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HOMOGENIZATION OF THE DIRICHLET PROBLEM FOR A SYSTEM OF QUASILINEAR ELLIPTIC EQUATIONS IN A DOMAIN WITH FINE-GRAINED BOUNDARY

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ABSTRACT. – We study the Dirichlet problem for a system of nonlinear elliptic equations of Leray–Lions type in a sequence of domains $\Omega^{(s)}$, $s = 1, 2, \ldots$, with fine-grained boundaries. Under appropriate structure conditions on the system and the geometry of $\Omega^{(s)}$, we prove that the sequence of solutions of the problem converges in suitable topologies to the solution of a limit problem which contains an additional term of capacity type. We construct the limit problem.

RÉSUMÉ. – Nous étudions le problème de Dirichlet pour un système d'équations non linéaires élliptiques de type Leray–Lions dans une suite de domaines $\Omega^{(s)}$, $s = 1, 2, \ldots$, avec des frontières finement granulées. Sous des conditions de structure appropriées que nous imposons sur le système et sur la géometrie de $\Omega^{(s)}$, nous démontrons que la suite de solutions du problème tend dans des topologies appropriées vers la solution d'un problème limite qui contient un terme additionnel de type capacitaire. Nous construisons le problème limite.

1. Introduction

Let Ω be a bounded domain in $\mathbf{R}^n$ and for each given positive integer s let there be defined a system $\{F_i^{(s)}; i = 1, \ldots, I(s)\}$ of closed, pairwise disjoint sets lying inside Ω. In the domain $\Omega^{(s)} = \Omega \setminus \bigcup_{i=1}^{I(s)} F_i^{(s)}$, we consider the boundary value problem for a system of nonlinear elliptic equations

$$\sum_{l=1}^{n} \frac{\partial}{\partial x_l} A_l^j\left(x, \frac{\partial u}{\partial x}\right) = 0 \quad \text{in } \Omega^{(s)};\ j = 1, \ldots, N, \tag{1}$$

$$u(x) = f(x) \quad \text{on } \partial\Omega^{(s)}, \tag{2}$$

where $f(x) = (f_1(x), \ldots, f_N(x))^T$ is a given vector-function in $\bar{\Omega}$, $u(x) = (u_1(x), \ldots, u_N(x))^T$, $\frac{\partial u}{\partial x} = (\frac{\partial u_j}{\partial x_l})_{j=1,\ldots,N;\ l=1,\ldots,n}$ is the gradient of u, $A^j = A^j(x, p) =$

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$(A_l^i(x, p))_{l=1,\dots,n}$ is a vector function from $\Omega \times \mathbf{R}^{Nn}$ into $\mathbf{R}^n$ with $p = (p_l^i)_{i=1,\dots,N;\ l=1,\dots,n} \in \mathbf{R}^{Nn}$, $\partial\cdot$ denotes the boundary of a set $\cdot$. We shall assume throughout that $A_l^i = A_l^i(x, p)$ has the representation

$$A_l^j(x, p) = \sum_{i=1}^{N} a_{li}(x, p)\, p_i^j \quad \text{for } p \in \mathbf{R}^{Nn}.$$

For an open set $\cdot$, we shall denote by $W_m^1(\cdot, N)$ the direct product of N copies of the Sobolev space $W_m^1(\cdot)$ and by $\overset{\circ}{W}{}_m^1(\cdot, N)$ the set of vector-functions from $W_m^1(\cdot, N)$ which vanish on $\partial\cdot$. For a vector $v \in \mathbf{R}^M$ with components v_i $(i = 1, \dots, M)$, we denote its Euclidean norm by $|v|$, i.e., $|v| = (\sum_{i=1}^M v_i^2)^{1/2}$.

