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Analyse non linéaire

On minimal surfaces with free boundaries in given homotopy classes

by

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ABSTRACT. — Let S be a smooth compact imbedded surface in $\mathbb{R}^3$ and let B be the unit disc in $\mathbb{R}^2$. We consider the problem of finding a surface that minimizes area among all surfaces which have the topological type of a disc and which have boundaries in a given nontrivial homotopy class H of curves $\gamma: \partial B \to S$. We show that H can be decomposed into finitely many homotopy classes $H_1, \ldots, H_k$ for which the problem is solvable.

RÉSUMÉ. — Soit S une surface compacte régulière dans $\mathbb{R}^3$ et soit B un disque dans $\mathbb{R}^2$. On étudie le problème de trouver une surface qui minimise la superficie entre les surfaces qui sont topologiquement équivalentes à un disque et qui ont des frontières dans une classe d'homotopie H nontriviale des courbes $\gamma: \partial B \to S$. On prouve qu'on peut décomposer H dans des classes d'homotopie $H_1, H_2, \ldots, H_k$ non triviales pour lesquelles il existe une solution du problème étudié.

1. INTRODUCTION AND RESULTS

In this work, we consider the problem of finding a surface that minimizes area among all surfaces which have the topological type of a disc and

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which have boundaries in a given nontrivial homotopy class. In his book [2; p. 213], R. Courant described a difficulty that makes the treatment of such free boundary problems rather difficult. Namely, the boundary values of an arbitrary minimizing sequence need not converge uniformly. The main purpose of this paper is to show how one can find minimizing sequences having uniformly convergent boundary values. We believe that this method can be applied to many other free boundary problems for minimal surfaces, harmonic mappings or H-surfaces. The idea for this work was born in a discussion with S. Hildebrandt and F. Tomi in the « Oberseminar Analysis » at the University of Bonn.

Now, we have to introduce some notations and assumptions. By B, we denote the unit disc in $\mathbb{R}^2$. For $v \in H^{1,2}(B)$, we set

$$D(v) = 1/2 . \int_B |\nabla v|^2 dx .$$

We consider a two-dimensional embedded connected compact C^∞-surface $S \subset \mathbb{R}^3$. Two continuous curves $\gamma_i : \partial B \to S$ are homotopic, if there is a continuous mapping $h: [0,1] \times \partial B \to S$ such that $\gamma_i(\sigma) = h(i, \sigma)$, for $i = 0, 1$ and all $\sigma \in \partial B$. By $\Pi_0(S)$, we denote the set of all homotopy classes of continuous curves $\gamma : \partial B \to S$. We suppose that

$$\Pi(S) = \Pi_0(S) \setminus \{ O \} \neq \phi , \tag{1.1}$$

where O is the homotopy class containing the constant curves. A tupel $(H_1, H_2, \ldots, H_k)$ of $H_j \in \Pi_0(S)$ belongs to the set $Z_0(H)$ of all decompositions of a homotopy class $H \in \Pi_0(S)$, if there are $\gamma_j \in H_j$ such that the curve

$$\gamma(e^{i\theta}) = \begin{cases} \gamma_j(e^{i(\theta - \theta_{j-1})/\Delta\theta}) , & \text{if} \quad \theta \in [\theta_{j-1}, \theta_j] , \\ \gamma_j(e^{i(\theta + \theta_j - 2\pi)/\Delta\theta + i\pi}) , & \text{if} \quad \theta \in [2\pi - \theta_j, 2\pi - \theta_{j-1}] , \end{cases}$$

belongs to H, where $\Delta\theta = \pi/k$ and $\theta_j = j . \Delta\theta$. For $H \in \Pi_0(S)$, we set

$$Z(H) = \{ (H_1, H_2, \ldots, H_k) \in Z_0(H) \mid H_j \neq O, \quad \text{for} \quad j = 1, 2, \ldots, k \} ,$$
$$M(H) = \{ v \in C^0(\overline{B}) \cap H^{1,2}(B) \mid v|_{\partial B} \in H \} ,$$
$$d_H = \inf \{ D(v) \mid v \in M(H) \} .$$

