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EXISTENCE OF SOLUTIONS FOR COMPRESSIBLE FLUID MODELS OF KORTEWEG TYPE

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ABSTRACT. – The purpose of this work is to prove existence and uniqueness results of suitably smooth solutions for an isothermal model of capillary compressible fluids derived by J.E. Dunn and J. Serrin (1985), which can be used as a phase transition model.

We first study the well-posedness of the model in spaces with critical regularity indices with respect to the scaling of the associated equations. In a functional setting as close as possible to the physical energy spaces, we prove global existence of solutions close to a stable equilibrium, and local in time existence for solutions when the pressure law may present spinodal regions. Uniqueness is also obtained.

Assuming a lower and upper control of the density, we also show the existence of weak solutions in dimension 2 near equilibrium. Finally, referring to the work of Z. Xin (1998) in the non-capillary case, we describe some blow-up properties of smooth solutions with finite total mass.

RÉSUMÉ. – On s'intéresse ici à des résultats d'existence et d'unicité de solutions pour un modèle de fluides compressibles isothermes avec capillarité. Ce modèle de transition de phase a été dérivé par J.E. Dunn et J. Serrin (1985).

Pour commencer, on montre que le problème de Cauchy est bien posé dans des espaces à régularité critique pour le *scaling* des équations. Pour des données initiales proches d'un état d'équilibre stable, on obtient l'existence globale (et l'unicité) de solutions dans un cadre fonctionnel aussi proche que possible de l'espace d'énergie physique. Pour des lois de pression plus générales (pouvant être décroissantes), on prouve des résultats locaux en temps.

En supposant que l'on dispose d'un minorant strictement positif et d'une borne supérieure pour la densité, on obtient l'existence de solutions faibles en dimension 2 pour des données initiales proches de l'équilibre. Enfin, en adaptant un travail de Z. Xin pour les fluides sans capillarité, on établit l'explosion de solutions régulières à masse totale finie.
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1. Introduction

Let us consider a fluid of density $\rho \geqslant 0$, velocity field $\mathbf{u} \in \mathbb{R}^d$ $(d \geqslant 2)$, entropy density s, energy density e, and temperature $\theta = (\partial e/\partial s)_\rho$. We are interested in the following model of compressible capillary fluid, which can be derived from a Cahn–Hilliard like free energy (see the pioneering work by J.E. Dunn and J. Serrin in [11], and also [1,6,12])

$$\partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0, \tag{1}$$

$$\partial_t(\rho \mathbf{u}) + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) = \operatorname{div}(\mathbf{S} + \mathbf{K}), \tag{2}$$

$$\partial_t\left(\rho\left(e + \frac{\mathbf{u}^2}{2}\right)\right) + \operatorname{div}\left(\rho \mathbf{u}\left(e + \frac{\mathbf{u}^2}{2}\right)\right) = \operatorname{div}(\alpha \nabla \theta) + \operatorname{div}\big((\mathbf{S} + \mathbf{K}) \cdot \mathbf{u}\big), \tag{3}$$

where the viscous stress tensor $\mathbf{S}$ and the Korteweg stress tensor $\mathbf{K}$ read as

$$\mathbf{S}_{i,j} = \big(\lambda \operatorname{div}\mathbf{u} - P(\rho, e)\big)\delta_{i,j} + 2\mu \mathbf{D}(\mathbf{u})_{i,j}, \tag{4}$$

$$\mathbf{K}_{i,j} = \frac{\kappa}{2}\big(\Delta \rho^2 - |\nabla \rho|^2\big)\delta_{i,j} - \kappa \partial_i \rho \partial_j \rho, \tag{5}$$

$\mathbf{D}(\mathbf{u})_{i,j} = (\partial_i u_j + \partial_j u_i)/2$ being the strain tensor, and (λ, μ) the constant viscosity coefficients of the fluid. We require that λ and μ satisfy $\mu > 0$ and $\lambda + 2\mu > 0$, which in particular covers the case when λ and μ satisfy Stokes' law $d\lambda + 2\mu = 0$. The thermal conduction coefficient α is a given non negative function of the temperature θ and the surface tension coefficient $\kappa > 0$ is assumed to be constant. In view of the first principle of thermodynamics, the entropy density s solves

$$\partial_t(\rho s) + \operatorname{div}(\rho s \mathbf{u}) = \frac{1}{\theta}\big(\operatorname{div}(\alpha \nabla \theta) + \mathbf{K} : \mathbf{D}(\mathbf{u}) + 2\mu \mathbf{D}(\mathbf{u}) : \mathbf{D}(\mathbf{u}) + \lambda |\operatorname{div}\mathbf{u}|^2\big). \tag{6}$$

As a reasonable starting point of our analysis, we consider the scaled Van der Waals equation of state

$$P(\rho) = a\rho\theta\left(\frac{8}{3-\rho} - \frac{3\rho}{\theta}\right), \tag{7}$$

where a is a positive constant, and the critical density ρ_c and temperature θ_c are equal to 1. Depending on the fixed temperature θ, the pressure is a nondecreasing function of the density ρ or may present decreasing regions (spinodal regions) for some values of ρ, which are thermodynamically unstable. The above equation of state (7) ensures the presence of two basic states, a "liquid" one, and a "gaseous" one. Let us as in [20] put emphasis on the existence of steady solutions connecting a gas phase to a liquid phase through a smoothly varying density profile. When initial conditions involve densities in the unstable (spinodal) region, the two phases are expected to spontaneously separate. For details on the derivation of the above Korteweg like model, we refer to [1,11,12,16].

In what follows, we do not consider thermal fluctuations so that the pressure p is a function of ρ only. The corresponding isothermal model which was also considered in [13,20] then reads as

$$\partial_t\rho + \operatorname{div}(\rho\mathbf{u}) = 0, \tag{8}$$

$$\partial_t(\rho\mathbf{u}) + \operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u}) - \mu\Delta\mathbf{u} - (\lambda+\mu)\nabla\operatorname{div}\mathbf{u} + \nabla P(\rho) = \operatorname{div}\mathbf{K} + \rho\mathbf{f}, \tag{9}$$

where $\mathbf{f}$ is an exterior forcing term, supplemented with initial conditions

$$\rho_{|t=0} = \rho_0 \geqslant 0 \quad \text{and} \quad \rho\mathbf{u}_{|t=0} = \mathbf{m}_0. \tag{10}$$

In a bounded domain Ω, we would have to precise the boundary conditions, namely homogeneous Dirichlet conditions for the velocity: $\mathbf{u}_{|\partial\Omega} = 0$ and Neumann conditions for the density: $\partial_n\rho_{|\partial\Omega} = 0$. In order to simplify the presentation, we will focus on the whole space case $\mathbb{R}^d$ ($d \geqslant 2$) and study the well-posedness of (8) (9) for an initial density close enough to an equilibrium density $\bar\rho > 0$, or at least bounded away from vacuum, which is a major difficulty in most of compressible fluid models.

Before getting into the heart of mathematical results, we first derive the physical energy bounds of the above system in the case $\mathbf{f} \equiv 0$ to simplify the presentation. Let $\bar\rho > 0$ be a constant reference density, and π defined by

$$\pi(s) = s\left(\int_{\bar\rho}^{s} \frac{P(z)}{z^2}\,dz - \frac{P(\bar\rho)}{\bar\rho}\right), \tag{11}$$

so that $P(s) = s\pi'(s) - \pi(s)$, $\pi'(\bar\rho) = 0$, and

$$\partial_t\pi(\rho) + \operatorname{div}\big(\mathbf{u}\pi(\rho)\big) + P(\rho)\operatorname{div}\mathbf{u} = 0 \quad \text{in } \mathcal{D}'\big((0,T)\times\mathbb{R}^d\big). \tag{12}$$

Notice that π is convex as far as P is non decreasing (since $P'(s) = s\pi''(s)$), which is the case for γ-type pressure laws. Multiplying the equation of momentum conservation by $\mathbf{u}$ and integrating by parts over $\mathbb{R}^d$, we obtain the following energy estimate

$$\begin{aligned}&\int_{\mathbb{R}^d}\left(\frac{1}{2}\rho\mathbf{u}^2 + \big(\pi(\rho) - \pi(\bar\rho)\big) + \frac{\kappa}{2}|\nabla\rho|^2\right)(t)\,d\mathbf{x}\\ &\quad + \int_0^t ds\int_{\mathbb{R}^d}\big(\mu\,\mathbf{D}(\mathbf{u}):\mathbf{D}(\mathbf{u}) + (\lambda+\mu)|\operatorname{div}\mathbf{u}|^2\big)\,d\mathbf{x}\\ &\quad\quad \leqslant \int_{\mathbb{R}^d}\left(\frac{|\mathbf{m}_0|^2}{2\rho_0} + (\pi(\rho_0) - \pi(\bar\rho)) + \frac{\kappa}{2}|\nabla\rho_0|^2\right)d\mathbf{x}.\end{aligned} \tag{13}$$

Indeed, in order to compute formally the contribution to energy of the capillary tensor $\mathbf{K}$, we observe that

$$\operatorname{div}\mathbf{K} = \kappa\,\rho\nabla\Delta\rho. \tag{14}$$

In view of the above expression, this model can be understood as a diffuse interface model, in which surface tension takes place between level sets of the continuously varying density. As a matter of fact, the right hand side (14) can be rewritten up to a gradient term as the product between $\nabla\rho$ and $\Delta\rho$, which, roughly speaking, respectively represent the normal direction and the curvature of the level sets of the density. As observed for instance in [1], formal analyses show that the sharp interface limit leads to the classical two-fluid problem. We obtain indeed

$$-\int_{\mathbb{R}^d} \mathbf{u}\cdot \operatorname{div}\mathbf{K}\,d\mathbf{x} = \int_{\mathbb{R}^d} \kappa \operatorname{div}(\rho\mathbf{u})\,\Delta\rho\,d\mathbf{x} = \kappa\int_{\mathbb{R}^d} \partial_t\nabla\rho\cdot\nabla\rho\,d\mathbf{x} = \kappa\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d}\frac{|\nabla\rho|^2}{2}\,d\mathbf{x}.$$

It follows that assuming that the total energy is finite

$$\mathcal{E}_0 = \int_{\mathbb{R}^2}\left(\frac{1}{2}\rho_0\mathbf{u}_0^2 + \big(\pi(\rho_0)-\pi(\bar\rho)\big) + \frac{\kappa}{2}|\nabla\rho_0|^2\right) < +\infty, \tag{15}$$

we have the *a priori* bounds

$$\pi(\rho)-\pi(\bar\rho) \quad \text{and} \quad \rho|\mathbf{u}|^2 \in L^\infty(0,\infty;L^1(\mathbb{R}^d)), \tag{16}$$

$$\nabla\rho \in L^\infty\big(0,\infty;L^2(\mathbb{R}^d)\big)^d \quad \text{and} \quad \nabla\mathbf{u}\in L^2\big((0,\infty)\times\mathbb{R}^d\big)^{d^2}. \tag{17}$$

Let us emphasize at this point that the above *a priori bounds* do not provide any L^∞ control on the density from below or from above. Indeed, even in dimension $d=2$, $H^1(\mathbb{R}^d)$ functions are not necessarily locally bounded. Thus, vacuum patches are likely to form in the fluid in spite of the presence of capillary forces, which are expected to smooth out the density.

2. Mathematical results

We wish to prove existence and uniqueness results of solutions to (8) (9) in functional spaces very close to energy spaces. In the case $\kappa=0$ and $p(\rho)=a\rho^\gamma$, with $a>0$ and $\gamma>1$, P.-L. Lions proved in [17,18] the global existence of weak solutions “à la Leray” $(\rho,\mathbf{u})$ to (8) (9) for $\gamma\geqslant 3d/(d+2)$ and initial data $(\rho_0,\mathbf{m}_0)$ such that

$$\pi(\rho_0)-\pi(\bar\rho) \quad \text{and} \quad \frac{|\mathbf{m}_0|^2}{\rho_0}\in L^1(\mathbb{R}^d), \tag{18}$$

where we agree that $\mathbf{m}_0=0$ on $\{\mathbf{x}\in\mathbb{R}^d/\rho_0(\mathbf{x})=0\}$. More precisely, he obtains the existence of global weak solutions $(\rho,\mathbf{u})$ to (8)–(10) such that

- $\rho-\bar\rho\in L^\infty(0,\infty;L^2_\gamma(\mathbb{R}^d))$ (where $L^2_p(\mathbb{R}^d)$ spaces are Orlicz spaces defined in [18]),
- $\mathbf{u}\in L^2(0,\infty;\ \dot H^1(\mathbb{R}^d))^d$ ($\dot H^s$ being defined in Section 3),

with in addition

- $\rho\in C([0,\infty);\ L^p_{loc}(\mathbb{R}^d))$ if $1\leqslant p<\gamma$,
- $\rho|\mathbf{u}|^2\in L^\infty(0,\infty;\ L^1(\mathbb{R}^d))$, $\rho\mathbf{u}\in C([0,\infty);\ L^{2\gamma/(\gamma+1)}_{loc}(\mathbb{R}^d)$-weak),
- $\rho\in L^q_{loc}([0,\infty)\times\mathbb{R}^d)$ for $q=\gamma-1+2\gamma/d$.

Moreover, the energy inequality (13) holds for almost every $t\geqslant 0$.

Notice that the main difficulty for proving Lions' theorem consists in strong compactness properties of the density ρ in L^p_{loc} spaces required to pass to the limit in the pressure term $p(\rho) = a\rho^\gamma$. In the capillary case $\kappa > 0$, more a priori bounds are available for the density, which belongs to $L^\infty(0, \infty; \dot{H}^1(\mathbb{R}^d))$. Hence, one can easily pass to the limit in the pressure term. However, in the remaining quadratic terms involving gradients of the density $\nabla\rho \otimes \nabla\rho$ (see (5)), we have been unable to pass to the limit.

Let us mention now that the existence of strong solutions is known since the works by H. Hattori and D. Li [13,14]. Notice that high order regularity in Sobolev spaces H^s is required, namely the initial data $(\rho_0, \mathbf{m}_0)$ are assumed to belong to $H^s \times H^{s-1}$ with $s \geqslant d/2 + 4$. Moreover, they considered convex pressure profiles, which cannot cover the case of Van der Waals' equation of state.

Here we want to investigate the well-posedness of the problem in *critical spaces*, that is, in spaces which are invariant by the scaling of Korteweg's system. Recall that such an approach is now classical for incompressible Navier–Stokes equations (see, for example, [7] and the references therein) and yields local well-posedness (or global well-posedness for small data) in spaces with minimal regularity.

Let us explain precisely the scaling of Korteweg's system. We can easily verify that, if $(\rho, \mathbf{u})$ solves (8) (9), so does $(\rho_\lambda, \mathbf{u}_\lambda)$, where

$$\rho_\lambda(t,x) = \rho(\lambda^2 t, \lambda x) \quad \text{and} \quad \mathbf{u}_\lambda(t,x) = \lambda \mathbf{u}(\lambda^2 t, \lambda x),$$

provided the pressure law P has been changed into $\lambda^2 P$.

DEFINITION 1. – *We will say that a functional space is critical with respect to the scaling of the equation if the associated norm is invariant under the transformation* $(\rho, \mathbf{u}) \mapsto (\rho_\lambda, \mathbf{u}_\lambda)$ (*up to a constant independent of* λ).

This suggests us to choose initial data $(\rho_0, \mathbf{u}_0)$ in spaces whose norm is invariant by $(\rho_0, \mathbf{u}_0) \mapsto (\rho_0(\lambda\cdot), \lambda \mathbf{u}_0(\lambda\cdot))$.

A natural candidate is the homogeneous Sobolev space $\dot{H}^{d/2} \times (\dot{H}^{d/2-1})^d$, but since $\dot{H}^{d/2}$ is not included in L^∞, we cannot expect to get L^∞ control on the density when $\rho_0 \in \dot{H}^{d/2}$. This is the reason why, instead of the classical homogeneous Sobolev spaces $\dot{H}^s(\mathbb{R}^d)$, we will consider homogeneous Besov spaces with the same derivative index $B^s = B^s_{2,1}(\mathbb{R}^d)$ (for the corresponding definitions, we refer to Section 3). One of the nice property of B^s spaces for critical exponents s is that $B^{d/2}$ is an algebra embedded in L^∞. This allows to control the density from below and from above, without requiring more regularity on derivatives of ρ.

Since a global in time approach does not seem to be accessible for general data, we will mainly consider the global well-posedness problem for initial data close enough to stable equilibria (Section 4). More precisely, we will state the following theorem:

THEOREM 1. – *Let* $\bar{\rho} > 0$ *be such that* $P'(\bar{\rho}) > 0$. *Suppose that the initial density fluctuation* $\rho_0 - \bar{\rho}$ *belongs to* $B^{d/2} \cap B^{d/2-1}$, *that the initial velocity* $\mathbf{u}_0$ *is in* $(B^{d/2-1})^d$ *and that the exterior forcing term* $\mathbf{f}$ *is in* $L^1(\mathbb{R}^+; B^{d/2-1})^d$. *Then there exists a constant* $\eta > 0$ *depending only on* $\kappa, \mu, \lambda, \rho, P'(\bar{\rho})$ *and* d, *such that, if*

$$\|\rho_0 - \bar{\rho}\|_{B^{d/2-1} \cap B^{d/2}} + \|\mathbf{u}_0\|_{B^{d/2-1}} + \|\mathbf{f}\|_{L^1(B^{d/2-1})} \leqslant \eta,$$

then (8)–(10) *has a unique global solution* $(\rho, \mathbf{u})$ *such that the density fluctuation* $(\rho - \bar{\rho}) \in C(\mathbb{R}^+; B^{d/2-1} \cap B^{d/2}) \cap L^1(\mathbb{R}^+; B^{d/2+1} \cap B^{d/2+2})$ *and the velocity* $\mathbf{u} \in C(\mathbb{R}^+; B^{d/2-1})^d \cap L^1(\mathbb{R}^+; B^{d/2+1})^d$.

In Section 5, we get a local in time existence result for initial densities bounded away from zero, which does not require any stability assumption on the pressure law, and thus applies to Van der Waals' law. The precise statement reads as follows:

THEOREM 2. – *Suppose that the forcing term* $\mathbf{f}$ *belongs to* $L^1_{loc}(\mathbb{R}_+; B^{d/2-1})^d$, *that the initial velocity* $\mathbf{u}_0$ *belongs to* $(B^{d/2-1})^d$, *and that the initial density* ρ_0 *satisfies* $(\rho_0 - \bar{\rho}) \in B^{d/2}$ *and* $\rho_0 \geqslant c$ *for a positive constant* c. *Then there exists* $T > 0$ *such that* (8)–(10) *has a unique solution* $(\rho, \mathbf{u})$ *satisfying* $(\rho - \bar{\rho}) \in C([0,T]; B^{d/2}) \cap L^1([0,T]; B^{d/2+2})$ *and* $\mathbf{u} \in C([0,T]; B^{d/2-1})^d \cap L^1([0,T]; B^{d/2+1})^d$.

