Ann. Inst. Henri Poincaré,
Vol. 2, n° 3, 1985, p. 237-260. *Analyse non linéaire*

Some remarks on quasi-variational inequalities and the associated impulsive control problem

by

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ABSTRACT. — We study Quasi-Variational Inequalities:

$$(1) \qquad \begin{cases} \text{Max}\,(\mathrm{A}u - f, u - \mathrm{M}u) = 0\,, \\ u\,|_{\partial\Omega} = \varphi\,, \end{cases}$$

where

$$(2) \qquad \mathrm{M}u = k + \inf_{\substack{\xi \geq 0 \\ x+\xi \in \bar{\Omega}}} \{ c_0(\xi) + u(x + \xi) \}\,.$$

In general, (1) has no solution, we prove here that (1) has a unique maximum subsolution that we caracterize. Then we compare the implicit obstacle (2) and the obstacle:

$$\mathrm{M}_+ u = k + \operatorname*{infess}_{\substack{\xi \geq 0 \\ x+\xi \in \Omega}} \{ c_0(\xi) + u(x + \xi) \}$$

and we finally show that, under general assumptions, the solution of (1) is Holder continuous.

Key-words: Quasi-Variational Inequalities, Implicit obstacle, maximum subsolution, Holder continuity, impulsive control.

RÉSUMÉ. — Nous étudions les Inéquations Quasi-Variationnelles :

$$(1) \qquad \begin{cases} \text{Max}\,(\mathrm{A}u - f, u - \mathrm{M}u) = 0 \quad \text{dans} \quad \Omega\,, \\ u\,|_{\partial\Omega} = \varphi\,, \end{cases}$$

où

$$(2) \qquad \mathrm{M}u = k + \inf_{\substack{\xi \geq 0 \\ x+\xi \in \bar{\Omega}}} \{ c_0(\xi) + u(x + \xi) \}\,.$$

En général (1) n'a pas de solution, nous montrons ici que (1) admet une unique sous-solution maximale que nous caractérisons. Nous comparons ensuite l'obstacle implicite (2) et l'obstacle :

$$M_+u = k + \operatorname*{infess}_{\substack{\xi \geq 0 \\ x+\xi \in \Omega}} \{ c_0(\xi) + u(x + \xi) \} ,$$

et nous finissons par montrer que, sous des hypothèses générales, la solution de (1) est hölderienne.

Mots-clefs : Inéquations Quasi-Varationnelles, obstacle implicite, sous-solution maximale, continuité Holdérienne, contrôle impulsionnel.

We study here the Quasi-Variational Inequalities (Q. V. I.) and the associated stochastic impulsive control problem:

$$\text{(1)} \qquad \begin{cases} \operatorname{Max}(Au - f, u - Mu) = 0 \quad \text{on} \quad \Omega, \\ u|_{\partial\Omega} = \varphi , \end{cases}$$

where Ω is a regular (C^3) connected open set of $\mathbb{R}^N$ and M is given by:

$$\text{(2)} \qquad Mu = k + \inf_{\substack{\xi \geq 0 \\ x+\xi \in \overline{\Omega}}} \{ c_0(\xi) + u(x + \xi) \}$$

The problem (1) was introduced in A. Bensoussan and J. L. Lions [*1*] (more recent results may be found in [*3*]). A typical result is the following if we assume $\varphi \equiv 0$, $f \geq 0$ and c_0 increasing, then (1) has a unique solution which is in $W^{1,\infty}$.

Here we will relax these assumptions and answer the following questions. If we take general φ and f, (1) has in general no solution because of a difficulty involving the boundary condition: there is no reason *a priori* that Mu should be above φ on $\partial\Omega$. And if in general there is no solution of (1), the question is to determine in which sense (1) might be solved and whether the corresponding solution is the optimal cost function for the associated impulse control problem which is meaningful without any condition.

If the solution of (1) is not continuous, then formula (2) has no clear meaning and one generally defines the implicit obstacle by:

$$M_+u = k + \operatorname*{infess}_{\substack{\xi \geq 0 \\ x+\xi \in \Omega}} \{ c_0(\xi) + u(x + \xi) \}$$

But it is not clear (and to our knowledge it has never been checked before !) that $Mu = M_+u$ (at least for $u \in C(\overline{\Omega})$). We also answer here that question.

The last question that we study is to find general local regularity results, say in $C^{0,\alpha}$, which do not involve the boundary data φ. Such a regularity has already been proved by J. Frehse and U. Mosco [9]. Here we state an analogous result but our proof is completely different from the one in [9].

In a first part we show that (1) has a unique maximal subsolution which is the solution of:

$$\text{(3)} \qquad \begin{cases} \operatorname{Max}(Au - f, u - Mu) = 0, \\ u|_{\partial\Omega} = \varphi \wedge Mu. \end{cases}$$

Here we cannot look after solutions in H^1 since this space is not adapted to the obstacle Mu and, in particular, $\varphi \wedge Mu \notin H^{1/2}(\partial\Omega)$. But under a general assumption introduced in B. Perthame [16], $\varphi \wedge Mu$ and Mu are continuous so that we can deal with continuous solution of (3) called viscosity solution. This kind of solution (a particular case of the notion introduced in M. G. Crandall and P. L. Lions [7], P. L. Lions [13] [14]) allows us to solve (3) with the classical argument of B. Hanouzet and J. L. Joly [10]. Remark that in Appendix 2 we prove equivalence between different notions of solution of the obstacle problem. In particular it appears that the viscosity solution is also the classical solution in H^1_{loc}.

We prove in a second part that this analytical solution is the one we should expect in view of the associated impulsive control problem. This is a verification analogous to the one in [3] [17] which clearly shows that the above notion is the correct one.

We give in a third part some comparison results on Mu and M_+u. There are two main results: if Ω is sufficiently smooth (C^N) then $Mu = M_+u$ a. e. and if Mu or M_+u is continuous then $Mu = M_+u$ everywhere (remark Mu and M_+u are defined pointwise and not almost everywhere).

Finally we prove that when c_0 is continuous and $c_0(\xi) \leqq C|\xi|^\alpha$, α small enough, then the solution of (1) is in $C^{0,\alpha}_{loc}$. The method is to replace Mu by an other obstacle which is in $C^{0,\alpha}_{loc}$ and which is built locally with the help of general properties of Mu. Remark that we can give an example with very regular data where u is only $W^{1,\infty}_{loc}$ (cf. B. Perthame [18]), and that in general no regularity up to the boundary holds.

I. Q. V. I. WITHOUT THE EXISTENCE OF A SUBSOLUTION

1. Assumptions and main results.

This section is devoted to the Q. V. I.:

$$\text{(1)} \qquad \begin{cases} \operatorname{Max}(Au - f, u - Mu) = 0, \\ u|_{\partial\Omega} = \varphi, \end{cases}$$

We prove (1) has a unique maximum subsolution and that it is the solution of:

$$\text{(3)}\qquad \begin{cases} \operatorname{Max}\,(\mathrm{A}u - f,\, u - \mathrm{M}u) = 0\,, \\ u\,|_{\partial\Omega} = \varphi \wedge \mathrm{M}u\,. \end{cases}$$

Here we call subsolution of (1) any $u \in \mathrm{C}(\overline{\Omega})$, satisfying:

$$\begin{cases} \mathrm{A}u \leqq f & \text{in} \quad \mathscr{D}'(\Omega)\,, \\ u \leqq \mathrm{M}u & \text{in} \quad \overline{\Omega}\,, \\ u \leqq \varphi & \text{on} \quad \partial\Omega\,, \end{cases}$$

where we have set:

$$\mathrm{A}u = -\, a_{ij}\frac{\partial^2 u}{\partial \mathrm{X}_i \partial \mathrm{X}_j} + b_i \frac{\partial u}{\partial \mathrm{X}_i} + cu\,.$$

We assume:

$$\text{(5)}\qquad a_{ij},\, b_i,\, c,\, f \in \mathrm{W}^{2,\infty}(\mathbb{R}^{\mathrm{N}})$$

$$\text{(6)}\qquad \exists \nu > 0,\quad a_{ij}\xi_i\xi_j \geqq \nu\,|\,\xi\,|^2 \quad \forall \xi = (\xi_1, \ldots, \xi_{\mathrm{N}}) \in \mathbb{R}^{\mathrm{N}}\,,$$

$$\text{(7)}\qquad \exists \lambda > 0, \qquad c \geqq \lambda\,,$$

and

$$\text{(2)}\qquad \mathrm{M}u = k + \inf_{\substack{\xi \geqq 0 \\ x+\xi \in \overline{\Omega}}} \{\, c_0(\xi) + u(x + \xi)\,\}\,,$$

where k is a positive constant, $c_0 : (\mathbb{R}^+)^{\mathrm{N}} \to \mathbb{R}^+$ is a lower semi-continuous sub-additive function with $c_0(0) = 0$ and $\xi \geqq 0$ means $\xi = (\xi_1, \ldots, \xi_{\mathrm{N}})$ with $\xi_i \geqq 0$.