We note that

$$d_H \leqq d_{H_1} + d_{H_2} , \qquad \forall (H_1, H_2) \in Z(H) . \tag{1.2}$$

THEOREM 1. — Suppose that $H \in \Pi(S)$. Then, there are a decomposition $(H_1, H_2, \ldots, H_k) \in Z(H)$ and $u_j \in M(H_j)$ such that

$$d_H = d_{H_1} + d_{H_2} + \ldots + d_{H_k} \tag{1.3}$$
$$D(u_j) = d_{H_j} , \qquad \text{for} \quad j = 1, 2, \ldots, k . \tag{1.4}$$

Additions to Theorem 1. — Let H, H_j and u_j be as in Theorem 1. The results on classical minimal surfaces (cf. [*5*] [*8*] [*10*] [*11*] and the literature cited there) imply that $u_j \in C^\infty(\overline{B})$ and that

$$|u_{j,x_1}| - |u_{j,x_2}| = 0 = u_{j,x_1} . u_{j,x_2}, \qquad \text{in} \quad B, \tag{1.5}$$

$$\Delta u_j = 0, \qquad \text{in} \quad B, \tag{1.6}$$

$$u_{j,r} \perp S, \qquad \text{on} \quad \partial B, \tag{1.7}$$

where $x = (x_1, x_2) = (r . \cos\theta, r . \sin\theta)$. Moreover, one can use the method of S. Hildebrandt & J. C. C. Nitsche [*6*] and A. Küster [*7*] in order to estimate the length of $u_1(\partial B) \cup u_2(\partial B) \cup \ldots \cup u_k(\partial B)$ only in dependence on S and an upper bound for d_H.

THEOREM 2. — Pick a $c \in]0, \infty[$. Then, there are at most finitely many $H \in \Pi_0(S)$ for which $d_H \leqq c$.

From Theorem 1 and 2, one easily derives the following four existence results.

COROLLARY 1. — Suppose that $H \in \Pi(S)$ and that the « Douglas-Criterion »

$$d_H < d_{H_1} + d_{H_2}, \qquad \forall (H_1, H_2) \in Z(H), \tag{1.8}$$

holds. Then, there is a $u \in M(H)$ such that

$$D(u) = d_H. \tag{1.9}$$

COROLLARY 2. — There is an $H^* \in \Pi(S)$ and a $u^* \in M(H^*)$ such that

$$0 < D(u^*) = d_{H^*} \leqq d_H, \qquad \forall H \in \Pi(S).$$

COROLLARY 3. — Suppose that

$$d_H < 2 . d_{H^*},$$

for some $H \in \Pi(S)$. Then, there is a $u \in M(H)$ that solves (1.9).

COROLLARY 4. — Suppose that Γ is a Jordan arc in $\mathbb{R}^3 \backslash S$ which is not contractable, in $\mathbb{R}^3 \backslash S$. Let $\Pi(S, \Gamma) \subset \Pi(S)$ be the set of all homotopy classes of curves which are linked with Γ. Then, there is an $H' \in \Pi(S, \Gamma)$ and a $u' \in M(H')$ such that

$$0 < D(u') = d_{H'} \leqq d_H, \qquad \forall H \in \Pi(S, \Gamma).$$

Corollary 4 has been stated by R. Courant in his book [*2*; p. 213]. There he also gave an idea for its proof. This has been performed exactly by S. Hildebranct [*4*], not only for minimal surfaces, but also for H-surfaces. We have to admit that Courant's and Hildebrandt's result holds also for non-smooth surfaces S, while our method cannot work for arbitrary non-smooth surfaces S.

A. Küster attracted my attention to the work of N. Davids [*16*] which generalizes Courant's method. By the Alexander duality, one can characterize homology classes by means of linking conditions. Thus, he obtained a result similar to our Theorem 1, but only for homology classes.

It makes more physical and geometrical sense to consider such problems for surfaces that have to be contained in one connectivity component U of $\mathbb{R}^3 \setminus S$. This has been done by W. H. Meeks & S. T. Yau [*9*]. Just recently, F. Tomi [*15*] and H. W. Alt & F. Tomi [*1*] gave rather strong results on the structure of set of the minimal solutions. In [*1*], [*9*] and [*15*], it is supposed that the inward mean curvature of U is nonnegative. This excludes the possibility, that the minimal solutions touch ∂U. Apart from the restriction on the competing surfaces, the problems considered in [*1*] [*9*] [*15*] are similar to those of Corollary 2 and 4. We want to point out that the conclusions of Theorem 1 and 2 can be generalized to the situations considered in [*1*] [*9*] [*15*].