In Section 6, we show that the problem is still locally well-posed in more general scaling invariant Besov spaces of type $B^s_{p,1}$ which are not related to energy spaces (namely $\rho - \bar{\rho}$ is assumed to be in $B^{d/p}_{p,1}$ and $\mathbf{u}_0$ to be in $(B^{d/p-1}_{p,1})^d$). No stability assumption on the pressure is required, but we have to suppose that the density is close to a constant (see Theorem 5). Let us observe that working with $p > d$ allows to consider initial velocities in $B^s_{p,1}$ spaces with negative exponents s, which is in particular relevant for oscillating initial data.

Finally, we will investigate blow-up properties of smooth solutions without smallness assumptions on the data, like in the work of Z. Xin [22], and study sufficient conditions for the existence of weak solutions close to equilibria in dimension $d = 2$.

Notation. In all the paper, C will stand for a "harmless" constant, and we will sometimes use the notation $A \lesssim B$ equivalently to $A \leqslant CB$.

3. Littlewood–Paley theory and Besov spaces

3.1. Littlewood–Paley decomposition

The homogeneous Littlewood–Paley decomposition relies upon a dyadic partition of unity. We can use for instance any $\varphi \in C^\infty(\mathbb{R}^d)$, supported in $\mathcal{C} \stackrel{\text{def}}{=} \{\xi \in \mathbb{R}^d, 3/4 \leqslant |\xi| \leqslant 8/3\}$ such that

$$\sum_{\ell \in \mathbb{Z}} \varphi(2^{-\ell}\xi) = 1 \quad \text{if } \xi \neq 0.$$

Denoting $h = \mathcal{F}^{-1}\varphi$, we then define the dyadic blocks by

$$\Delta_\ell u \stackrel{\text{def}}{=} \varphi(2^{-\ell}D)u = 2^{\ell d} \int_{\mathbb{R}^d} h(2^\ell \mathbf{y}) u(\mathbf{x} - \mathbf{y})\, d\mathbf{y} \quad \text{and} \quad S_\ell u = \sum_{k \leqslant \ell - 1} \Delta_k u.$$

The formal decomposition

$$u = \sum_{\ell \in \mathbb{Z}} \Delta_\ell u \tag{19}$$

is called homogeneous Littlewood–Paley decomposition. Let us observe that the above formal equality does not hold in $\mathcal{S}'(\mathbb{R}^d)$ for two reasons:

(i) The right-hand side does not necessarily converge in $\mathcal{S}'(\mathbb{R}^d)$.

(ii) Even if it does, the equality is not always true in $\mathcal{S}'(\mathbb{R}^d)$ (consider the case $u=1$).

Nevertheless, (19) holds true modulo polynomials (see [21]).

Furthermore, the above dyadic decomposition has nice properties of quasi-orthogonality: with our choice of φ, we have

$$\Delta_k\Delta_\ell u\equiv 0 \quad \text{if } |k-\ell|\geqslant 2, \quad \text{and} \quad \Delta_k(S_{\ell-1}u\Delta_\ell u)\equiv 0 \quad \text{if } |k-\ell|\geqslant 5. \tag{20}$$

3.2. Homogeneous Besov spaces

DEFINITION 2. – *For $s\in\mathbb{R}$, $p\in[1,+\infty]$, $q\in[1,+\infty]$ and $u\in\mathcal{S}'(\mathbb{R}^d)$, we set*

$$\|u\|_{B^s_{p,q}}\stackrel{\text{def}}{=}\left(\sum_{\ell\in\mathbb{Z}}\left(2^{s\ell}\|\Delta_\ell u\|_{L^p}\right)^q\right)^{1/q}.$$

A difficulty due to the choice of homogeneous spaces arises at this point. Indeed, $\|\cdot\|_{B^s_{p,q}}$ cannot be a norm on $\{u\in\mathcal{S}'(\mathbb{R}^d),\ \|u\|_{B^s_{p,q}}<+\infty\}$ because $\|u\|_{B^s_{p,q}}=0$ means that u is a polynomial. This enforces us to adopt the following definition for homogeneous Besov spaces (see [5] for more details):

DEFINITION 3. – *Let $s\in\mathbb{R}$, $p\in[1,+\infty]$ and $q\in[1,+\infty]$. Denote $m=[s-d/p]$ if $s-d/p\notin\mathbb{Z}$ or $q>1$ and $m=s-d/p-1$ otherwise. If $m<0$, then we define $B^s_{p,q}$ as*

$$B^s_{p,q}=\left\{u\in\mathcal{S}'\left(\mathbb{R}^d\right)\ |\ \|u\|_{B^s_{p,q}}<\infty \text{ and } u=\sum_{\ell\in\mathbb{Z}}\Delta_\ell u \text{ in } \mathcal{S}'\left(\mathbb{R}^d\right)\right\}.$$

If $m\geqslant 0$, we denote by $\mathcal{P}_m[\mathbb{R}^d]$ the set of polynomials of degree less than or equal to m and we set

$$B^s_{p,q}=\left\{u\in\mathcal{S}'\left(\mathbb{R}^d\right)/\mathcal{P}_m\left[\mathbb{R}^d\right]\ |\ \|u\|_{B^s_{p,q}}<\infty \text{ and } u=\sum_{\ell\in\mathbb{Z}}\Delta_\ell u \text{ in } \mathcal{S}'\left(\mathbb{R}^d\right)\mathcal{P}_m\left[\mathbb{R}^d\right]\right\}.$$

Remark 1. – The above definition is a natural generalization of the homogeneous Sobolev or Hölder spaces: one can show that $B^s_{\infty,\infty}$ is the homogeneous Hölder space $\dot{C}^s$ and that $B^s_{2,2}$ is the homogeneous Sobolev space $\dot{H}^s$.

In the sequel, we will use only Besov spaces $B^s_{p,q}$ with $q=1$ and we will denote them by B^s_p or even by B^s if there is no ambiguity on the index p.

3.3. Basic properties of Besov spaces

PROPOSITION 1. – *The following properties hold*:

(i) *Density*: *if $p<+\infty$ and $|s|\leqslant d/p$, then C^∞_0 is dense in B^s_p.*

(ii) *Derivation*: *there exists a universal constant C such that*

$$C^{-1}\|u\|_{B^s_p}\leqslant\|\nabla u\|_{B^{s-1}_p}\leqslant C\|u\|_{B^s_p}.$$

(ii′) *Fractional derivation*: *let* $\Lambda \stackrel{\text{def}}{=} \sqrt{-\Delta}$ *and* $\sigma \in \mathbb{R}$. *Then the operator* Λ^σ *is an isomorphism from* B^s_p *to* $B^{s-\sigma}_p$.

(iii) *Sobolev embeddings*: *if* $p_1 < p_2$ *then* $B^s_{p_1} \hookrightarrow B^{s-d(1/p_1-1/p_2)}_{p_2}$ (*where* $\hookrightarrow$ *means continuous embedding*).

(iv) *Algebraic properties*: *for* $s > 0$, $B^s_p \cap L^\infty$ *is an algebra.*

(v) *Interpolation:* $(B^{s_1}_p, B^{s_2}_p)_{\theta,1} = B^{\theta s_1+(1-\theta)s_2}_p$.

In Section 6, we will make extensive use of the space $B^{d/p}_p$. Note that, if $p < +\infty$, then $B^{d/p}_p$ is an algebra included in the space C_0 of continuous functions which tend to 0 at infinity. Note also that $B^{d/p}_p \times (B^{d/p-1}_p)^d$ is invariant by the scaling of Korteweg's system.

In Sections 4 and 5, we will focus on the case $p = 2$. Note that the following inclusion chain

$$B^{d/2}_{2,1} \hookrightarrow \dot{H}^{d/2} = B^{d/2}_{2,2} \hookrightarrow B^{d/2}_{2,\infty}$$

shows us that $\dot{H}^{d/2}$ is very close to $B^{d/2}_{2,1}$. But $B^{d/2}_{2,1}$ has two additional nice properties: to be an algebra and to be a subset of C_0.

3.4. Besov–Chemin–Lerner spaces

The study of non stationary PDE's usually requires spaces of type $L^r_T(X) \stackrel{\text{def}}{=} L^r(0,T; X)$ for appropriate Banach spaces X. In our case, we expect X to be a Besov space, so that it is natural to localize the equations through Littlewood–Paley decomposition. We then get estimates for each dyadic block and perform integration in time. But, in doing so, we obtain bounds in spaces which are not of type $L^r(0,T; B^s_p)$. This approach was initiated in [9] and naturally leads to the following definitions:

DEFINITION 4. – *Let* $(\rho, p) \in [1,+\infty]^2$, $T \in]0,+\infty]$ *and* $s \in \mathbb{R}$. *We set*

$$\|u\|_{\widetilde{L}^\rho_T(B^s_p)} \stackrel{\text{def}}{=} \sum_{\ell\in\mathbb{Z}} 2^{\ell s} \left(\int_0^T \|\Delta_\ell u(t)\|^\rho_{L^p}\, dt \right)^{1/\rho}.$$

Noticing that Minkowski's inequality yields $\|u\|_{L^\rho_T(B^s_p)} \leqslant \|u\|_{\widetilde{L}^\rho_T(B^s_p)}$, we define $\widetilde{L}^\rho_T(B^s_p)$ spaces as follows

$$\widetilde{L}^\rho_T\big(B^s_p\big) \stackrel{\text{def}}{=} \big\{u \in L^\rho_T\big(B^s_p\big) \mid \|u\|_{\widetilde{L}^\rho_T(B^s_p)} < +\infty\big\}.$$

Let us observe that $L^1_T(B^s_p) = \widetilde{L}^1_T(B^s_p)$ but that the embedding $\widetilde{L}^\rho_T(B^s_p) \subset L^\rho_T(B^s_p)$ is strict if $\rho > 1$.

We will denote by $\widetilde{C}_T(B^s_p)$ the subset of functions of $\widetilde{L}^\infty_T(B^s_p)$ which are continuous on $[0,T]$ with values in B^s_p.

Throughout the paper, the notation $\widetilde{L}^\rho_T(B^s_p \cap B^{s'}_{p'})$ (respectively $\widetilde{L}^\rho_T(B^s_p \times B^{s'}_{p'})$) will stand for $\widetilde{L}^\rho_T(B^s_p) \cap \widetilde{L}^\rho_T(B^{s'}_{p'})$ (respectively $\widetilde{L}^\rho_T(B^s_p) \times \widetilde{L}^\rho_T(B^{s'}_{p'})$). Moreover, in the case $T = +\infty$, the T will be omitted. For example, $\widetilde{L}^\rho(B^s_p)$ means $\widetilde{L}^\rho_{+\infty}(B^s_p)$.

We will often use the following interpolation property

$$\|u\|_{\widetilde{L}^{\rho}_T(B^s_p)} \leqslant \|u\|^{\theta}_{\widetilde{L}^{\rho_1}_T(B^{s_1}_p)} \|u\|^{1-\theta}_{\widetilde{L}^{\rho_2}_T(B^{s_2}_p)} \quad \text{with } \frac{1}{\rho} = \frac{\theta}{\rho_1} + \frac{1-\theta}{\rho_2} \text{ and } s = \theta s_1 + (1-\theta)s_2,$$

and the following embeddings

$$\widetilde{L}^{\rho}_T\big(B^{d/p}_p\big) \hookrightarrow L^{\rho}_T(C_0) \quad \text{and} \quad \widetilde{C}_T\big(B^{d/p}_p\big) \hookrightarrow C\big([0,T]\times\mathbb{R}^d\big).$$

The $\widetilde{L}^{\rho}_T(B^s_p)$ spaces suit particularly well to the study of smoothing properties of the heat equation. In [7], J.-Y. Chemin proved the following proposition

PROPOSITION 2. – *Let $p \in [1,+\infty]$ and $1 \leqslant \rho_2 \leqslant \rho_1 \leqslant +\infty$. Let u solve*

$$\begin{cases} \partial_t u - \nu\Delta u = f, \\ u_{|t=0} = u_0. \end{cases}$$

Then there exists $C > 0$ depending only on d, ν, ρ_1 and ρ_2 such that

$$\|u\|_{\widetilde{L}^{\rho_1}_T(B^{s+2/\rho_1}_p)} \leqslant C\|u_0\|_{B^s_p} + C\|f\|_{\widetilde{L}^{\rho_2}_T(B^{s-2+2/\rho_2}_p)}.$$

In Sections 4, 5 and 6, we will point out similar smoothing properties for the linearized Korteweg system.

Let us now state properties of $\widetilde{L}^{\rho}_T(B^s_p)$ spaces with respect to the product.

PROPOSITION 3. – *If $s > 0$, $1/\rho_2 + 1/\rho_3 = 1/\rho_1 + 1/\rho_4 = 1/\rho \leqslant 1$, $u \in L^{\rho_1}_T(L^\infty) \cap \widetilde{L}^{\rho_3}_T(B^s_p)$ and $v \in L^{\rho_2}_T(L^\infty) \cap \widetilde{L}^{\rho_4}_T(B^s_p)$, then $uv \in \widetilde{L}^{\rho}_T(B^s_p)$ and*

$$\|uv\|_{\widetilde{L}^{\rho}_T(B^s_p)} \lesssim \|u\|_{L^{\rho_1}_T(L^\infty)}\|v\|_{\widetilde{L}^{\rho_4}_T(B^s_p)} + \|v\|_{L^{\rho_2}_T(L^\infty)}\|u\|_{\widetilde{L}^{\rho_3}_T(B^s_p)}.$$

If $s_1, s_2 \leqslant d/p$, $s_1 + s_2 > 0$, $1/\rho_1 + 1/\rho_2 = 1/\rho \leqslant 1$, $u \in \widetilde{L}^{\rho_1}_T(B^{s_1}_p)$ and $v \in \widetilde{L}^{\rho_2}_T(B^{s_2}_p)$, then $uv \in \widetilde{L}^{\rho}_T(B^{s_1+s_2-d/p}_p)$ and

$$\|uv\|_{\widetilde{L}^{\rho}_T(B^{s_1+s_2-d/p}_p)} \lesssim \|u\|_{L^{\rho_1}_T(B^{s_1}_p)}\|v\|_{L^{\rho_2}_T(B^{s_2}_p)}.$$

This proposition is a straightforward adaptation of the corresponding results for usual homogeneous Besov spaces (see [8]).

We finally need a composition lemma in $\widetilde{L}^{\rho}_T(B^s_p)$ spaces.

LEMMA 1. – *Let $s > 0$, $p \in [1,+\infty]$ and $u \in \widetilde{L}^{\rho}_T(B^s_p) \cap L^\infty_T(L^\infty)$.*

(i) *Let $F \in W^{[s]+2,\infty}_{loc}(\mathbb{R}^d)$ such that $F(0) = 0$. Then $F(u) \in \widetilde{L}^{\rho}_T(B^s_p)$. More precisely, there exists a function C depending only on s, p, d and F such that*

$$\|F(u)\|_{\widetilde{L}^{\rho}_T(B^s_p)} \leqslant C\big(\|u\|_{L^\infty_T(L^\infty)}\big)\|u\|_{\widetilde{L}^{\rho}_T(B^s_p)}.$$

(ii) *If v also belongs to $\widetilde{L}^{\rho}_T(B^s_p) \cap L^\infty_T(L^\infty)$ and $G \in W^{[s]+3,\infty}_{loc}(\mathbb{R}^d)$, then $G(v) - G(u)$ belongs to $\widetilde{L}^{\rho}_T(B^s_p)$ and there exists a function C depending only on s, p, d and G, and such that*

$$\begin{aligned}&\big\|G(v)-G(u)\big\|_{\widetilde{L}^\rho_T(B^s_p)}\\&\quad\leqslant C\big(\|u\|_{L^\infty_T(L^\infty)},\|v\|_{L^\infty_T(L^\infty)}\big)\big(\|v-u\|_{\widetilde{L}^\rho_T(B^s_p)}(1+\|u\|_{L^\infty_T(L^\infty)}+\|v\|_{L^\infty_T(L^\infty)})\\&\qquad+\|v-u\|_{L^\infty_T(L^\infty)}(\|u\|_{\widetilde{L}^\rho_T(B^s_p)}+\|v\|_{\widetilde{L}^\rho_T(B^s_p)})\big).\end{aligned}$$

Proof. – For (i), one just has to use the proof of [2] and replace L^2 norms with L^p norms. For (ii), we use the following identity

$$G(v)-G(u)=(v-u)\int_0^1 H\big(u+\tau(v-u)\big)\,d\tau+G'(0)(v-u),$$

where $H(w)=G'(w)-G'(0)$, and we conclude by using (i) and Proposition 3. □

4. Global solutions near equilibrium

In this section, we want to prove global existence and uniqueness of suitably smooth solutions to the Korteweg system (8) (9) in the functional spaces $\widetilde{L}^\rho_T(B^s_2)$ which are very close to the physical energy spaces. Given a reference density $\bar\rho$ such that the stability condition $P'(\bar\rho)>0$ is satisfied, we introduce the density fluctuation $q=(\rho-\bar\rho)/\bar\rho$ and the scaled momentum $\mathbf{m}=\rho\mathbf{u}/\bar\rho$. We also define the scaled viscosity coefficients $\bar\mu=\mu/\bar\rho$ and $\bar\lambda=\lambda/\bar\rho$ and the scaled surface tension coefficient $\bar\kappa=\bar\rho\kappa$. Assuming that the density ρ is bounded away from zero, we rewrite the Korteweg system (8) (9) as follows

$$\partial_t q+\operatorname{div}\mathbf{m}=0, \tag{21}$$

$$\partial_t\mathbf{m}-\bar\mu\Delta\mathbf{m}-(\bar\lambda+\bar\mu)\nabla\operatorname{div}\mathbf{m}-\bar\kappa\nabla\Delta q+P'(\bar\rho)\nabla q=\mathbf{G}(q,\mathbf{m})+\mathbf{f}, \tag{22}$$

$$(q,\mathbf{m})_{|t=0}=(q_0,\mathbf{m}_0), \tag{23}$$

where we define $\mathbf{G}=\mathbf{G}_1+\mathbf{G}_2+\mathbf{G}_3+\mathbf{G}_4+\mathbf{G}_5$ by

$$\begin{aligned}&\mathbf{G}_1(q,\mathbf{m})=-\operatorname{div}\left(\frac{\mathbf{m}\otimes\mathbf{m}}{1+q}\right),\qquad \mathbf{G}_2(q,\mathbf{m})=-\nabla H(q),\\&\mathbf{G}_3(q,\mathbf{m})=-\bar\mu\Delta\left(\frac{q\mathbf{m}}{1+q}\right)-(\bar\lambda+\bar\mu)\nabla\operatorname{div}\left(\frac{q\mathbf{m}}{1+q}\right),\\&\mathbf{G}_4(q,\mathbf{m})=\frac{\bar\kappa}{2}\nabla\big(\Delta q^2-|\nabla q|^2\big)-\bar\kappa\operatorname{div}(\nabla q\otimes\nabla q)=\bar\kappa q\nabla\Delta q,\end{aligned}$$

and

$$\mathbf{G}_5(q,\mathbf{m})=\mathbf{f}q,$$

H being defined by $H(q)=(P(\bar\rho(1+q))-P(\bar\rho)-P'(\bar\rho)q\bar\rho)/\bar\rho$.