We will also assume that the boundary data satisfies:

$$\text{(8)}\qquad \varphi \in \mathrm{C}(\partial\Omega)\,,$$

$$\text{(9)}\qquad \mathrm{M}_0\varphi = k + \inf_{\xi \geqslant 0,\, x+\xi \in \partial\Omega} \{\, c_0(\xi) + \varphi(x + \xi)\,\} \in \mathrm{C}(\overline{\Omega})\,,$$

These assumptions will allow us to deal with continuous solutions of (1) or (3) which are called viscosity solutions. Recall that P. L. Lions [*13*] [*14*] has introduced the notion of viscosity solution of general second order equations adapting to second order equations the notion introduced by M. G. Grandall and P. L. Lions [*7*] for first order equations. A very particular application of this notion yield that the following obstacle problem:

$$\text{(10)}\qquad \begin{cases} \operatorname{Max}\,(\mathrm{A}u - f,\, u - \Psi) = 0\,, \\ u\,|_{\partial\Omega} = \varphi \wedge \Psi\,, \end{cases}$$

has a unique viscosity solution $u \in \mathrm{C}(\overline{\Omega})$ if the obstacle $\Psi \in \mathrm{C}(\overline{\Omega})$. In this particular setting u is nothing else that the limit in $\mathrm{C}(\overline{\Omega})$ of all regularized obstacle problems with nice standard solutions. (The different notions of solution of (10) are collected in Appendix 2 where we prove the equivalence of these notions).

In the following we will call viscosity solution of (3) a continuous function u such that $u|_{\partial\Omega} = \varphi \wedge Mu$, $Mu \in C(\overline{\Omega})$ and u is a viscosity solution of (10) with $\Psi = Mu$.

Remarks. — 1) Below it will be proved that, under assumption (9) and with the definition of Mu in (2), we always have $Mu \in C(\overline{\Omega})$ when $u \in C(\overline{\Omega})$ and $u|_{\partial\Omega} = \varphi \wedge Mu$.

2) In the particular case treated here, the notion of viscosity solution is equivalent to $u \in C(\overline{\Omega})$, $Mu \in C(\overline{\Omega})$ and:

$$\begin{cases} Au \leqq f \quad \text{in} \quad \mathscr{D}'(\Omega), \quad u \leqq Mu \quad \text{on} \quad \Omega, \\ \text{on the open set } \mathcal{O} = \{ u < Mu \} \cap \Omega, \\ u \in W^{2,\infty}_{\text{loc}}(\mathcal{O}) \quad \text{and} \quad Au = f \text{ a. e.}, \end{cases}$$

moreover in Appendix 2 it is proved that $u \in H^1_{\text{loc}}$.

3) Exactly as in B. Perthame [*16*] where the assumption has been introduced to get the continuity of the solution, one easily checks that (9) may be replaced by:

$$(9') \qquad Mw \in C(\overline{\Omega}),$$

where w is any continuous super solution of (1) such that $w|_{\partial\Omega} = \varphi$.

Then we can state the:

THEOREM 1. — *Under assumptions* (5)-(9), *equation* (1) *has a unique maximal subsolution u and it is the unique solution of* (3).

If $\varphi \in W^{3,\infty}(\partial\Omega)$ *and* $M_0\varphi \in W^{1,\infty}_{\text{loc}}(\Omega)$ (*resp. is locally semi-concave*) *then* $u \in W^{1,\infty}_{\text{loc}}(\Omega)$ (*resp.* $W^{2,\infty}_{\text{loc}}$).

Let us recall that a function $u \in C(\overline{\Omega})$ is said to be locally semi-concave if for each open set $\mathcal{O} \subset \overline{\mathcal{O}} \subset \Omega$ there exist a constant C such that:

$$\frac{\partial^2 u}{\partial \chi^2} \leqq C \quad \text{in} \quad \mathscr{D}'(\mathcal{O}), \qquad \forall \chi, \qquad |\chi| = 1,$$

or, in other words: $u(x) - \frac{1}{2} C |x|^2$ is concave on convex subsets of $\mathcal{O}$.

The end of this section is devoted to the proof of Theorem 1, it is divided in three parts: first we build a decreasing process that is uniformly converging to a solution of (3) (this is a variant of the « usual » proof due to B. Hanouzet and J. L. Joly [*10*]). Here the difficulty comes from the fact that the boundary value of the process changes at each step. The concavity of the operator which associates $\varphi \wedge Mu$ to u enables us to conclude. In the second part we state the uniqueness result and finally we prove the regularity.

2. Existence of a solution.

Throughout section I we will assume that $f \geqq 0$, $\varphi > 0$. Indeed (7) enables us to make such an assumption by adding constants to u and φ and a positive function to f. We can now define the following process: $u_0 \in C(\overline{\Omega})$ is the solution of:

$$(11) \qquad \begin{cases} Au_0 = f \\ u_0 |_{\partial\Omega} = \varphi, \end{cases}$$

then we define u_1 as the solution of:

$$\begin{cases} \text{Max}\,(Au_1 - f, u_1 - Mu_0) = 0, \\ u_1 |_{\partial\Omega} = \varphi \wedge Mu_0, \end{cases}$$

and by induction,

$$\begin{cases} \text{Max}\,(Au_n - f, u_n - Mu_{n-1}) = 0, \\ u_n |_{\partial\Omega} = \varphi \wedge Mu_{n-1}. \end{cases}$$

The existence of this sequence is justified by the:

LEMMA 1. — *For each* $n \geqq 0$ $Mu_n \in C(\overline{\Omega})$.

Proof. — It is easily checked (cf. [*16*]) that under assumption (9) Mu_0 is continuous on $\overline{\Omega}$. So, we assume by induction that for $n \geqq 0$, Mu_n is continuous and we prove that Mu_{n+1} is also continuous. We know (see [*16*]) that Mu_{n+1} is lower semi-continuous. Let us prove it is upper semi-continuous.

Let: $Mu_{n+1}(x_0) = k + c_0(\xi_0) + u_{n+1}(x_0 + \xi_0)$, $\xi_0 \geqq 0$; if $x_0 + \xi_0 \in \Omega$, then, on a neighbourhood of x_0 we have:

$$Mu_{n+1}(x) \leqq k + c_0(\xi_0) + u_{n+1}(x + \xi_0)$$

and the result is proved. If $x_0 + \xi_0 \in \partial\Omega$, we will prove that:

$$(12) \qquad u_{n+1}(x_0 + \xi_0) = \varphi(x_0 + \xi_0) < Mu_n(x_0 + \xi_0),$$

and this prove that Mu_{n+1} is upper semi-continuous at x_0 since:

$$\begin{cases} Mu_n(x) \leqq M_0\varphi(x) \quad \text{on} \quad \overline{\Omega}, \\ Mu_n(x_0) = M_0\varphi(x_0). \end{cases}$$

But (12) is deduced from the:

LEMMA 2. — *If* $v \in C(\overline{\Omega})$ *and* $Mv(x_0) = k + c_0(\xi_0) + v(x_0 + \xi_0)$, $\xi_0 \geqq 0$, then $v(x_0 + \xi_0) = Mv(x_0 + \xi_0) - k$.

Indeed take $v = u_{n+1}$ in lemma 2, then:

$$u_{n+1}(x_0 + \xi_0) < Mu_{n+1}(x_0 + \xi_0) \leqq Mu_n(x_0 + \xi_0),$$

since u_n is a decreasing sequence). But $x_0 + \xi \in \Gamma$ and thus we have

$$u_{n+1}(x_0 + \xi_0) = \varphi(x_0 + \xi_0) < \mathrm{M}u_n(x_0+\xi_0),$$

and (12) is proved, concluding the proof of Lemma 1.

Proof of Lemma 2. — Notice that, if $\eta \geqq 0$, $x_0 + \xi_0 + \eta \in \overline{\Omega}$, then:

$$\begin{aligned}\mathrm{M}v(x_0) &= k + c_0(\xi_0) + v(x_0 + \xi_0)\\ &\leqq k + c_0(\xi_0 + \eta) + v(x_0 + \xi_0 + \eta)\\ &\leqq k + c_0(\xi_0) + c_0(\eta) + v(x_0 + \xi_0 + \eta)\end{aligned}$$

and, taking the infimum over all $\eta \geqq 0$, we find:

$$v(x_0 + \xi_0) \leqq \mathrm{M}v(x_0 + \xi_0) - k,$$

since equality holds for $\eta = 0$, Lemma 2 is proved.