Now, let us make some remarks on surfaces S which are diffeomorphic to a sphere, i. e. for which $\Pi(S) = \phi$. Then, the problems considered in this work do not make sense, any more. Nevertheless, it has been conjectured by J. C. C. Nitsche [*12*] that there exist three different non constant minimal surfaces that satisfy (1.5)-(1.7). For a quadrilateral, this has been proven by B. Smyth [*13*]. In the general case, the existence of one nontrivial minimal surface satisfying (1.5)-(1.7) has been established by M. Struwe [*14*]. In the case that the bounded connectivity component of $\mathbb{R}^3 \setminus S$ is convex, M. Grüter & J. Jost [*3*] showed that there exists such a minimal surface which is embedded.

2. PROOFS

DEFINITION. — For $v \in C^0(\overline{B})$ and $\Omega \subset \overline{B}$, we set

$$l(v, \Omega) = \sup \left\{ \sum_{m} \sum_{j=1}^{k_m} | v(\gamma_m(\theta_{m,j+1})) - v(\gamma_m(\theta_{m,j})) | \right\},$$

where the supremum is taken over all regular injective arcs $\gamma_m : [0, 1] \to \Omega$ that satisfy

$$\gamma_m(]0, 1[) \cap \gamma_n(]0, 1[) = \phi,$$

for $m \neq n$, and over all $\emptyset = \theta_{m,1} < \theta_{m,2} < \ldots < \theta_{m,k} = 1$. In the case that $\Omega = \partial B$, we write $l(v)$ instead of $l(v, \partial B)$.

We begin by stating two simple properties of S.

LEMMA 1. — Pick a $c \in]0, \infty[$. Then, there are at most finitely many $H \in \Pi_0(S)$ for which there exists a $v \in M(H)$ satisfying $l(v) \leq c$.

LEMMA 2. — Let U be the bounded connectivity component of $\mathbb{R}^3 \backslash S$, let N be the outer normal of S with respect to U and set

$$\rho(q) = \text{dist}(q, S), \qquad \forall q \in \mathbb{R}^3,$$
$$\Phi(s, p) = p + s.N(p), \qquad \forall s \in \mathbb{R}, \qquad \forall p \in S.$$

Then, there is a $\lambda > 0$ such that $\Phi :]-2\lambda, 2\lambda[\times S \to \{ q \in \mathbb{R}^3 \mid \rho(q) < 2\lambda \}$ is a C^∞-diffeomorphism. Moreover, there is a C^∞-mapping

$$P : \{ q \in \mathbb{R}^3 \mid \rho(q) < 2\lambda \} \to S$$

such that

$$P \circ \Phi(s, p) = p \qquad \text{and} \qquad \rho \circ \Phi(s, p) = |s|,$$

for all $s \in]-2\lambda, 2\lambda[$ and all $p \in S$.

LEMMA 3. — There is a $c_d > 0$ such that

$$c_d \leqq d_H, \qquad \forall H \in \Pi(S).$$

Proof of Lemma 3. — Let us suppose that Lemma 3 is wrong. Then, there are sequences of $H_\nu \in \Pi(S)$ and of $u_\nu \in M(H_\nu)$ such that

$$\Delta u_\nu = 0, \qquad \text{in} \quad B, \qquad \forall \nu \in \mathbb{N}, \tag{2.1}$$
$$D(u_\nu) \to 0. \tag{2.2}$$

In the case that $\rho \circ u_\nu < \lambda$, in B, one can use the mapping P to show that $H_\nu = 0$. This and the conformal invariance of (2.1) and (2.2) imply that we can choose the sequence (u_ν) in such a way that $\rho \circ u_\nu(0) \geqq \lambda$, $\forall \nu \in \mathbb{N}$. There is a $u \in H^{1,2}(B) \cap C^\infty(B)$ and a subsequence of (u_ν) that converges to u, in the sense of $C^\infty(B)$ and weakly in the sense of $H^{1,2}$ (B). This u has to satisfy

$$D(u) = 0 \qquad \text{and} \qquad \rho \circ u(0) \geqq \lambda, \tag{2.3}$$
$$\rho \circ u = 0, \qquad \text{on} \qquad \partial B. \tag{2.4}$$

As (2.3) and (2.4) contradict each other, Lemma 3 must be true.