In Section 4.1, we study the linearized system around $(q,\mathbf{m})=(0,0)$, which turns out to have the same smoothing properties as the heat equation. Finally, we prove in Section 4.2 our main global theorem, estimating the right-hand side $\mathbf{G}$ of (22) in terms of suitable norms of $(q,\mathbf{m})$. Notice that in Sections 4 and 5, B^s will stand for $B^s_{2,1}$.

4.1. Estimates for the linearized system

This section is devoted to the linearized isothermal system of Korteweg type around $(q,\mathbf{m})=(0,0)$. This system reads

$$\begin{cases}\partial_t q+\operatorname{div}\mathbf{m}=F,\\ \partial_t\mathbf{m}-\bar\mu\Delta\mathbf{m}-(\bar\lambda+\bar\mu)\nabla\operatorname{div}\mathbf{m}-\bar\kappa\nabla\Delta q+\beta\nabla q=\mathbf{G}.\end{cases}\tag{LNSK1}$$

The term $\beta\nabla q$ corresponds to the linearized pressure (that is $\beta=P'(\bar\rho)>0$). Our purpose is to prove estimates for (LNSK1) in Besov spaces closely related to energy spaces. We get:

PROPOSITION 4. – *Let $s\in\mathbb{R}$, $1\leqslant r_1\leqslant r\leqslant+\infty$ and $T\in]0,+\infty]$. If $(q_0,\mathbf{m}_0)\in(B^s\cap B^{s-1})\times(B^{s-1})^d$ and $(F,\mathbf{G})\in\widetilde{L}_T^{r_1}((B^{s-2+2/r_1}\cap B^{s-3+2/r_1})\times(B^{s-3+2/r_1})^d)$ then the linear system* (LNSK1) *has a unique solution $(q,\mathbf{m})\in\widetilde{C}_T((B^s\cap B^{s-1})\times(B^{s-1})^d)\cap\widetilde{L}_T^r((B^{s+2/r}\cap B^{s-1+2/r})\times(B^{s-1+2/r})^d)$. Moreover, there exists a constant C depending only on r, r_1, $\bar\mu$, $\bar\lambda$, $\bar\kappa$ and β such that the following inequality holds*:

$$\begin{aligned}&\|q\|_{\widetilde{L}_T^r(B^{s+2/r}\cap B^{s-1+2/r})}+\|\mathbf{m}\|_{\widetilde{L}_T^r(B^{s-1+2/r})}\\&\quad\leqslant C\big(\|q_0\|_{B^s\cap B^{s-1}}+\|\mathbf{m}_0\|_{B^{s-1}}+\|\mathbf{F}\|_{\widetilde{L}_T^{r_1}(B^{s-2+2/r_1}\cap B^{s-3+2/r_1})}+\|\mathbf{G}\|_{\widetilde{L}_T^{r_1}(B^{s-3+2/r_1})}\big).\end{aligned}$$

Proof. – Denote by $W(t)$ the semi-group associated to (LNSK1). According to Duhamel's formula,

$$\begin{pmatrix}q(t)\\ \mathbf{m}(t)\end{pmatrix}=W(t)\begin{pmatrix}q_0\\ \mathbf{m}_0\end{pmatrix}+\int_0^t W(t-s)\begin{pmatrix}F(s)\\ \mathbf{G}(s)\end{pmatrix}ds.\tag{24}$$

Let us first consider the case $F\equiv0$ and $\mathbf{G}\equiv0$ and denote $(q_\ell(t),\mathbf{m}_\ell(t))^t=W(t)(\Delta_\ell q_0,\Delta_\ell\mathbf{m}_0)^t$. Then, we have the following lemma

LEMMA 2. – *There exist two positive constants c and C depending only on $\bar\lambda,\bar\mu,\bar\kappa$ and β such that for all $\ell\in\mathbb{Z}$,*

$$\begin{aligned}&\|\mathbf{m}_\ell(t)\|_{L^2}+\|\nabla q_\ell(t)\|_{L^2}+\|q_\ell(t)\|_{L^2}\\&\quad\leqslant Ce^{-c2^{2\ell}t}\big(\|\Delta_\ell\mathbf{m}_0\|_{L^2}+\|\nabla\Delta_\ell q_0\|_{L^2}+\|\Delta_\ell q_0\|_{L^2}\big).\end{aligned}$$

Proof. – We apply the operator Δ_ℓ to (LNSK1) in the case $F\equiv\mathbf{G}\equiv0$ and get

$$\partial_t q_\ell+\operatorname{div}\mathbf{m}_\ell=0,\tag{25}$$

$$\partial_t\mathbf{m}_\ell-\bar\mu\Delta\mathbf{m}_\ell-(\bar\lambda+\bar\mu)\nabla\operatorname{div}\mathbf{m}_\ell-\bar\kappa\nabla\Delta q_\ell-\beta\nabla q_\ell=0.\tag{26}$$

In view of Eq. (25), integrations by parts yield

$$\int_{\mathbb{R}^d}\Delta q_\ell\operatorname{div}\mathbf{m}_\ell\,d\mathbf{x}=\frac12\frac{d}{dt}\|\nabla q_\ell\|_{L^2}^2\quad\text{and}$$

$$\int_{\mathbb{R}^d}q_\ell\operatorname{div}\mathbf{m}_\ell\,d\mathbf{x}=-\frac12\frac{d}{dt}\|q_\ell\|_{L^2}^2.$$

Thus, taking scalar product of (26) with $\mathbf{m}_\ell$, we get

$$\frac{1}{2}\frac{d}{dt}\big(\|\mathbf{m}_\ell\|_{L^2}^2+\beta\|q_\ell\|_{L^2}^2+\bar\kappa\|\nabla q_\ell\|_{L^2}^2\big)+\bar\mu\|\nabla\mathbf{m}_\ell\|_{L^2}^2+(\bar\lambda+\bar\mu)\|\operatorname{div}\mathbf{m}_\ell\|_{L^2}^2=0. \quad (27)$$

In order to obtain a second energy estimate, we take the scalar product of $\mathbf{m}_\ell$ with the gradient of (25), which yields

$$\int_{\mathbb{R}^d}\mathbf{m}_\ell\cdot\partial_t\nabla q_\ell\,d\mathbf{x}-\|\operatorname{div}\mathbf{m}_\ell\|_{L^2}^2=0. \quad (28)$$

Taking the scalar product of (26) with ∇q_ℓ, we obtain

$$\int_{\mathbb{R}^d}\nabla q_\ell\cdot\partial_t\mathbf{m}_\ell\,d\mathbf{x}+\bar\kappa\|\Delta q_\ell\|_{L^2}^2+\beta\|\nabla q_\ell\|_{L^2}^2\leqslant C\|\nabla\mathbf{m}_\ell\|_{L^2}\|\nabla^2 q_\ell\|_{L^2}. \quad (29)$$

Summing (28) and (29), we deduce

$$\frac{d}{dt}\bigg(\int_{\mathbb{R}^d}\mathbf{m}_\ell\cdot\nabla q_\ell\,d\mathbf{x}\bigg)+\frac{\bar\kappa}{2}\|\nabla^2 q_\ell\|_{L^2}^2+\beta\|\nabla q_\ell\|_{L^2}^2\leqslant C\|\nabla\mathbf{m}_\ell\|_{L^2}^2. \quad (30)$$

Let $\alpha>0$ be a constant to be chosen later and denote

$$h_\ell^2=\|\mathbf{m}_\ell\|_{L^2}^2+\bar\kappa\|\nabla q_\ell\|_{L^2}^2+\beta\|q_\ell\|_{L^2}^2+2\alpha\int_{\mathbb{R}^d}\mathbf{m}_\ell\cdot\nabla q_\ell\,d\mathbf{x}.$$

As a result, from (30) and (27), we derive for some positive constant c_0

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}h_\ell^2+c_0\big(\|\nabla\mathbf{m}_\ell\|_{L^2}^2+\alpha\big\|\nabla^2 q_\ell\big\|_{L^2}^2+\alpha\|\nabla q_\ell\|_{L^2}^2\big)\\&\quad\leqslant C\alpha\|\nabla\mathbf{m}_\ell\|_{L^2}^2. \quad (31)\end{aligned}$$

Now choosing α suitably small, we deduce that

$$\frac{1}{\delta}h_\ell^2\leqslant\|\mathbf{m}_\ell\|_{L^2}^2+\bar\kappa\|\nabla q_\ell\|_{L^2}^2+\beta\|q_\ell\|_{L^2}^2\leqslant\delta h_\ell^2, \quad (32)$$

for some positive δ. Thus, there exists a constant $c>0$ such that

$$\frac{1}{2}\frac{d}{dt}h_\ell^2+c2^{2\ell}h_\ell^2\leqslant 0,$$

so that the proof of Lemma 2 is complete. □

Proof of Proposition 4 (continued). – In view of Lemma 2 and formula (24), we have

$$\begin{aligned}&\|\Delta_\ell\mathbf{m}(t)\|_{L^2}+\|\nabla\Delta_\ell q(t)\|_{L^2}+\|\Delta_\ell q(t)\|_{L^2}\\&\quad\leqslant Ce^{-c2^{2\ell}t}\big(\|\Delta_\ell\mathbf{m}_0\|_{L^2}+\|\nabla\Delta_\ell q_0\|_{L^2}+\|\Delta_\ell q_0\|_{L^2}\big)\end{aligned}$$

$$+\int_0^t e^{-c2^{2\ell}(t-\tau)}\big(\|\Delta_\ell \mathbf{G}(\tau)\|_{L^2}+\|\nabla\Delta_\ell F(\tau)\|_{L^2}+\|\Delta_\ell F(\tau)\|_{L^2}\big)\,d\tau,$$

so routine computations yield Proposition 4. □

4.2. Global existence and uniqueness

Let us first introduce functional spaces needed in the main global existence result. We will prove existence in the space

$$E=\Big(\widetilde{C}\big(B^{d/2-1}\cap B^{d/2}\big)\cap L^1\big(B^{d/2+1}\cap B^{d/2+2}\big)\Big)\Big(\widetilde{C}\big(B^{d/2-1}\big)\cap L^1\big(B^{d/2+1}\big)\Big)^d,$$

and uniqueness in the larger space

$$\widetilde{E}=\widetilde{C}\Big(B^{d/2}\times\big(B^{d/2-1}\big)^d\Big)\cap L^2\Big(B^{d/2+1}\times\big(B^{d/2}\big)^d\Big).$$

We denote by $\|\cdot\|_{\widetilde{E}}$ and $\|\cdot\|_E$ the corresponding norms. Since $B^{d/2}$ is a Banach space, it is easy to verify that $\widetilde{E}$ and E are also Banach spaces. We now turn to our main global existence theorem

THEOREM 3. – *Let $\bar\rho>0$ be such that $P'(\bar\rho)>0$. Suppose that the initial density fluctuation q_0 belongs to $B^{d/2}\cap B^{d/2-1}$, that the initial momentum $\mathbf{m}_0$ is in $B^{d/2-1}$ and that the forcing term $\mathbf{f}$ is in $L^1(\mathbb{R}^+;B^{d/2-1})^d$. Then there exists a constant $\eta>0$ depending only on $\bar\kappa$, $\bar\mu$, $\bar\lambda$, $\bar\rho$, P such that, if*

$$\|q_0\|_{B^{d/2-1}\cap B^{d/2}}+\|\mathbf{m}_0\|_{B^{d/2-1}}+\|\mathbf{f}\|_{L^1(B^{d/2-1})}\leqslant\eta,$$

then (21)–(23) *has a unique global solution $(q,\mathbf{m})$ in $\widetilde{E}$. In addition, $(q,\mathbf{m})$ belongs to E.*

Proof. – Let us denote $(q_L,\mathbf{m}_L)$ the "free" solution of the linearized system

$$\begin{pmatrix}q_L(t)\\ \mathbf{m}_L(t)\end{pmatrix}=W(t)\begin{pmatrix}q_0\\ \mathbf{m}_0\end{pmatrix}+\int_0^t W(t-s)\begin{pmatrix}0\\ \mathbf{f}(s)\end{pmatrix}ds.$$

We define the functional Ψ_{q_L,m_L} in a neighborhood of 0 in E by

$$\Psi_{q_L,m_L}(\bar q,\bar{\mathbf{m}})=\int_0^t W(t-s)\begin{pmatrix}0\\ \mathbf{G}(q_L+\bar q,\mathbf{m}_L+\bar{\mathbf{m}})(s)\end{pmatrix}ds.\tag{33}$$

To prove the existence part of the theorem, we just have to show that Ψ_{q_L,m_L} has a fixed point in E.

First step: *stability of* $B(0,R)$.

We start by proving that the ball $B(0,R)$ of E is stable under Ψ_{q_L,m_L} provided R is small enough. Denote $q=q_L+\bar q$ and $\mathbf{m}=\mathbf{m}_L+\bar{\mathbf{m}}$. According to Proposition 4, we have

$$\|(q_L,\mathbf{m}_L)\|_E\leqslant C\eta, \tag{34}$$

$$\|\Psi_{q_L,m_L}(\bar q,\bar{\mathbf{m}})\|_E\leqslant C\|\mathbf{G}(q,\mathbf{m})\|_{L^1(B^{d/2-1})}. \tag{35}$$

Making the assumption

$$\|q\|_{L^\infty(\mathbb{R}^+\times\mathbb{R}^d)}\leqslant 1/2, \tag{$\mathcal{H}$}$$

and using Proposition 3 and Lemma 1, we deduce the following estimates:

$$\|\mathbf{G}_1(q,\mathbf{m})\|_{L^1(B^{d/2-1})}\leqslant C\|\mathbf{m}\|_{L^1(B^{d/2+1})}\|\mathbf{m}\|_{\widetilde L^\infty(B^{d/2-1})}\big(1+\|q\|_{\widetilde L^\infty(B^{d/2})}\big),$$
$$\|\mathbf{G}_2(q,\mathbf{m})\|_{L^1(B^{d/2-1})}\leqslant C\|q\|_{L^1(B^{d/2+1})}\|q\|_{\widetilde L^\infty(B^{d/2-1})},$$
$$\|\mathbf{G}_3(q,\mathbf{m})\|_{L^1(B^{d/2-1})}\leqslant C\big(\|q\|_{L^1(B^{d/2+2})}\|\mathbf{m}\|_{\widetilde L^\infty(B^{d/2-1})}+\|q\|_{\widetilde L^\infty(B^{d/2})}\|\mathbf{m}\|_{L^1(B^{d/2+1})}\big),$$
$$\|\mathbf{G}_4(q,\mathbf{m})\|_{L^1(B^{d/2-1})}\leqslant C\|q\|_{L^1(B^{d/2+2})}\|q\|_{\widetilde L^\infty(B^{d/2})},$$
$$\|\mathbf{G}_5(q,\mathbf{m})\|_{L^1(B^{d/2-1})}\leqslant C\|\mathbf{f}\|_{L^1(B^{d/2-1})}\|q\|_{\widetilde L^\infty(B^{d/2})},$$

since by interpolation, we have

$$\|Z\|^2_{L^2(B^s)}\leqslant\|Z\|_{\widetilde L^\infty(B^{s-1})}\|Z\|_{L^1(B^{s+1})}.$$

In the second inequality above, we also used that $H(q)=q\widetilde H(q)$ for a smooth function $\widetilde H$ such that $\widetilde H(0)=0$. Therefore, assuming that $R\leqslant 1$, we obtain

$$\begin{aligned}\|\Psi_{q_L,m_L}(\bar q,\bar{\mathbf{m}})\|_E&\leqslant C\|(q_L+\bar q,\mathbf{m}_L+\bar{\mathbf{m}})\|_E\big(\|(q_L+\bar q,\mathbf{m}_L+\bar{\mathbf{m}})\|_E+\eta\big)\\&\leqslant C\big((C+1)\eta+R\big)^2.\end{aligned}\tag{36}$$

Let c be a constant such that $\|\cdot\|_{B^{d/2}}\leqslant c$ implies $\|\cdot\|_{L^\infty}\leqslant 1/5$. We choose (R,η) such that

$$R\leqslant\inf\big((5C)^{-1},c,1\big)\ \text{and}\ \eta\leqslant\inf(R,c)/(C+1),\ \text{so that}\ \mathcal{H},\ \text{is satisfied.}\tag{37}$$

From (36), we finally deduce that $\Psi_{q_L,m_L}(B(0,R))\subset B(0,R)$.

Second step: *Contraction properties.*

Consider two elements $(\bar q_1,\bar{\mathbf{m}}_1)$ and $(\bar q_2,\bar{\mathbf{m}}_2)$ in $B(0,R)$, and denote $q_i=q_L+\bar q_i$ and $\mathbf{m}_i=\mathbf{m}_L+\bar{\mathbf{m}}_i$ for $i=1,2$. According to (33) and to Proposition 4, we have

$$\big\|\Psi_{q_L,m_L}(\bar q_2,\bar{\mathbf{m}}_2)-\Psi_{q_L,m_L}(\bar q_1,\bar{\mathbf{m}}_1)\big\|_E\leqslant C\big\|\mathbf{G}(q_2,\mathbf{m}_2)-\mathbf{G}(q_1,\mathbf{m}_1)\big\|_{L^1(B^{d/2-1})}.\tag{38}$$

Under assumption $(\mathcal{H})$ for q_1 and q_2, we obtain estimates for $\mathbf{G}(q_2,\mathbf{m}_2)-\mathbf{G}(q_1,\mathbf{m}_1)$. Indeed, we just have to apply Proposition 3 and Lemma 1 to

$$\begin{aligned}&\mathbf{G}_1(q_2,\mathbf{m}_2)-\mathbf{G}_1(q_1,\mathbf{m}_1)\\&\quad=\operatorname{div}\left(\mathbf{m}_1\otimes\mathbf{m}_1\left(\frac{q_2}{1+q_2}-\frac{q_1}{1+q_1}\right)-\frac{\mathbf{m}_2\otimes(\mathbf{m}_2-\mathbf{m}_1)+(\mathbf{m}_2-\mathbf{m}_1)\otimes\mathbf{m}_1}{1+q_2}\right),\end{aligned}$$

$$
\begin{aligned}
&\mathbf{G}_2(q_2,\mathbf{m}_2)-\mathbf{G}_2(q_1,\mathbf{m}_1)=-\nabla\big((q_2-q_1)\widetilde{H}(q_2)+q_1\big(\widetilde{H}(q_2)-\widetilde{H}(q_1)\big)\big),\\
&\mathbf{G}_3(q_2,\mathbf{m}_2)-\mathbf{G}_3(q_1,\mathbf{m}_1)\\
&\quad=-\big(\bar\mu\Delta+(\bar\lambda+\bar\mu)\nabla\mathrm{div}\big)\left((\mathbf{m}_2-\mathbf{m}_1)\frac{q_2}{1+q_2}+\mathbf{m}_1\left(\frac{q_2}{1+q_2}-\frac{q_1}{1+q_1}\right)\right),\\
&\mathbf{G}_4(q_2,\mathbf{m}_2)-\mathbf{G}_4(q_1,\mathbf{m}_1)=\kappa(q_2-q_1)\nabla\Delta q_2+\kappa q_1\nabla\Delta(q_2-q_1),\\
&\mathbf{G}_5(q_2,\mathbf{m}_2)-\mathbf{G}_5(q_1,\mathbf{m}_1)=\mathbf{f}(q_2-q_1).
\end{aligned}
$$

This leads to the following inequality

$$
\begin{aligned}
&\big\|\Psi_{q_L,m_L}(\bar q_2,\bar{\mathbf{m}}_2)-\Psi_{q_L,m_L}(\bar q_1,\bar{\mathbf{m}}_1)\big\|_E\\
&\quad\leqslant C\|(\bar q_2-\bar q_1,\bar{\mathbf{m}}_2-\bar{\mathbf{m}}_1)\|_E\\
&\qquad\times\big(\|(\bar q_1,\bar{\mathbf{m}}_1)\|_E+\|(\bar q_2,\bar{\mathbf{m}}_2)\|_E+2\|(q_L,\mathbf{m}_L)\|_E+\|\mathbf{f}\|_{L^1_T(B^{d/2-1})}\big).
\end{aligned}
$$

Now, if (R,η) satisfies (37) (for a greater constant C if needed), we deduce

$$
\big\|\Psi_{q_L,m_L}(\bar q_2,\bar{\mathbf{m}}_2)-\Psi_{q_L,m_L}(\bar q_1,\bar{\mathbf{m}}_1)\big\|_E\leqslant\frac{4}{5}\big\|(\bar q_2-\bar q_1,\bar{\mathbf{m}}_2-\bar{\mathbf{m}}_1)\big\|_E
$$

and the proof of the existence part of Theorem 3 is achieved.