Remarks. — 1) This also proves Remark 1 in I.1: take $u_{n+1} \equiv u_n \equiv u$ in the proof of Lemma 1.

2) The argument used above is very similar to the one introduced in L. Caffarelli and A. Friedman [*5*] [*6*].

As mentionned in the proof of Lemma 1, the sequence u_n is decreasing, but $u_n \geqq 0$ since $f \geqq 0$, $\varphi \geqq 0$, so that u_n converges to some function u; moreover we have:

PROPOSITION 1. — *The sequence u_n converges uniformly to $u \in \mathrm{C}(\overline{\Omega})$ which is a viscosity solution of (3) and $0 \leqq u \leqq u_0$.*

Proof. — The proof below is adapted from B. Hanouzet and J. L. Joly [*10*]. Notice that (3) is for example deduced from the « stability » of viscosity solutions of second order equations by uniform convergence. Let us prove the uniform convergence of u_n: choose $\mu \in]0, 1[$ such that $u \leqq \dfrac{k \wedge \varphi}{\varphi}$, $\mu \leqq \dfrac{k}{\| u_0 \|_{\mathrm{L}^\infty}}$ and assume that for some $\theta \in [0, 1]$ and some $n \geqq 0$ we have:

$$u_n - u_{n+1} \leqq \theta u_n,$$

then, since M is a concave mapping:

$$(1 - \theta)\mathrm{M}u_n + \theta.\mathrm{M}0 = (1 - \theta)\mathrm{M}u_n + \theta k \leqq \mathrm{M}u_{n+1}$$

Let us call z the solution of the obstacle problem (10) with

$$\varphi = (1 - \theta)\mathrm{M}u_n + \theta k$$

and v_0 the solution of

$$\left\{\begin{array}{l}\mathrm{Max}\,(\mathrm{A}v_0 - f, v_0 - k) = 0,\\ v_0 \,|_{\partial\Omega} = \varphi \wedge k,\end{array}\right. \tag{13}$$

then, applying the maximum principle (regularizing Mu_n if necessary) and noticing that:

$$(1-\mathcal{O})\varphi \wedge Mu_n + \mathcal{O}\varphi \wedge k \leqq \varphi \wedge \{(1-\mathcal{O})Mu_n + \mathcal{O}.k\}$$

one easily checks that:

$$(1-\mathcal{O})u_{n+1} + \mathcal{O}.v_0 \leqq z \leqq u_{n+2}.$$

But the choice of μ shows that: $\mu u_{n+1} \leqq \mu u_0 \leqq v_0$ and thus we obtain: $u_{n+1} - u_{n+2} \leqq \mathcal{O}(1-\mu)u_{n+1}$.

Since $u_0 - u_1 \leqq u_0$, iterating the above inequalities we obtain:

$$u_n - u_{n+1} \leqq (1-\mu)^n \| u_0 \|_{L^\infty},$$

and this proves Proposition 1.

Remark. — Of course this also proves that Mu_n converges uniformly to Mu hence u is a viscosity solution of (3).

3. Uniqueness.

We must prove two results: that the solution of (3) is unique and also that it is the maximum subsolution. This last point is clear, if a function $w \in C(\overline{\Omega})$ satisfies:

$$\begin{aligned} Aw &\leqq f \quad \text{in} \quad \mathcal{D}'(\Omega), \\ w &\leqq Mw \quad \text{in} \quad \overline{\Omega}, \\ w &\leqq \varphi \quad \text{on} \quad \Gamma, \end{aligned}$$

(this actually means that w is a viscosity subsolution of (1) if $Mw \in C(\Omega)$) then, recalling the result of P. L. Lions [*13*], $w \leqq u_0$ since for example w is a viscosity subsolution of (11). We deduce that:

$$\begin{aligned} &w \leqq Mw \leqq Mu_0, \\ &w \leqq \varphi \wedge Mw \leqq \varphi \wedge Mu_0 \quad \text{on} \quad \Gamma, \end{aligned}$$

and so $w \leqq u_1$. An easy induction proves that $w \leqq u_n$ for each $n \geqq 0$ and so $w \leqq u$.

PROPOSITION 2. — *Under the assumptions of theorem 1, if $w \in C(\overline{\Omega})$ is a viscosity solution of (3), then $w = u$.*

Proof. — We already know that $w \leqq u$. First we show that $w \geqq 0$: we know (by the remark in I 2°) that $Mw \in C(\overline{\Omega})$ so we can build a sequence $\Psi_n \in C^\infty(\overline{\Omega})$ such that $\Psi_n \underset{n\to\infty}{\rightrightarrows} Mw$ in $C(\overline{\Omega})$. Let us denote by w_n the solution of (10) for $\Psi = \Psi_n$ so that $w_n \underset{n\to\infty}{\longrightarrow} w$ in $C(\overline{\Omega})$ and $w_n \in W^{2,\infty}_{loc}(\Omega)$ (see [*16*]). Take $x_n \in \overline{\Omega}$ so that: $\underset{x\in\Omega}{\text{Min}}\, w_n(x) = w_n(x_n)$; as

$$w_n(x_n) \underset{n\to\infty}{\longrightarrow} \underset{x\in\Omega}{\text{Min}}\, w(x) < \underset{x\in\Omega}{\text{Min}}\, Mw(x),$$

we may assume (at least if n is large enough) that $w_n(x_n) < Mw_n(x_n)$.

If $x_n \in \Gamma$ we deduce that $w_n(x_n) = \varphi(x_n) \geqq 0$, if not the maximum principle shows that $w_n(x_n) \geqq 0$ and in both cases $w_n \geqq 0$ and thus $w \geqq 0$.

We now prove Proposition 2 with the help of B. Hanouzet and J. L. Joly method. We have:

$$u - w \leqq u,$$

by the same arguments as in Proposition 1 we get

$$u - w \leqq (1-\mu)^n u, \qquad \forall n \geqq 0,$$

and so $u \leqq w$ and Proposition 2 is proved.

4. **Regularity.**

Here we assume that $M_0\varphi \in W^{1,\infty}_{loc}(\Omega) \cap C(\overline{\Omega})$ (Resp. that $M_0\varphi$ is locally semi-concave) and we prove that $u \in W^{1,\infty}_{loc}(\Omega)$ (resp. $W^{2,\infty}_{loc}(\Omega)$).

We only sketch the proof since it is nearly the same as in [*16*]. One introduces:

$$F = \{ y \in \Omega \mid \exists x \leqq y, Mu(x) = k + c_0(y-x) + u(y) \},$$

and one easily proves that there exists some open set G (for the topology of $\overline{\Omega}$) such that:

$$F \subset G \subset \overline{G} \subset \{ u < Mu \}$$

and so $u \in W^{1,\infty}(\mathring{G})$ (resp. $u \in W^{2,\infty}(\mathring{G})$). Now, as Mu « takes its values » in F and with the regularity of $M_0\varphi$ one can prove that Mu is in $W^{1,\infty}_{loc}(\Omega)$ (resp. locally semi-concave) and applying classical results on the obstacle problem this proves that $u \in W^{1,\infty}_{loc}(\Omega)$ (Resp. $u \in W^{2,\infty}_{loc}(\Omega)$), and the proof of Theorem 1 is completed.

Remarks. — 1) In particular if Ω is convex the solution belongs to $W^{2,\infty}_{loc}$ when $\varphi \in W^{3,\infty}(\partial\Omega)$. Indeed $Mu_0 \in C(\Omega)$ and is locally semi-concave and this is enough by remark 3 in I.1 (cf. [*16*], [*18*]).

2) Of course these results extend to Hamilton-Jacobi-Bellman equations (see P. L. Lions [*12*], L. C. Evans and P. L. Lions [*8*]). The associated Q. V. I. is:

$$\begin{cases} \text{Max}\,(\underset{1\leq i\leq m}{\text{Max}}\,(A^i u - f^i), u - Mu) = 0, \\ u|_{\partial\Omega} = \varphi. \end{cases}$$

(For results on this problem see S. Lehnart [*11*], B. Perthame [*16*]).

II. THE IMPULSIVE CONTROL PROBLEM

Our purpose, here, is to give the stochastic interpretation of u, the solution of (3), in terms of control of diffusion processes. Two remarks are to be developped below: first, despite of the jumps the process is constrained to stay in $\overline{\Omega}$; next, the change of boundary data (from φ to $\varphi \wedge Mu$) does not induce any change on the optimal cost function.