Remark. — A similar argument has been used also by S. Hildebrandt [*4*].

We pick a $t > 0$ and an $H \in \Pi(S)$ and set

$$M(H, t) = \{ v \in M(H) \mid l(v) \leqq t \}$$
$$D_H(t) = \begin{cases} \inf \{ D(v) \mid v \in M(H, t) \}, & \text{if} \quad M(H, t) \neq \phi, \\ \infty, & \text{if} \quad M(H, t) = \phi, \end{cases}$$
$$d_H(t) = \inf \{ D_{H_1}(t) + D_{H_2}(t) + \ldots + D_{H_k}(t) \mid (H_1, H_2, \ldots, H_k) \in Z(H) \}.$$

We note that d_H and D_H are non-increasing functions of t, and that

$$d_H \leqq d_H(t) \leqq D_H(t), \qquad \forall t > 0, \tag{2.5}$$
$$d_H(t) \leqq d_{H_1}(t) + d_{H_2}(t) + \ldots + d_{H_k}(t), \ \forall t > 0, \ \forall (H_1, \ldots, H_k) \in Z(H), \tag{2.6}$$
$$D_H(t) \to d_H, \qquad \text{for} \qquad t \to \infty. \tag{2.7}$$

From (2.5), Lemma 1 and Lemma 3, one easily derives

LEMMA 4. — Suppose that $M(H, t) \neq \phi$. Then, there are a decomposition $(H_1, H_2, \ldots, H_k) \in Z(H)$ and a $\delta > 0$ such that

$$d_H(t) = D_{H_1}(t) + D_{H_2}(t) + \ldots + D_{H_k}(t),$$
$$d_{H_j}(t) = D_{H_j}(t) \leqq D_{H_{1,j}}(t) + D_{H_{2,j}}(t) - \delta,$$

for $j = 1, 2, \ldots, k$ and all $(H_{1,j}, H_{2,j}) \in Z(H_j)$.

PROPOSITION 1. — Suppose that $M(H, t) \neq \phi$ and that there is a $\delta > 0$ such that

$$d_H(t) = D_H(t) \leqq D_{H_1}(t) + D_{H_2}(t) - \delta, \qquad \forall (H_1, H_2) \in Z(H). \quad (2.8)$$

Then, there exists a solution $u \in M(H, t)$ of

$$D(u) = d_H(t) = D_H(t). \quad (2.9)$$

PROPOSITION 2. — Let H, t, δ and u be as in Proposition 1. Then, there is a $t_0 > 0$, depending only on S and an upper bound for $D(u)$ such that

$$d_H(t) = d_H(t_0), \qquad \text{if} \qquad t \geqq t_0. \quad (2.10)$$

Proof of Theorem 1 and 2. — Theorem 1 follows from (2.5)-(2.7), Lemma 4, Proposition 1 and Proposition 2. For Theorem 2, we have to use Lemma 1 and Lemma 3, additionally.

Proof of Proposition 1. — We set

$$t^* = \inf\{ \liminf_{\nu \to \infty} l(u_\nu) \mid u_\nu \in M(H, t),\ D(u_\nu) \to d_H(t) \}.$$

We can find a sequence of $u_\nu \in M(H, t)$ such that

$$l(u_\nu) \to t^* \leqq t, \quad (2.11)$$
$$D(u_\nu) \to d_H(t), \quad (2.12)$$
$$l(u_\nu, C_m) \geqq t^*/4, \qquad \forall m \in \{1, 2, 3\}, \quad \forall \nu \in \mathbb{N}, \quad (2.13)$$

where C_1, C_2 and C_3 are the connected components of $\partial B \setminus \{ e^0, e^{2i\pi/3}, e^{4i\pi/3} \}$. By the smoothness of S, we can find a $c > 0$ and an $\varepsilon > 0$ such that, for all $p_0, p_1 \in S$ satisfying $| p_0 - p_1 | \leqq \varepsilon$, there is a $C^\infty([0, 1])$-curve α: $[0, 1] \to S$ satisfying

$$\alpha(i) = p_i, \qquad \text{for} \qquad i = 0, 1, \quad (2.14)$$
$$\int_0^1 | \dot{\alpha}(t) | \, dt \leqq c . | p_0 - p_1 | . \quad (2.15)$$