Notice that in view of (34) and (37), $(q,\mathbf{m})$ satisfies

$$
\|q\|_{L^\infty(\mathbb{R}^+\times\mathbb{R}^d)}\leqslant 2/5\quad\text{and}\quad\|q\|_{\widetilde{L}^\infty_T(B^{d/2})}\leqslant 2/(5C).
$$

The solution $(q,\mathbf{m})$ obviously belongs to the space $\widetilde{E}^2_\infty$ defined in Section 6.2 (we have $\widetilde{E}=\widetilde{E}^2_\infty$). Changing C into a greater constant if necessary, we therefore can apply Lemma 4 to get uniqueness in $\widetilde{E}$.

5. Local solutions away from vacuum

In this section, we want to show local well-posedness for the Korteweg system with initial data $(\rho_0,\mathbf{m}_0)$ such that $(\rho_0-\bar\rho,\mathbf{m}_0)$ in $B^{d/2}\times(B^{d/2-1})^d$. Let us emphasize that no smallness assumption is required: we just need the initial density to be bounded away from zero. The pressure P may be any (possibly decreasing) smooth function of ρ ($P\in W^{[d/2]+3,\infty}_{loc}$ is enough).

It is convenient to rewrite (8) (9) in terms of q and $\mathbf{m}$ by using the same scaled coefficients as in Section 4

$$
\partial_t q+\mathrm{div}\,\mathbf{m}=0,\tag{39}
$$

$$
\begin{aligned}
&D_t\mathbf{m}-\bar\mu\,\mathrm{div}\left(\frac{\nabla\mathbf{m}}{1+q}\right)-(\bar\lambda+\bar\mu)\nabla\left(\frac{\mathrm{div}\,\mathbf{m}}{1+q}\right)-\bar\kappa\nabla\big((1+q)\Delta q\big)\\
&\quad=\Gamma(q,\mathbf{m})+\mathbf{f},
\end{aligned}\tag{40}
$$

where $\Gamma=\Gamma_1+\Gamma_2+\Gamma_3+\Gamma_4+\Gamma_5$ is defined by

$$
\Gamma_1(q,\mathbf{m})=-\nabla\big(P\big(\bar\rho(1+q)\big)-P(\bar\rho)\big)/\bar\rho,
$$

$$\Gamma_2(q,\mathbf{m}) = -\operatorname{div}\left(\frac{\mathbf{m}\otimes\mathbf{m}}{1+q}\right),$$
$$\Gamma_3(q,\mathbf{m}) = \bar{\mu}\operatorname{div}\left(\mathbf{m}\otimes\nabla\left(\frac{1}{1+q}\right)\right) + (\bar{\lambda}+\bar{\mu})\nabla\left(\mathbf{m}\cdot\nabla\left(\frac{1}{1+q}\right)\right),$$
$$\Gamma_4(q,\mathbf{m}) = -\bar{\kappa}\operatorname{div}(\nabla q\otimes\nabla q) - \frac{\bar{\kappa}}{2}\nabla|\nabla q|^2,$$
$$\Gamma_5(q,\mathbf{m}) = q\mathbf{f}.$$

The proof of our local existence theorem relies upon the study of the linearized system around $(1+q,0)$ for a given q such that $1+q$ is bounded away from zero, whereas the first order linearized pressure term is dropped. The purpose of Section 5.1 is to derive estimates for such a system. Well-posedness for (39) (40) is obtained in Section 5.2 through an iterative method.

5.1. Estimates for the linearized system

We now study the following linearized system

$$\begin{cases} \partial_t q + \operatorname{div}\mathbf{m} = F, \\ \partial_t\mathbf{m} - \bar{\mu}\operatorname{div}(a\nabla\mathbf{m}) - (\bar{\lambda}+\bar{\mu})\nabla(a\operatorname{div}\mathbf{m}) - \bar{\kappa}\nabla(b\Delta q) = \mathbf{G}, \end{cases} \tag{LNSK2}$$

where $\mathbf{m}$ is a vector field in $\mathbb{R}^d$, and a, b are scalar functions, bounded and bounded away from zero

$$0 < c_1 \leqslant a \leqslant M_1 < +\infty, \quad 0 < c_2 \leqslant b \leqslant M_2 < +\infty \quad \text{on } [0,T]. \tag{41}$$

Our purpose is to prove estimates for (LNSK2) in Besov spaces closely related to energy spaces. We obtain

PROPOSITION 5. – *Let* $1\leqslant r_1\leqslant r\leqslant +\infty$, $(q_0,\mathbf{m}_0)\in B^{d/2}\times(B^{d/2-1})^d$ *and* $(F,\mathbf{G})\in \widetilde{L}^{r_1}_T(B^{d/2-2+2/r_1}\times(B^{d/2-3+2/r_1})^d)$. *Suppose* (41), ∇b *and* ∇a *belong to* $\widetilde{L}^2_T(B^{d/2})$, $\partial_t b\in L^1(0,T;L^\infty)$. *Let* $(q,\mathbf{m})\in \widetilde{L}^r_T(B^{d/2+2/r}\times(B^{d/2-1+2/r})^d)\cap\widetilde{L}^2_T(B^{d/2+1}\times(B^{d/2})^d)$ *be a solution of the system* (LNSK2). *Then there exists a constant* C *depending only on* $r, r_1, \bar{\lambda}, \bar{\mu}, \bar{\kappa}, c_1, c_2, M_1$, *and* M_2 *such that the following inequality holds*:

$$\begin{aligned} &\|(\nabla q,\mathbf{m})\|_{\widetilde{L}^r_T(B^{d/2-1+2/r})}\big(1 - C\|\nabla b\|_{L^2_T(L^\infty)}\big) \\ &\quad\leqslant C\big(\|(\nabla q_0,\mathbf{m}_0)\|_{B^{d/2-1}} + \|(\nabla F,\mathbf{G})\|_{L^{r_1}_T(B^{d/2-3+2/r_1})} + \|\partial_t b\|_{L^1_T(L^\infty)}\|\nabla q\|_{\widetilde{L}^\infty_T(B^{d/2-1})} \\ &\qquad + \|(\nabla q,\mathbf{m})\|_{\widetilde{L}^2_T(B^{d/2})}\big(\|\nabla b\|_{\widetilde{L}^2_T(B^{d/2})} + \|\nabla a\|_{\widetilde{L}^2_T(B^{d/2})}\big)\big). \end{aligned}$$

Proof. – It is just a matter of showing appropriate estimates for $\Delta_\ell q$ and $\Delta_\ell\mathbf{m}$. Denoting $q_\ell = \Delta_\ell q$, $\mathbf{m}_\ell = \Delta_\ell\mathbf{m}$, $F_\ell = \Delta_\ell F$, $\mathbf{G}_\ell = \Delta_\ell\mathbf{G}$, and applying Δ_ℓ to (LNSK2), we get

$$\partial_t q_\ell + \operatorname{div}\mathbf{m}_\ell = F_\ell, \tag{42}$$
$$\partial_t\mathbf{m}_\ell - \bar{\mu}\operatorname{div}(a\nabla\mathbf{m}_\ell) - (\bar{\lambda}+\bar{\mu})\nabla(a\operatorname{div}\mathbf{m}_\ell) - \bar{\kappa}\nabla(b\Delta q_\ell) = \mathbf{G}_\ell + \mathbf{R}_\ell, \tag{43}$$

where

$$r_\ell = -\bar\mu\,\mathrm{div}([a,\Delta_\ell]\nabla\mathbf{m}) - (\bar\lambda+\bar\mu)\nabla([a,\Delta_\ell]\mathrm{div}\,\mathbf{m}) - \bar\kappa\nabla([b,\Delta_\ell]\Delta q).$$

Using integrations by parts and Eq. (42), we obtain

$$-\int_{\mathbb{R}^d}\mathbf{m}_\ell\nabla(b\Delta q_\ell)\,d\mathbf{x} = \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d} b|\nabla q_\ell|^2 d\mathbf{x} - \int_{\mathbb{R}^d}\left(\mathrm{div}\,\mathbf{m}_\ell(\nabla q_\ell\cdot\nabla b) + \frac{|\nabla q_\ell|^2}{2}\partial_t b + b\nabla q_\ell\cdot\nabla F_\ell\right)d\mathbf{x}.$$

We now take the scalar product of (43) with $\mathbf{m}_\ell$ and use the previous identity, which yields

$$\frac{1}{2}\frac{d}{dt}\left(\|\mathbf{m}_\ell\|_{L^2}^2 + \bar\kappa\int_{\mathbb{R}^d} b|\nabla q_\ell|^2 d\mathbf{x}\right) + \int_{\mathbb{R}^d}\left(\bar\mu|\nabla\mathbf{m}_\ell|^2 + (\bar\lambda+\bar\mu)|\mathrm{div}\,\mathbf{m}_\ell|^2\right)a\,d\mathbf{x}$$
$$= \int_{\mathbb{R}^d}\left((\mathbf{G}_\ell+\mathbf{R}_\ell)\cdot\mathbf{m}_\ell + \bar\kappa\left(\mathrm{div}\,\mathbf{m}_\ell(\nabla b\cdot\nabla q_\ell) + \frac{|\nabla q_\ell|^2}{2}\partial_t b + b\nabla q_\ell\cdot\nabla F_\ell\right)\right)d\mathbf{x}. \quad (44)$$

In order to obtain a second estimate, we take the scalar product of the gradient of (42) with $\mathbf{m}_\ell$, the scalar product of (43) with ∇q_ℓ and sum both inequalities. We obtain

$$\frac{d}{dt}\int_{\mathbb{R}^d}\nabla q_\ell\cdot\mathbf{m}_\ell\,d\mathbf{x} + \int_{\mathbb{R}^d}\bar\kappa\,b\,(\Delta q_\ell)^2 d\mathbf{x}$$
$$= \int_{\mathbb{R}^d}\big((\mathbf{G}_\ell+\mathbf{R}_\ell)\cdot\nabla q_\ell + |\mathrm{div}\,\mathbf{m}_\ell|^2 + \mathbf{m}_\ell\cdot\nabla F_\ell - \bar\mu\,a\nabla\mathbf{m}_\ell : \nabla^2 q_\ell$$
$$- (\bar\lambda+\bar\mu)a\Delta q_\ell\,\mathrm{div}\,\mathbf{m}_\ell\big)\,d\mathbf{x}. \quad (45)$$

Let $\alpha > 0$ be suitably small, and define

$$k_l^2 \stackrel{\mathrm{def}}{=} \|\mathbf{m}_\ell\|_{L^2}^2 + \int_{\mathbb{R}^d}\big(\bar\kappa b|\nabla q_\ell|^2 + 2\alpha\nabla q_\ell\cdot\mathbf{m}_\ell\big)\,d\mathbf{x} \quad \text{and} \quad \bar\nu = \inf(\bar\mu, \bar\lambda+2\bar\mu).$$

Using (44), (45) and (41), we deduce that when

$$\alpha \leqslant \frac{\bar\nu}{2}\left(\frac{M_1(\bar\mu+|\bar\lambda+\bar\mu|)^2}{2c_2\bar\kappa} + 1\right)^{-1},$$

we have

$$\frac{1}{2}\frac{d}{dt}k_\ell^2 + \frac{1}{2}\int_{\mathbb{R}^d}\big(a\bar\nu\|\nabla\mathbf{m}_\ell\|_{L^2}^2 + \alpha\bar\kappa b|\Delta q_\ell|^2\big)\,d\mathbf{x}$$
$$\leqslant \big(\|\mathbf{G}_\ell\|_{L^2} + \|\mathbf{R}_\ell\|_{L^2}\big)\big(\alpha\|\nabla q_\ell\|_{L^2} + \|\mathbf{m}_\ell\|_{L^2}\big) + \|\nabla F_\ell\|_{L^2}\big(\alpha\|\mathbf{m}_\ell\|_{L^2} + \|\nabla q_\ell\|_{L^2}\big)$$
$$+ \frac{1}{2}\|\partial_t b\|_{L^\infty}\|\nabla q_\ell\|_{L^2}^2 + \bar\kappa\|\nabla b\|_{L^\infty}\|\nabla q_\ell\|_{L^2}\|\nabla\mathbf{m}_\ell\|_{L^2}. \quad (46)$$

Using (41), we clearly have for $\alpha > 0$ small enough

$$\frac{1}{2}k_\ell^2 \leqslant \|\mathbf{m}_\ell\|_{L^2}^2 + \bar{\kappa}\int_{\mathbb{R}^d} b|\nabla q_\ell|^2 \leqslant \frac{3}{2}k_\ell^2, \tag{47}$$

so that using the above energy estimate, the spectral localization of $(\nabla q_\ell, \mathbf{m}_\ell)$ and (41), we get for some positive K

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}k_\ell^2 + K2^{2l}k_\ell^2\\ &\quad\leqslant C\big(k_\ell\big(\|\mathbf{G}_\ell\|_{L^2} + \|\nabla F_\ell\|_{L^2} + \|\mathbf{R}_\ell\|_{L^2}\big) + \|\partial_t b\|_{L^\infty}\|\nabla q_\ell\|_{L^2}^2 + 2^\ell k_\ell^2\|\nabla b\|_{L^\infty}\big).\end{aligned} \tag{48}$$

Integrating with respect to time yields

$$\begin{aligned}k_\ell(t) \leqslant e^{-K2^{2\ell}t}k_\ell(0) + C\int_0^t e^{-K2^{2\ell}(t-\tau)}\big(&\|\partial_t b(\tau)\|_{L^\infty}\|\nabla q_\ell(\tau)\|_{L^2}\\ &+ \|\nabla F_\ell(\tau)\|_{L^2} + \|\mathbf{G}_\ell(\tau)\|_{L^2} + \|\mathbf{R}_\ell(\tau)\|_{L^2} + 2^\ell k_\ell(\tau)\|\nabla b(\tau)\|_{L^\infty}\big)\,d\tau.\end{aligned} \tag{49}$$

Using convolution inequalities, we easily get

$$\begin{aligned}\|k_\ell\|_{L^r([0,T])} \leqslant C\big(&2^{-2\ell/r}k_\ell(0) + 2^{-2\ell(1+1/r-1/r_1)}\|(\nabla F_\ell, \mathbf{G}_\ell)\|_{L_T^{r_1}(L^2)}\\ &+ 2^{-2\ell/r}\|\mathbf{R}_\ell\|_{L_T^1(L^2)} + 2^{-2\ell/r}\|\nabla q_\ell\|_{L_T^\infty(L^2)}\|\partial_t b\|_{L_T^1(L^\infty)}\\ &+ \|\nabla b\|_{L_T^2(L^\infty)}\|k_\ell\|_{L^r([0,T])}\big).\end{aligned} \tag{50}$$

We first use (47) and (41) to infer that there exists a constant $C > 0$ such that

$$C^{-1}k_\ell \leqslant \|\nabla q_\ell\|_{L^2} + \|\mathbf{m}_\ell\|_{L^2} \leqslant Ck_\ell,$$

then multiply both sides of (50) by $2^{\ell(d/2-1+2/r)}$ and sum over $\mathbb{Z}$, which yields

$$\begin{aligned}&\|(\nabla q, \mathbf{m})\|_{\widetilde{L}_T^r(B^{d/2-1+2/r})}\big(1 - C\|\nabla b\|_{L_T^2(L^\infty)}\big)\\ &\quad\leqslant \|(\nabla q_0, \mathbf{m}_0)\|_{B^{d/2-1}} + \|(\nabla F, \mathbf{G})\|_{\widetilde{L}_T^{r_1}(B^{d/2-3+2/r_1})}\\ &\qquad+ \|\nabla q\|_{\widetilde{L}_T^\infty(B^{d/2-1})}\|\partial_t b\|_{L_t^1(L^\infty)} + \sum_{q\in\mathbb{Z}} 2^{\ell(d/2-1)}\|\mathbf{R}_\ell\|_{L_T^1(L^2)}.\end{aligned}$$

We then obtain the desired inequality thanks to Lemma 5 in the appendix. □

Remark 2. – When $r_1 = 1$, estimates of Proposition 5 clearly enable us to prove the existence and uniqueness of a solution $(q, \mathbf{m})$ to (LNSK2) in the space $L_T^1(B^{d/2+2} \times (B^{d/2+1})^d) \cap \widetilde{C}_T(B^{d/2} \times (B^{d/2-1})^d)$ as long as

$$\|\nabla a\|_{\widetilde{L}_T^2(B^{d/2})} + \|\nabla b\|_{\widetilde{L}_T^2(B^{d/2})} + \|\partial_t b\|_{L_T^1(L^\infty)} \leqslant C^{-1}.$$

This stems from a basic duality method.