1. The optimal cost function.

Let (Ω, F, F_t, P, w_t) be a standard space composed by a probability space (Ω, F, P) with a right-continuous increasing filtration of complete sub-σ algebra F_t and a Browian motion w_t in $\mathbb{R}^N$, F_t adapted. For any sequences $\mathcal{O}^1 < \mathcal{O}^2 < \ldots < \mathcal{O}^n < \ldots, \mathcal{O}^n \xrightarrow[n\to\infty]{} \infty$, of stopping times and $\xi^1, \ldots, \xi^n, \ldots$ of $F_{\mathcal{O}_n}$-random variables in $(\mathbb{R}^+)^n$ we can define by induction:

$$(14) \qquad \begin{cases} dy_x^n(t) = \sigma(y_x^n(t))dw_t - b(y_x^n)(t))dt, \ t \geqq \mathcal{O}^n \\ y_x^n(\mathcal{O}^n) = y_x^{n-1}(\mathcal{O}^n) + \xi^n , \end{cases}$$

and:

$$y_x^0(0) = x$$

we set:

$$y_x(t) = y_t = y_x^n(t), \qquad \mathcal{O}^n \leqq t < \mathcal{O}^{n+1} .$$

We will say that such a system $\mathscr{A}$ is admissible if $y_x(\mathcal{O}^n) \in \overline{\Omega}$ whenever $y_x^{n-1}(\mathcal{O}^n) \in \overline{\Omega}$. Then we define

$$\tau = \inf \{ t \geqq 0, y_x(t) \notin \overline{\Omega} \} ,$$

and the cost function of this system is:

$$(15) \quad j(x, \mathscr{A}) = \mathrm{E} \left\{ \int_0^\tau f(y_s)e^{-\lambda s}ds + \sum_{n=1}^{\infty} (k + c_0(\xi^n))e^{-\lambda \mathcal{O}^n} + \varphi(y_x(\tau))e^{-\lambda\tau} \right\}$$

and the optimal cost function is:

$$(16) \qquad v(x) = \inf_{\mathscr{A} \text{ adm.}} j(x, \mathscr{A}) .$$

Remark. — 1) With our definition of admissible systems we know that $y_x(\tau) \in \partial\Omega$, and so (15) is meaningful.

2) Here we have taken $c(x) \equiv \lambda$ to simplify notations but the following results still hold with any $c(x) > 0$.

THEOREM 2. — *Under the assumptions of Theorem 1, the solution u of (3) is the optimal cast function given by (16)*

Proof. — *i*) $u \leqq v$: This is clear in view of A. Bensoussan and J. L. Lions [*4*] or P. L. Lions [*13*]: u is solution of the Q. V. I. (1) with $\tilde{\varphi} = \varphi \wedge \mathrm{M}u$ so that it is the optimal cost function for a cost function defined by (15) with $\tilde{\varphi}$ in place of φ. These cost functions are less than $j(x, \mathscr{A})$ since $\tilde{\varphi} \leqq \varphi$ so that $u \leqq v$.

ii) $u \geqq v$: Here we give an optimal admissible system as in [*3*] [*17*]. Let $\hat{\xi}$ be measurable function in $(\mathbb{R}^+)^{\mathrm{N}}$ such that:

$$\mathrm{M}u(x) = k + c_0(\hat{\xi}(x)) + u(x + \hat{\xi}(x)) \qquad \forall x \in \overline{\Omega}.$$

we can find a standard space $(\Omega, \mathrm{F}, \mathrm{F}_t, \mathrm{P}, w_t)$ where we solve:

$$\begin{cases} dy_x^0(t) = \sigma(y_x^0(t))dw_t + b(y_x^0(t))dt, \\ y_x^0(0) = x, \end{cases}$$

let $\hat{\mathcal{O}}_1$ be the first time when y_x^0 belongs to $\mathscr{C} = \{ u = \mathrm{M}u \}$ ($\hat{\mathcal{O}}^1 = +\infty$ if $\mathscr{C}$ is not reached), $\hat{\xi}_1 = \hat{\xi}(y_x^0(\mathcal{O}_1))$ ($\hat{\xi}_1$ is anything measurable if $\hat{\mathcal{O}}_1 = +\infty$) and by induction

$$\begin{cases} dy_x^n(t) = \sigma(y_x^n(t))dw(t) + b(y_x^n(t))dt, \\ y_x^n(\hat{\mathcal{O}}^n) = y_x^{n-1}(\hat{\mathcal{O}}^n) + \hat{\xi}^n, \end{cases}$$

and $\hat{\mathcal{C}}_{n+1}$ is the first time y_x^n enters $\mathscr{C}$ and $\hat{\xi}^{n+1} = \hat{\xi}(y_x^n(\hat{\mathcal{O}}^{n+1}))$.

From [*2*] we know the:

LEMMA 3. — *For any* $n \geqq 0$, *one has*:

$$u(y_x(\hat{\mathcal{O}}^n \wedge \tau))e^{-\lambda\hat{\mathcal{O}}^n \wedge \tau} = \mathrm{E}\left\{ \int_{\hat{\mathcal{O}}^n \wedge \tau}^{\hat{\mathcal{O}}^{n+1} \wedge \tau} f(y_x(s))e^{-\lambda s}ds + u(y_x^-(\hat{\mathcal{O}}^{n+1} \wedge \tau))e^{-\lambda\hat{\mathcal{O}}^{n+1} \wedge \tau} \mid \mathrm{F}_{\hat{\mathcal{O}}^n} \right\} \quad \text{a. e.} \tag{17}$$

Remarks. — 1°) $y_x(\hat{\mathcal{O}}^n)$ and $y_x(\tau)$ are in the set $\{ u < \mathrm{M}u \}$, indeed whenever $y_x(s)$ reaches $\{ u = \mathrm{M}u \}$ it jumps to come back in $\{ u < \mathrm{M}u \}$. In particular $u(y_x(\tau))1_{(\tau<\infty)} = \varphi(y_x(\tau))1_{(\tau<\infty)}$.

2°) One checks as in [*3*] that $\hat{\mathcal{O}}^n \underset{n\to\infty}{\longrightarrow} +\infty$; indeed for each n, $u(y_x^{n-1}(\hat{\mathcal{O}}^n)) - u(y_x^{n-1}(\hat{\mathcal{O}}^n) + \hat{\xi}_n) \geqq k$, and, since u is continuous, $|\hat{\xi}^n| \geqq \mathrm{L} > 0$. Since $\hat{\xi}^n \geqq 0$ we obtain $\hat{\xi}^1 + \ldots + \hat{\xi}^n \geqq \mathrm{L}\sqrt{n}$ and the formula:

$$y_x(\hat{\mathcal{O}}^n) = x + \int_0^{\hat{\mathcal{O}}^n} \sigma(y_x(s))dw_s - \int_0^{\hat{\mathcal{O}}^n} b(y_x(s))ds + \hat{\xi}^1 + \ldots + \hat{\xi}^n$$

on $\{ \hat{\mathcal{O}}^n < \infty \}$ shows that $\hat{\mathcal{O}}^n \underset{n\to\infty}{\longrightarrow} +\infty$.

In (17) we remark that:

$$u[y_x^-(\hat{\mathcal{O}}^{n+1} \wedge \tau)]1_{(\hat{\mathcal{O}}^n \leqq \tau)} = u[y_x(\hat{\mathcal{O}}^{n+1})]1_{(\hat{\mathcal{O}}^{n+1} \leqq \tau)} + [k + c_0(\xi^{n+1})]1_{(\hat{\mathcal{O}}^{n+1} \leqq \tau)} + u[y_x(\tau)]1_{(\hat{\mathcal{O}}^n \leqq \tau < \hat{\mathcal{O}}^{n+1})},$$

but we have:

$$e^{-\lambda\hat{\mathcal{C}}^{n+1}}1_{\hat{\mathcal{O}}^{n+1}\leqq\tau}=e^{-\lambda\hat{\mathcal{C}}^{n+1}}1_{\hat{\mathcal{C}}^{n+1}\leqq\tau<\infty}=e^{-\lambda\hat{\mathcal{O}}^{n+1}}1_{\hat{\mathcal{O}}^{n+1}<\infty}=e^{-\lambda\hat{\mathcal{O}}^{n+1}}$$

since $\hat{\mathcal{O}}^n \leqq \tau$ whenever $\hat{\mathcal{O}}^n < \infty$. Thus adding (17) for each $n \geqq 0$ and using remark 2, we get

$$u(x)=\mathrm{E}\left\{\int_0^\infty f(y_x(s))e^{-\lambda s}ds+\sum_{n=1}^{\infty}[k+c_0(\hat{\xi}^n)]e^{-\lambda\hat{\mathcal{O}}^n}+\varphi(y_x(\tau))e^{-\lambda\tau}\right\},$$

and this means that $u \geqq w$ and theorem 2 is proved.

