-N)/(N-2)}$ and $\Delta_{S^{N-1}}$ is the Laplace-Beltrami operator on S^{N-1}. Consider a C^∞ function ρ such that $\rho \in L^\infty(\mathbb{R})$, $\rho \equiv 0$ on $(-\infty, 0)$, $\rho' > 0$ on $(0, +\infty)$ and $j(r) = \int_0^r \rho(\tau)\, d\tau$. From convexity and monotonicity we have

$$(1.39) \qquad s^2 \frac{d^2}{ds^2} \int_{S^{N-1}} j(w' - \omega'_\delta)\, d\sigma \geqq 0.$$

As $\int_{S^{N-1}} j(w' - \omega'_\delta)\, d\sigma \leqq C \int_{S^{N-1}} |w' - \omega'_\delta|\, d\sigma$ and as $w'(s)$ and $\tilde{\omega}'_\delta(s, .)$ converges to $\nu(\lambda)$ in $L^1(S^{N-1})$ as s tends to 0 we deduce that $\int_{S^{N-1}} j(w' - \omega'_\delta)\, d\sigma = 0$ on $(0, R^{N-2}/(N-2)]$ and $w' \leqq \tilde{\omega}'_\delta$ or

$$(1.40) \qquad v_{\nu(\lambda)}(r) \leqq \omega_\delta(r, \sigma) \leqq \omega(r, \sigma) + \delta/4$$

which implies

$$(1.41) \qquad \nu(\lambda) \leqq \varliminf_{x \to 0} |x|^{N-2} \omega(x) \leqq \varliminf_{x \to 0} |x|^{N-2} u(x)$$

and we get (1.31).

Remark 1.2. — If u satisfies (i) then $v_\gamma(x) \leqq u(x)$ in $B_R \setminus \{0\}$.

Remark 1.3. — If u is a radial solution of (1.29), $u \geqq 0$, in $B_R \setminus \{0\}$, then a simple adaptation of the proof of Theorem 1.1 shows that $|x|^{N-2} u(x)$ admits a limit in $[0, +\infty]$ as x tends to 0.

The 2-dimensional version of Theorem 1.1 is the following

THEOREM 1.2. — *Assume* $N=2$, $f \in L^1(\Omega)$ *is radial near* 0 *and* $u \in C^2(\Omega')$ *is a nonnegative solution of* (1.29) *in* Ω'. *Then*

— *If* $a_g^+ = 0$ *the alternative of Theorem* 1.1 *holds with* $|x|^{2-N}$ *replaced by* $Ln(1/|x|)$.

— *If* $a_g^+ > 0$, *we have the following alternative*

(i) *either there exists a nonnegative real number* $\gamma \in [0, 2/a_g^+)$ *such that* $u(r, .)/Ln(1/r)$ *converges in measure on* S^1 *to* γ *as* r *tends to* 0,

(ii) *or*

$$\lim_{x \to 0} u(x)/Ln(1/|x|) \geqq 2/a_g^+. \tag{1.43}$$

Proof. — *Case* 1. — Assume $a_g^+ = 0$. We define $\nu(\lambda)$ as

$$\lim_{r \to 0} (Ln(1/r))^{-1} \inf(u(r, .), v_\lambda(r)) = \nu(\lambda). \tag{1.44}$$

As $\nu(\lambda)$ is nondecreasing and v_λ exists for every $\lambda > 0$ we can proceed as in the proof of Theorem 1.1 if $\lim_{\lambda \to +\infty} \nu(\lambda) = \gamma < +\infty$. If $\lim_{\lambda \to +\infty} \nu(\lambda) = +\infty$ we introduce $\tilde{\omega}_\delta$ and $v_{\nu(\lambda)} = w$ as in Theorem 1.1 and make the following change of variable

$$\begin{gathered} t = Ln(1/r) \\ w'(t) = w(r), \qquad \tilde{\omega}'_\delta(t, \sigma) = \tilde{\omega}_\delta(r, \sigma), \qquad f'(t) = f(r). \end{gathered} \tag{1.45}$$

Hence w' and $\tilde{\omega}'_\delta$ satisfies

$$\begin{gathered} (\tilde{\omega}'_\delta)_{tt} + (\tilde{\omega}'_\delta)_{\theta\theta} \leqq e^{-2t}(g(\omega'_\delta) + f') \\ w'_{tt} = e^{-2t}(g(w') + |f'|) \end{gathered} \tag{1.46}$$

on $(T, +\infty) \times S^1$ and with the same function j as before

$$\frac{d^2}{dt^2} \int_{S^1} j(w' - \omega'_\delta)\, d\theta \geqq 0. \tag{1.47}$$

As $t^{-1}(w' - \omega'_\delta)$ converges to 0 in $L^1(S^1)$ we deduce that $j(w' - \omega'_\delta) = 0$ and we get finally

$$\lim_{x \to 0} u(x)/Ln(1/|x|) = +\infty. \tag{1.48}$$

Case 2. — Assume $a_g^+>0$ and set $\gamma=\lim_{\lambda\uparrow 2/a_g^+}\nu(\lambda)$. Clearly $\gamma\leqq 2/a_g^+$. If $\gamma<2/a_g^+$ we can proceed as in Theorem 1.1. If $\gamma=2/a_g^+$ we get as in Case 1

$$\inf(u(x), v_\lambda(x))\geqq v_{\nu(\lambda)}(x)-\frac{\delta}{4} \tag{1.49}$$

for any $\lambda\leqq\dfrac{2}{a_g^+}$ and $x\in B_R\setminus\{0\}$. We can take in particular $\lambda=\dfrac{2}{a_g^+}=\nu(\lambda)$ and we get (ii).

2. SINGULAR SOLUTIONS OF $\Delta u=\pm g(u)$

The first application of Theorem 1.1 is the following

THEOREM 2.1. — *Assume* $N\geqq 3$, *g is a nondecreasing locally Lipschitz continuous function satisfying* (1.1) *and* $u\in C^2(\Omega')$ *is a nonnegative solution of*

$$\Delta u=g(u) \tag{2.1}$$

in Ω'. *Then* $|x|^{N-2}u(x)$ *admits a limit in* $[0,+\infty]$ *as x tends to* 0.

Proof. — From Theorem 1.1 we can assume that there exist $\gamma\in[0,+\infty)$ and a sequence $\{r_n\}$ converging to 0 such that

$$\lim_{n\to+\infty} r_n^{N-2}u(r_n,.)=\gamma \quad a.e.\ \text{in } S^{N-1}. \tag{2.2}$$

Case 1. — Assume $\gamma>0$. For $\varepsilon>0$ set w_ε the solution of

$$\begin{aligned} \Delta w_\varepsilon=g(w_\varepsilon) \quad &\text{in } \Gamma_{\varepsilon,R}=\{x\in\mathbb{R}^N:\varepsilon<|x|<R\}\\ w_\varepsilon=u \quad &\text{on } \partial B_\varepsilon\\ w_\varepsilon=\max_{x\in\partial B_R}u(x) \quad &\text{on } \partial B_R\end{aligned} \tag{2.3}$$

(we may assume that $\bar{B}_R\subset\Omega$). From maximum principle $u\leqq w_\varepsilon$ in $\Gamma_{\varepsilon,R}$. Let $u^s=u+w_\varepsilon(R)$, then

$$-\Delta u^s+g(u^s)\geqq 0 \tag{2.4}$$

and finally $u\leqq w_\varepsilon\leqq u^s$ in $\Gamma_{\varepsilon,R}$ and there exists a sequence $\{\varepsilon_n\}$ converging to 0 and a function $w\in C^2(\bar{B}_R\setminus\{0\})$ satisfying $-\Delta w+g(w)=0$ in $B_R\setminus\{0\}$ such that $\{w_{\varepsilon_n}\}$ converges to w in the C^1_{loc}-topology of $\bar{B}_R\setminus\{0\}$.