Now, suppose that the u_ν are not equicontinuous, on ∂B, with respect to $\nu \in \mathbb{N}$. Then, we can use (2.13) and the Courant-Lebesgue-Lemma [*2*; p. 103] in order to determine a $\sigma \in]0, t^*/4]$, a subsequence (u_μ) of (u_ν) and a sequence of balls with radius $r_\mu \in]0, 1/2]$ and center at $x_\mu \in \partial B$ that satisfy

$$l(u_\mu, \partial B_{r_\mu}(x_\mu) \cap B) \rightarrow 0, \tag{2.16}$$

$$l(u_\mu, \partial B \cap B_{r_\mu}(x_\mu)) \geqq \sigma, \qquad \forall \mu \in \mathbb{N} \tag{2.17}$$

$$l(u_\mu, \partial B \backslash B_{r_\mu}(x_\mu)) \geqq t^*/2, \qquad \forall \mu \in \mathbb{N}. \tag{2.18}$$

Following S. Hildebrandt & J. C. C. Nitsche [5], one can use (2.11), (2.12) and (2.14)-(2.18) in order to find sequences of $(H_{1,\mu}, H_{2,\mu}) \in Z_0(H)$ and of $u_{i,\mu} \in M(H_{i,\mu})$ that satisfy

$$\limsup_{\mu \to \infty} D(u_{1,\mu}) + D(u_{2,\mu}) \leqq d_H(t), \tag{2.19}$$

$$\limsup_{\mu \to \infty} \; l(u_{1,\mu}) + \; l(u_{2,\mu}) \leqq t^* \leqq t, \tag{2.20}$$

$$3t^*/4 \geqq \liminf_{\mu \to \infty} l(u_{2,\mu}) \geqq \limsup_{\mu \to \infty} l(u_{2,\mu}) \geqq \sigma > 0. \tag{2.21}$$

In the case that $H_{2,\mu} \neq O$, for infinitely many $\mu \in \mathbb{N}$, (2.19)-(2.21) contradict (2.8). In the case that $H_{2,\mu} = O$ (i. e. $H_{1,\mu} = H$), for infinitely many $\mu \in \mathbb{N}$, (2.19)-(2.21) contradict the definition of t^*. Thus, we have proven that the u_ν are equicontinuous, on ∂B, with respect to $\nu \in \mathbb{N}$.

This and (2.12) imply that there is a subsequence (u_μ) of (u_ν) such that

$$u_\mu \rightarrow u,$$

in the sense of $C^0(\partial B)$ and weakly in the sense of $H^{1,2}(B)$. Now, it is easily verified that $u \in M(H, t)$ and that u satisfies (2.9).

Proof of Proposition 2. — With the aid of the mapping Φ of Lemma 2, for each $\varepsilon \in]0, \lambda/4]$ and each $s \in]0, \lambda]$, one can construct a mapping $F_{\varepsilon,s} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ that satisfies

$$F_{\varepsilon,s}(q) = q, \qquad \text{if} \quad \rho(q) \geqq s + 2\varepsilon, \tag{2.22}$$

$$\rho \circ F_{\varepsilon,s}(q) = \max\{0, \rho(q) - \varepsilon\}, \qquad \text{if} \quad \rho(q) \leqq s, \tag{2.23}$$

$$|\nabla F_{\varepsilon,s}(q)| \leqq 1 + c_1 . \varepsilon, \qquad \text{if} \quad \rho(q) \leqq s, \tag{2.24}$$

$$|\nabla F_{\varepsilon,s}(q)| \leqq c_1, \qquad \forall q \in \mathbb{R}^3, \tag{2.25}$$

for some constant $c_1 \geqq 1$ depending only on S. Hence, there is a constant $c_2 > 0$ depending only on S and an upper bound for $D(u)$ such that, for each $\varepsilon \in]0, \lambda/4]$, there is an $s_\varepsilon \in]0, \lambda]$ satisfying

$$D(u_\varepsilon) \leqq D(u) + c_2 . \varepsilon, \qquad \text{for} \quad u_\varepsilon = F_{\varepsilon, s_\varepsilon} \circ u. \tag{2.26}$$