III. SOME REMARKS ON THE OPERATOR M

The operator M which defines the Q. V. I. is not always given by (2). Indeed when one deals with discontinuous solution the operator Mu has no clear meaning and the Q. V. I. is generally defined with an operator M_+:

$$M_+u = k + \underset{\substack{\xi\geqq 0\\ x+\xi\in\Omega}}{\operatorname{infess}} \{ c_0(\xi) + u(x+\xi) \}.$$

From a stochastical viewpoint it is also natural to consider admissible systems such that the jumps ξ at a point x are constrained to satisfy $x+\xi\in\Omega$ and no longer $x+\xi\in\overline{\Omega}$. It will rise an other operator, defined for $x\in\Omega$:

$$M_-u = k + \inf_{\substack{\xi\geqq 0\\ x+\xi\in\Omega}} \{ c_0(\xi) + u(x+\xi) \}.$$

M_+ is defined for any bounded from below measurable function and M_- for $u\in C(\Omega)$. Here we give the relation between M, M_-, M_+.

1. Regularity of M, M_-, M_+.

PROPOSITION 3. — *Let* $u\in C(\overline{\Omega})$ *then* Mu *is lower semi-continuous on* $\overline{\Omega}$ (l.s.c.), M_-u *is upper semi-continuous* (u. s. c.) *and* $Mu \leqq M_-u$ *on* Ω.

Proof. — The regularity of Mu and the inequality are clear. The other results are based on the following remark: take $u_n\in C(\overline{\Omega})$ a decreasing sequence such that $u_n \underset{n\to\infty}{\longrightarrow} u$ pointwise on Ω, then M_-u_n decreases to M_-u. Now if we choose u_n such that $u_{n|\partial\Omega} = \| u_n \|_{L^\infty} > \| u \|_{L^\infty}$ and $u_n < \| u_n \|_{L^\infty}$ on Ω, then it is easily checked that M_-u_n is continuous and so, that Mu is u. s. c.

PROPOSITION 4. — *If* u *is* u. s. c. *then* M_-u *is* u. s. c. *moreover, if* c_0 *is continuous and* u *is* u. s. c. *then* $M_-u = M_+u$ *on* Ω.

Proof. — The proof of Proposition 3 directly proves the first part of

Proposition 4. Moreover, if c_0 is continuous and u_n is chosen as above it is clear that $M_+u_n = M_-u_n$ so that:

$M_-u \leqq M_+u \leqq M_+u_n \xrightarrow[n\to\infty]{} M_+u$ on Ω, and Proposition 4 is proved.

2. Relation between M and M_- (or M_+).

PROPOSITION 5. — *If c_0 is continuous, $u \in C(\overline{\Omega})$ and Ω is of class C^N then* $Mu = M_+u$ a. e.

Proof of Proposition 5. — Let us introduce some notations: F will be the set of points $x \in \Omega$ where possibly $Mu(x) \neq M_+u(x)$, and will denote by $T\Gamma(y)$ the tangent hyperplane to $\partial\Omega$ at y and $\nu(y)$ the unit outward normal to $\partial\Omega$ at y. Then we have:

$F \subset G = \{ x \in \Omega, \exists y > x, y \in \partial\Omega, (e_i, y-x) = 0$ for some i and $x \in T\Gamma(y) \}$,

where $(e_i)_{1 \leqq i \leqq N}$ is the canonical basis of $\mathbb{R}^N$. This result has been proved in [*15*], but it is not precise enough. In fact we have $F \subset \bigcup_{n<N} G_n$ where:

$G_1 = \{ x \in \Omega, \exists y \in \partial\Omega, y > x, \exists i \leqq N$ such that:

$$(e_i, y - x) = 0, \nu(y) \parallel e_i, (e_j, y - x) \neq 0 \quad \forall j \neq i \}$$

$G_k = \{ x \in \Omega, \exists y \in \partial\Omega, y > x, \exists i_1 < i_2 < \ldots < i_k$ such that:

$$(e_{i_j}, y-x) = 0, (e_i, y-x) \neq 0 \ i \neq i_j, 1 \leqslant j \leqq k; \qquad \nu(y) \in \text{vect}\,(e_{i_1}, \ldots, e_{i_k}) \}.$$

Let us prove that $F \subset \bigcup_{n<N} G_n$: take some $x_0 \in F$, as $x_0 \in G$ there exists $y > x$, $y \in \partial\Omega$ and some e_{i_1} such that $(e_{i_1}, y - x) = 0$. We may always choose $e_{i_2} \ldots e_{i_k} \ldots e_{i_n}$ such that: $(e_{i_j}, y-x) = 0$ for $1 \leqq j \leqq k$, $(e_{i_j}, y-x) \neq 0$ for $j > k$. Now assume that $x \notin G_k$, it means that for some $1 > k$, $(e_{i_l}, \nu(y)) \neq 0$. Since $y > x$ then $y \pm \varepsilon\, e_{i_l} > x$ for ε small enough, and either $y - \varepsilon\, e_{i_l} \in \Omega$, or $y + \varepsilon\, e_{i_l} \in \Omega$ for ε small enough. In both cases we get that $M_+u(x_0) = Mu(x_0)$, indeed we know that $M_+u(x_0) \geqq Mu(x_0)$ and

$$\begin{aligned} M_+u(x_0) &\leqq k + c_0(y + \varepsilon\, e_{i_l} - x_0) + u(y + \varepsilon e_{i_l}) \\ &\leqq k + c_0(y - x_0) + u(y) \qquad (\text{as} \quad \varepsilon \to 0) \\ &\leqq Mu(x_0). \end{aligned}$$

This yields a contradiction and thus $x \in G_k$. We now conclude with the:

LEMMA 4. — *For* $1 \leqq k \leqq N$, *meas* $(G_k) = 0$.

STEP 1. — We prove that G_1 is of zero measure and it is enough to prove that

$$H_1 = \{ x \in \Omega, \exists y \in \partial\Omega, y > x,$$

$\nu(y)$ is parallel to e_N and $(e_N, y - x) = 0\}$ is of zero mesure. Since the set of $y \in \partial\Omega$ such that $\nu(y)$ is parallel to e_N is compact, it can be covered by a finite number of open sets $(\Gamma_i)_{i\in I}$

$$\Gamma_i = \{ y, y_N = f_i(y_1 \ldots y_{N-1}), \quad (y_1 \ldots y_{N-1}) \in \mathcal{O}_i \}$$

where $f_i \in C^N$ and $\mathcal{O}_i$ is an open set.

Denoting $y' = (y_1 \ldots y_{N-1})$ it remains to prove that for these functions $f_i \in C^N(\theta_i)$ the sets:

$$\begin{gathered} H = \{ x \in \Omega, \exists y = (y', y_N), y > x, y_N = f_i(y') \\ D f_i(y') = 0 \quad \text{and} \quad (e_N, y - x) = 0 \} \end{gathered}$$

is of zero measure. But:

$$H \subset \{ y + (\alpha', 0), | \alpha' | \leqq \operatorname{diam}(\Omega), \quad y \in \partial\Omega$$

and
$$y_N = f_i(y') \quad \text{and} \quad \nu(y) = \pm e_N \} .$$

Denoting by $m_{\mathbb{R}}$ the Lebesgue measure on $\mathbb{R}$ we have, applying Sard's Theorem:

$$m_{\mathbb{R}} \{ z = f_i(y'), y' \in \mathcal{O}_i, D f_i(y') = 0 \} = 0 ,$$

thus H is of zero measure and we conclude the first step: meas $(G_1) = 0$.

STEP 2. — In the same way we prove that G_k is of zero measure for $k > 1$: it is enough to prove that H_k is of zero measure where:

$$H_k = \{ x \in \Omega, \exists y \in \partial\Omega, y > x, \nu(y) \in \operatorname{Vect}(e_N, \ldots, e_{N-k+1})$$

and
$$(e_j, y - x) = 0 \ \forall j, \ N - k + 1 \leqq j \leqq N \} .$$

By the implicit function Theorem and changing e_N into some $e_j, j \geqq N - k + 1$, if necessary we have locally:

$$\Gamma_i = \{ y, y_N = f_i(y'), y' \in \mathcal{O}_i \} ,$$

and it is enough to study the set:

$$\begin{gathered} H = \{ x \in \Omega, \exists y > x, y_N = f_i(y'), \frac{\partial f_i}{\partial y_j} = 0 \\ \forall j \leqq N - k ; (e_j, y - x) = 0 \ \forall j > N - k \}. \end{gathered}$$

But its measure is less than:

$$(2 \operatorname{diam} \Omega)^{N-k} m_{\mathbb{R}^k} \left\{ (f_i(y'), y_{N-1}, \ldots, y_{N-k+1}), y' \in \mathcal{O}_i \text{ and } \frac{\partial f_i}{\partial y_j} = 0 \ \forall j \leqq N - k \right\},$$

this last term is the measure of the critical values of the function

$$F(y_1 \ldots y_{N-1}) = (f_i(y_1 \ldots y_{N-1}), y_{N-1}, \ldots, y_{N-k+1})$$

and so is zero by Sard's theorem. This concludes the proof of Lemma 4 and of Proposition 5.