Moreover

(2.5) $$u \leqq w \leqq u^1 = u + \max_{\partial B_R} u(x)$$

From Remark 1.2 $\overline{\lim_{x \to 0}} |x|^{N-2} w(x) = \gamma$, hence we deduce from Serrin and Ni's results [12] that w is radial and from (2.2) and (2.5)

(2.6) $$\lim_{n \to +\infty} r_n^{N-2} w(r_n) = \gamma.$$

If $w'(s) = w'(r^{N-2}/(N-2)) = r^{N-2} w(r)$, then

(2.7) $$s^2 w'_{ss} = k(N) s^{N/(N-2)} g(w'/s(N-2))$$

we deduce that $s \to w'(s) - k(N)(N-2)^2/(2N) s^{N/(N-2)} g(0)$ is convex and

(2.8) $$\lim_{r \to 0} r^{N-2} w(r) = \gamma = \lim_{x \to 0} |x|^{N-2} u(x).$$

Case 2. — Assume $\gamma = 0$. For $\varepsilon > 0$ and $\nu > 0$ set $w_{\varepsilon,\nu}$ the solution of

(2.9) $$\begin{aligned} \Delta w_{\varepsilon,\nu} &= g(w_{\varepsilon,\nu}) \quad \text{in } \Gamma_{\varepsilon,R} \\ w_{\varepsilon,\nu} &= u + \nu\varepsilon^{2-N} \quad \text{on } \partial B_\varepsilon \\ w_{\varepsilon,\nu} &= \max_{x \in \partial B_R} (u(x) + \nu |x|^{2-N}) \quad \text{on } \partial B_R. \end{aligned}$$

As in case 1 we have

(2.10) $$u(x) \leqq w_{\varepsilon,\nu}(x) \leqq u(x) + \nu |x|^{2-N} + w_{\varepsilon,\nu}(R)$$

in $\Gamma_{\varepsilon,R}$. For $0 < \nu' < \nu$ let $v_{\nu'}$ be the radial solution of $-\Delta v_{\nu'} + g(v_{\nu'}) = C(N)\nu' \delta_0$ in $D'(B_R)$ such that $v_{\nu'} = 0$ on ∂B_R. As $\lim_{x \to 0} |x|^{N-2} v_{\nu'}(x) = \nu'$ we deduce that for ε small enough $v_{\nu'} < w_{\varepsilon,\nu}$ on ∂B_ε and finally

(2.11) $$w_{\varepsilon,\nu} \geqq v_{\nu'}$$

in $\Gamma_{\varepsilon,R}$ and as in Case 1 there exists a subsequence $\{\varepsilon_n\}$ such that $\lim \varepsilon_n = 0$ and a function w^ν satisfying $-\Delta w^\nu + g(w^\nu) = 0$ in B_R such that $w_{\varepsilon,\nu}$ converges to w^ν in the C^1_{loc} topology of $\bar{B}_R \setminus \{0\}$ and we have

(2.12) $$\max(u, v_{\nu'}) \leqq w^\nu \leqq u + \nu |x|^{2-N} + \max_{\partial B_R} u(x).$$

Applying again [12] we deduce that w^ν is radial and as in Case 1 we get that

(2.13) $$\overline{\lim_{x \to 0}} |x|^{N-2} u(x) \leqq \lim_{x \to 0} |x|^{N-2} w^\nu(x) = \nu.$$

As v is arbitrary $\lim_{x \to 0} |x|^{N-2} u(x) = 0$ and u can be extended to Ω as a C^2 solution of (2.1) in Ω.

In the same way we can prove the two dimensional case

THEOREM 2.2. — *Assume* $N=2$ *and* g *is a nondecreasing locally Lipschitz continuous function defined on* $\mathbb{R}^+$. *If* $u \in C^2(\Omega')$ *is a nonnegative solution of* (2.1) *in* Ω', *we have the following*:

– *if* $a_g^+ = 0$ $u(x)/\mathrm{L}n(1/|x|)$ *admits a limit in* $[0, +\infty]$ *as* x *tends to* 0;

– *if* $a_g^+ > 0$ *and* g *satisfies*

(2.14) *for any* $a \geqq 0$ $\lim_{r \to +\infty} e^{-ar} g(r)$ *exists in* $[0, +\infty]$,

$u(x)/\mathrm{L}n(1/|x|)$ *admits a limit in* $[0, 2/a_g^+]$ *as* x *tends to* 0.

Proof. — If $a_g^+ = 0$ we proceed as in Theorem 2.1. If $a_g^+ = +\infty$ and g satisfies (2.14), u can be extended to Ω as a C^2 solution of (2.1) in Ω [21]. If $0 < a_g^+ < +\infty$ we have two cases

(i) either there exists $\gamma \in [0, 2/a_g^+)$ and a sequence $\{r_n\}$ converging to 0 such that

$$\lim_{n \to +\infty} u(r_n, .)/\mathrm{L}n(1/r_n) = \gamma \quad a.e. \text{ in } S^1 \tag{2.15}$$

(ii) or $\varliminf_{x \to 0} u(x)/\mathrm{L}n(1/|x|) \geqq 2/a_g^+$.

In case (i) we have $\lim_{x \to 0} u(x)/\mathrm{L}n(1/|x|) = \gamma$ as in Theorem 2.1. In case (ii) we have an *a priori* estimate thanks to (2.14) [21]:

$$u(x) \leqq \left(\frac{2}{a_g^+} + \varepsilon\right) \mathrm{L}n(1/|x|) + \mathrm{B}(\varepsilon) \tag{2.16}$$

near 0 for any $\varepsilon > 0$. This clearly implies

$$\lim_{x \to 0} u(x)/\mathrm{L}n(1/|x|) = 2/a_g^+. \tag{2.17}$$

THEOREM 2.3. — *Assume* $N \geqq 3$, g *is a continuous function defined on* $[0, +\infty)$ *such that* $\varliminf_{r \to +\infty} g(r)/r = K$ *for some* $K > -\infty$ *and* $u \in C^2(\Omega')$ *is a nonnegative solution of*

$$-\Delta u = g(u) \tag{2.18}$$

in Ω'. *Then there exists* $\gamma \in [0, +\infty)$ *such that*

(2.19)
$$\lim_{x \to 0} |x|^{1-N} \int_{|y|=|x|} |\gamma - |x|^{N-2} u(y)| \, dS = 0,$$

$g(u) \in L^1_{loc}(\Omega)$ and u solves

(2.20)
$$-\Delta u = g(u) + C(N)\gamma\delta_0$$

in $D'(\Omega)$. If we assume moreover that

(2.21)
$$\int_0^1 \inf(g(\alpha r^{2-N}), g(\beta r^{2-N}))\, r^{N-1}\, dr = +\infty$$

for any α, $\beta > 0$, then $\gamma = 0$.

Proof. — The fact that $g(u) \in L^1_{loc}(\Omega)$ and u satisfies (2.20) for some $\gamma \geqq 0$ is proved in [5]. If $\bar{u}(r)$ [res. $\overline{g(u)}(r)$] is the spherical average of u [resp. $g(u)$] then

(2.22)
$$\Delta\bar{u} = \overline{g(u)}$$

in $B_R \setminus \{0\} \subset \Omega'$ and we deduce from Lemma 1.2 that

(2.23)
$$\lim_{x \to 0} |x|^{1-N} \int_{|y|=|x|} |\gamma' - |x|^{N-2} u(y)| \, dS = 0$$

for some $\gamma' \geqq 0$ and $\bar{u}$ solves

(2.24)
$$-\Delta\bar{u} = \overline{g(u)} + C(N)\gamma'\delta_0$$

in $D'(B_R)$. Whence $\gamma = \gamma'$. Let us assume now that $\gamma > 0$ and g satisfies (2.21) for any α, $\beta > 0$. As $r^{N-2} u(r,.)$ converges to γ in $L^1(S^{N-1})$ it converges in measure and for any $\eta \in (0, |S^{N-1}|)$ there exists $r_0 \in (0, R)$ such that for any $r \in (0, r_0)$ there exists a measurable subset $\omega(r) \subset S^{N-1}$ such that $|\omega(r)| \geqq \eta$ and $|r^{N-2} u(r,\sigma) - \gamma| < \gamma/2$ for $\sigma \in \omega(r)$. As $g(r) \geqq K'r - L$ and $u \in L^1_{loc}(B_R)$ there is no loss of generality to assume that $g(r) \geqq 0$ on $(0, +\infty)$, hence

(2.25)
$$\int_{B_{r_0}} g(u)\, dx = \int_0^{r_0} \int_{S^{N-1}} g(u)\, r^{N-1}\, d\sigma\, dr \geqq \int_0^{r_0} \int_{\omega(r)} g(u)\, r^{N-1}\, d\sigma\, dr.$$

For $\rho \in (0, r_0]$ and $\sigma \in \omega(\rho)$, $\frac{\gamma}{2}\rho^{2-N} \leqq u(\rho,\sigma) < 2\gamma\rho^{2-N}$ and as g is continuous, $g(u(\rho,\sigma)) \geqq \inf\left(g\left(\frac{\gamma}{2}\rho^{2-N}\right), g(2\gamma\rho^{2-N})\right)$. As g satisfies (2.21) we get

$$\int_{B_{r_0}} g(u)\,dx \geqq \eta \int_0^{r_0} \inf\left(g\left(\frac{\gamma}{2}r^{2-N}\right), g(2\gamma r^{2-N})\right) r^{N-1}\,dr = +\infty, \tag{2.26}$$

contradiction. Hence $\gamma = 0$.