In this work we investigate the possibilities of approximating the problem (1)–(2) in the perforated domain $\Omega^{(s)}$ by a new homogenized problem in Ω whose solution is the limit of the sequence u_s of solutions of (1)–(2) as $s \to \infty$. In the scalar case, i.e., when $N = 1$, this problem has been studied by Skrypnik under various conditions on the geometry of $\Omega^{(s)}$ in many papers among which we cite [16–18] (the geometric restrictions in this paper are the weakest among those considered by the author so far). These works represent the nonlinear version of the theory invented by Marchenko and Khruslov (see [10]) for the study of linear elliptic boundary value problems in domains with fine-grained boundaries; the limit problem in this theory contains an additional term involving a function of capacity type. We note that a different approach has been elaborated by Cioranescu and Murat [3] and further developed in many other papers (see for instance [2,5–7], and the references therein); it does not require any conditions on the perforation and the limit problem contains an additional term involving a Borel measure which may assume infinite values. At the present moment many powerful methods in homogenization have been developed and an abundant literature on various aspects of the theory exist in form of monographs [1,4,10,17,12,19], just to cite a few.

Some considerable difficulties arise at key turning points of our work, due to the nature of the system of nonlinear equations that we are dealing with, in particular in the appropriate choice of some auxiliary model problems whose solutions (satisfying some special boundary conditions) and their a priori estimates are of paramount importance in the asymptotic analysis of problem (1)–(2); they are needed for the construction of the asymptotic expansion of the solutions of (1)–(2) and the test function used for the derivation of the limit problem. The results of this work have been announced in [14].

The work is organized as follows. In Section 2, we formulate the conditions ensuring that any solution u_s of problem (1)–(2) exists for each s and the sequence is bounded in $W_m^1(\Omega^{(s)}, N)$ (in particular the system has a Leray–Lions structure), using Moser's iteration technics we prove that it is uniformly bounded. For a background on Moser's method, we refer to his celebrated paper [11] and the work of Serrin [15]. Next we introduce an auxiliary model problem which plays a crucial role in the sequel and the geometric conditions on $\Omega^{(s)}$ under which we investigate (1)–(2), we formulate our main result. In Section 3, we derive some sharp pointwise and integral estimates for solutions of the model problem, some of which have been obtained by us in [13]. In Section 4, we construct an appropriate asymptotic expansion with a remainder term for the sequence

of solutions u_s of problem (1)–(2), we prove that the sequence converges strongly in $W_p^1(\Omega, N)$ for $p \in (1, m)$, and that the remainder term converges strongly to zero in $W_m^1(\Omega, N)$. In Section 5, we construct the problem of which the limit u_0 of u_s is a solution.

2. Hypotheses and formulation of main results

1) We assume that the functions A_l^j are Caratheodory functions in $\bar{\Omega}$, i.e., they are defined for all $x \in \bar{\Omega}$ and $p \in \mathbf{R}^{Nn}$, continuous in p for a.e. $x \in \bar{\Omega}$ and measurable in x for any $p \in \mathbf{R}^{Nn}$.

2) The functions A_l^j satisfy the following structure conditions: there exist the constants ν_1, ν_2, ν_3 and m, with $2 \leqslant m < n$ such that for every vector $\xi = (\xi_1, \ldots, \xi_n)$

$$\sum_{i,l=1}^{n} a_{li}(x, p)\xi_l\xi_i \geqslant \nu_1 |p|^{m-2}|\xi|^2, \tag{3}$$

and for every $p, q \in \mathbf{R}^{Nn}$; $p = (p^j)_{j=1,\ldots,N}$, $q = (q^j)_{j=1,\ldots,N}$ we have

$$\sum_{j=1}^{N} (p^j - q^j)\big(A^j(x, p) - A^j(x, q)\big) \geqslant \nu_2 |p - q|^m, \tag{4}$$

$$\big|A^j(x, p) - A^j(x, q)\big| \leqslant \big[1 + \nu_3\big(|p|^{m-2} + |q|^{m-2}\big)\big]|p - q|, \quad j = 1, \ldots, N. \tag{5}$$

We shall often use the symbol C for nonessential constants depending on the data.