It is quite easier to get the following comparison Proposition in which we consider $M_{-}u$ and Mu are defined for $x \in \Omega$:

PROPOSITION 6. — *If* $u \in C(\bar{\Omega})$, *and* c_0 *is a non-negative, sub-additive function such that* $c_0(0) \equiv 0$, *then the set of discontinuity points in* Ω *of* $M_{-}u$ *is also the set of discontinuity points in* Ω *of* Mu *and it is also the set of points where* $M_{-}u(x) \neq Mu(x)$.

COROLLARY. — *If* $u \in C(\bar{\Omega})$, $M_{-}u = Mu$ a. e. *in Baire Sense.*

This is because we know that the set of discontinuity of a l. s. c. (or u. s. c.) function is rare in Baire sense.

Proposition 6 is clearly deduced from the following:

LEMMA 5. — *Under the assumptions of Proposition 6*:

$$\liminf_{x \to x_0} M_{-}u(x) = Mu(x_0), \qquad \limsup_{x \to x_0} Mu(x) = M_{-}u(x_0).$$

Let us prove the second equality which is the most difficult one: we know that $\limsup_{x \to x_0} Mu(x) \leqq M_{-}u(x_0)$, and we shall prove that:

$$\liminf_{\substack{x \to x_0 \\ x > x_0}} Mu(x) \geqq M_{-}u(x_0).$$

Take $x_n > x_0$ and $x_n \underset{n \to \infty}{\longrightarrow} x_0$, $Mu(x_n) = k + c_0(\xi_n) + u(x_n + \xi_n)$, $\xi_n \geqq 0$. Extracting a subsequence if necessary we assume: $\xi_n \underset{n \to \infty}{\longrightarrow} \xi$. Since $x_n + \xi_n - x_0 \geqq x_n - x_0 > 0$, we can find an $\varepsilon_n \in \mathbb{R}^N$, $\varepsilon_n \underset{n \to \infty}{\longrightarrow} 0$, $x_n - x_0 + \xi_n + \varepsilon_n \geqq 0$ and $x_0 + \xi_n + \varepsilon_n \in \Omega$. We have then:

$$\begin{aligned} M_{-}u(x_0) &\leqq c_0(x_n - x_0 + \xi_n + \varepsilon_n) + u(x_n + \xi_n + \varepsilon_n) \\ &\leqq \liminf_{n \to \infty} \{ c_0(x_n - x_0 + \varepsilon_n) + c_0(\xi_n) + u(x_n + \xi_n) \} \\ &\leqq \liminf_{n \to \infty} Mu(x_n) \end{aligned}$$

Remark. — 1) Here we have used the three properties of c_0: c_0 is subadditive, $c_0(\xi) \underset{\xi \to 0}{\longrightarrow} 0$ and c_0 is l. s. c. To prove the first equality of Proposition 6 we only need c_0 to be l. s. c.

2) A useful consequence of Proposition 6 is that the solution of (3) is also the solution of:

$$\begin{cases} \text{Max}\,(Au - f, u - M_{+}u) = 0 \\ u\,|_{\partial\Omega} = \varphi \wedge M_{+}u, \end{cases}$$

in other words u is also the solution of the impulsive control problem where the admissible systems are defined in such away that the process $y_x(t)$ stays in Ω (and no longer in $\bar{\Omega}$) and the stopping time τ is the first exit time from Ω (and no longer from $\overline{\Omega}$). This remark also applies to the results proved in [*16*] for Q. V. I. associated to Hamilton-Jacobi-Bellman equations.

IV. HOLDER CONTINUITY OF THE SOLUTION OF Q. V. I.

In this section we prove a general Holder continuity result for the Q. V. I.:

$$(18) \quad \begin{cases} a(u, v-u) \geqq (f, v-u), \\ \forall v \in \mathrm{H}_0^1,\ v \leqq \mathrm{M}_+ u\ ;\ u \in \mathrm{H}_0^1,\ u \leqq \mathrm{M}_+ u, \end{cases}$$

where $f \geqq 0$ and $a(.,.)$ is the bilinear form associated with A:

$$(\mathrm{A}u, v) = a(u, v)$$

Indeed if we do not assume (9) the general theory developped in A. Bensoussan and J. L. Lions [*3*] shows that (18) has a unique solution in H_0^1, but it is not continuous in general since our assumptions on c_0 (see (2)) are too weak. Here we give a proof of the Holder continuity of u under assumption:

$$(19) \qquad c_0 \text{ is continuous on } (\mathbb{R}^+)^{\mathrm{N}}, \qquad c_0(\xi) \leqq \mathrm{C}\,|\,\xi\,|^{\alpha}, \qquad \forall \xi \geqq 0,$$

where α is an exponent to be precised in Theorems 3 and 4. Such an assumption has already been introduced in A. Bensoussan, J. Frehse and U. Mosco [*4*], or J. Frehse and U. Mosco [*9*] to prove the continuity of Q. V. I. with quadratic growth. The method used here is completly different of the one in [*4*] [*9*]. It consists in interpreting u as the solution of an obstacle problem where the obstacle Ψ has some special property. Using a regularity result for linear second order elliptic equations on open sets satisfying an exterior cone condition we prove that the solution is Holder continuous. The precise result is the following:

THEOREM 3. — *Under assumptions* (*5*)-(*7*) *there exist some* α_0, $0 < \alpha_0 < 1$, *such that if* (*19*) *holds with* $0 < \alpha < \alpha_0$ *then the solution of* (*18*) *belongs to* $\mathrm{C}^{0,\alpha}_{\mathrm{loc}}(\Omega)$.

The proof of this Theorem is divided in two parts: first we prove that the solution u of (18) is u. s. c., then we prove the Hölder continuity of u.

Remark. — Of course this result extends to the more general case where f is eventually negative but (18) has a subsolution. It also extends to any boundary data.

1. Selection of an u. s. c. solution.

Here we make the assumptions of Theorem 3. We know that u is obtained as the limit of the decreasing process as in I: u_0 is the solution of $\mathrm{A}u_0 = f$, $u_0 \in \mathrm{C}(\overline{\Omega})$. Then u_n is the solution of:

$$\begin{cases} a(u_n, v-u_n) \geqq (f, v-u_n), \\ u_n \in \mathrm{H}_0^1,\ u_n \leqq \mathrm{M}_+ u_{n-1}\ ;\quad \forall v \in \mathrm{H}_0^1 \quad v \leqq \mathrm{M}_+ u_{n-1}\,. \end{cases}$$

At each step we can select a u. s. c. solution u_n. Indeed if u_{n-1} is u. s. c. then $M_-u_{n-1} = M_+u_{n-1}$ by Proposition 5 since c_0 is continuous (in particular it doesn't depend on the choice of the u. s. c. selection) and we know M_-u_{n-1} is u. s. c. Now take a decreasing sequence of obstacles $\Psi_p \searrow_{p\to\infty} Mu_{n-1}$ the solution u^p of the obstacle problem for Ψ_p is continuous and decreases with p to u_n (moreover $u^p \underset{p\to\infty}{\rightharpoonup} u_n$ in H_0^1 by the general theory in [3]) so that we can select an u. s. c. solution u_n at each step.

Finally since u_n is decreasing to u, u has a u. s. c. representant.

2. Holder continuity of u in theorem 3.

Take a sequence $u^n \searrow_n u$ in Ω, $u^n \in C(\overline{\Omega})$ and assume $u^n|_\Gamma = \| u^n \|_{L^\infty}$ is large enough so that M_-u^n is continuous and $M_-u^n \searrow M_-u$. We prove in what follows that each solution v_n of the obstacle problem for $\Psi = M_-u^n$ belongs to $C^{0,\alpha}_{loc}$ with uniform bounds on compact subsets of Ω.