Under an assumption of monotonicity on g we get a much more accurate result:

PROPOSITION 2.1. — *Assume* $N \geqq 3$, *g is a nondecreasing locally Lipschitz continuous function defined on* $[0, +\infty)$ *and* $u \in C^2(\Omega')$ *is a nonnegative solution of* (2.18) *in* Ω'. *Assume also that* $\bar{B}_R \subset \Omega$ *and that there exists a radial continuous function* Φ *defined in* $\bar{B}_R \setminus \{0\}$ *and satisfying*

$$\begin{aligned} -\Delta\Phi &\geqq g(\Phi) \quad \text{in } D'(B_R \setminus \{0\}), \\ \Phi &\geqq u \quad \text{in } \bar{B}_R \setminus \{0\}. \end{aligned} \tag{2.27}$$

Then $|x|^{N-2} u(x)$ *converges to some nonnegative real number* γ *when* x *tends to* 0.

Proof. — From Remark 1.3 $|x|^{N-2}\Phi(x)$ converges to some $\gamma' \geqq 0$ as x tends to 0. If $\gamma' = 0$ then $\lim_{x \to 0} |x|^{N-2} u(x) = 0$. Let us assume that $\gamma' > 0$. From Brezis and Lions' result

$$-\Delta\Phi = -\{\Delta\Phi\} + C(N)\gamma'\delta_0$$

with $-\{\Delta\Phi\} \in L^1_{loc}(B_R)$ which implies that $g(\Phi) \in L^1(B_R)$ and g satisfies (1.1). From Theorem 2.3 there exists $\gamma \in [0, \gamma']$ such that $r^{N-2} u(r, .)$ converges to γ in $L^1(S^{N-1})$ as r tends to 0. We consider now the sequence of functions $\{u^N\}$ defined by $u^0 = \Phi$ and for $N \geqq 1$

$$\begin{aligned} -\Delta u^N &= g(u^{N-1}) + C(N)\gamma\delta_0 \quad \text{in } D'(B_R) \\ u^N &= \Phi \quad \text{on } \partial B_R. \end{aligned} \tag{2.28}$$

Then u^N is radial and $u \leqq u^N \leqq u^{N-1} < \Phi$. It is clear that $\{u^N\}$ converges in $C^1_{loc}(\bar{B}_R \setminus \{0\})$ to a radial function $\bar{u}$ which satisfies

$$-\Delta\bar{u} = g(\bar{u}) + C(N)\gamma\delta_0 \quad \text{in } D'(B_R) \tag{2.29}$$

and $\bar{u} \geqq u$. As a consequence of Lemma 1.2 $\lim_{x \to 0} |x|^{N-2}\bar{u}(x) = \gamma$. From Remark 1.2 $\varliminf_{x \to 0} |x|^{N-2} u(x) = \gamma$ which ends the proof.

Remark 2.1. — The hypothesis of radiality of Φ which is rather restrictive can be withdrown if we know that $\varliminf_{x \to 0} u(x) = +\infty$ and

$\Phi \geqq \sup_{|x|=R} u(x)$. In that case we can consider the following iterative scheme with $\Phi^0 = \Phi$ and

$$(2.30)\qquad \begin{aligned} -\Delta\Phi^N &= g(\Phi^{N-1}) + C(N)\gamma'\delta_0 \quad \text{in } \mathbf{D}'(B_R) \\ \Phi^N &= \sup_{|x|=R} u(x) \quad \text{on } \partial B_R. \end{aligned}$$

Then $u \leqq \Phi^N \leqq \Phi^{N-1} \leqq \Phi$ and $\{\Phi^N\}$ converges in $C^1_{loc}(\bar{B}_R \setminus \{0\})$ to some Φ^- satisfying

$$(2.31)\qquad \begin{aligned} -\Delta\Phi^- &= g(\Phi^-) + C(N)\gamma'\delta_0 \quad \text{in } \mathbf{D}'(B_R) \\ \Phi^- &= \sup_{|x|=R} u(x) \quad \text{on } \partial B_R \end{aligned}$$

and $\Phi^- \geqq u$. As $\underline{\lim}_{x \to 0} \Phi^-(x) = +\infty$ we deduce from Serrin and Ni' results [12] that Φ^- is radial and we can apply Lemma 1.2.

PROPOSITION 2.2. — *Assume* $N \geqq 3$, *g is a nondecreasing locally Lipschitz continuous function defined on* $[0, +\infty)$ *satisfying for some* $q > N/2$.

$$(2.32)\qquad \sup(g'(\varphi), g'(\psi)) \in L^q_{loc}(\Omega)$$

for any φ *and* ψ *continuous and nonnegative in* Ω' *such that* $g(\varphi)$ *and* $g(\psi) \in L^1_{loc}(\Omega)$. *If* $u \in C^2(\Omega')$ *is a nonnegative solution of* (2.18) *in* Ω', *then* $|x|^{N-2} u(x)$ *converges to some nonnegative real number* γ *as x tends to* 0.

Proof. — From Theorem 2.3 we have (2.20) for some $\gamma \geqq 0$ and $g(u) \in L^1_{loc}(\Omega)$.

Case 1. — $\gamma = 0$. Without any restriction we can assume that $u > \varepsilon$ in $\bar{B}_R \setminus \{0\} \subset \Omega'$ and we write (2.20) as

$$(2.33)\qquad \Delta u + du + g(0) = 0$$

in $B_R \setminus \{0\}$ where $d(x) = (g(u) - g(0))/u$. As $g(u) \in L^1(B_R)$ (2.32) implies that $d \in L^q(B_R)$ and we deduce from [18] that either u has a removable singularity at 0 or

$$(2.34)\qquad 0 < \underline{\lim}_{x \to 0} |x|^{N-2} u(x) < \overline{\lim}_{x \to 0} |x|^{N-2} u(x) < +\infty,$$

which is impossible as $\gamma = 0$.

Case 2. — $\gamma > 0$. Let v_γ be the solution of

$$(2.35)\qquad \begin{aligned} -\Delta v_\gamma &= g(v_\gamma) + C(N)\gamma\delta_0 \quad \text{in } \mathbf{D}'(B_R), \\ v_\gamma &= 0 \quad \text{on } \partial B_R, \end{aligned}$$

v_γ is constructed using an increasing sequence of approximate solutions as in [11], $0 \leqq v_\gamma \leqq u$ in $B_R \setminus \{0\}$ and v_γ is radial. Let w be $u - v_\gamma$, then

$$\Delta w + dw = 0 \tag{2.36}$$

in $B_R \setminus \{0\}$ with $d = (g(u) - g(v_\gamma))/(u - v_\gamma) \in L^q(B_R)$. Then we deduce from [18] that either w has a removable singularity at 0 or

$$0 < \varliminf_{x \to 0} |x|^{N-2} w(x) \leqq \varlimsup_{x \to 0} |x|^{N-2} w(x) \tag{2.37}$$

which is impossible as

$$\gamma = \lim_{x \to 0} |x|^{N-2} v_\gamma(x) = \varliminf_{x \to 0} |x|^{N-2} u(x). \tag{2.38}$$

Remark 2.2. — Under the hypotheses of Proposition 2.2 two nonnegative solutions $u_i (i = 1, 2)$ of

$$-\Delta u = g(u) + C(N)\gamma\delta_0 \tag{2.39}$$

in $\mathbf{D}'(\Omega)$ are such that $u_1 - u_2 \in L^\infty_{loc}(\Omega)$. As for the solvability of (2.39) we have

PROPOSITION 2.3. — *Assume* $N \geqq 3$, Ω *is bounded with a* C^1 *boundary* $\partial\Omega$ *and* g *is a nondecreasing function defined on* $[0, +\infty)$, *satisfying* (1.1) *and* $g(r) = o(r)$ *near* 0. *Then there exists* $\gamma^* \in (0, +\infty]$ *with the following properties:*

(i) *for any* $\gamma \in [0, \gamma^*)$ *there exists at least one nonnegative function* $u \in C^1(\bar{\Omega} \setminus \{0\})$ *vanishing on* $\partial\Omega$ *solution of* (2.39) *in* $\mathbf{D}'(\Omega)$,

(ii) *for* $\gamma > \gamma^*$ *no such* u *exists.*

Proof. — *Step* 1. *Assume* $\Omega = B_R$. — A function u vanishing on ∂B_R is a radial solution of (2.40) in $\mathbf{D}'(B_R)$ if and only if the function $v(t) = u(r)$, with $t = r^{2-N}$, satisfies

$$\begin{aligned} & v_{tt} + \frac{1}{(N-2)^2} t^{-2(N-1)/(N-2)} g(v) = 0 \quad \text{on } (R^{2-N}, +\infty), \\ & v(R^{2-N}) = 0, \\ & \lim_{t \to +\infty} v(t)/t = \gamma. \end{aligned} \tag{2.40}$$

As v is concave the last condition is equivalent to

$$\lim_{t \to +\infty} v_t(t) = \gamma. \tag{2.41}$$

For $\alpha>0$, let v^α be the solution of the initial value problem defined on a maximal interval $[R^{2-N}, T^*)$

$$v^\alpha_{tt}+\frac{1}{(N-2)^2}t^{-2(N-1)/(N-2)}g(v^\alpha)=0 \quad \text{on } (R^{2-N}, T^*), \tag{2.42}$$
$$v^\alpha(R^{2-N})=0,$$
$$v^\alpha_t(R^{2-N})=\alpha.$$