We shall call the vector function $u \in W_m^1(\Omega^{(s)}, N)$ a weak solution of problem (1)–(2) if, $u \in f + \overset{\circ}{W}{}_m^1(\Omega^{(s)}, N)$ with f being a function which belongs to $W_m^1(\Omega, N)$, and for any vector function $\varphi(x) = (\varphi_1(x), \ldots, \varphi_N(x)) \in \overset{\circ}{W}{}_m^1(\Omega^{(s)}, N)$, the integral identity

$$\sum_{j=1}^{N}\sum_{l=1}^{n} \int_{\Omega^{(s)}} A_l^j\left(x, \frac{\partial u}{\partial x}\right)\frac{\partial \varphi_j}{\partial x_l}\,dx = 0 \tag{6}$$

holds.

From the results of Leray and Lions [9], we have

THEOREM 1. – *Under the conditions* 1) *and* 2), *if* $f \in W_m^1(\Omega, N)$, *then the problem* (1)–(2) *has at least one solution* $u_s \in f + \overset{\circ}{W}{}_m^1(\Omega^{(s)}, N)$ *and there exists a positive constant* M_1 *independent of* s, *such that for all* s, *the inequality*

$$\|u_s\|_{W_m^1(\Omega^{(s)}, N)} \leqslant M_1, \tag{7}$$

holds.

Let u_s be one of the solution of problem (1)–(2) satisfying inequality (7). We extend u_s to Ω by setting $u_s(x) = f(x)$ in $\Omega \setminus \Omega^{(s)}$. The resulting function that we denote without

fear of confusion by u_s belongs to $W_m^1(\Omega, N)$ and satisfies the inequality

$$\|u_s\|_{W_m^1(\Omega,N)} \leqslant M_1 + \|f\|_{W_m^1(\Omega,N)}.$$

From this inequality and the weak compactness of bounded sets in $W_m^1(\Omega, N)$, we may assume, by passing if necessary to a subsequence, that u_s converges weakly to a function u_0 in $W_m^1(\Omega, N)$.

In our investigations the boundedness of the sequence u_s will be needed. We show that in the following

THEOREM 2. – *Let the conditions* 1) *and* 2) *be satisfied and let f be a function belonging to $W_q^1(\Omega, N)$ with $q > n$. Then there exists a constant M independent of s such that u_s satisfies the estimate,*

$$\operatorname{vrai}\max_{x\in\Omega}\left|u_s(x)\right| \leqslant M. \tag{8}$$

Before proving the theorem, we note that the membership of f to $W_q^1(\Omega, N)$ with $q > n$ and the boundedness of Ω imply that f is continuous in $\bar{\Omega}$ and

$$\sup_{x\in\Omega}\left|f(x)\right| \leqslant C(\operatorname{meas}\Omega)^{1/n-1/q}\left\|\frac{\partial f}{\partial x}\right\|_{L_q(\Omega)}, \tag{9}$$

with the constant C depending only on n, N and q.

Proof. – We use a modification of Moser's method. Let $w_s(x) = |u_s(x) - f(x)|$. It is clear that $w_s \in \overset{\circ}{W}{}_m^1(\Omega^{(s)})$. We consider the positive real numbers r, k and K such that $r \geqslant 2$, $0 < k < K$. We define the truncated function $\zeta : \mathbf{R}^n \ni x \mapsto \zeta(x) \in \mathbf{R}$ as

$$\zeta(x) = \begin{cases} k & \text{if } 0 \leqslant |x| \leqslant k, \\ |x| & \text{if } k < |x| < K, \\ K & \text{if } K \leqslant |x|, \end{cases}$$

and set

$$w_{sk}^{(K)}(x) = \zeta\left(u_s(x) - f(x)\right).$$

The function $w_{sk}^{(K)}$ so defined is clearly bounded and since ζ is Lipshitz, standard arguments as in the scalar case (see, e.g., [8, Chapter 2, § 3]) show that $w_{sk}^{(K)} \in W_m^1(\Omega)$. Let us denote by $\chi_k^{(K)}$ the characteristic function of the set $\{x \in \Omega^{(s)}\colon k < |u_s - f| < K\}$. We consider the function $\varphi(x) = (\varphi_1(x), \ldots, \varphi_N(x))$, with

$$\varphi_j(x) = \left(u_{sj}(x) - f_j(x)\right)\left[w_{sk}^{(K)}(x)\right]^r, \quad j = 1, \ldots, N.$$