For any $x_0 \in \Omega$ we denote by $d(x_0)$ the function $d(x_0) = \inf\limits_{y\in\partial\Omega} d(x_0, y)$. Since c_0 is subadditive and by (19) we have for all $y \in \Omega$, $y \leqq x_0$:

$$\begin{aligned} M_-u^n(y) &\leqq M_-u^n(x_0) + c_0(x_0 - y) \\ &\leqq M_-u^n(x_0) + C\,| x_0 - y |^\alpha \\ &\leqq M_-u^n(x_0) + \operatorname{Max}\left(C, \frac{\| u_0 \|_{L^\infty}}{d^\alpha(x_0)}\right) | x_0 - y |^\alpha . \end{aligned}$$

In the following we will denote by:

$$C(x_0) = \operatorname{Max}\left(C, \frac{\| u_0 \|_{L^\infty}}{d^\alpha(x_0)}\right)$$

and we define two open sets:

$$\begin{aligned} Q(x_0) &= \{ y \in \Omega, y < x_0, | y - x_0 | < d(x_0) \} \\ K(x_0) &= \{ y \in \Omega, | y - x_0 | < d(x_0), y \notin \overline{Q}(x_0) \} . \end{aligned}$$

By the appendix 1 if α is small enough $\left(\alpha \leqq \alpha\left(\frac{\sqrt{2}}{2}\right) \text{ in Theorem 4}\right)$ we can solve:

$$\begin{cases} Av^1 = f,\ v^1 \in C^{0,\alpha}(\overline{Q(x_0)}) \cap C^\infty(Q(x_0)), \\ v^1 |_{\partial Q(x_0)} = Mu^n(x_0) + C(x_0)\,| x_0 - y |^\alpha , \end{cases}$$

$$\begin{cases} Av^2 = f,\ v^2 \in C^{0,\alpha}(K(x_0)) \cap C^\infty(K(x_0)), \\ v^2 |_{\partial K(x_0)} = Mu^n(x_0) + c(x_0)\,| x_0 - y |^\alpha , \end{cases}$$

then we set:

$$v_{x_0} = \begin{cases} v^1 & \text{on} \quad \overline{Q(x_0)} \\ v^2 & \text{on} \quad \overline{K(x_0)}, \\ Mu^n(x_0) + C(x_0)d^\alpha(x_0) & \text{on} \quad \Omega \backslash (Q(x_0) \cup K(x_0)). \end{cases}$$

The Appendix 1 shows that $v_{x_0}(y)$ is in $C^{0,\alpha}$ if α is small enough and $\| v_{x_0} \|_{C^{0,\alpha}} \leqq D(x_0)$, where $D(x_0)$ only depends on $d(x_0)$, and so the obstacle Ψ defined below is in $C^{0,\alpha}_{loc} \cap C(\overline{\Omega})$:

$$\Psi = (\inf_{x_0 \in \Omega} v_{x_0}) \wedge u_0,$$

and we conclude by the:

LEMMA 6. — *The solution v_n of the obstacle problem for the obstacle* $M_- u^n$ *is the solution of*:

$$\begin{cases} \text{Max}\,(Aw - f, w - \Psi) = 0, & w \in C(\overline{\Omega}) \cap C^{0,\alpha}(\Omega), \\ w\,|_{\partial\Omega} = 0. \end{cases}$$

Proof. — For any $x \in \Omega$ we have:

$$w(x) \leqq \Psi(x) \leqq v_x(x) = Mu^n(x),$$

hence w is a subsolution of (18) and $w \leqq v_n$. On the other hand for any $x_0 \in \Omega$ we have:

$$\begin{cases} Aw_n \leqq f & \text{on} \quad Q(x_0) \\ v_n\,|_{\partial Q(x_0)}(y) \leqq Mu^n\,|_{\partial Q(x_0)}(y) \leqq Mu^n(x_0) + C(x_0)\,|\,x_0 - y\,|^\alpha, \end{cases}$$

$$\begin{cases} Av_n \leqq f \quad \text{on} \quad K(x_0), \\ v_n\,|_{\partial K(x_0)}(y) \leqq Mu^n(y) \leqq Mu^n(x_0) + C(x_0)\,|\,x_0 - y\,|^\alpha, \quad \text{if} \quad y \leqq x_0, \\ v_n\,|_{\partial K(x_0)}(y) \leqq \|\,u_0\,\|_{L^\infty} \leqq C(x_0)\,|\,x_0 - y\,|^\alpha \quad \text{if} \quad y > x_0. \end{cases}$$

This implies $v_n \leqq v_{x_0}$, $\forall x_0$ i.e. $v_n \leqq \Psi$ and then $v_n \leqq w$; and Lemma 6 is proved.

Now we deduce Theorem 3 from Lemma 6: we notice that the Hölder estimates on v_n do not depend on u^n and since v_n converges to u, Theorem 3 is proved.

APPENDIX I

CONSTRUCTION OF A BARRIER FUNCTION FOR AN OPEN SET SATISFYING THE EXTERIOR CONE CONDITION

In section IV we need to solve linear elliptic second order equations for open sets satisfying only the exterior cone condition. The purpose of this appendix is to show how to construct a barrier function in such cases (which do not seem to be classical). The results below are due to K. Miller *[15]* where one can find another way to prove them.

In this section we are given an operator A and a function f satisfying assumptions (and notations) of section I. We denote $B = \| bi \|_{L^\infty(\mathbb{R}^N)}$ and by C_μ the cone:

$$C_\mu = \left\{ x, \frac{x_N}{r} < -\mu \right\}, \qquad 0 \leqq \mu < 1,$$

where $r = |x|$, and we construct a barrier function at the point 0 for the open set:

$$\mathcal{O} = \{ x, |x| < R_0 \} \backslash \overline{C}_\mu .$$

The existence of a barrier function and its regularity are given in the:

THEOREM 4. — *Let $\mu \in [0, 1[$ then there exist $\alpha \in]0, 1[$ such that for $h \in C^{0,\alpha}(\mathbb{R}^N)$ there is a function $v \in C^\infty(\mathcal{O}) \cap C^{0,\alpha}(\overline{\mathcal{O}})$ such that*:

$$\left\{ \begin{array}{l} Av \geqq f, \\ v|_{\partial\mathcal{O}} \geqq h|_{\partial\mathcal{O}}, \qquad v(0) = h(0), \end{array} \right. \tag{20}$$

moreover α and $\| v \|_{C^{0,\alpha}}$ only depends on R_0, μ, A, f and $\| h \|_{C^{0,\alpha}}$.

Proof. — We may assume $h(0) = 0$ and we look for v satisfying:

$$v(x) = r^\alpha g\left(\frac{x_N}{r}\right), \qquad 0 < \alpha < 1 .$$

We will divide the proof in three parts: computation of Av, lower bound for Av, existence of g and α.

1. Computation of Av.

We assume $g \in C^\infty([-\mu, +1], \mathbb{R}^+)$ and we compute:

$$\frac{\partial v}{\partial x_i} = \alpha r^{\alpha-2} g\left(\frac{x_N}{r}\right) x_i + r^{\alpha-3} g'\left(\frac{x_N}{r}\right) \{ \delta_{Ni} r^2 - x_N x_i \}$$

$$\begin{aligned} \frac{\partial^2 v}{\partial x_i \partial x_j} &= \alpha(\alpha-2) r^{\alpha-4} x_i x_j g\left(\frac{x_N}{r}\right) + \alpha r^{\alpha-2} \partial ij g\left(\frac{x_N}{r}\right) \\ &+ g'\left(\frac{x_N}{r}\right) \{ \alpha r^{\alpha-5} x_i (\partial_{Nj} r^2 - x_N x_j) \\ &\qquad + (\alpha-3) r^{\alpha-5} x_j (\delta_{Ni} r^2 - x_N x_i) + r^{\alpha-3} (2\delta_{Ni} x_j - x_N \delta ij - x_i \delta_{jN}) \} \\ &\qquad + r^{\alpha-6} g''\left(\frac{x_N}{r}\right) \{ \delta_{Ni} r^2 - x_N x_i \} \{ \delta_{Nj} r^2 - x_N x_j \} . \end{aligned}$$

Introducing $y = \left(-\frac{x_N x_1}{r^2}, \ldots, \frac{-x_N x_{N-1}}{r^2}, 1 - \frac{x_N^2}{r^2} \right)$ we get:

$$
\begin{aligned}
Av = g\left(\frac{x_N}{r}\right) n^\alpha c(x) + \alpha g\left(\frac{x_N}{r}\right)^{\alpha-2} \left\{ b_i x_i - (\alpha-2) a_{ij} \frac{x_i x_j}{r^2} - a_{ij}\delta_{ij} \right\} \\
+ r^{\alpha-2} g'\left(\frac{x_N}{r}\right) \left\{ r b_i \cdot y_i - \alpha a_{ij} \frac{x_i}{r} y_j - (\alpha-3) a_{ij} \frac{x_j}{r} y_i - a_{ij}\left(2\delta_{Ni} \frac{x_j}{r} - \frac{x_N}{r} \delta_{ij} \right.\right. \\
\left.\left. - \frac{x_i}{r} \delta_{jN} \right) \right\} - r^{\alpha-2} g''\left(\frac{x_N}{r}\right) a_{ij} y_i y_j
\end{aligned}
$$