If $T^*<+\infty$ then $\lim_{t\uparrow T^*} v^\alpha(t)=0$ as a consequence of concavity and there exists $T\in(R^{2-N}, T^*)$ such that $v_t(T)=0$. If $T^*=+\infty$ and $\lim_{t\to+\infty} v_t(t)=0$ then the same relation holds with $T=+\infty$. As a consequence if no solution v^α of (2.42) satisfies (2.41) with $\gamma>0$ we have

$$(N-2)^2\alpha=\int_{R^{2-N}}^{T} t^{-2(N-1)/(N-2)}g(v^\alpha(t))\,dt \tag{2.43}$$

and the right-hand side of (2.43) is majorized by $\int_{R^{2-N}}^{+\infty} t^{-2(N-1)/(N-2)}g(\alpha(t-R^{2-N}))\,dt$, which implies

$$(N-2)^2\alpha R^{-N}<\int_0^{+\infty}(t+1)^{-2(N-1)/(N-2)}g(\alpha R^{2-N}t)\,dt, \tag{2.44}$$

or

$$(N-2)^2R^{-2}<\int_0^{+\infty}t(t+1)^{-2(N-1)/(N-2)}\frac{g(\alpha R^{2-N}t)}{\alpha R^{2-N}t}\,dt. \tag{2.45}$$

For $\varepsilon>0$ there exists $\eta>0$ such that $\alpha R^{2-N}t<\eta$ implies $g(\alpha R^{2-N}t)<\varepsilon\alpha R^{2-N}t$. Hence the right-hand side of (2.45) is majorized by

$$\frac{R^{N-2}}{\alpha}\int_{R^{N-2}\eta/\alpha}^{+\infty}(t+1)^{-2(N-1)/(N-2)}g(\alpha R^{2-N}t)\,dt$$
$$+\varepsilon\int_0^{R^{N-2}\eta/\alpha}t(t+1)^{-2(N-1)/(N-2)}\,dt$$

or

$$\alpha^{2(N-1)/(N-2)}\int_\eta^{+\infty}(R^{N-2}s+\alpha)^{-2(N-1)/(N-2)}g(s)\,ds$$
$$+\varepsilon\int_0^{+\infty}t(t+1)^{-2(N-1)/(N-2)}\,dt.$$

Consequently

$$(2.46)\qquad \lim_{\alpha \to 0} \int_0^{+\infty} t(t+1)^{-2(N-1)/(N-2)} \frac{g(\alpha R^{2-N} t)}{\alpha R^{2-N} t} dt = 0$$

contradicting (2.45). As a consequence there exists $\alpha^* > 0$ such that for any $\alpha \in (0, \alpha^*)$ the solution v^α of (2.42) is defined on $[R^{2-N}, +\infty)$ and satisfies (2.41) for some $\gamma > 0$.

Step 2. *The general case.* — There exists $R > 0$ such that $\Omega \subset B_R$. If $\tilde{\gamma} > 0$ is such that there exists a solution v to (2.40), then for any $\gamma \in [0, \tilde{\gamma}]$ the sequence $\{u_n\}$ defined by $u_0 = 0$ and for $n \geqq 1$

$$(2.47)\qquad \begin{aligned} -\Delta u^n &= g(u^{n-1}) + C(N)\gamma\delta_0 \quad \text{in } \mathbf{D}'(\Omega), \\ u^n &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

increases, is majorized by v in Ω and converges to some u which vanishes on $\partial\Omega$ and satisfies (2.39) in $\mathbf{D}'(\Omega)$. For the same reasons, the set of $\gamma > 0$ such that there exists a nonnegative solution of (2.39) vanishing on $\partial\Omega$ is an interval.

Remark 2.3. — If $\underline{\lim}_{r \to +\infty} g(r)/r > 0$ it is proved in [11] that $\gamma^* < +\infty$. If we no longer assume that $\lim_{r \to 0} g(r)/r = 0$ it can be proved that for any $\nu_0 > 0$ there exists $R_0 > 0$ such that for any $\Omega \subset B_{R_0}$ and any $\gamma \in [0, \nu_0)$ there exists a solution u of (2.39) in $\mathbf{D}'(\Omega)$.

The two-dimensional version of Theorem 2.3 is the following

THEOREM 2.4. — *Assume* $N = 2$, g *is a continuous function defined on* $[0, +\infty)$ *such that* $\underline{\lim}_{r \to +\infty} g(r)/r > -\infty$ *and* $u \in C^2(\Omega')$ *is a nonnegative solution of* (2.18) *in* Ω'. *Then there exists* $\gamma \in [0, +\infty)$ *such that*

$$(2.48)\qquad \lim_{x \to 0} |x|^{-1} \int_{|y| = |x|} |\gamma - u(y)/\mathrm{Ln}(1/|x|)| \, dS = 0,$$

$g(u) \in L^1_{\mathrm{loc}}(\Omega)$ *and* u *solves*

$$(2.49)\qquad -\Delta u = g(u) + 2\pi\gamma\delta_0$$

in $\mathbf{D}'(\Omega)$. *If we assume moreover that*

$$(2.50)\qquad \int_0^1 \inf(g(\alpha\, \mathrm{Ln}(1/r)), g(\beta\, \mathrm{Ln}(1/r)) \, r \, dr = +\infty$$

for any α, $\beta > 0$, *then* $\gamma = 0$.

Remark 2.4. — When $a_g^+=0$, Proposition 2.2 which holds in the case $N=2$ with $|x|^{2-N}$ replaced by $\mathrm{Ln}(1/|x|)$ provides an interesting criterion for proving that

$$\lim_{x\to 0} u(x)/\mathrm{Ln}(1/|x|)=\gamma \tag{2.51}$$

for some $\gamma\geqq 0$. Proposition 2.1 is also valid in the case $N=2$ (with the same modifications).

We introduce now a class new of g's defined on $[0, +\infty)$ which are those satisfying

$$\forall\sigma>0,\quad \lim_{r\to+\infty} e^{-\sigma r} g(r)=l(\sigma) \text{ exists in } [0,+\infty], \tag{2.52}$$

and we have [20]

$$a_g^+=\sup\{\sigma>0: l(\sigma)=+\infty\}=\inf\{\sigma>0: l(\sigma)=0\}. \tag{2.53}$$

THEOREM 2.5. — *Assume* $N=2$, g *is a continuous function defined on* $[0,+\infty)$ *satisfying* $\underline{\lim}_{r\to+\infty} g(r)/r>-\infty$ *and* (2.52) *with* $a_g^+<+\infty$ *and* $u\in C^2(\Omega')$ *is a nonnegative solution of* (2.18) *in* Ω' *and assume also*

(i) *either* $a_g^+=0$,

(ii) *or* $a_g^+>0$ *and* $\displaystyle\int_0^1 g\left(\frac{2}{a_g^+}\mathrm{Ln}(1/r)\right) r\,dr=+\infty$.

Then there exists $\gamma\in\left[0,\dfrac{2}{a_g^+}\right)$ *such that* $u-\gamma\,\mathrm{Ln}\dfrac{1}{r}$ *is locally bounded in* Ω.

Proof. — The main ingredient for proving this is a theorem due to John and Nirenberg ([9], Th. 7.21) that we recall

«Let $u\in W^{1,1}(G)$ where $G\subset\Omega$ is convex and suppose that there exists a constant K such that

$$\int_{G\cap B_r} |\nabla u|\,dx\leqq K r \quad \text{for any ball } B_r, \tag{2.54}$$

then there exist positive constant μ_0 and C such that

$$\int_G \exp\left(\frac{\mu}{K}|u-u_G|\right)dx\leqq C(\mathrm{diam}(G))^2 \tag{2.55}$$

where $\mu=\mu_0|G|(\mathrm{diam}(G))^{-2}$ and $u_G=\dfrac{1}{|G|}\displaystyle\int_G u\,dx$».

From Theorem 2.4 there exists $\gamma\geqq 0$ such that $u(r,.)/\mathrm{Ln}(1/r)$ converges to γ in $L^1(S^1)$ as r tends to 0 and $g(u)\in L^1_{loc}(\Omega)$. Set $w=u-\gamma\,\mathrm{Ln}(1/|x|)$, then

$$-\Delta w = g(u) \tag{2.56}$$

in $D'(\Omega)$. It is now classical that $\nabla w \in M^2_{\text{loc}}(\Omega)$ where $M^2(G)$ is the usual Marcinkiewicz space over G. If we take $G = \bar{B}_R \subset \Omega$ then ∇w satisfies (2.54) for some $K > 0$, which implies

$$\int_{B_\rho} e^{\alpha w}\, dx \leqq C(\rho) \tag{2.57}$$

for some $\alpha > 0$ and $0 < \rho \leqq R$.