5.2. Local existence and uniqueness

First we define the functional spaces needed in our local existence and uniqueness theorem. We will prove existence in the space

$$F_T \stackrel{\text{def}}{=} L^1_T\big(B^{d/2+2} \times \big(B^{d/2+1}\big)^d\big) \cap \widetilde{C}_T\big(B^{d/2} \times \big(B^{d/2-1}\big)^d\big)$$

endowed with the norm

$$\|(q,\mathbf{m})\|_{F_T} = \|q\|_{L^1_T(B^{d/2+2})} + \|q\|_{\widetilde{L}^\infty_T(B^{d/2})} + \|\mathbf{m}\|_{L^1_T(B^{d/2+1})} + \|\mathbf{m}\|_{\widetilde{L}^\infty_T(B^{d/2-1})},$$

and uniqueness in the larger space

$$\widetilde{F}_T \stackrel{\text{def}}{=} \widetilde{L}^2_T\big(B^{d/2+1} \times \big(B^{d/2}\big)^d\big) \cap \widetilde{C}_T\big(B^{d/2} \times \big(B^{d/2-1}\big)^d\big)$$

endowed with the norm

$$\|(q,\mathbf{m})\|_{\widetilde{F}_T} = \|q\|_{\widetilde{L}^2_T(B^{d/2+1})} + \|q\|_{\widetilde{L}^\infty_T(B^{d/2})} + \|\mathbf{m}\|_{\widetilde{L}^2_T(B^{d/2})} + \|\mathbf{m}\|_{\widetilde{L}^\infty_T(B^{d/2-1})}.$$

THEOREM 4. – *Suppose that the exterior forcing term* $\mathbf{f}$ *belongs to* $(L^1_T(B^{d/2-1}))^d$*, that the initial momentum* $\mathbf{m}_0$ *belongs to* $(B^{d/2-1})^d$*, and that the initial density* ρ_0 *satisfies* $(\rho_0 - \bar\rho) \in B^{d/2}$ *and* $\rho_0 \geqslant c$ *for a positive constant* c*. Then, there exists* $T > 0$ *such that the system* (39) (40) *with initial data* $((\rho_0 - \bar\rho)/\bar\rho, \mathbf{m}_0)$ *has a unique solution* $(q,\mathbf{m})$ *in* $\widetilde{F}_T$*. In addition,* $(q,\mathbf{m})$ *belongs to* F_T*.*

Proof. – The existence part of the theorem is proved by an iterative method. We define a sequence $\{(q^n,\mathbf{m}^n)\}_{n\in\mathbb{N}}$ as follows: the first term $(q^0,\mathbf{m}^0)$ is taken to be the solution of the heat equation

$$\partial_t\begin{pmatrix} q^0 \\ \mathbf{m}^0\end{pmatrix} - \Delta\begin{pmatrix} q^0 \\ \mathbf{m}^0\end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{f}\end{pmatrix}, \qquad \begin{pmatrix} q^0 \\ \mathbf{m}^0\end{pmatrix}_{|t=0} = \begin{pmatrix} q_0 \\ \mathbf{m}_0\end{pmatrix}, \tag{51}$$

with $q_0 = (\rho_0 - \bar\rho)/\bar\rho$. Assuming that $(q^n,\mathbf{m}^n)$ belongs to F_T, we then define $q^{n+1} = q^0 + \bar q^{n+1}$ and $\mathbf{m}^{n+1} = \mathbf{m}^0 + \bar{\mathbf{m}}^{n+1}$ with $(\bar q^{n+1}, \bar{\mathbf{m}}^{n+1})$ solution of the following linear system

$$\begin{cases} \partial_t \bar q^{n+1} + \operatorname{div}\bar{\mathbf{m}}^{n+1} = -\Delta q^0 - \operatorname{div}\mathbf{m}^0, \\ \partial_t\bar{\mathbf{m}}^{n+1} - \bar\mu\operatorname{div}\left(\dfrac{\nabla\bar{\mathbf{m}}^{n+1}}{1+q^n}\right) - (\bar\lambda+\bar\mu)\nabla\left(\dfrac{\operatorname{div}\bar{\mathbf{m}}^{n+1}}{1+q^n}\right) \\ \qquad - \bar\kappa\nabla((1+q^n)\Delta\bar q^{n+1}) = \Gamma(q^n,\mathbf{m}^n) + H^0(q^n,\mathbf{m}^n), \\ \bar q^{n+1}_{|t=0} = 0, \quad \bar{\mathbf{m}}^{n+1}_{|t=0} = 0, \end{cases} \tag{52}$$

where

$$H^0(q^n,\mathbf{m}^n) = -\Delta\mathbf{m}^0 + \bar\mu\operatorname{div}\left(\frac{\nabla\mathbf{m}^0}{1+q^n}\right) + (\bar\lambda+\bar\mu)\nabla\left(\frac{\operatorname{div}\mathbf{m}^0}{1+q^n}\right) + \bar\kappa\nabla\big((1+q^n)\Delta q^0\big).$$

First step: *uniform bounds in* F_T

We want now to show that $(q^n, \mathbf{m}^n)$ is uniformly bounded in F_T. Denote $E_0 = \|q_0\|_{B^{d/2}} + \|\mathbf{m}_0\|_{B^{d/2-1}}$. Let $\varepsilon \in (0,1)$. In view of Proposition 2, we can choose $T \in]0, \varepsilon]$ such that

$$\begin{aligned} &\|\mathbf{f}\|_{L^1_T(B^{d/2-1})} \leqslant \varepsilon, \\ &\|q^0\|_{L^1_T(B^{d/2+2})} + \|\mathbf{m}^0\|_{L^1_T(B^{d/2+1})} \leqslant \varepsilon, \\ &\|q^0\|_{\widetilde{L}^\infty_T(B^{d/2})} + \|\mathbf{m}^0\|_{\widetilde{L}^\infty_T(B^{d/2-1})} \leqslant C(E_0+1). \end{aligned} \tag{$\mathcal{H}_\varepsilon$}$$

We are going to show that if ε is chosen suitably small, we have for all $n \in \mathbb{N}$,

$$\big\|(\bar{q}^n, \bar{\mathbf{m}}^n)\big\|_{F_T} \leqslant \sqrt{\varepsilon}. \tag{$\mathcal{P}_n$}$$

Since $\bar{q}^0 = 0$ and $\bar{\mathbf{m}}^0 = 0$, $(\mathcal{P}_0)$ is true. Suppose that $(\mathcal{P}_n)$ is fulfilled and that $\sqrt{\varepsilon}$ is less than $c/(4C_1\bar{\rho})$ (where C_1 is the norm of the embedding $B^{d/2} \hookrightarrow L^\infty$). From the fact that $q^n(t) - q_0 = -\int_0^t \operatorname{div} \mathbf{m}^n(\tau)\, d\tau$ and $(\mathcal{H}_\varepsilon)$, we gather

$$\begin{aligned} \big\|q^n - q_0\big\|_{L^\infty([0,T]\times\mathbb{R}^d)} &\leqslant C_1\big(\big\|\operatorname{div} \mathbf{m}^0\big\|_{L^1_T(B^{d/2})} + \big\|\operatorname{div} \bar{\mathbf{m}}^n\big\|_{L^1_T(B^{d/2})}\big), \\ &\leqslant C_1(\varepsilon + \sqrt{\varepsilon}), \\ &\leqslant c/2\bar{\rho}. \end{aligned}$$

We thus have

$$\frac{c}{2\bar{\rho}} - 1 \leqslant q^n \leqslant \frac{\|\rho_0\|_{L^\infty}}{\bar{\rho}} \quad \text{on } [0,T], \tag{53}$$

which entails, according to Lemma 1,

$$\left\|\frac{1}{1+q^n} - 1\right\|_{\widetilde{L}^\infty_T(B^{d/2})} \leqslant C\|q^n\|_{\widetilde{L}^\infty_T(B^{d/2})}, \tag{54}$$

$$\left\|\nabla\left(\frac{1}{1+q^n}\right)\right\|_{\widetilde{L}^2_T(B^{d/2})} = \left\|\nabla\left(\frac{q^n}{1+q^n}\right)\right\|_{\widetilde{L}^2_T(B^{d/2})} \leqslant C\|q^n\|_{\widetilde{L}^2_T(B^{d/2+1})}. \tag{55}$$

Apply Proposition 5 to (52), and use that $\partial_t q^n = -\operatorname{div} \mathbf{m}^n$. This yields

$$\begin{aligned} &\big\|(\bar{q}^{n+1}, \bar{\mathbf{m}}^{n+1})\big\|_{F_T}\left(1 - C\left(\|\nabla q^n\|_{\widetilde{L}^2_T(B^{d/2})} + \left\|\nabla\left(\frac{1}{1+q^n}\right)\right\|_{\widetilde{L}^2_T(B^{d/2})}\right.\right. \\ &\qquad \left.\left. + \big\|\operatorname{div} \mathbf{m}^n\big\|_{L^1_T(B^{d/2})}\right)\right) \\ &\quad \leqslant C\big(\big\|\Gamma(q^n, \mathbf{m}^n)\big\|_{L^1_T(B^{d/2-1})} + \big\|H^0(q^n, \mathbf{m}^n)\big\|_{L^1_T(B^{d/2-1})} \\ &\qquad + \big\|\Delta q^0\big\|_{L^1_T(B^{d/2})} + \big\|\operatorname{div} \mathbf{m}^0\big\|_{L^1_T(B^{d/2})}\big). \end{aligned} \tag{56}$$

Next, we use Proposition 3, Lemma 1, (53), (54) and (55) to estimate the right-hand side of (56). The following bounds hold:

$$\|\Gamma_1(q^n,\mathbf{m}^n)\|_{L^1_T(B^{d/2-1})} \leqslant CT\|q^n\|_{\widetilde{L}^\infty_T(B^{d/2})},$$
$$\|\Gamma_2(q^n,\mathbf{m}^n)\|_{L^1_T(B^{d/2-1})} \leqslant C\|\mathbf{m}^n\|^2_{\widetilde{L}^2_T(B^{d/2})}\Big(1+\|q^n\|_{\widetilde{L}^\infty_T(B^{d/2})}\Big),$$
$$\|\Gamma_3(q^n,\mathbf{m}^n)\|_{L^1_T(B^{d/2-1})} \leqslant C\|\mathbf{m}^n\|_{\widetilde{L}^2_T(B^{d/2})}\|q^n\|_{\widetilde{L}^2_T(B^{d/2+1})},$$
$$\|\Gamma_4(q^n,\mathbf{m}^n)\|_{L^1_T(B^{d/2-1})} \leqslant C\|\nabla q^n\|^2_{\widetilde{L}^2_T(B^{d/2})},$$
$$\|\Gamma_5(q^n,\mathbf{m}^n)\|_{L^1_T(B^{d/2-1})} \leqslant C\|q^n\|_{\widetilde{L}^\infty_T(B^{d/2})}\|\mathbf{f}\|_{L^1_T(B^{d/2-1})}.$$

We also have

$$\left\|\nabla\left(\frac{\nabla\mathbf{m}^0}{1+q^n}\right)\right\|_{L^1_T(B^{d/2-1})} \leqslant C\|\nabla\mathbf{m}^0\|_{L^1_T(B^{d/2})}\big(1+\|q^n\|_{\widetilde{L}^\infty_T(B^{d/2})}\big),$$

$$\|\nabla((1+q^n)\Delta q^0)\|_{L^1_T(B^{d/2-1})} \leqslant C\|\Delta q^0\|_{L^1_T(B^{d/2})}\big(1+\|q^n\|_{\widetilde{L}^\infty_T(B^{d/2})}\big).$$

Therefore, using (55), the above computations, (56) and $(\mathcal{P}_n)$, we deduce that

$$\|(\bar{q}^{n+1},\bar{\mathbf{m}}^{n+1})\|_{F_T}\big(1-C\sqrt{\varepsilon(E_0+1)}\big) \leqslant C\varepsilon(1+E_0)^2.$$

Choosing $\varepsilon \leqslant (4C^2(1+E_0))^{-1}$, this implies $(\mathcal{P}_{n+1})$. The sequence $\{(q^n,\mathbf{m}^n)\}_{n\in\mathbb{N}}$ is therefore bounded in the space F_T. Moreover, (53) holds for all $n\in\mathbb{N}$.

Second step: *convergence of the sequence in F_T.*

Next, we are going to show that $(q^n,\mathbf{m}^n)$ converges strongly in F_T to a solution $(q,\mathbf{m})$ of (39) (40). We denote $\delta q^n = q^{n+1}-q^n$ and $\delta\mathbf{m}^n = \mathbf{m}^{n+1}-\mathbf{m}^n$. According to (51) and (52), we have

$$\begin{cases} \partial_t\delta q^n + \operatorname{div}\delta\mathbf{m}^n = 0,\\ \partial_t\delta\mathbf{m}^n - \bar{\mu}\operatorname{div}\left(\dfrac{\nabla\delta\mathbf{m}^n}{1+q^n}\right) - (\bar{\lambda}+\bar{\mu})\nabla\left(\dfrac{\operatorname{div}\delta\mathbf{m}^n}{1+q^n}\right) - \bar{\kappa}\nabla\big((1+q^n)\Delta\delta q^n\big)\\ \quad = \Gamma(q^n,\mathbf{m}^n) - \Gamma(q^{n-1},\mathbf{m}^{n-1}) + \mathbf{H}(q^{n-1},\mathbf{m}^{n-1},q^n,\mathbf{m}^n),\\ \delta q^n{}_{|t=0} = 0, \quad \delta\mathbf{m}^n{}_{|t=0} = 0, \end{cases} \tag{57}$$

with

$$\mathbf{H}(q^{n-1},\mathbf{m}^{n-1},q^n,\mathbf{m}^n) = -\bar{\mu}\operatorname{div}\left(\frac{\delta q^{n-1}\nabla\mathbf{m}^n}{(1+q^n)(1+q^{n-1})}\right)$$
$$- (\bar{\lambda}+\bar{\mu})\nabla\left(\frac{\delta q^{n-1}\operatorname{div}\mathbf{m}^n}{(1+q^n)(1+q^{n-1})}\right) + \bar{\kappa}\nabla\big(\delta q^{n-1}\Delta q^n\big).$$

We keep the same assumptions on ε as in the first step. According to Proposition 5, (53), $(\mathcal{P}_n)$ and (55), we thus get

$$\|(\delta q^n,\delta\mathbf{m}^n)\|_{F_T} \leqslant C\big(\|\Gamma(q^n,\mathbf{m}^n) - \Gamma(q^{n-1},\mathbf{m}^{n-1})\|_{L^1_T(B^{d/2-1})}$$
$$+ \|\mathbf{H}(q^{n-1},\mathbf{m}^{n-1},q^n,\mathbf{m}^n)\|_{L^1_T(B^{d/2-1})}\big). \tag{58}$$

Denote $\delta\Gamma_i^n \stackrel{\text{def}}{=} \Gamma_i(q^n,\mathbf{m}^n) - \Gamma_i(q^{n-1},\mathbf{m}^{n-1})$. Thanks to Proposition 3, to Lemma 1, (53), (54) and (55), we obtain the following estimates:

$$\|\delta\Gamma_1^n\|_{L_T^1(B^{d/2-1})} \lesssim T\big(1+\|q^n\|_{\widetilde{L}_T^\infty(B^{d/2})}+\|q^{n-1}\|_{\widetilde{L}_T^\infty(B^{d/2})}\big)\|\delta q^{n-1}\|_{\widetilde{L}_T^\infty(B^{d/2})},$$

$$\begin{aligned}\|\delta\Gamma_2^n\|_{L_T^1(B^{d/2-1})} \lesssim{}& \big(1+\|q^n\|_{\widetilde{L}_T^\infty(B^{d/2})}\big)\big(\|\mathbf{m}^{n-1}\|_{\widetilde{L}_T^2(B^{d/2})}\\ &+\|\mathbf{m}^n\|_{\widetilde{L}_T^2(B^{d/2})}\big)\|\delta\mathbf{m}^{n-1}\|_{\widetilde{L}_T^2(B^{d/2})}\\ &+\|\mathbf{m}^{n-1}\|^2_{\widetilde{L}_T^2(B^{d/2})}\big(1+\|q^{n-1}\|_{\widetilde{L}_T^\infty(B^{d/2})}\big)\\ &\times\big(1+\|q^n\|_{\widetilde{L}_T^\infty(B^{d/2})}\big)\|\delta q^{n-1}\|_{\widetilde{L}_T^\infty(B^{d/2})},\end{aligned}$$

$$\begin{aligned}\|\delta\Gamma_3^n\|_{L_T^1(B^{d/2-1})} \lesssim{}& \|q^n\|_{\widetilde{L}_T^2(B^{d/2+1})}\|\delta\mathbf{m}^{n-1}\|_{\widetilde{L}_T^2(B^{d/2})}+\|\mathbf{m}^{n-1}\|_{\widetilde{L}_T^2(B^{d/2})}\|\delta q^{n-1}\|_{\widetilde{L}_T^2(B^{d/2+1})}\\ &+\|\mathbf{m}^{n-1}\|_{\widetilde{L}_T^2(B^{d/2})}\big(\|q^{n-1}\|_{\widetilde{L}_T^2(B^{d/2+1})}\\ &+\|q^n\|_{\widetilde{L}_T^2(B^{d/2+1})}\big)\|\delta q^{n-1}\|_{\widetilde{L}_T^\infty(B^{d/2})},\end{aligned}$$

$$\|\delta\Gamma_4^n\|_{L_T^1(B^{d/2-1})} \lesssim \|\nabla\delta q^{n-1}\|_{\widetilde{L}_T^2(B^{d/2})}\big(\|\nabla q^{n-1}\|_{\widetilde{L}_T^2(B^{d/2})}+\|\nabla q^n\|_{\widetilde{L}_T^2(B^{d/2})}\big),$$

$$\|\delta\Gamma_5^n\|_{L_T^1(B^{d/2-1})} \lesssim \|f\|_{L_T^1(B^{d/2-1})}\|\delta q^{n-1}\|_{\widetilde{L}_T^\infty(B^{d/2})},$$

$$\begin{aligned}&\left\|\frac{\delta q^{n-1}\nabla\mathbf{m}^n}{(1+q^n)(1+q^{n-1})}\right\|_{L_T^1(B^{d/2})}\\ &\quad\lesssim \big(1+\|q^{n-1}\|_{\widetilde{L}_T^\infty(B^{d/2})}\big)\big(1+\|q^n\|_{\widetilde{L}_T^\infty(B^{d/2})}\big)\|\nabla\mathbf{m}^n\|_{L_T^1(B^{d/2})}\|\delta q^{n-1}\|_{\widetilde{L}_T^\infty(B^{d/2})},\end{aligned}$$

$$\|\delta q^{n-1}\Delta q^n\|_{L_T^1(B^{d/2})} \lesssim \|\Delta q^n\|_{L_T^1(B^{d/2})}\|\delta q^{n-1}\|_{\widetilde{L}_T^\infty(B^{d/2})}.$$

Using $(\mathcal{H}_\varepsilon)$ and $(\mathcal{P}_n)$ in the above computations, then (58), we thus get

$$\|(\delta q^n,\delta\mathbf{m}^n)\|_{F_T} \leqslant C\sqrt{\varepsilon}(E_0+1)^3.$$

Now, if we choose an ε such that $\varepsilon \leqslant (4C^2)^{-1}(E_0+1)^{-6}$ holds, $(q_n,\mathbf{m}_n)_{n\in\mathbb{N}}$ is clearly a Cauchy sequence and thus converges in F_T to a limit $(q,\mathbf{m})$ which satisfies (53). The verification that the limit is solution of (39) (40) in the sense of distributions is a straightforward application of Proposition 3. □

Third step: *uniqueness in* $\widetilde{F}_T$

Consider the solution $(q,\mathbf{m})$ built in the previous part and suppose that $(q',\mathbf{m}') \in \widetilde{L}_T^2(B^{d/2+1}\times(B^{d/2})^d)\cap\widetilde{C}_T(B^{d/2}\times(B^{d/2-1})^d)$ also solves (39) (40) with initial data $(q_0,\mathbf{m}_0)$. Denote $\delta q = q'-q$ and $\delta\mathbf{m}=\mathbf{m}'-\mathbf{m}$. We have

$$\begin{cases}\partial_t\delta q+\operatorname{div}\delta\mathbf{m}=0,\\ \partial_t\delta\mathbf{m}-\bar\mu\operatorname{div}\left(\dfrac{\nabla\delta\mathbf{m}}{1+q}\right)-(\bar\lambda+\bar\mu)\nabla\left(\dfrac{\operatorname{div}\delta\mathbf{m}}{1+q}\right)-\bar\kappa\nabla\big((1+q)\Delta\delta q\big)\\ \quad=\Gamma(q',\mathbf{m}')-\Gamma(q,\mathbf{m})-H(q',\mathbf{m}',q,\mathbf{m}),\\ \delta q_{|t=0}=0,\quad \delta\mathbf{m}_{|t=0}=0.\end{cases}\tag{59}$$

Let $T^* \in [0, T]$ be the greatest time such that (53) is satisfied by q' on $[0, T^*]$. Continuity for q' in $C([0, T]; B^{d/2})$ implies that $0 < T^* \leqslant T$.