2. A lower bound for Av.

We assume now $g \geqq 0$, $g' \geqq 0$, $g'' \leqq 0$ and we put $t = \frac{x_N}{r} \in [-1, 1]$, then we get:

$$
\begin{aligned}
Av \geqq -\alpha C_0 g(t) r^{\alpha-2} + r^{\alpha-2} g'(t) \Big\{ -2\,BR_0 + (2\alpha - 3) a_{ij} \frac{x_i x_j}{r^2} t \\
+ (1-\alpha) a_{Nj} \frac{x_j}{r} + (1-\alpha) a_{iN} \frac{x_i}{r} + tr\,(a_{ij}) t - r^{\alpha-2} g''(t) \nu (1 - t^2),
\end{aligned}
$$

now we put $x = (x', x_N)$ and $d = (a_{ij})_{1 \leqq i,j \leqq N-1}$ so that the coefficient of $r^{\alpha-2} g'(t)$ is equal to:

$$
\begin{aligned}
-2\,BR_0 + (2\alpha - 3)\left(d \frac{x'}{r}, \frac{x'}{r} \right) t + \sum_{i=1}^{N-1} (a_{Ni} + a_{iN}) \Big[\frac{x_i}{r}(1-\alpha) \\
+ t^2 \frac{x_i}{r}(2\alpha - 3) \Big] + tr\,(a_{ij}) t + a_{NN} [2(1-\alpha)t + (2\alpha - 3)t^3] \\
\geqq -2\,BR_0 - Ct^+(1-t) - C\sqrt{1-t} + tr\,(a_{ij}) t + a_{NN} t [2 - 2\alpha + (2\alpha - 3)t^2] \\
\geqq -2\,BR_0 - C\sqrt{1-t} + tr\,(a_{ij}) t^+ - a_{NN} t^+ \\
\geqq -2\,BR_0 - C\sqrt{1-t} + \nu t^+,
\end{aligned}
$$

here C denotes different constants depending only on (a_{ij}) and we assume $N \geqq 2$.

Finally we get:

$$
Av \geqq -\alpha C_0 r^{\alpha-2} g(t) + r^{\alpha-2} g'(t) \left\{ -\frac{\nu}{2} + \nu t^+ - C\sqrt{1-t} \right\} - r^{\alpha-2} g''(t)(1 - t^2).
$$

It remains to find some g for which this expression is large enough.

3. Existence of g.

LEMMA 7. — *For* $\mu \in [0, 1[$ *there exist* $\alpha \in]0, 1[$ *and* $g_\mu \in C^\infty([-\mu, 1], \mathbb{R}^+)$, $g_\mu \geqq 1$, $g'_\mu \geqq 0$, $g''_\mu \geqq 0$ *such that*:

$$
-\alpha C_0 g_\mu(t) + g'_\mu(t) \left\{ -\frac{\nu}{2} + \nu t^+ - C\sqrt{1-t} \right\} - g''_\mu(t)(1 - t^2) > 0. \tag{21}
$$

Proof of lemma 7. — We look for g_μ such that:

$$
g_\mu(t) = 1 + e^{+K\mu} - e^{-Kt}
$$

then (21) becomes:

$$
\begin{aligned}
-\alpha C_0 g_\mu(t) + g'_\mu(t) \left\{ -\frac{\nu}{2} + \nu t^+ - C\sqrt{1-t} \right\} - g''_\mu(t)(1 - t^2) \\
\geqq -\alpha C_0 g_\mu(t) + K e^{-Kt} \left\{ -\frac{\nu}{2} + \nu t^+ - C\sqrt{1-t} + K(1-t)^2 \right\}
\end{aligned}
$$

and a straightforward study of the function between the brackets shows that, if K is large enough (depending on μ and (a_{ij})) it is positive. Choosing α small enough we get lemma 7.

We can now conclude the proof of Theorem 4; by lemma 7 and choosing $g = Cg_\mu$ we have: $Av \geqq Cr^{\alpha-2}C(\mu)$ where $C(\mu)$ is the minimum in (21), $C(\mu) > 0$; for C large enough (20) is satisfied and the dependance of $\| v \|_{C^{0,\alpha}}$ on μ, A, f and $\| h \|_{C^{0,\alpha}}$ is clear.

Remark. — Of course for each β, $0 < \beta \leqq \alpha$, and $h \in C^{0,\beta}$, $(\mathbb{R}^N)$ we can find a barrier function in $C^{0,\beta}(\overline{\mathcal{C}})$.

APPENDIX 2

DIFFERENT NOTIONS OF SOLUTION FOR THE OBSTACLE PROBLEM

Throughout this paper we have used the notion of viscosity solution for the obstacle problem. An other definition is given in a remark of Section I. On the other hand we could use the usual definition of variational inequalities. The purpose of this Appendix is to prove the equivalence between these three notions. Namely, we have, with the notations and assumptions of section I and IV:

PROPOSITION 7. — *Let u and* Ψ *belong to* $C(\bar{\Omega})$ *then the following are equivalent:*

i) u is a viscosity solution of:

$$\operatorname{Max}(Au - f, u - \psi) = 0,$$

ii)
$$\begin{cases} Au \leq f \quad \text{in} \quad \mathscr{D}'(\Omega), \quad u \leq \Psi \quad \text{on} \quad \bar{\Omega}, \\ \text{on the open set } \mathcal{O} = \{ u < \Psi \}, \quad u \in W^{2,\infty}_{loc}(\mathcal{O}), \\ \text{and} \quad Au = f, \end{cases}$$

iii) $u \in H^1_{loc}(\Omega)$ and u is a solution of the variational inequality associated to Ψ i. e.:

$$\begin{cases} a(u, v - u) \geq (f, v - u) \\ \forall v \in H^1_{loc}(\Omega), \ v \leq \Psi, \ u \leq \Psi \quad \text{and} \quad v = u \quad \text{on a neighbourhood on } \partial\Omega. \end{cases} \tag{22}$$

iv) u is the limit of solutions of all regularized obstacle problem with nice standard solution (actually this limit holds in $C(\bar{\Omega})$ and weakly in $H^1(\Omega')$ for all $\Omega' \subset \bar{\Omega}' \subset \Omega$). In other words if we consider u_ε solution of:

$$\begin{cases} \operatorname{Max}(Au_\varepsilon - f, u_\varepsilon - \Psi_\varepsilon) = 0, \qquad u_\varepsilon \in C(\bar{\Omega}) \cap W^{2,\infty}_{loc}(\Omega), \\ u_\varepsilon|_{\partial\Omega} = \varphi_\varepsilon \end{cases} \tag{23}$$

with $\Psi_\varepsilon \geq \varphi_\varepsilon$ on $\partial\Omega$, $\Psi_\varepsilon \to \Psi$ in $C(\Omega)$ and $\varphi_\varepsilon \to \varphi$ in $C(\partial\Omega)$, then $u_\varepsilon \to u$ in $C(\bar{\Omega})$ and weakly in $H^1(\Omega')$.

Proof. — The equivalence between (*i*) and (*ii*) is a simple consequence of the notion of viscosity solution as defined in P. L. Lions [*13*].

The equivalence between (*i*) and (*iv*) is easily deduced from the stability (under convergence in $C(\bar{\Omega})$) of the notion of viscosity solution and the existence results for (23) which are proved in [*2*] and [*16*]. Remark that the family (u_ε) is bounded in $H^1(\Omega')$ (and thus the weak convergence holds) since we have:

$$Au_\varepsilon \leq f,$$

and for $\xi \in \mathscr{D}_+(\Omega)$ and C large enough:

$$\int_\Omega Au_\varepsilon \xi^2(u_\varepsilon + C) \leq \int_\Omega \xi^2 f(u_\varepsilon + C),$$

$$\nu \int_\Omega |Du_\varepsilon|^2 \xi^2 - \int_\Omega C |Du_\varepsilon| \xi^2(u_\varepsilon + C) - \int_\Omega C\xi |D\xi| . |Du_\varepsilon| (u_\varepsilon + C)$$
$$\leq \int_\Omega f \xi^2(u_\varepsilon + C) - \int_\Omega c(x) u_\varepsilon(u_\varepsilon + C)$$

and a simple minoration yields:

$$\frac{\nu}{2}\int_{\Omega} | Du_{\varepsilon} |^2 \xi^2 \leqq C,$$

(here C denotes different constants independant of ε). This concludes the equivalence between (*i*) and (*iv*).

Using the estimates proved above the equivalence (*iii*) $\Leftrightarrow$ (*iv*) is clear since the solution of the variational inequality (22) is stable under uniform convergence of the obstacle (see [*2*]).

Remark. — The regularity stated in (*ii*) for u on the set $\mathcal{O} = \{ u < \Psi \}$ is not optimal. It depends of course on the regularity of f: if $f \in L^2$, $u \in H^2_{loc}$; if $f \in L^\infty$ $u \in W^{2,p}_{loc}$ $\forall p < \infty$ ... etc...

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(Manuscrit reçu le 12 septembre 1984)