Owing to the properties of $w_{sk}^{(K)}$ we see that $\varphi \in \overset{\circ}{W}{}_m^1(\Omega^{(s)}, N)$. Thus φ is an admissible test function in the integral identity (6). We have

$$\frac{\partial\varphi_j}{\partial x_l} = \frac{\partial(u_{sj} - f_j)}{\partial x_l}\left[w_{sk}^{(K)}\right]^r + r\sum_{i=1}^{N}(u_{sj} - f_j)(u_{si} - f_i)\left[w_{sk}^{(K)}\right]^{r-2}\frac{\partial(u_{si} - f_i)}{\partial x_l}\chi_k^{(K)}. \tag{10}$$

Let us write the identity (6) in the form

$$\sum_{j=1}^{N}\sum_{l=1}^{n}\int\limits_{\Omega^{(s)}} A_l^i\left(x,\frac{\partial(u_s-f)}{\partial x}\right)\frac{\partial\varphi_j}{\partial x_l}\,dx$$
$$=\sum_{j=1}^{N}\sum_{l=1}^{n}\int\limits_{\Omega^{(s)}}\left[A_l^i\left(x,\frac{\partial(u_s-f)}{\partial x}\right)-A_l^i\left(x,\frac{\partial u_s}{\partial x}\right)\right]\frac{\partial\varphi_j}{\partial x_l}\,dx. \tag{11}$$

Substituting formula (10) in (11) and using the condition 2), we get

$$\int\limits_{\Omega^{(s)}}\left|\frac{\partial(u_s-f)}{\partial x}\right|^m\left[w_{sk}^{(K)}\right]^r(x)\,dx+r\int\limits_{\Omega^{(s)}}\left|\frac{\partial(u_s-f)}{\partial x}\right|^{m-2}\left|\frac{\partial w_{sk}^{(K)}}{\partial x}\right|^2\left[w_{sk}^{(K)}\right]^r\chi_k^{(K)}\,dx$$
$$\leqslant C\{H_1+H_2+H_3\},$$

where

$$H_1=(r+1)\int\limits_{\Omega^{(s)}}\left|\frac{\partial f}{\partial x}\right|\left|\frac{\partial(u_s-f)}{\partial x}\right|\left[w_{sk}^{(K)}\right]^r dx,$$

$$H_2=(r+1)\int\limits_{\Omega^{(s)}}\left|\frac{\partial f}{\partial x}\right|\left|\frac{\partial(u_s-f)}{\partial x}\right|^{m-1}\left[w_{sk}^{(K)}\right]^r dx,$$

$$H_3=(r+1)\int\limits_{\Omega^{(s)}}\left|\frac{\partial f}{\partial x}\right|\left|\frac{\partial u_s}{\partial x}\right|^{m-2}\left|\frac{\partial(u_s-f)}{\partial x}\right|\left[w_{sk}^{(K)}\right]^r dx.$$

By Young's inequality, we have

$$H_1\leqslant C_{\varepsilon_1}(r+1)^{\frac{m}{m-1}}\int\limits_{\Omega^{(s)}}\left|\frac{\partial f}{\partial x}\right|^{\frac{m}{m-1}}\left[w_{sk}^{(K)}\right]^r dx+\varepsilon_1\int\limits_{\Omega^{(s)}}\left|\frac{\partial(u_s-f)}{\partial x}\right|^m\left[w_{sk}^{(K)}\right]^r dx,$$

$$H_2\leqslant C_{\varepsilon_2}(r+1)^m\int\limits_{\Omega^{(s)}}\left|\frac{\partial f}{\partial x}\right|^m\left[w_{sk}^{(K)}\right]^r dx+\varepsilon_2\int\limits_{\Omega^{(s)}}\left|\frac{\partial(u_s-f)}{\partial x}\right|^m\left[w_{sk}^{(K)}\right]^r dx,$$