We now apply Proposition 5 to (59) using bounds of step 1 and (53) for q. Unlike in the second step, our assumptions on $(q', \mathbf{m}')$ only provide us with bounds for H in $\widetilde{L}^2_{T^*}(B^{d/2-2})$. This leads to the following estimate:

$$\|(\delta q, \delta\mathbf{m})\|_{\widetilde{F}_{T^*}} \lesssim \|\Gamma(q', \mathbf{m}') - \Gamma(q, \mathbf{m})\|_{L^1_{T^*}(B^{d/2-1})} + \|H(q', \mathbf{m}', q, \mathbf{m})\|_{\widetilde{L}^2_{T^*}(B^{d/2-2})}.$$

Using the same estimates as in step 2 for $\|\Gamma(q', \mathbf{m}') - \Gamma(q, \mathbf{m})\|_{L^1_{T^*}(B^{d/2-1})}$, and the fact that

$$\begin{aligned}
&\|H(q', \mathbf{m}', q, \mathbf{m})\|_{\widetilde{L}^2_{T^*}(B^{d/2-2})} \\
&\quad \lesssim \left(1 + \|q\|_{\widetilde{L}^\infty_{T^*}(B^{d/2})}\right)\left(1 + \|q'\|_{\widetilde{L}^\infty_{T^*}(B^{d/2})}\right)\left(\|\mathbf{m}'\|_{\widetilde{L}^2_{T^*}(B^{d/2})} + \|\nabla q'\|_{\widetilde{L}^2_{T^*}(B^{d/2})}\right),
\end{aligned}$$

we finally gather

$$\begin{aligned}
\|(\delta q, \delta\mathbf{m})\|_{\widetilde{F}_{T^*}} \lesssim{}& \left(1 + \|q\|_{\widetilde{L}^\infty_{T^*}(B^{d/2})}\right)\left(1 + \|q'\|_{\widetilde{L}^\infty_{T^*}(B^{d/2})}\right) \\
&\times \Big(T^* + \|f\|_{L^1_{T^*}(B^{d/2-1})} + K\Big(\|\mathbf{m}\|_{\widetilde{L}^2_{T^*}(B^{d/2})} \\
&+ \|\nabla q\|_{\widetilde{L}^2_{T^*}(B^{d/2})} + \|\mathbf{m}'\|_{\widetilde{L}^2_{T^*}(B^{d/2})} \\
&+ \|\nabla q'\|_{\widetilde{L}^2_{T^*}(B^{d/2})}\Big)\Big)\|(\delta q, \delta\mathbf{m})\|_{\widetilde{F}_{T^*}},
\end{aligned}$$

with $K(z) = z + z^2$. This obviously entails $\delta q \equiv 0$ and $\delta\mathbf{m} \equiv \mathbf{0}$ on a suitably small interval $[0, T']$ with $T' > 0$.

Using the same arguments as for the proof of Lemma 4, we can now conclude that the two solutions coincide on the whole interval $[0, T]$. □

6. Local strong solutions near equilibrium

In this section, we want to show that local well-posedness for Korteweg system when the density is close to a constant also holds in spaces of type B^s_p with $p \neq 2$, that is, in spaces which are not related to physical energy spaces. Recall that this approach was extensively used for the study of incompressible Navier–Stokes equations (see [7] and the references enclosed).

This viewpoint enables us to get well-posedness even if the initial velocity belongs to a space B^s_p such that the regularity index s is negative, which in particular is relevant for oscillating initial velocities.

To avoid tedious discussions about the definition of $q\mathbf{f}$, we suppose from now on, that $\mathbf{f} \equiv \mathbf{0}$ (see Remark 3 at the end of the section). Using the same notations as in Section 4, the Korteweg system rewrites as follows:

$$\partial_t q + \operatorname{div}\mathbf{m} = 0, \tag{60}$$

$$\partial_t \mathbf{m} - \bar{\mu}\Delta\mathbf{m} - (\bar{\lambda} + \bar{\mu})\nabla\operatorname{div}\mathbf{m} - \bar{\kappa}\nabla\Delta q = \mathbf{G}(q, \mathbf{m}), \tag{61}$$

$$(q, \mathbf{m})_{|t=0} = (q_0, \mathbf{m}_0), \tag{62}$$

where we define $\mathbf{G} = \mathbf{G}_1 + \mathbf{G}_2 + \mathbf{G}_3 + \mathbf{G}_4$ by

$$\mathbf{G}_1(q, \mathbf{m}) = -\mathrm{div}\left(\frac{\mathbf{m} \otimes \mathbf{m}}{1+q}\right),$$
$$\mathbf{G}_2(q, \mathbf{m}) = -\nabla\big(P(\bar\rho(1+q)) - P(\bar\rho)\big)/\bar\rho,$$
$$\mathbf{G}_3(q, \mathbf{m}) = -\bar\mu\,\mathrm{div}\left(\nabla\left(\frac{q}{1+q}\right) \otimes \mathbf{m} + \frac{q}{1+q}\nabla\mathbf{m}\right) - (\bar\lambda + \bar\mu)\nabla\left(\nabla\left(\frac{q}{1+q}\right)\cdot\mathbf{m} + \frac{q}{1+q}\mathrm{div}\,\mathbf{m}\right),$$

and

$$\mathbf{G}_4(q, \mathbf{m}) = -\frac{\bar\kappa}{2}\nabla\big(|\nabla q|^2\big) - \bar\kappa\,\mathrm{div}\big(q\nabla^2 q\big).$$

Let us emphasize that no stability assumption on the pressure law is required: we just have to assume that P is suitably smooth ($P \in W^{[d/p]+3,\infty}_{loc}$ is enough).

In Section 6.1 we study the linearized system around $(0, 0)$ where we drop the first order linearized pressure term. Local well-posedness for $(q_0, \mathbf{m}_0)$ is then proved in Section 6.2 through a fixed-point argument.

6.1. Estimates for the linearized pressure-less system

This section is devoted to the proof of estimates for the following linear system

$$\begin{cases} \partial_t q + \mathrm{div}\,\mathbf{m} = F, \\ \partial_t \mathbf{m} - \bar\mu\Delta\mathbf{m} - (\bar\lambda + \bar\mu)\nabla\mathrm{div}\,\mathbf{m} - \bar\kappa\nabla\Delta q = \mathbf{G}. \end{cases} \tag{LNSK3}$$

The main result of this section is the following proposition:

PROPOSITION 6. – *Let $s \in \mathbb{R}$, $p \in [1, +\infty]$, $1 \leqslant \rho_1 \leqslant +\infty$ and $T \in]0, +\infty]$. If $(q_0, \mathbf{m}_0) \in B^s_p \times (B^{s-1}_p)^d$ and $(F, \mathbf{G}) \in \widetilde{L}^{\rho_1}_T(B^{s-2+2/\rho_1}_p \times (B^{s-3+2/\rho_1}_p)^d)$, then the linear system* (LNSK3) *has a unique solution $(q, \mathbf{m}) \in \widetilde{C}_T(B^s_p \times (B^{s-1}_p)^d) \cap \widetilde{L}^{\rho_1}_T(B^{s+2/\rho_1}_p \times (B^{s-1+2/\rho_1}_p)^d)$. Moreover, for all $\rho \in [\rho_1, +\infty]$, there exists a constant C depending only on $\bar\mu, \bar\lambda, \bar\kappa$, ρ, ρ_1 and d such that the following inequality holds*:

$$\|q\|_{\widetilde{L}^\rho_T(B^{s+2/\rho}_p)} + \|\mathbf{m}\|_{\widetilde{L}^\rho_T(B^{s-1+2/\rho}_p)} \leqslant C\big(\|q_0\|_{B^s_p} + \|\mathbf{m}_0\|_{B^{s-1}_p} + \|F\|_{\widetilde{L}^{\rho_1}_T(B^{s-2+2/\rho_1}_p)} + \|\mathbf{G}\|_{\widetilde{L}^{\rho_1}_T(B^{s-3+2/\rho_1}_p)}\big).$$

Proof. – Apply operator Δ to the first equation and operators div and curl (with $\mathrm{curl}\,\mathbf{g} := \partial_j\mathbf{g}^i - \partial_i\mathbf{g}^j$) to the second one. Denoting $\bar\nu = \bar\lambda + 2\bar\mu$, we obtain

$$\begin{cases} \partial_t\Delta q + \Delta\mathrm{div}\,\mathbf{m} = \Delta F, \\ \partial_t\mathrm{div}\,\mathbf{m} - \bar\nu\Delta\mathrm{div}\,\mathbf{m} - \bar\kappa\Delta^2 q = \mathrm{div}\,\mathbf{G}, \\ \partial_t\mathrm{curl}\,\mathbf{m} - \bar\mu\Delta\mathrm{curl}\,\mathbf{m} = \mathrm{curl}\,\mathbf{G}. \end{cases} \tag{63}$$

Proposition 2 gives the following estimates for the third equation, which decouples from the first two equations

$$\|\mathrm{curl}\,\mathbf{m}\|_{\widetilde{L}^\rho_T(B^{s-2+2/\rho}_p)} \leqslant C\big(\|\mathrm{curl}\,\mathbf{m}_0\|_{B^{s-2}_p} + \|\mathrm{curl}\,\mathbf{G}\|_{\widetilde{L}^{\rho_1}_T(B^{s-4+2/\rho_1}_p)}\big). \tag{64}$$

The following lemma points out a smoothing effect for the first two equations:

LEMMA 3. – *Let* $s \in \mathbb{R}$, $(p,\rho_1) \in [1,+\infty]^2$ *and* $T \in]0,+\infty]$. *Suppose that* $(c_0, v_0) \in (B^s_p)^2$ *and* $(h,k) \in (\widetilde{L}^{\rho_1}_T(B^{s-2+2/\rho_1}_p))^2$. *Then the system*

$$\begin{cases} \partial_t c + \Delta v = h, \\ \partial_t v - \bar{\nu}\Delta v - \bar{\kappa}\Delta c = k, \\ (c,v)_{|t=0} = (c_0, v_0), \end{cases} \tag{65}$$

has a unique solution (c,v) *in* $(\widetilde{C}_T(B^s_p) \cap \widetilde{L}^{\rho_1}_T(B^{s+2/\rho_1}_p))^2$. *Moreover, for all* $\rho \in [\rho_1,+\infty]$, *there exists* $C > 0$ *depending only on* $\bar{\nu}, \bar{\kappa}, \rho, \rho_1$ *such that*

$$\|(c,v)\|_{\widetilde{L}^{\rho}_T(B^{s+2/\rho}_p)} \leqslant C\big(\|(c_0,v_0)\|_{B^s_p} + \|(h,k)\|_{\widetilde{L}^{\rho_1}_T(B^{s-2+2/\rho_1}_p)}\big).$$

Using (64), Lemma 3 with $c_0 = \Delta q_0$, $v_0 = \operatorname{div}\mathbf{m}_0$, $h = \Delta f$, $k = \operatorname{div} g$ and noticing that $\Delta\mathbf{m} = \nabla\operatorname{div}\mathbf{m} + \operatorname{div}\operatorname{curl}\mathbf{m}$, Proposition 6 is now obvious. □

Proof of Lemma 3. – Denoting by $U(t)$ the semi-group associated to (65), we deduce from Duhamel's formula that

$$\begin{pmatrix} c(t) \\ v(t) \end{pmatrix} = U(t)\begin{pmatrix} c_0 \\ v_0 \end{pmatrix} + \int_0^t U(t-s)\begin{pmatrix} h(s) \\ k(s) \end{pmatrix} ds,$$

with $U(t) = e^{-tA(D)}$ and

$$A(\xi) = \begin{pmatrix} 0 & -|\xi|^2 \\ \bar{\kappa}|\xi|^2 & \bar{\nu}|\xi|^2 \end{pmatrix}.$$

Straightforward computations show that

$$e^{-tA(\xi)} = e^{-\frac{t\bar{\nu}|\xi|^2}{2}} \begin{pmatrix} h_1(t,\xi) + \frac{\bar{\nu}}{2}h_2(t,\xi) & h_2(t,\xi) \\ -\bar{\kappa}h_2(t,\xi) & h_1(t,\xi) - \frac{\bar{\nu}}{2}h_2(t,\xi) \end{pmatrix}$$

with

$$h_1(t,\xi) = \cos(\nu'|\xi|^2 t), \quad h_2(t,\xi) = \frac{\sin(\nu'|\xi|^2 t)}{\nu'}, \quad \text{if } \bar{\nu}^2 < 4\bar{\kappa},$$

$$h_1(t,\xi) = 1, \quad h_2(t,\xi) = t|\xi|^2, \quad \text{if } \bar{\nu}^2 = 4\bar{\kappa},$$

$$h_1(t,\xi) = \cosh(\nu'|\xi|^2 t), \quad h_2(t,\xi) = \frac{\sinh(\nu'|\xi|^2 t)}{\nu'}, \quad \text{if } \bar{\nu}^2 > 4\bar{\kappa}$$

and $\nu' = \sqrt{|\bar{\kappa} - \bar{\nu}^2/4|}$.

Let $\widetilde{\varphi}$ be a smooth function supported in $\{\xi \in \mathbb{R}^d \mid |\xi|^{\pm 1} \leqslant 2\}$ and such that $\widetilde{\varphi} \equiv 1$ on $\operatorname{Supp}\varphi$. Denote $a_{ij}(t,\xi)$ the coefficients of the matrix $e^{-tA(\xi)}$ and

$$H^q_{ij}(t,\mathbf{x}) = (2\pi)^{-d}\int e^{ix\cdot\xi} a_{ij}(t,\xi)\widetilde{\varphi}(2^{-q}\xi)\,d\xi.$$

We assume that the following inequality holds:

$$\|H_{ij}^q\|_{L^1} \leqslant Ce^{-c\min(1,4\bar\kappa/\bar\nu^2)2^{2q}\bar\nu t}, \tag{66}$$

where C depends only on $\bar\nu$, d and $\bar\kappa$, and c is a universal constant. Since

$$U(t)\begin{pmatrix}\Delta_q c\\ \Delta_q v\end{pmatrix}(\mathbf{x}) = \begin{pmatrix}(H_{11}^q(t,\cdot)\star\Delta_q c)(\mathbf{x}) + (H_{12}^q(t,\cdot)\star\Delta_q v)(\mathbf{x})\\ (H_{21}^q(t,\cdot)\star\Delta_q c)(\mathbf{x}) + (H_{22}^q(t,\cdot)\star\Delta_q v)(\mathbf{x})\end{pmatrix},$$

estimate (66) yields

$$\|U(t)(\Delta_q c, \Delta_q v)\|_{L^p} \leqslant Ce^{-c\min(1,4\bar\kappa/\bar\nu^2)2^{2q}\bar\nu t}\big(\|\Delta_q c\|_{L^p} + \|\Delta_q v\|_{L^p}\big).$$

Now, we complete the proof of Lemma 3 coming back to the definition of Besov spaces, and using convolution inequalities.

In order to prove (66), we first remark that $\|H_{ij}^q\|_{L^1} = \|h_{ij}^q\|_{L^1}$ with

$$h_{ij}^q(t,\mathbf{y}) = (2\pi)^{-d}\int e^{iy\cdot\eta} a_{ij}\big(t, 2^q\eta\big)\widetilde{\varphi}(\eta)\,d\eta.$$

All the functions h_{ij}^q are of the type

$$h^q(t,\mathbf{x}) = \int e^{ix\cdot\xi} f\big(2^{2q}|\xi|^2 t\big)\widetilde{\varphi}(\xi)\,d\xi, \tag{67}$$

for a function $f \in C^\infty(\mathbb{R}^+)$. Using integrations by parts and Leibniz' formula, we get for all $\alpha \in \mathbb{N}^d$,

$$(-i\mathbf{x})^\alpha h^q(\mathbf{x}) = \sum_{\beta\leqslant\alpha}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\int e^{ix\cdot\xi}\partial^\beta f\big(2^{2q}|\xi|^2 t\big)\partial^{\alpha-\beta}\widetilde{\varphi}(\xi)\,d\xi. \tag{68}$$

Next, from Faà-di-Bruno's formula, we deduce that

$$\partial^\beta f\big(2^{2q}|\xi|^2 t\big) = \sum_{\substack{\gamma_1+\cdots+\gamma_m=\beta\\ |\gamma_i|\geqslant 1}} f^{(m)}\big(2^{2q}|\xi|^2 t\big)\big(2^{2q}t\big)^m\left(\prod_{j=1}^m \partial^{\gamma_j}\big(|\xi|^2\big)\right). \tag{69}$$

Let us suppose first that $\bar\nu^2 < 4\bar\kappa$. Then, we just have to prove that

$$\|h^q\|_{L^1} \leqslant Ce^{-c2^{2q}\bar\nu t} \tag{70}$$

for $f(u) = e^{i\nu' u}e^{-\bar\nu u/2}$ and $\nu' = \sqrt{\bar\kappa - \bar\nu^2/4}$. We have

$$f^{(m)}(u) = \left(i\nu' - \frac{\bar\nu}{2}\right)^m e^{i\nu' u}e^{-\bar\nu u/2},$$

so that $|f^{(m)}(u)| \leqslant (\nu' + \bar\nu/2)^m e^{-\bar\nu u/2}$. Using (68), (69) and that $\operatorname{Supp}\widetilde{\varphi} \subset \{\xi \in \mathbb{R}^d \mid |\xi|^{\pm 1} \leqslant 2\}$, we prove the existence of constants $C_{\alpha,\beta,m}$ such that

$$\left|\mathbf{x}^\alpha h^q(\mathbf{x})\right| \leqslant \sum_{\beta\leqslant\alpha}\sum_{m=1}^{|\beta|} C_{\alpha,\beta,m}\left(2^{2q}t\right)^m e^{-\bar\nu t 2^{2q}/8}.$$

For any constant $c < 1$ and $m \in \mathbb{N}$, there exists C_m such that $u^m e^{-u} \leqslant C_m e^{-cu}$. This clearly yields (70).