Case 1. — *Assume* $a_g^+ = 0$. Then for any $\varepsilon > 0$ we have

$$|g(r)| \leqq K_\varepsilon e^{\varepsilon r} \tag{2.58}$$

for some $K_\varepsilon > 0$ and any $r \geqq 0$. From (2.57) we have

$$\int_{B_\rho} e^{\alpha u} |x|^{\alpha\gamma}\, dx \leqq C(\rho). \tag{2.59}$$

If $\gamma > 0$ we have for p, $\sigma > 1$ and $\lambda > 0$

$$\int_{B_\rho} e^{p\varepsilon u}\, dx \leqq \left(\int_{B_\rho} e^{\sigma p \varepsilon u} |x|^{\sigma\lambda}\, dx\right)^{1/\sigma} \left(\int_{B_\rho} |x|^{-\sigma'\lambda}\, dx\right)^{1/\sigma'} \tag{2.60}$$

($\sigma' = \sigma/(\sigma - 1)$). We set $\sigma p \varepsilon = \alpha$, $\sigma\lambda = \alpha\gamma$, hence $\lambda = \gamma p \varepsilon$, $\sigma = \dfrac{\alpha}{p\varepsilon}$ and $\sigma'\lambda = \alpha\gamma p\varepsilon/(\alpha - p\varepsilon)$.

Hence for any $p > 1$ we can take ε small enough so that $\sigma'\lambda < 2$ and $\sigma > 1$. As a consequence $g(u) \in L^p(B_\rho)$ and $w \in L^\infty(B_\rho)$. If $\gamma = 0$, (2.59) implies that $g(u) \in L^p(B_\rho)$ for any $p \in [1, \infty)$ and $u \in L^\infty(B_\rho)$.

Case 2. — *Assume* $a_g^+ > 0$ *and* $\displaystyle\int_0^1 g\left(\frac{2}{a_g^+} \operatorname{Ln}(1/r)\right) r\, dr = +\infty$.

Step 1. — $0 \leqslant \gamma < \dfrac{2}{a_g^+}$. Assume the contrary that is $\gamma \geqq \dfrac{2}{a_g^+}$. As $a_g^+ > 0$ we have $\lim_{r \to +\infty} g(r) = +\infty$ and from Remark 1.2

$$u(x) > v_\gamma(x), \tag{2.61}$$

where v_γ satisfies

$$-\Delta v_\gamma + g(v_\gamma) = 2\pi\gamma\delta_0 \tag{2.62}$$

in $D'(B_R)$, $v_\gamma = 0$ on ∂B_R. As a consequence [21] $\lim_{x \to 0} u(x) = +\infty$ and for $|x| < R'$ small enough

(2.63) $$-\Delta u \geqq 2\pi\gamma\delta_0$$

in $D'(B_{R'})$. As a consequence $u(x) \geqq \gamma \operatorname{Ln}\left(\frac{1}{|x|}\right) - l$, which implies $\int_{B_{R'}} g(u)\,dx = +\infty$, contradiction.

Step 2. — We claim that for any $\alpha > 0$ there exist $\rho \in (0, R]$ such that (2.57) holds. We fix $0 < R' < R$ and write $w = w_1 + w_2$ where w_1 is harmonic in $B_{R'}$ and take the value w on $\partial B_{R'}$ and w_2 satisfies

(2.64) $$-\Delta w_2 = g(u)$$

in $B_{R'}$ and $w_2 = 0$ on $\partial B_{R'}$. As $\nabla w_1 \in L^2(B_{R'})$ we deduce

(2.65) $$\|\nabla w_1\|_{M^2(B_\rho)} \underset{\rho \to 0}{\to} 0$$

and for w_2 we have

(2.66) $$\|\nabla w_2\|_{M^2(B_{R'})} \leqq C \|g(u)\|_{L^1(B_{R'})}$$

where C is independent of R′. As a consequence we get

(2.67) $$\lim_{\rho \to 0} \|\nabla w\|_{M^2(B_\rho)} = 0$$

and the constant K in (2.55) can be taken as small as we want provided $G = B_\rho$ and u is replaced by w. This implies that for any $\alpha > 0$ we can find $\rho \in (0, R)$ such that (2.57) holds.

Step 3: *End of the proof.* — From the definition of a_g^+, for any $\varepsilon > 0$ there exists $K_\varepsilon > 0$ such that

(2.68) $$|g(r)| \leqq K_\varepsilon e^{(a_g^+ + \varepsilon) r}$$

for $r \geqq 0$, and we have from (2.59)

(2.69) $$\int_{B_\rho} e^{p(a_g^+ + \varepsilon) u}\,dx \leqq \left(\int_{B_\rho} e^{\sigma p (a_g^+ + \varepsilon) u} |x|^{\sigma\lambda}\,dx\right)^{1/\sigma} \left(\int_{B_\rho} |x|^{-\sigma'\lambda}\,dx\right)^{1/\sigma'}.$$

We take $\sigma p(a_g^+ + \varepsilon) = \alpha$, $\sigma\lambda = \alpha\gamma$ [we assume $\gamma > 0$ other-while $g(u) \in L^p_{loc}(\Omega)$ for any $p > 1$ and $w \in L^\infty_{loc}(\Omega)$] and $\lambda = \gamma p(a_g^+ + \varepsilon)$, $\sigma = \alpha / p(a_g^+ + \varepsilon)$ and $\lambda\sigma' = \alpha\gamma p(a_g^+ + \varepsilon)/(\alpha - p(a_g^+ + \varepsilon))$. As $\gamma a_g^+ < 2$ there exist $p > 1$, $\varepsilon > 0$, $\alpha > 0$ such that $\sigma'\lambda < 2$ which implies $g(u) \in L^p_{loc}(\Omega)$ and we end the proof as in Case 1.

Remark 2.5. — If $a_g^+ = +\infty$ then $\gamma=0$ from Theorem 2.4. In that case it is unlikely that Theorem 2.5 still holds. However we conjecture that $\lim_{x \to 0} u(x)/\mathrm{Ln}(1/|x|)=0$.

Concerning the existence of solutions of (2.49) the following result can be proved as in Proposition 2.3.

PROPOSITION 2.4. — *Assume* $N=2$, Ω *is bounded with a* C^1 *boundary* $\partial\Omega$ *and* g *is a nondecreasing function defined on* $[0, +\infty)$ *such that* $a_g^+ \in (0, +\infty]$ *and* $g(r)=o(r)$ *near* 0. *Then there exists* $\gamma^* \in (0, 2/a_g^+]$ *with the following properties*:

(i) *for any* $\gamma \in [0, \gamma^*)$ *there exists at least one nonnegative function* $u \in C^1(\bar{\Omega}\setminus\{0\})$ *vanishing on* $\partial\Omega$ *solution of* (2.49) *in* $\mathbf{D}'(\Omega)$,

(ii) *for* $\gamma > \gamma^*$ *no such* u *exists.*

Remark 2.6. — If $g(r)=e^{ar}$ it is easy to see that γ^* exists only if diam. (Ω) is small enough. Moreover in that case $\gamma^* < \dfrac{2}{a_g^+} = \dfrac{2}{a}$.

3. SINGULARITIES OF $\Delta u = u(\mathrm{Ln}^+ u)^\alpha$

Our first result deals with the one-dimensional case

THEOREM 3.1. — *Assume* $u \in C^2(0, R)$ *is a nonnegative solution of*

$$u_{rr} = u(\mathrm{Ln}^+ u)^\alpha \quad \text{in } (0, R). \tag{3.1}$$

Then:

— *if* $0<\alpha<2$,

$u(r)$ *admits a finite limit as* r *tends to* 0;

— *if* $\alpha>2$,

(i) *either* $u(r)$ *admits a finite limit as* r *tends to* 0,

(ii) *or*

$$\begin{cases} u(r)=\sqrt{e}\, e^{\gamma(\alpha) r^{2/(2-\alpha)}}(1+O(r^{2/(\alpha-2)})), \\ u_r(r) = -\sqrt{e}(\gamma(\alpha))^{\alpha/2} r^{\alpha/(2-\alpha)} e^{\gamma(\alpha) r^{2/(2-\alpha)}}(1+O(r^{2/(\alpha-2)})), \end{cases} \tag{3.2}$$

near 0 *where*

$$\gamma(\alpha) = \left(\frac{2}{\alpha-2}\right)^{2/(\alpha-2)}. \tag{3.3}$$

From (3.1) u is convex and $u(r)$ admits a limit in $\mathbb{R}^+ \cup \{+\infty\}$ as r tends to 0. If this limit is larger than 1, (3.1) is equivalent to

$$v_{rr} + v_r^2 = v^\alpha \tag{3.4}$$

on some interval $(0, R')$ with the transformation $u = e^v$. Theorem 3.1 is an immediate consequence of the following result

LEMMA 3.1. — *Assume $v \in C^2(0, R')$ is a nonnegative solution of* (3.4) *in* $(0, R')$. *Then*

— *if* $0 < \alpha \leqq 2$, *v remains bounded near* 0;