$$H_3\leqslant C_{\varepsilon_3}(r+1)^m\int\limits_{\Omega^{(s)}}\left|\frac{\partial f}{\partial x}\right|^m\left[w_{sk}^{(K)}\right]^r dx+\varepsilon_3\int\limits_{\Omega^{(s)}}\left|\frac{\partial(u_s-f)}{\partial x}\right|^m\left[w_{sk}^{(K)}\right]^r dx$$
$$+C(r+1)^m\int\limits_{\Omega^{(s)}}\left|\frac{\partial f}{\partial x}\right|^{m-1}\left[w_{sk}^{(K)}\right]^r dx.$$

Choosing ε_1, ε_2 and ε_3 sufficiently small in these inequalities, we get

$$\int\limits_{\Omega^{(s)}}\left|\frac{\partial(u_s-f)}{\partial x}\right|^m\left[w_{sk}^{(K)}\right]^r dx+r\int\limits_{\Omega^{(s)}}\left|\frac{\partial(u_s-f)}{\partial x}\right|^{m-2}\left|\frac{\partial w_{sk}^{(K)}}{\partial x}\right|\left[w_{sk}^{(K)}\right]^r\chi_k^{(K)}\,dx$$
$$\leqslant C(r+1)^m\left\{\int\limits_{\Omega^{(s)}}\left|\frac{\partial f}{\partial x}\right|^{\frac{m}{m-1}}\left[w_{sk}^{(K)}\right]^r dx+\int\limits_{\Omega^{(s)}}\left|\frac{\partial f}{\partial x}\right|^m\left[w_{sk}^{(K)}\right]^r dx\right.$$

$$+\int_{\Omega^{(s)}}\left|\frac{\partial f}{\partial x}\right|^{m-1}\left[w_{sk}^{(K)}\right]^{r}dx\Bigg\}. \tag{12}$$

Let us denote the integrals in the right-hand side of this inequality by H_{11}, H_{12} and H_{13} respectively. Using Young's and Hölder's inequalities, we estimate these integrals as follows.

$$H_{11}\leqslant C\left\{\int_{\Omega^{(s)}}\left|\frac{\partial f}{\partial x}\right|^{q}dx\right\}^{\frac{m}{q(m-1)}}\left\{1+\int_{\Omega^{(s)}}\left[w_{sk}^{(K)}\right]^{(r+m)p_1}\right\}^{\frac{1}{p_1}},$$

where $p_1=\frac{q(m-1)}{q(m-1)-m}$.

$$H_{12}\leqslant C\left\{\int_{\Omega^{(s)}}\left|\frac{\partial f}{\partial x}\right|^{q}dx\right\}^{\frac{m}{q}}\left\{1+\int_{\Omega^{(s)}}\left[w_{sk}^{(K)}\right]^{(r+m)p_2}\right\}^{\frac{1}{p_2}},$$

where $p_2=\frac{q}{q-m}$.

$$H_{13}\leqslant C\left\{\int_{\Omega^{(s)}}\left|\frac{\partial f}{\partial x}\right|^{q}dx\right\}^{\frac{m-1}{q}}\left\{1+\int_{\Omega^{(s)}}\left[w_{sk}^{(K)}\right]^{(r+m)p_3}dx\right\}^{\frac{1}{p_3}},$$

where $p_3=\frac{q}{q-m+1}$.