When $\bar\nu^2 = 4\bar\kappa$, we must verify (70) for $f(u) = ue^{-\bar\nu u/2}$ and $f(u) = e^{-\bar\nu u/2}$. This is obvious in view of (69) and Leibniz' formula. When $\bar\nu^2 > 4\bar\kappa$, we must verify (70) for

$$f(u) = \exp\left(-\frac{\bar\nu}{2}\left(1 \pm \sqrt{1 - \frac{4\bar\kappa}{\bar\nu^2}}\right)u\right).$$

Using again (69) we thus get

$$\left|\mathbf{x}^\alpha h^q(\mathbf{x})\right| \leqslant Ce^{-c\bar\nu t 2^{2q}(1\pm\sqrt{1-4\bar\kappa/\bar\nu^2})} \leqslant Ce^{-c'(\bar\kappa/\bar\nu)2^{2q}t}$$

and we conclude to (66). □

6.2. Local well-posedness for an initial density close to a constant

In this section, we agree that B^s stands for B^s_p. Let us introduce the functional spaces needed in the local existence theorem. We will prove existence in

$$E^p_T = \widetilde{C}_T\left(B^{d/p} \times \left(B^{d/p-1}\right)^d\right) \cap L^1_T\left(B^{d/p+2} \times \left(B^{d/p+1}\right)^d\right)$$

and uniqueness in

$$\widetilde{E}^p_T = \widetilde{C}_T\left(B^{d/p} \times \left(B^{d/p-1}\right)\right) \cap \widetilde{L}^2_T\left(B^{d/p+1} \times \left(B^{d/p}\right)^d\right).$$

We have the following result:

THEOREM 5. – *Let $p \in [1, +\infty[$. Then there exists $\eta > 0$ such that if $q_0 \in B^{d/p}$, $\mathbf{m}_0 \in (B^{d/p-1})^d$ and*

$$\|q_0\|_{B^{d/p}} \leqslant \eta,$$

then there exists $T > 0$ such that system (60)–(62) has a unique solution $(q, \mathbf{m})$ in $\widetilde{E}^p_T$. In addition, $(q, \mathbf{m})$ belongs to E^p_T.

Proof. – For $T > 0$ and $p \in [1, +\infty[$, we denote

$$\|(q, \mathbf{m})\|_{F^p_T} = \|q\|_{\widetilde{L}^\infty_T(B^{d/p})} + \|q\|_{L^1_T(B^{d/p+2})} + \|\mathbf{m}\|_{\widetilde{L}^2_T(B^{d/p})} + \|\mathbf{m}\|_{L^1_T(B^{d/p+1})}.$$

Let $(q_0, \mathbf{m}_0)$ be as in Theorem 5 and denote by $(q_L, \mathbf{m}_L)$ the solution of the linearized pressure-less system on the interval $[0, T]$:

$$\begin{cases} \partial_t q + \operatorname{div} \mathbf{m} = 0, \\ \partial_t \mathbf{m} - \bar{\mu}\Delta \mathbf{m} - (\bar{\lambda} + \bar{\mu})\nabla \operatorname{div} \mathbf{m} - \bar{\kappa}\nabla\Delta q = 0, \\ (q, \mathbf{m})_{|t=0} = (q_0, \mathbf{m}_0). \end{cases}$$

Denoting by $V(t)$ the semi-group generated by the above system, we have

$$(q_L, \mathbf{m}_L)(t) = V(t)(q_0, \mathbf{m}_0).$$

Let us define

$$\Phi_{q_L,m_L}(\bar{q}, \bar{\mathbf{m}}) \stackrel{\text{def}}{=} \int_0^t V(t-s)\big(0, \mathbf{G}(q_L + \bar{q}, \mathbf{m}_L + \bar{\mathbf{m}})(s)\big)\, ds.$$

In order to prove the existence part of the theorem, we just have to show that Φ_{q_L,m_L} has a fixed point in E_T^p. Since E_T^p is a Banach space, we are going to prove that Φ_{q_L,m_L} satisfies the hypotheses of Picard's theorem in a ball $B(0, R)$ of E_T^p for sufficiently small R.

1st step: *Stability of* $B(0, R)$

Denote $q = q_L + \bar{q}$ and $\mathbf{m} = \mathbf{m}_L + \bar{\mathbf{m}}$. According to Proposition 6, we have

$$\|\Phi_{q_L,m_L}(\bar{q}, \bar{\mathbf{m}})\|_{E_T^p} \leqslant C\|\mathbf{G}(q, \mathbf{m})\|_{L_T^1(B^{d/p-1})}. \tag{71}$$

Under the assumption

$$\|q\|_{L^\infty([0,T]\times\mathbb{R}^d)} \leqslant 1/2, \tag{$\mathcal{H}$}$$

and using Proposition 3 and Lemma 1, we state the following estimates:

$$\|\mathbf{G}_1(q, \mathbf{m})\|_{L_T^1(B^{d/p-1})} \lesssim (1 + \|q\|_{\widetilde{L}_T^\infty(B^{d/p})})\|\mathbf{m}\|^2_{\widetilde{L}_T^2(B^{d/p})}, \tag{72}$$

$$\|\mathbf{G}_2(q, \mathbf{m})\|_{L_T^1(B^{d/p-1})} \lesssim T\|q\|_{\widetilde{L}_T^\infty(B^{d/p})}, \tag{73}$$

$$\|\mathbf{G}_3(q, \mathbf{m})\|_{\widetilde{L}_T^1(B^{d/p-1})} \lesssim \|q\|_{\widetilde{L}_T^\infty(B^{d/p})}\|\nabla\mathbf{m}\|_{\widetilde{L}_T^1(B^{d/p})} + \|q\|_{\widetilde{L}_T^2(B^{d/p+1})}\|\mathbf{m}\|_{\widetilde{L}_T^2(B^{d/p})}, \tag{74}$$

$$\|\mathbf{G}_4(q, \mathbf{m})\|_{L_T^1(B^{d/p-1})} \lesssim \|\nabla q\|^2_{\widetilde{L}_T^2(B^{d/p})} + \|q\|_{\widetilde{L}_T^\infty(B^{d/p})}\|\nabla^2 q\|_{L_T^1(B^{d/p})}. \tag{75}$$

This leads to the following inequality:

$$\begin{aligned} \|\Phi_{q_L,m_L}(\bar{q}, \bar{\mathbf{m}})\|_{E_T^p} \leqslant{}& C\big(1 + \|q_L\|_{\widetilde{L}_T^\infty(B^{d/2})}\big)\big(\|(q_L, \mathbf{m}_L)\|_{F_T^p} + \|(\bar{q}, \bar{\mathbf{m}})\|_{F_T^p}\big) \\ &\times\big(T + \|(q_L, \mathbf{m}_L)\|_{F_T^p} + \|(\bar{q}, \bar{\mathbf{m}})\|_{F_T^p}\big). \end{aligned}$$

Let c be a constant such that $\|\cdot\|_{B^{d/2}} \leqslant c$ implies $\|\cdot\|_{L^\infty} \leqslant 1/5$. We choose

$$R = \min\big((10C)^{-1}, c, 1\big), \tag{76}$$

and suppose that $\|q_0\|_{B^{d/p}} \leqslant R/2$. Then, since $q_L \in C([0, T]; B^{d/p})$, we have $\|(q_L, \mathbf{m}_L)\|_{F_T^p} \leqslant R$ for T small enough so that $(\mathcal{H})$ is fulfilled. We also suppose that

$T \leqslant R/4$ and we get

$$\Phi_{q_L,m_L}\big(B(0,R)\big) \subset B(0,9R/10). \tag{77}$$

2nd step: *Contraction properties*

Suppose that $(\bar{q}_1, \bar{\mathbf{m}}_1)$ and $(\bar{q}_2, \bar{\mathbf{m}}_2)$ belong to the ball $B(0,R)$ of E_T^p, and denote $q_1 = q_L + \bar{q}_1$, $\mathbf{m}_1 = \mathbf{m}_L + \bar{\mathbf{m}}_1$, $q_2 = q_L + \bar{q}_2$ and $\mathbf{m}_2 = \mathbf{m}_L + \bar{\mathbf{m}}_2$. Using again Proposition 6, we get

$$\big\|\Phi_{q_L,m_L}(\bar{q}_2,\bar{\mathbf{m}}_2) - \Phi_{q_L,m_L}(\bar{q}_1,\bar{\mathbf{m}}_1)\big\|_{E_T^p} \leqslant C\big\|\mathbf{G}(q_2,\mathbf{m}_2) - \mathbf{G}(q_1,\mathbf{m}_1)\big\|_{L_T^1(B^{d/p-1})}.$$

Under assumption $(\mathcal{H})$ for q_1 and q_2, we can derive estimates for $\mathbf{G}_i(q_2,\mathbf{m}_2) - \mathbf{G}_i(q_1,\mathbf{m}_1)$. We apply Proposition 3 and Lemma 1 to the following identities:

$$\begin{aligned}
&\mathbf{G}_1(q_2,\mathbf{m}_2) - \mathbf{G}_1(q_1,\mathbf{m}_1)\\
&\quad = \operatorname{div}\big((\mathbf{m}_1\otimes\mathbf{m}_1)(\widetilde{q}_2 - \widetilde{q}_1) - \widetilde{q}_2\big(\mathbf{m}_2\otimes(\mathbf{m}_2-\mathbf{m}_1) + (\mathbf{m}_2-\mathbf{m}_1)\otimes\mathbf{m}_1\big)\big),\\
&\mathbf{G}_2(q_2,\mathbf{m}_2) - \mathbf{G}_2(q_1,\mathbf{m}_1) = -\nabla(P(\bar\rho(1+q_2)) - (P(\bar\rho(1+q_1))/\bar\rho,\\
&\mathbf{G}_3(q_2,\mathbf{m}_2) - \mathbf{G}_3(q_1,\mathbf{m}_1)\\
&\quad = -\bar\mu\operatorname{div}\Big(\nabla\widetilde{q}_1\otimes(\mathbf{m}_2-\mathbf{m}_1) + \widetilde{q}_1\nabla(\mathbf{m}_2-\mathbf{m}_1)\\
&\qquad + \nabla(\widetilde{q}_2-\widetilde{q}_1)\otimes\mathbf{m}_2 + (\widetilde{q}_2-\widetilde{q}_1)\nabla\mathbf{m}_2\Big) - (\bar\lambda+\bar\mu)\nabla(\cdots),\\
&\mathbf{G}_4(q_2,\mathbf{m}_2) - \mathbf{G}_4(q_1,\mathbf{m}_1) = -\frac{\bar\kappa}{2}\nabla\big(\nabla(q_2-q_1)\cdot\nabla(q_2+q_1)\big)\\
&\qquad\qquad - \bar\kappa\operatorname{div}\big((q_2-q_1)\nabla^2 q_2 + q_1\nabla^2(q_2-q_1)\big),
\end{aligned}$$

where $\widetilde{q}_i \overset{\text{def}}{=} q_i/(1+q_i)$. We finally get a constant C such that

$$\begin{aligned}
&\big\|\Phi_{q_L,m_L}(\bar{q}_2,\bar{\mathbf{m}}_2) - \Phi_{q_L,m_L}(\bar{q}_1,\bar{\mathbf{m}}_1)\big\|_{E_T^p}\\
&\quad \leqslant CT\|\bar{q}_2 - \bar{q}_1\|_{L_T^\infty(B^{d/p})} + C\|(\bar{q}_2-\bar{q}_1, \bar{\mathbf{m}}_2-\bar{\mathbf{m}}_1)\|_{F_T^p}\\
&\qquad \times\big(\|(q_1,\mathbf{m}_1)\|_{F_T^p} + \|(q_2,\mathbf{m}_2)\|_{F_T^p}\big).
\end{aligned} \tag{78}$$

We make the same assumption on R and T as in the first step (replacing C with a larger constant if necessary) and get

$$\big\|\Phi_{q_L,m_L}(\bar{q}_2,\bar{\mathbf{m}}_2) - \Phi_{q_L,m_L}(\bar{q}_1,\bar{\mathbf{m}}_1)\big\|_{E_T^p} \leqslant \frac{9}{10}\|(q_2-q_1,\mathbf{m}_2-\mathbf{m}_1)\|_{F_T^p}.$$

This completes the proof of the existence of a solution $(q,\mathbf{m})$ for (60)–(62) in E_T^p, which in addition satisfies

$$\|q\|_{L^\infty([0,T]\times\mathbb{R}^d)} \leqslant \frac{2}{5} \quad\text{and}\quad \|q\|_{\widetilde{L}_T^\infty(B^{d/2})} \leqslant \frac{1}{5C}. \tag{79}$$

3rd step: *Uniqueness*

Uniqueness is a straightforward application of the following lemma, which yields also the global uniqueness result of Section 4.

LEMMA 4. – *There exists a constant C depending only on d and p such that if $T \in]0,+\infty]$ and $(q_i, \mathbf{m}_i)$ $(i = 1, 2)$ are two solutions of* (60)–(62) *belonging to $\widetilde{E}_p^T$ and $(q_1, \mathbf{m}_1)$ satisfies* (79) *with the constant C, then $(q_2, \mathbf{m}_2) \equiv (q_1, \mathbf{m}_1)$ on $[0, T]$.*

Proof. – Let T^* be the largest time such that $(\mathcal{H})$ is satisfied by q_2. As $\|q_2(0)\|_{L^\infty} \leqslant 1/5$ and $q_2 \in C([0,T] \times \mathbb{R}^d)$, we have $T^* > 0$. Let $[0, T_m] \subset [0, T^*]$ be the biggest interval such that the two solutions coincide on $[0, T_m]$. Suppose that $T_m < T$. Let

$$\big(\widetilde{q}_i(t), \widetilde{\mathbf{m}}_i(t)\big) \stackrel{\text{def}}{=} q_i\big((t - T_m), \mathbf{m}_i(t - T_m)\big).$$

Continuity in time for q_i implies that $(\mathcal{H})$ is satisfied by $\widetilde{q}_1$ and $\widetilde{q}_2$ on an interval $[0, \varepsilon]$ for $\varepsilon > 0$ small enough. Moreover $\{(\widetilde{q}_i, \widetilde{\mathbf{m}}_i)\}_{1 \leqslant i \leqslant 2}$ belongs to $\widetilde{E}_\varepsilon^p \times \widetilde{E}_\varepsilon^p$.

We now use the decomposition

$$\mathbf{G}(q_i, \mathbf{m}_i) = \mathbf{G}_1(q_i, \mathbf{m}_i) + \mathbf{G}_2(q_i, \mathbf{m}_i) + \mathbf{G}_3'(q_i, \mathbf{m}_i) + \mathbf{G}_4'(q_i, \mathbf{m}_i) + \mathbf{G}_5'(q_i, \mathbf{m}_i)$$

with

$$\begin{aligned}
\mathbf{G}_3'(q_i, \mathbf{m}_i) &= -\bar{\mu}\Delta\left(\frac{q_i \mathbf{m}_i}{1+q_i}\right) - (\bar{\lambda} + \bar{\mu})\nabla\operatorname{div}\left(\frac{q_i \mathbf{m}_i}{1+q_i}\right),\\
\mathbf{G}_4'(q_i, \mathbf{m}_i) &= -\frac{\bar{\kappa}}{2}\nabla\big(|\nabla q_i|^2\big) + \bar{\kappa}\operatorname{div}\big(\nabla q_i \otimes \nabla q_i\big),\\
\mathbf{G}_5'(q_i, \mathbf{m}_i) &= -\bar{\kappa}\Delta(q_i \nabla q_i).
\end{aligned}$$

Proposition 3 provides us with estimates for $\mathbf{G}_1(q_i, \mathbf{m}_i)$, $\mathbf{G}_2(q_i, \mathbf{m}_i)$ and $\mathbf{G}_3'(q_i, \mathbf{m}_i)$ in $L_T^1(B^{d/p-1})$, and for $\mathbf{G}_4'(q_i, \mathbf{m}_i)$ and $\mathbf{G}_5'(q_i, \mathbf{m}_i)$ in $\widetilde{L}_T^2(B^{d/p-2})$. Thanks to Proposition 6, we get

$$\|(\widetilde{q}_2 - \widetilde{q}_1, \widetilde{\mathbf{m}}_2 - \widetilde{\mathbf{m}}_1)\|_{\widetilde{E}_\varepsilon^p} \leqslant Z(\varepsilon)\|(\widetilde{q}_2 - \widetilde{q}_1, \widetilde{\mathbf{m}}_2 - \widetilde{\mathbf{m}}_1)\|_{\widetilde{E}_\varepsilon^p},$$

with

$$Z(\varepsilon) = C\varepsilon + C\sup_{i\in\{1,2\}}\big(\|q_i\|_{\widetilde{L}_T^\infty(B^{d/p})} + \|q_i\|_{\widetilde{L}_T^2(B^{d/p+1})} + \|\mathbf{m}_i\|_{\widetilde{L}_T^2(B^{d/p})}\big).$$

From (79) for q_1 and the definition of the space $\widetilde{C}_T(B^{d/p})$, we infer that

$$\lim_{\varepsilon\to 0} Z(\varepsilon) = 2C\|q_1(T_m)\|_{B^{d/p}} \leqslant 4/5,$$

thus $Z(\varepsilon) < 1$ for ε small enough. We therefore get $(\widetilde{q}_2, \widetilde{\mathbf{m}}_2) = (\widetilde{q}_1, \widetilde{\mathbf{m}}_1)$ on $[0, \varepsilon]$. This achieves the proof of Lemma 4. □

Remark 3. – In Theorem 5, we supposed that the external forcing term $\mathbf{f}$ vanishes. We can easily show that the local existence and uniqueness still holds if $\mathbf{f}$ belongs to $L_T^1(B^{d/p-1})$ where $p < 2d$. Indeed, the usual product maps $B^{d/p} \times B^{d/p-1}$ into $B^{d/p-1}$, provided $p < 2d$.