— *if* $\alpha > 2$

(i) *either v remains bounded near* 0,

(ii) *or*

$$\left\{\begin{array}{l} r^{2/(\alpha-2)} v(r) = \gamma(\alpha) + \dfrac{1}{2} r^{2/(\alpha-2)} + O(r^{4/(\alpha-2)}) \\ r^{\alpha/(\alpha-2)} v_r(r) = -(\gamma(\alpha))^{\alpha/2} + O(r^{4/(\alpha-2)}). \end{array}\right. \tag{3.5}$$

Proof. — Assuming that u is unbounded near 0, then $\lim_{r \to 0} u(r) = +\infty = \lim_{r \to 0} v(r)$ and v is decreasing near 0. So we can define

$$\left\{\begin{array}{l} \rho = v \in [\sigma, +\infty), \\ h(\rho) = v_r^2, \end{array}\right. \tag{3.6}$$

and (3.5) become

$$\frac{1}{2} h_\rho + h = \rho^\alpha \quad \text{in } [\sigma, +\infty). \tag{3.7}$$

Hence $h(\rho) = h(\sigma) e^{2(\sigma-\rho)} + 2 e^{-2\rho} \int_\sigma^\rho s^\alpha e^{2s} ds.$

As

$$\int_\sigma^\rho s^\alpha e^{2s} ds = \frac{1}{2} [s^\alpha e^{2s}]_\sigma^\rho - \frac{\alpha}{4} [s^{\alpha-1} e^{2s}]_\sigma^\rho + \frac{\alpha(\alpha-1)}{4} \int_\sigma^\rho s^{\alpha-2} e^{2s} ds$$

and

$$\frac{e^{-2\rho}}{\rho^\alpha} \int_\sigma^\rho s^{\alpha-2} e^{2s} ds = O\left(\frac{1}{\rho^2} + \frac{1}{\rho^\alpha}\right)$$

we get

$$\frac{h(\rho)}{\rho^\alpha} = 1 - \frac{\alpha}{2\rho} + O\left(\frac{1}{\rho^2} + \frac{1}{\rho^\alpha}\right) \tag{3.8}$$

as ρ goes to $+\infty$, which implies

$$\lim_{r \to 0} \frac{v_r(r)}{v^{\alpha/2}(r)} = -1 \tag{3.9}$$

Integrating (3.9) implies that $v^{(2-\alpha)/2}(r)$ (if $0<\alpha<2$) or $\mathrm{Ln}\, v(r)$ (if $\alpha=2$) remains bounded near 0 which is a contradiction. So we are left *with the case* $\alpha>2$, $\lim_{r \to 0} v(r) = +\infty$. From (3.8) we have

$$\frac{v_r}{v^{\alpha/2}} = -1 + \frac{\alpha}{4v} + O\left(\frac{1}{v^2}\right), \tag{3.10}$$

near 0, which implies $\lim_{r \to 0} r^{2/(\alpha-2)} v(r) = \left(\frac{2}{a-2}\right)^{2/(\alpha-2)} = \gamma(a)$. As a consequence $\frac{1}{v(r)} = \frac{1+o(1)}{\gamma(\alpha)} r^{2/(\alpha-2)}$ and (3.10) becomes

$$\frac{v_r}{v^{\alpha/2}} = -1 + \frac{1+o(1)}{\gamma(\alpha)} \frac{\alpha}{4} r^{2/(\alpha-2)} \tag{3.11}$$

Integrating (3.11) on $(0, r)$ for r small yields

$$v(r) = \gamma(\alpha)\, r^{2/(2-\alpha)}\left(1 + \frac{1+o(1)}{2\gamma(\alpha)} r^{2/(\alpha-2)}\right), \tag{3.12}$$

which implies, with (3.10),

$$\frac{v_r}{v^{\alpha/2}} = -1 + \frac{\alpha}{4\gamma(\alpha)} r^{2/(\alpha-2)} + O\left(r^{4/(\alpha-2)}\right). \tag{3.13}$$

Reasoning as before we get

$$v(r) = \gamma(\alpha)\, r^{2/(2-\alpha)} + \frac{1}{2} + O\left(r^{2/(\alpha-2)}\right) \tag{3.14}$$

near 0 and

$$r^{\alpha/(\alpha-2)} v_r(r) = -(\gamma(\alpha))^{\alpha/2} + O\left(r^{4/(\alpha-2)}\right). \tag{3.15}$$

We assume now that Ω is an open subset of $\mathbb{R}^N$, $N \geqq 2$, containing 0, $\Omega' = \Omega \setminus \{0\}$ and we consider the following equation in Ω'

$$\Delta u = u\,(\mathrm{Ln}^+ u)^\alpha \tag{3.16}$$

where $u \in C^2(\Omega')$ is nonnegative.

LEMMA 3.2. — *If* $\alpha>2$ *and* $\bar{B}_R \subset \Omega$; *then there exists a constant* $C = C(\alpha, N, R, \mathrm{dist}(\partial B_R, \partial\Omega)$ *such that*

(3.17) $$u(x) \leq e^{C|x|^{2/(2-\alpha)}} \quad \text{in } \bar{B}_R \setminus \{0\}.$$

Proof. — We define $\beta(t) = t(\mathrm{Ln}^+ t)^\alpha$, $j(t) = \int_0^t \beta(s)\,ds$ and $\tau(t) = \int_t^{+\infty} \frac{dt}{\sqrt{j(s)}}$. As $\tau(2) < +\infty$ we deduce from Vazquez's result that the equation (3.16) satisfies the a priori interior estimate property [19]: if $x_0 \in \Omega'$ and if the cube $Q_\rho(x_0) = \{x \in \mathbb{R}^N : \sup_{1 \leq i \leq N} |x^i - x_0^i| < \rho\}$ is included in Ω', then for any $a \in (0, 1)$ there exists a constant $\mu = \mu(a) > 0$ such that

(3.18) $$u(x_0) \leq \frac{N}{a} \tau^{-1}(\mu\rho).$$

So the main point is to get a precise estimate on τ^{-1}. If $s_0 > e^{\alpha/2}$ and $C(s_0) = \frac{1}{2} - \frac{\alpha}{4\,\mathrm{Ln}\,s_0}$ it is easy to check that

$$j(t) > C(s_0)\, t^2 (\mathrm{Ln}\, t)^\alpha \qquad \text{for} \quad t > s_0.$$

If $C_0 = \frac{2}{(\alpha - 2)\sqrt{C(s_0)}}$, then $\tau(s) < C_0 (\mathrm{Ln}\, s)^{(2-\alpha)/2}$ for $s > s_0$ and

(3.19) $$\tau^{-1}(y) \leq e^{C_0^{2/(\alpha-2)} y^{2/(2-\alpha)}}.$$

for $0 < y < \tau(s_0)$. For $|x| < \frac{\sqrt{N}}{2} R$, $Q_{\frac{2|x|}{\sqrt{N}}}(x) \subset B_R$. We set

$$R_0 = \min\left(\frac{1}{2} R, \frac{1}{2} \frac{\tau(s_0)}{\mu}\right)$$

and for $|x| \leq R_0$ we can apply (3.18), (3.19) which gives

(3.20) $$u(x) \leq \frac{N}{a} e^{((C_0 \sqrt{N})/2)^{2/(\alpha-2)} |x|^{2/(2-\alpha)}}.$$

The estimate in $B_R \setminus B_{R_0}$ is obtained from (3.18) with a simple compactness argument and we get (3.17).

LEMMA 3.3. — *Assume* $N \geq 2$, $\alpha > 0$ *and* $v \in C^2(\bar{B}_R \setminus \{0\})$ *is a nonnegative solution of*

(3.21) $$v_{rr} + \frac{N-1}{r} v_r + v_r^2 = v^\alpha \quad \textit{in} \ (0, R)$$

such that $\lim_{r \to 0} v(r) = +\infty$. *Then for any* $\varepsilon > 0$ *there exists* $r(\varepsilon) \in (0, R)$ *such that*

$$(3.22) \qquad -\frac{N-1}{r v^{\alpha/2}} - 1 < \frac{v_r}{v^{\alpha/2}} \leqq -1 + \varepsilon \quad \textit{in } (0, r(\varepsilon)).$$