The works cited in the Additions to Theorem 1 imply that u is a classical minimal surface, in particular that $u \in C^\infty(B)$. Hence, by Sard's theorem, there is a null-set $\Lambda \subset]0, \lambda/4]$ such that

$$\nabla(\rho \circ u) \neq 0, \qquad \text{in} \quad \{x \in B \mid \rho \circ u(x) = \varepsilon\}, \qquad \forall \varepsilon \in]0, \lambda/4] \backslash \Lambda.$$

Now, pick an $\varepsilon \in]0, \lambda/4] \backslash \Lambda$. The implicit function theorem implies that there are simply connected open subsets $U_{\varepsilon,1}, U_{\varepsilon,2}, \ldots, U_{\varepsilon,j}$ of B such that

$$\Omega_\varepsilon = B \setminus \bigcup_{j=1}^{j_\varepsilon} U_{\varepsilon,j}$$

is $(j_\varepsilon + 1)$-fold connected and that

$$\rho \circ u \leqq \varepsilon, \qquad \text{in} \quad \Omega_\varepsilon, \tag{2.27}$$

$$\rho \circ u \leqq \varepsilon, \qquad \text{on} \quad \partial\Omega_\varepsilon. \tag{2.28}$$

Moreover, we can choose the $U_{\varepsilon,j}$ in such a way that

$$\Omega_\varepsilon \subset \Omega_{\varepsilon'}, \qquad \text{if} \quad \varepsilon > \varepsilon'. \tag{2.29}$$

Let $\tau_{\varepsilon,j} : \partial B \to \partial U_{\varepsilon,j}$ be curves that parametrize $\partial U_{\varepsilon,j}$ and let $H_{\varepsilon,j} \in \Pi_0(S)$ be the homotopy classes generated by $u_\varepsilon \circ \tau_{\varepsilon,j}$. The above considerations, in particular (2.23) and (2.27) imply that

$$(H_{\varepsilon,1}, H_{\varepsilon,2}, \ldots, H_{\varepsilon,j}) \in Z_0(H). \tag{2.30}$$

By the coarea formula,

$$\int_0^{\lambda/4} l(u, \partial\Omega_\varepsilon) d\varepsilon \leqq D(u).$$

Hence, there is a $c_3 > 0$ depending only on S and an upper bound for $D(u)$ and an $\varepsilon \in]0, \lambda/4] \setminus \Lambda$ such that

$$l(u, \partial\Omega_\varepsilon) \leqq c_3. \tag{2.31}$$

Now, we set

$$t_0 = c_1 . \max(c_2, c_3).$$

By (2.31), there is an $\varepsilon_0 \in [0, \lambda/4]$ and a sequence of $\varepsilon_\nu \in [\varepsilon_0, \lambda/4] \setminus \Lambda$ tending to ε_0 such that

$$l(u, \partial\Omega_{\varepsilon_\nu}) \leqq t_0, \qquad \forall \nu \in \mathbb{N}, \tag{2.32}$$

$$l(u, \partial\Omega_\varepsilon) \geqq t_0, \qquad \forall \varepsilon \in]0, \varepsilon_0] \setminus \Lambda. \tag{2.33}$$

From (2.25), (2.26), (2.29), (2.33) and the coarea formula, we obtain the estimate

$$\begin{aligned}
1/2 . \int_{B\setminus\Omega_{\varepsilon_\nu}} |\nabla u_{\varepsilon_\nu}|^2 dx &\leqq D(u_{\varepsilon_\nu}) - 1/2 . \int_{\Omega_{\varepsilon_\nu}} |\nabla u_{\varepsilon_\nu}|^2 dx \\
&\leqq D(u) + c_2 . \varepsilon_\nu - (2 . c_1)^{-1} . \int_{\Omega_{\varepsilon_\nu}} |\nabla u|^2 dx \\
&\leqq D(u) + c_2 . \varepsilon_\nu - c_1^{-1} . \int_0^{\varepsilon_\nu} l(u, \partial\Omega_\varepsilon) d\varepsilon \\
&\leqq D(u) + c_2 . \varepsilon_\nu - \varepsilon_0 . t_0/c_1 \\
&\to D(u) + c_2 . \varepsilon_0 - \varepsilon_0 . t_0/c_1 .
\end{aligned} \tag{2.34}$$

Now, it is easily seen that (2.30), (2.32), (2.34) and the Riemann mapping theorem imply the conclusion of Proposition 2.

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