7. Further remarks

7.1. Weak solutions in dimension 2

We now focus on the 2-dimensional problem and study existence results of weak solutions. A weak solution $(\rho, \mathbf{u})$ of (8)–(10) in $\mathcal{D}'(\mathbb{R}_+ \times \mathbb{R}^2)$ is required to satisfy the finite energy a priori bounds (16) (17) for initial data verifying (15). More precisely, we require that for all test functions $(\psi, \phi) \in C_0^\infty([0,\infty) \times \mathbb{R}^2) \times C_0^\infty([0,\infty) \times \mathbb{R}^2)^2$,

$$\int_{\mathbb{R}^2} \rho\psi(t)\,d\mathbf{x} = \int_{\mathbb{R}^2} \rho_0\psi(0)\,d\mathbf{x} - \int_0^t ds \int_{\mathbb{R}^2} \rho\mathbf{u}\cdot\nabla\psi\,d\mathbf{x},$$

$$\begin{aligned}\int_{\mathbb{R}^2} \rho\mathbf{u}(t)\cdot\phi(t)\,d\mathbf{x} = \int_{\mathbb{R}^2} \rho_0\mathbf{u}_0\cdot\phi(0)\,d\mathbf{x} + \int_0^t ds \int_{\mathbb{R}^2} \Big(&\rho\mathbf{u}\otimes\mathbf{u} : \mathbf{D}(\phi) - \mu\,\mathbf{D}(\mathbf{u}) : \mathbf{D}(\phi)\\ &- (\lambda+\mu)\operatorname{div}\mathbf{u}\operatorname{div}\phi + p(\rho)\operatorname{div}\phi\\ &+ \kappa\Big(\frac{|\nabla\rho|^2}{2}\operatorname{div}\phi + \nabla\rho\otimes\nabla\rho : \mathbf{D}(\phi) - \frac{\rho^2}{2}\Delta\operatorname{div}\phi\Big)\Big)\,d\mathbf{x}.\end{aligned}$$

As pointed out in the introduction, we are not able to prove a global existence theorem like in [17] when $\kappa \neq 0$ because of the quadratic terms $\nabla\rho\otimes\nabla\rho$, even though $\nabla\rho$ is a priori bounded in $L^\infty((0,T); L^2(\mathbb{R}^d))^d$. We focus now on weak solutions in dimension $d = 2$ near a stable equilibrium, i.e. solutions $(\rho, \mathbf{u})$ close to $(\bar\rho, 0)$, where $\bar\rho > 0$ satisfies $P'(\bar\rho) > 0$. Let us introduce

$$\delta(t) = \left\| \frac{\rho(t) - \bar\rho}{\bar\rho} \right\|_{L^\infty}, \tag{80}$$

which measures the density fluctuation in L^∞. Unfortunately, such an a priori bound does not seem to be available for finite energy initial data. However, in view of the results of the preceding sections, it is valid for suitably smooth initial data and small enough time, which motivates the following result

PROPOSITION 7. – *There exists* $\eta > 0$ *such that as long as*

$$\mathcal{E}_0 + \sup_{t\in[0,T]} \delta(t) \leqslant \eta,$$

there exists a weak solution $(\rho, \mathbf{u})$ *on* $(0,T)$ *of* (NSK) *such that* $\rho - \bar\rho \in L^2(0,T; \dot H^1(\mathbb{R}^2) \cap \dot H^2(\mathbb{R}^2))$, $\mathbf{u} \in L^2(0,T; \dot H^1(\mathbb{R}^2))^2 \cap L^\infty(0,T; L^2(\mathbb{R}^2))^2$.

Proof. – First, multiplying the equation of momentum conservation by $\nabla\rho^2$ and integrating by part, we easily infer

$$\frac{d}{dt}\int_{\mathbb{R}^2} \mathbf{m}\cdot\nabla\rho^2\,d\mathbf{x} + \frac{\kappa}{2}\int_{\mathbb{R}^2} (\Delta\rho^2)^2\,d\mathbf{x}$$

$$= \int_{\mathbb{R}^2} \big(2\rho|\operatorname{div}\mathbf{m}|^2 + 2\kappa|\nabla\rho|^4 + \nabla^2\rho^2 : (\rho\mathbf{u}\otimes\mathbf{u}) - (\lambda+2\mu)\Delta\rho^2 \operatorname{div}\mathbf{u} - 2\rho P'(\rho)|\nabla\rho|^2\big)\,d\mathbf{x}, \tag{81}$$

hence, denoting $Z(s) = 2sP'(s)$, we obtain

$$\frac{d}{dt}\int_{\mathbb{R}^2} \mathbf{m}\cdot\nabla\rho^2\,d\mathbf{x} + \int_{\mathbb{R}^2}\Big(Z(\bar\rho)|\nabla\rho|^2 + \frac{\kappa}{4}\big|\Delta\rho^2\big|^2\Big)\,d\mathbf{x}$$
$$\leqslant C\big(1+\|\rho\|_{L^\infty}^3\big)\|\nabla\mathbf{u}\|_{L^2}^2 + C\|\rho\|_{L^\infty}\|\mathbf{u}\cdot\nabla\rho\|_{L^2}^2 + C\|\nabla\rho\|_{L^4}^4 + C\|\rho\|_{L^\infty}^2\|\mathbf{u}\|_{L^4}^4 + C\|Z(\rho)-Z(\bar\rho)\|_{L^\infty}\|\nabla\rho\|_{L^2}^2.$$

Let us now recall Gagliardo–Nirenberg's inequality

$$\|f\|_{L^4}^2 \leqslant C\|f\|_{L^2}\|\nabla f\|_{L^2}. \tag{82}$$

Thus, we obtain assuming that $\delta(t) < 1/2$

$$\|\mathbf{u}\|_{L^4}^4 \leqslant C\|\mathbf{u}\|_{L^2}^2\|\nabla\mathbf{u}\|_{L^2}^2,$$

$$\|\nabla\rho\|_{L^4}^2 \leqslant C\big\|\rho^{-1}\big\|_{L^\infty}^2\big\|\nabla\rho^2\big\|_{L^4}^2 \leqslant C\big(1+\delta(t)^2\big)\big\|\nabla\rho^2\big\|_{L^2}\big\|\Delta\rho^2\big\|_{L^2}$$
$$\leqslant C\big(1+\delta(t)^3\big)\|\nabla\rho\|_{L^2}\big\|\Delta\rho^2\big\|_{L^2},$$
$$\|\mathbf{u}\cdot\nabla\rho\|_{L^2}^2 \leqslant \|\mathbf{u}\|_{L^4}^2\|\nabla\rho\|_{L^4}^2 \leqslant C\big(1+\delta(t)^3\big)\|\mathbf{u}\|_{L^2}\|\nabla\mathbf{u}\|_{L^2}\|\nabla\rho\|_{L^2}\big\|\Delta\rho^2\big\|_{L^2},$$
$$\|\mathbf{u}\|_{L^2} \leqslant C(1-\delta(t))^{-1/2}\|\sqrt{\rho}\mathbf{u}\|_{L^2}.$$

Hence, for $\delta(t) < \eta$ (where $\eta > 0$ depends on Z'), we have for some $C_0 > 0$

$$\frac{d}{dt}\int_{\mathbb{R}^2}\mathbf{m}\cdot\nabla\rho^2\,d\mathbf{x} + C_0\big(\|\nabla\rho\|_{L^2}^2 + \|\Delta\rho^2\|_{L^2}^2\big)$$
$$\leqslant C\|\nabla\mathbf{u}\|_{L^2}^2 + C\|\mathbf{u}\|_{L^2}\|\nabla\mathbf{u}\|_{L^2}\|\nabla\rho\|_{L^2}\big\|\Delta\rho^2\big\|_{L^2} + C\|\nabla\rho\|_{L^2}^2\big\|\Delta\rho^2\big\|_{L^2}^2 + C\|\mathbf{u}\|_{L^2}^2\|\nabla\mathbf{u}\|_{L^2}^2,$$
$$\leqslant C\|\nabla\mathbf{u}\|_{L^2}^2\big(1+\|\mathbf{u}\|_{L^2}^2\big) + C\|\nabla\rho\|_{L^2}^2\big\|\Delta\rho^2\big\|_{L^2}^2$$
$$\leqslant C\|\nabla\mathbf{u}\|_{L^2}^2\big(1+\|\sqrt{\rho}\mathbf{u}\|_{L^2}^2\big) + \|\nabla\rho\|_{L^2}^2\big\|\Delta\rho^2\big\|_{L^2}^2,$$
$$\leqslant C\mathcal{E}_0\big(\|\nabla\mathbf{u}\|_{L^2}^2 + \big\|\Delta\rho^2\big\|_{L^2}^2\big) + C\|\nabla\mathbf{u}\|_{L^2}^2.$$

Therefore, as soon as $\mathcal{E}_0$ is small enough, we have for some constant $C_1 > 0$

$$\frac{d}{dt}\int_{\mathbb{R}^2}\mathbf{m}\cdot\nabla\rho^2\,d\mathbf{x} + C_1\big(\|\nabla\rho\|_{L^2}^2 + \big\|\Delta\rho^2\big\|_{L^2}^2\big) \leqslant C\|\nabla\mathbf{u}\|_{L^2}^2. \tag{83}$$

Let $\alpha > 0$ and define w_α by

$$w_\alpha(t) = \frac{1}{2}\|\sqrt{\rho}\mathbf{u}\|_{L^2}^2 + \big\|\pi(\rho)-\pi(\bar\rho)\big\|_{L^1} + \frac{\kappa}{2}\|\nabla\rho\|_{L^2}^2 + \alpha\int_{\mathbb{R}^d}\mathbf{m}\cdot\nabla\rho^2\,d\mathbf{x}.$$

Then, choosing α small enough, we deduce that

$$\frac{1}{2}w_\alpha(t) \leqslant w_0(t) \leqslant \frac{3}{2}w_\alpha(t),$$

and from the energy bounds and (83) that

$$\frac{d}{dt}w_\alpha(t) + C_2\big(\|\nabla \mathbf{u}\|_{L^2}^2 + \|\nabla\rho\|_{L^2}^2 + \big\|\Delta\rho^2\big\|_{L^2}^2\big) \leqslant 0, \tag{84}$$

so that we have the claimed a priori bounds. Now considering a suitable approximate problem, we can easily pass to the limit in quadratic terms like $\nabla\rho \otimes \nabla\rho$, since $\nabla\rho$ is bounded in $L^2(0,T;H^1(\mathbb{R}^d))$, and also in $C([0,T];H^{-1}(\mathbb{R}^d))$, as can be seen by writing a linear transport equation on $\nabla\rho$. □

7.2. Blow-up of solutions with compactly supported density

Let us finally make a few remarks based upon the work of Z. Xin [22] in the non-capillary case. We consider the full Navier–Stokes Korteweg system (1)–(3) (with energy equation), for which Hattori and Li [14] proved global existence of H^s solutions (for $s > 0$ large) close enough to constant states $|(\rho_0, \mathbf{m}_0, \theta_0) - (\bar\rho, 0, \bar\theta)| \ll 1$ when the heat conduction parameter α is positive. We expect that the preceding results, namely global well-posedness of the Korteweg system, still hold in scale invariant Besov spaces for solutions close to constant states.

In contrast to the preceding approach where ρ is close to a constant, we assume now that the initial data satisfy

$$A(0) \stackrel{\text{def}}{=} \int_{\mathbb{R}^d}\left(\rho_0\frac{|\mathbf{u}_0 - \mathbf{x}|^2}{2} + \rho_0 e_0\right)d\mathbf{x} < +\infty,$$

which is the case for instance if ρ_0 has compact support. For the sake of simplicity, we consider a perfect gas law $p = \rho R\theta$, $e = c_v\theta$, and set $\gamma = 1 + R/c_v$. When $\gamma \in (1, 1+2/d]$, we define $\sigma(t) = (1+t)$, whereas when $\gamma \in (1+2/d, \infty)$, we take $\sigma(t) = t$. Then, we deduce from easy computations (see [22]) that

$$\frac{d}{dt}A(t) = \sigma(t)\int_{\mathbb{R}^d}(2\rho e - dp)\,d\mathbf{x},$$

where

$$A(t) = \int_{\mathbb{R}^d}\left(\rho\frac{|\mathbf{u}\sigma(t) - \mathbf{x}|^2}{2} + \sigma(t)^2\rho e\right)(t)\,d\mathbf{x} + \kappa\frac{(d+2)}{2}\int_0^t \sigma(s)\int_{\mathbb{R}^d}|\nabla\rho(s,\mathbf{x})|^2\,ds\,d\mathbf{x}.$$

Using the fact that $p = (\gamma-1)\rho e$ and denoting $\delta = 2 - d(\gamma - 1)$, we obtain

$$\dot A(t) \leqslant \delta\frac{A(t)}{\sigma(t)}, \quad \text{hence } A(t) \leqslant A(0)\sigma(t)^{\max(\delta,0)}.$$

Therefore, we have

$$\sigma(t)^{\min(2,d(\gamma-1))}\int_{\mathbb{R}^d} p\,d\mathbf{x}\leqslant(\gamma-1)A(0).\tag{85}$$

As a consequence, we obtain in the isothermal case $\theta=\bar\theta$

$$RM\bar\theta\sigma(t)^{\min(2,d(\gamma-1))}\leqslant(\gamma-1)A(0),$$

denoting by M the total mass, which proves that solutions blow up after some critical time T_0 as soon as $A(0)$ is finite.

Let us remark that similar observations can be done in the isentropic case, as well as in the non-isentropic case when the thermal diffusion is neglected (i.e. $\alpha=0$). In order to get blow up estimates, we have to consider initial densities compactly supported in $\mathbb{R}^d$ (see [22]), and observe that the support of the density does not grow as time evolves. Then, blow up estimates stem from estimate (85) and Hölder's inequality.

Appendix

This section is devoted to a commutation lemma that we used to prove Proposition 5.

LEMMA 5. – *Suppose* $A\in\widetilde{L}^2_T(B^{d/2+1})$ *and* $B\in\widetilde{L}^2_T(B^{d/2})$. *Then the following estimate holds on* $[0,T]$:

$$\|\partial_k[A,\Delta_\ell]B\|_{L^1_T(L^2)}\leqslant Cc_\ell 2^{-\ell(d/2-1)}\|A\|_{\widetilde{L}^2_T(B^{d/2+1})}\|B\|_{\widetilde{L}^2_T(B^{d/2-1})},$$

where C *depends only on* d, *and* $\sum_{\ell\in\mathbb{Z}}c_\ell\leqslant 1$.

Proof. – The proof of the above lemma requires some paradifferential calculus. We have to recall here that paradifferential calculus enables to define a generalized product between distributions, which is continuous in many functional spaces where the usual product does not make sense (see the pioneering work of J.-M. Bony in [4]). The paraproduct between u and v is defined by

$$T_u v\stackrel{\text{def}}{=}\sum_{\ell\in\mathbb{Z}}S_{\ell-1}u\Delta_\ell v.$$

We thus have the following formal decomposition (modulo a polynomial):

$$uv=T_uv+T'_vu,\quad \text{with } T'_vu\stackrel{\text{def}}{=}\sum_{\ell\in\mathbb{Z}}S_{\ell+2}v\Delta_\ell u.$$

Coming back to the proof of Lemma 5, we split $\partial_k[A,\Delta_\ell]B$ into

$$\partial_k[A,\Delta_\ell]B=\partial_k T'_{\Delta_\ell B}A-\partial_k\Delta_\ell T'_BA+[T_A,\Delta_\ell]\partial_kB+T_{\partial_kA}\Delta_\ell B-\Delta_\ell T_{\partial_kA}B.$$

From now on, we agree that $(c_\ell)_{\ell\in\mathbb{Z}}$ denotes a positive sequence such that $\sum_{\ell\in\mathbb{Z}} c_\ell \leqslant 1$. According to (20) and to the definition of T', we have

$$\partial_k T'_{\Delta_\ell B} A = \sum_{m\geqslant \ell-2} \partial_k (S_{m+2}\Delta_\ell B\,\Delta_m A).$$

Thus, in view of Bernstein's lemma and the definition of $\widetilde{L}^\rho_T(B^s)$ spaces,

$$\begin{aligned}\|\partial_k T'_{\Delta_\ell B} A\|_{L^1_T(L^2)} &\lesssim \sum_{m\geqslant \ell-2} 2^m 2^{\ell d/2} \|\Delta_\ell B\|_{L^2_T(L^2)} \|\Delta_m A\|_{L^2_T(L^2)},\\ &\lesssim 2^{\ell d/2}\|\Delta_\ell B\|_{L^2_T(L^2)} \sum_{m\geqslant \ell-2} 2^{-md/2}\big(2^{m(d/2+1)}\|\Delta_m A\|_{L^2_T(L^2)}\big),\\ &\lesssim 2^{-\ell(d/2-1)}\big(2^{\ell(d/2-1)}\|\Delta_\ell B\|_{L^2_T(L^2)}\big) \sum_{m\geqslant \ell-2} 2^{m(d/2+1)}\|\Delta_m A\|_{L^2_T(L^2)},\\ &\lesssim c_\ell 2^{-\ell(d/2-1)}\|B\|_{\widetilde{L}^2_T(B^{d/2-1})}\|A\|_{\widetilde{L}^2_T(B^{d/2+1})}.\end{aligned}$$

We use classical estimates for the paraproduct to bound the second term of the right-hand side (see [7] and [8]). We get

$$\|T'_B A\|_{L^1_T(B^{d/2})} \lesssim \|B\|_{L^2_T(B^{d/2-1})}\|A\|_{L^2_T(B^{d/2+1})}.$$

Using spectral localization of Δ_ℓ and definition of $\widetilde{L}^\rho_T(B^s)$ spaces, we get

$$\|\partial_k\Delta_\ell T'_B A\|_{L^1_T(L^2)} \lesssim c_\ell 2^{-\ell(d/2-1)}\|B\|_{\widetilde{L}^2_T(B^{d/2-1})}\|A\|_{\widetilde{L}^2_T(B^{d/2+1})}.$$

According to (20), the third term reads

$$[T_A, \Delta_\ell]\partial_k B = \sum_{|m-\ell|\leqslant 4} [S_{m-1}A, \Delta_\ell]\Delta_m\partial_k B.$$

Applying first order Taylor's formula, we get for $\mathbf{x}\in\mathbb{R}^d$,

$$\begin{aligned}&[S_{m-1}A, \Delta_\ell]\Delta_m\partial_k B(\mathbf{x})\\ &\quad = 2^{-\ell}\int_{\mathbb{R}^d}\int_0^1 h(\mathbf{y})\big(\mathbf{y}\cdot S_{m-1}\nabla A\big(\mathbf{x}-2^{-\ell}\tau\mathbf{y}\big)\big)\Delta_m\partial_k B\big(\mathbf{x}-2^{-\ell}\mathbf{y}\big)\,d\tau\,d\mathbf{y}.\end{aligned}$$

Convolution inequality thus yields

$$\big\|[S_{m-1}A, \Delta_\ell]\Delta_m\partial_k B\big\|_{L^2} \lesssim 2^{-\ell}\|\nabla A\|_{L^\infty}\|\Delta_m\partial_k B\|_{L^2},$$

hence

$$\big\|[T_A, \Delta_\ell]\partial_k B\big\|_{L^1_T(L^2)} \lesssim c_\ell 2^{-\ell(d/2-1)}\|\nabla A\|_{L^2_T(L^\infty)}\|B\|_{\widetilde{L}^2_T(B^{d/2-1})}.$$

Finally, thanks to (20), we have

$$T_{\partial_k A}\Delta_\ell B = \sum_{|\ell-m|\leqslant 1} S_{m-1}\partial_k A\,\Delta_\ell\Delta_m B$$

so that

$$\|T_{\partial_k A}\Delta_\ell B\|_{L^1_T(L^2)} \leqslant \|\partial_k A\|_{L^2_T(L^\infty)}\|\Delta_\ell B\|_{L^2_T(L^2)}.$$

Classical estimates for the paraproduct yield

$$\|T_{\partial_k A}B\|_{L^1_T(B^{d/2-1})} \lesssim \|B\|_{\widetilde{L}^2_T(B^{d/2-1})}\|\partial_k A\|_{\widetilde{L}^2_T(B^{d/2})}$$

so that the proof of Lemma 5 is achieved. □

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