Proof. – From (3.21) it is clear that $v_r < 0$ on some $(0, r_0) \subset (0, R)$ and we get

$$(3.23) \qquad v_{rr} + v_r^2 \geqq v^\alpha \quad \text{in } (0, r_0).$$

Taking $v = \rho$ as a new variable and $h(\rho) = v_r^2$ as a new unknow we get as in Lemma 3.1

$$\frac{1}{2} h_\rho + h \geqq \rho^\alpha \qquad \text{for} \quad \rho \geqq \rho_0,$$

which implies $(e^{2\rho} h)_\rho \geqq 2 e^{2\rho} \rho^\alpha$ and by integration we get $\frac{h(\rho)}{\rho^\alpha} \geqq 1 - \varepsilon$ for any $\varepsilon > 0$ and $\rho > \rho(\varepsilon)$, that is

$$(3.24) \qquad \frac{v_r}{v^{\alpha/2}} \leqq -1 + \varepsilon \quad \text{in } (0, r(\varepsilon)),$$

where $r(\varepsilon)$ is small enough. As a consequence $\lim_{r \to 0} v_r(r) = -\infty$. If we set $\omega = v_r$ we get from (3.21)

$$(3.25) \qquad \omega_{rr} + \frac{N-1}{r} \omega_r + 2 \omega \omega_r - \frac{N-1}{r^2} \omega = \alpha \omega v^{\alpha - 1}.$$

As $\omega < 0$ on $(0, r_0)$, (3.25) implies

$$(3.26) \qquad \omega_{rr} + \left(\frac{N-1}{r} + 2\omega \right) \omega_r < 0 \quad \text{in } (0, r_0).$$

Hence if $\omega_r(r_1) \leqq 0$ for some $r_1 \in (0, r_0)$ we would have $\omega_r(r) < 0$ for $r \in (0, r_1)$ contradicting $\lim_{r \to 0} \omega(r) = -\infty$. As a consequence $\omega_r > 0$ and

$$(3.27) \qquad v_r^2 + \frac{N-1}{r} v_r - v^\alpha \leqq 0 \quad \text{in } (0, r_0).$$

A simple algebraic computation implies

$$(3.28) \qquad -\frac{N-1}{2r} - \sqrt{\left(\frac{N-1}{2r} \right)^2 + v^\alpha} \leqq v_r \leqq 0$$

and

$$\frac{v_r}{v^{\alpha/2}} \geqq -\frac{N-1}{rv^{\alpha/2}} - 1, \tag{3.29}$$

which ends the proof.

LEMMA 3.4. — *Assume* $N \geqq 2$, $\alpha > 1$ *and* $u \in C^2(\bar{B}_R \setminus \{0\})$ *is a nonnegative solution of*

$$u_{rr} + \frac{N-1}{r} u_r = u(\mathrm{L}n^+ u)^\alpha \quad in\ (0, R). \tag{3.30}$$

Then $\lim_{r \to 0} u(r)/\mu(r) = +\infty$ *if and only if* $\lim_{r \to 0} r^{2/\alpha} \mathrm{L}n\, u(r) = +\infty$.

Proof. — *Case* 1 : $N \geqq 3$. — We consider the following change of variable

$$s = r^{2-N}, \qquad \tilde{u}(s) = u(r); \tag{3.31}$$

$\tilde{u}$ satisfies

$$\tilde{u}_{ss} = \frac{1}{(N-2)^2} s^{-2((N-1)/(N-2))} \tilde{u}(\mathrm{L}n^+ \tilde{u})^\alpha \quad \text{in } (S, +\infty), \tag{3.32}$$

with $S = R^{2-N}$, and if $\lim_{r \to 0} r^{N-2} u(r) = +\infty$ we have

$$\lim_{r \to +\infty} \tilde{u}(s)/s = \lim_{s \to +\infty} \tilde{u}_s(s) = +\infty. \tag{3.33}$$

From convexity $\tilde{u}(s) \leqq s\, \tilde{u}_s(s)\ (1 + o(1))$ and

$$(\mathrm{L}n\, \tilde{u})^\alpha < (\mathrm{L}n\, s + \mathrm{L}n\, \tilde{u}_s + O(1))^\alpha \leqq (N-2)^2 (\mathrm{L}n\, s)^\alpha (\mathrm{L}n\, \tilde{u}_s)^\alpha$$

for s large enough; so (3.32) becomes

$$\tilde{u}_{ss} \leqq s^{-N/(N-2)} \tilde{u}_s (\mathrm{L}n\, \tilde{u}_s)^\alpha (\mathrm{L}n\, s)^\alpha. \tag{3.34}$$

As $\alpha > 1$

$$\int_\sigma^{+\infty} \frac{\tilde{u}_{ss}}{\tilde{u}_s (\mathrm{L}n\, \tilde{u}_s)^\alpha} ds = \frac{1}{\alpha - 1} (\mathrm{L}n\, \tilde{u}_s(\sigma))^{1-\alpha}$$

and

$$\int_\sigma^{+\infty} s^{-N/(N-2)} (\mathrm{L}n\, s)^\alpha\, ds < A\, \sigma^{-2/(N-2)} (\mathrm{L}n\, \sigma)^\alpha$$

for some constant A and σ large enough. As a consequence $\mathrm{L}n\, \tilde{u}_s(\sigma) \geqq B\, \sigma^{2/(N-2)(\alpha-1)} (\mathrm{L}n\, \sigma)^{\alpha/(1-\alpha)}$. A straightforward computation implies that for any $\varepsilon>0$ and for s large enough

$$\tilde{u}(s) \geqq e^{s^{(\varepsilon+2/(1-\alpha))/(N-2)}},$$

which means

$$\text{(3.35)} \qquad \mathrm{Ln}\, u(r) \geqq r^{\varepsilon+2/(1-\alpha)},$$

for r small enough and $\lim_{r \to 0} r^{2/\alpha} \mathrm{Ln}\, u(r) = +\infty$. Conversely $\lim_{r \to 0} r^{2/\alpha} \mathrm{Ln}\, u(r) = +\infty$ implies $\lim_{r \to 0} u(r)/\mu(r) = +\infty$ $(N \geqq 2)$.

Case 2: N = 2. – We make the following change of variable

$$\text{(3.36)} \qquad r = e^{-t}, \qquad \tilde{u}(t) = u(r),$$

and we get (with $T = \mathrm{Ln}\,(1/R)$)

$$\text{(3.37)} \qquad \tilde{u}_{tt} = e^{-2t} \tilde{u} (\mathrm{Ln}\, \tilde{u})^{\alpha} \quad \text{in } (T, +\infty).$$

If we assume $\lim_{r \to 0} u(r)/\mathrm{Ln}\,(1/r) = +\infty$ then

$$\lim_{t \to +\infty} \tilde{u}(t)/t = \lim_{t \to +\infty} \tilde{u}_t(t) = +\infty$$

(by convexity) and we get

$$\frac{\tilde{u}_{tt}}{\tilde{u}_t (\mathrm{Ln}\, \tilde{u}_t)} \leqq e^{-2t} t (\mathrm{Ln}\, t)^{\alpha} (1 + o(1)) \qquad \text{for} \quad t \gg T$$

and

$$\text{(3.38)} \qquad \mathrm{Ln}\, \tilde{u}_t(t) \geqq B\, t^{1/(1-\alpha)} (\mathrm{Ln}\, t)^{\alpha/(1-\alpha)} e^{-2t/(1-\alpha)}$$

for some $B>0$ and t large enough, which implies

$$\text{(3.39)} \qquad \tilde{u}(t) \geqq e^{(2/(\alpha-1)-\varepsilon)t},$$

for any $\varepsilon>0$ and t large. From (3.39) we get the result.

With lemmas 3.2-3.4 we can describe the behaviour of nonnegative radial solutions of (3.16) with a strong singularity at 0, when $\alpha>2$.

LEMMA 3.5. – *Assume* $N \geqq 2$, $\alpha > 2$ *and* $u \in C^2(\bar{B}_R \setminus \{0\})$ *is a nonnegative solution of* (3.30) *in* (0, R) *such that* $\lim_{r \to 0} u(r)/\mu(r) = +\infty$. *Then the following holds near* 0

$$\text{(3.40)} \qquad \begin{aligned} r^{2/(\alpha-2)} \mathrm{Ln}\, u(r) &= \gamma(\alpha) + \frac{\alpha-(N-1)(\alpha-2)}{2\alpha} r^{2/(\alpha-2)} + O(r^{4/(\alpha-2)}), \\ r^{\alpha/(\alpha-2)} (\mathrm{Ln}\, u(r))_r &= -(\gamma(\alpha))^{\alpha/2} + O(r^{4/(\alpha-2)}). \end{aligned}$$

Proof. — From the preceeding lemmas $\lim_{r\to 0} v_r(r)/v^{\alpha/2}(r) = -1$ where $v = \mathrm{Ln}\, u$. As a consequence

$$\lim_{r\to 0} r^{2/(\alpha-2)} v(r) = \gamma(\alpha)$$
$$\lim_{r\to 0} r^{\alpha/(\alpha-2)} v_r(r) = -(\gamma(\alpha))^{\alpha/2} \tag{3.41}$$

and $\frac{N-1}{r} v_r(r) = (-1+o(1))\frac{(N-1)(\alpha-2)}{2} v^{\alpha-1}(r)$ near 0. Pluging this estimate into equation (3.21) yields

$$v_{rr} + v_r^2 = v^\alpha + C(1+o(1))\, v^{\alpha-1} \tag{3.42}$$

with $C = (N-1)(\alpha-2)/2$. Taking again $\rho = v$ as the variable and $h(\rho) = v_r^2$ as the unknow implies

$$\frac{1}{2}(e^{2\rho} h(\rho))_\rho = \rho^\alpha e^{2\rho} + C(1+o(1))\, \rho^{\alpha-1} e^{2\rho}$$

and

$$\frac{h(\rho)}{\rho^\alpha} = 1 + (1+o(1))\left(C - \frac{\alpha}{2}\right)\frac{1}{\rho} \quad \text{as } \rho \to +\infty. \tag{3.43}$$

If we set $A = \frac{\alpha}{4} - \frac{C}{2} = \frac{\alpha-(N-1)(\alpha-2)}{4}$ we have $\frac{v_r}{v^{\alpha/2}} = -1 + \frac{1+o(1)}{v} A$, which implies $v(r) = \gamma(\alpha)\ (1+o(1))\ r^{2/(2-\alpha)}$ and finally

$$\frac{v_r}{v^{\alpha/2}} = -1 + \frac{1+o(1)}{\gamma(\alpha)} A\, r^{2/(\alpha-2)}. \tag{3.44}$$

Integrating (3.44) on $(0, r]$ for some small r implies

$$v(r) - \gamma(\alpha)\, r^{2/(2-\alpha)} = (1+o(1))(2A/\alpha).$$

As $v_r = -v^{\alpha/2}\left(1 + O\left(\frac{1}{v}\right)\right)$, we have $\frac{N-1}{r} v_r = -C v^{\alpha-1}\left(1 + 0\left(\frac{1}{v}\right)\right)$ and v satisfies

$$v_{rr} + v_r^2 = v^\alpha + C v^{\alpha-1} + O(v^{\alpha-2}); \tag{3.45}$$

using ρ and $h(\rho)$ yields

$$\frac{h(\rho)}{\rho^\alpha} = 1 + \frac{2C-\alpha}{2}\frac{1}{\rho} + O\left(\frac{1}{\rho^2}\right), \tag{3.46}$$

$$\frac{v_r}{v^{\alpha/2}} = -1 + \frac{A}{v} + O\left(\frac{1}{v^2}\right), \tag{3.47}$$

and, as $v=\gamma\, r^{2/(2-\alpha)}(1+O(r^{2/(\alpha-2)}))$,

$$\frac{v_r}{v^{\alpha/2}}=-1+\frac{A}{\gamma(\alpha)}r^{2/(\alpha-2)}+O(r^{4/(\alpha-2)}). \tag{3.48}$$

Integrating (3.48) gives $v(r)=\gamma(\alpha)r^{2/(2-\alpha)}+\dfrac{2A}{\alpha}+O(r^{2/(\alpha-2)})$ which implies (3.40).

Remark 3.1. — If $N\geqq 3$ and $\alpha=2\dfrac{N-1}{N-2}$, $\psi(r)=\gamma(\alpha)r^{2/(2-\alpha)}$ is a solution of (3.30) in $(0,+\infty)$.

We are now able to prove the main theorem of this section

THEOREM 3.2. — *Assume* $N\geqq 2$, $\alpha>0$ *and* $u\in C^2(\Omega')$ *is a nonnegative solution of* (3.16) *in* Ω'. *Then*

if $0<\alpha\leqq 2$:

(i) *either u can be extended to* Ω *as a* C^2 *solution of* (3.16) *in* Ω,

(ii) *or there exists* $\gamma>0$ *such that* $\lim_{x\to 0} u(x)/\mu(x)=\gamma$ *and u satisfies*

$$\Delta u=u(\mathrm{Ln}^+ u)^\alpha - C(N)\gamma\delta_0 \tag{3.49}$$

in $D'(\Omega)$;

if $\alpha>2$:

(iii) *either u behaves as in* (i) *or* (ii) *above*

(iv) *or* $u(x)=\gamma(\alpha,N)\,e^{\gamma(\alpha)|x|^{2/(2-\alpha)}}(1+O(|x|^{2/(\alpha-2)}))$

near 0 *with* $\gamma(\alpha)=\left(\dfrac{2}{\alpha-2}\right)^{2/(\alpha-2)}$ *and* $\gamma(\alpha,N)=e^{(\alpha-(N-1)(\alpha-2))/2\alpha}$.

Proof. — From Theorems 1.1, 1.2 we know that $u(x)/\mu(x)$ admits a limit in $(0,+\infty]$ as x tends to 0. If the limit is finite we get (i) or (ii) [(iii) if $\alpha>2$] and (3.49) from Theorems 1.1, 1.2 and Remark 1.1 (if the limit is 0 then u is regular as in Proposition 2.5). So let us assume that

$$\lim_{x\to 0} u(x)/\mu(x)=+\infty. \tag{3.50}$$

For any $c>0$ let φ_c be the solution of

$$\begin{aligned}&(\varphi_c)_{rr}+\frac{N-1}{r}(\varphi_c)_r=\varphi_c(\mathrm{Ln}^+\varphi_c)^\alpha \quad \text{in } (0,R),\\ &\lim_{r\to 0}\varphi_c(r)/\mu(r)=c, \qquad \varphi_c(R)=\min_{|x|=R} u(x),\end{aligned} \tag{3.51}$$

(we assume $B_R \subset \Omega$). It is clear that $0 \leqq \varphi_c \leqq u$ for $0<|x|<R$, $c \mapsto \varphi_c$ is increasing and $\lim_{c \to +\infty} \varphi_c = \varphi$ where φ satisfies

$$(3.52)\qquad \begin{aligned} &\varphi_{rr} + \frac{N-1}{r}\varphi_r = \varphi(Ln^+ \varphi)^\alpha \quad \text{in } (0, R),\\ &\lim_{r \to 0} \varphi(r)/\mu(r) = +\infty, \qquad \varphi(R) = \min_{|x|=R} u(x). \end{aligned}$$

Moreover $0 \leqq \varphi \leqq u$ in $B_R \setminus \{0\}$.

If $0<\alpha\leqq 2$ we can take R small enough such that $\varphi(R) > e$ and we construct in the same way as φ a function $\tilde{\varphi}$ such that $0 \leqq \tilde{\varphi} \leqq \varphi$ and

$$(3.53)\qquad \begin{aligned} &\tilde{\varphi}_{rr} + \frac{N-1}{r}\tilde{\varphi}_r = \tilde{\varphi}(Ln^+ \tilde{\varphi})^2 \quad \text{in } (0, R),\\ &\lim_{r \to 0} \tilde{\varphi}(r)/\mu(r) = +\infty, \qquad \tilde{\varphi}(R) = \varphi(R). \end{aligned}$$

From Lemma 3.4 $\lim_{r \to 0} r^{2/\alpha} Ln\, \tilde{\varphi}(r) = +\infty$. If we set $\zeta = Ln\, \tilde{\varphi}$, then Lemma 3.3 implies that $\lim_{r \to 0} \frac{\zeta_r}{\zeta}(r) = -1$ which implies by integration that ζ remains bounded near 0 and so does $\tilde{\varphi}$, a contradiction.

We assume now $\alpha > 2$. We define ψ_n as the solution of

$$(3.54)\qquad \begin{aligned} &(\psi_n)_{rr} + \frac{N-1}{r}(\psi_n)_r = \psi_n (Ln^+ \psi_n)^\alpha \quad \text{in } \left(\frac{1}{n}, R\right),\\ &\psi_n\left(\frac{1}{n}\right) = \max_{|x|=1/n} u(x), \qquad \psi_n(R) = \max_{|x|=R} u(x). \end{aligned}$$

Using Lemma 3.2 and the same device as in the proof of Proposition 2.5 we deduce that for some subsequence $\{\psi_{n_k}\}$ we have $\lim_{n_k \to \infty} \psi_{n_k} = \psi$ in the $C^1((0, R])$-topology and ψ satisfies

$$(3.55)\qquad \psi_{rr} + \frac{N-1}{r}\psi_r = \psi(Ln^+ \psi)^\alpha \quad \text{in } (0, R)$$

Moreover $0 \leqq u \leqq \psi$ in $B_R \setminus \{0\}$. Applying Lemma 3.5 to φ and ψ we get (iv).

Remark 3.2. — It is interesting to notice that if u is a positive solution of (3.16) with a strong singularity at 0, then $v = Ln\, u$ behaves like the explicit radial singular solution of the following *first order* equation in $\mathbb{R}^N \setminus \{0\}$ ($\alpha > 2$)

$$(3.56)\qquad |DU|^2 = U^\alpha$$

that is $U(x)=\gamma(\alpha)|x|^{2/(2-\alpha)}$.

Remark 3.3. — There is an alternative way to prove Theorem 3.2 in the case $\alpha>2$, it is to obtain Harnack type inequalities as in [23] and to use Lemmas 3.3-3.5 (*see* [16] for details). Unfortunately such inequalities are out of reach in the case $0<\alpha\leqq 2$ as Lemma 3.2 no longer holds.

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