Let $Q=\max\{p_1,p_2,p_3\}$. Since $f\in W_q^1(\Omega,N)$, and

$$\left|\frac{\partial w_{sk}^{(K)}}{\partial x}\right|\leqslant\left|\frac{\partial(u_s-f)(u_s-f)}{\partial x}\right|,$$

it follows from the above estimations of H_{1k}, $k=1,2,3$ and the inequality (12) that

$$\int_{\Omega^{(s)}}\left|\frac{\partial w_{sk}^{(K)}}{\partial x}\right|^{m}\left[w_{sk}^{(K)}\right]^{r}dx\leqslant C(r+1)^{m}\left\{1+\int_{\Omega^{(s)}}\left[w_{sk}^{(K)}\right]^{(r+m)Q}(x)\,dx\right\}^{1/Q}. \tag{13}$$

Since $w_{sk}^{(K)}$ is bounded, we have that $[w_{sk}^{(K)}]^{t}\in W_m^1(\Omega^{(s)})$ for any $t>1$. Thus (13) implies

$$\int_{\Omega^{(s)}}\left|\frac{\partial[w_{sk}^{(K)}]^{1+(r/m)}}{\partial x}\right|^{m}dx\leqslant C(r+1)^{m}\left\{1+\int_{\Omega^{(s)}}\left[w_{sk}^{(K)}\right]^{(r+m)Q}(x)\,dx\right\}^{1/Q}. \tag{14}$$

The Sobolev embedding $W_m^1(\Omega)\hookrightarrow L_{\frac{nm}{n-m}}(\Omega)$ holds. Thus for any $r>0$, we have

$$\int_{\Omega^{(s)}}\left[w_{sk}^{(K)}\right]^{r}dx\leqslant C\left\{\int_{\Omega^{(s)}}\left|\frac{\partial[w_{sk}^{(K)}]^{\frac{r(n-m)}{nm}}}{\partial x}\right|^{m}dx\right\}^{n/(n-m)}. \tag{15}$$

By rescaling r in (14), we obtain from (15) the inequality

$$\int_{\Omega^{(s)}} \left[w_{sk}^{(K)}\right]^r dx \leqslant C(r+1)^{\frac{2mn}{n-m}} \left\{1 + \int_{\Omega^{(s)}} \left[w_{sk}^{(K)}\right]^{\frac{r(n-m)}{n}Q}(x)\,dx\right\}^{\frac{n}{Q(n-m)}}. \tag{16}$$

Passing to the limit in this inequality as $k \to 0$ and $K \to \infty$, and applying Fatou's Lemma in the left-hand side and Lebesgue's dominated convergence theorem in the right-hand side, we get

$$\int_{\Omega^{(s)}} w_s^r\,dx \leqslant C(r+1)^{\frac{2mn}{n-m}} \left\{1 + \int_{\Omega^{(s)}} w_s^{\frac{r(n-m)}{n}Q}(x)\,dx\right\}^{\frac{n}{Q(n-m)}}. \tag{17}$$

Letting

$$r = r_i = \frac{nm}{n-m}\left(\frac{n}{Q(n-m)}\right)^i, \quad i = 0, 1, 2, \ldots,$$

and

$$I_i = 1 + \int_{\Omega^{(s)}} w_s^{r_i}(x)\,dx,$$

we obtain from (17) the recurrent inequality

$$I_i \leqslant C\left[\left(\frac{n}{(n-m)Q}\right)^{\frac{2nm}{n-m}}\right]^i I_{i-1}^{\sigma^{-1}}, \tag{18}$$

with $\sigma = \frac{Q(n-m)}{n}$. Iterating the inequality (18), we get

$$I_i^{\sigma^i} \leqslant C^{\sigma^i+\sigma^{i-1}+\cdots+\sigma} A^{i\sigma^i+(i-1)\sigma^{i-1}+\cdots+\sigma} I_0, \tag{19}$$

where $A = [\frac{n}{Q(n-m)}]^{\frac{2mn}{n-m}}$.

Let $U_M = \{x \in \Omega^{(s)}\colon\ w_s(x) \geqslant M\}$, with $M = \operatorname{vrai}\max_{x\in\Omega^{(s)}} w_s(x)$ and assume that $\operatorname{meas} U_M$, the Lebesgue measure of U_M is different from zero. We have

$$I_i \geqslant \int_{U_M} w_s^{r_i}(x)\,dx + 1 \geqslant M^{(\frac{nm}{n-m})^{\sigma^i}} \operatorname{meas} U_M.$$

Thus from (19), as i approaches infinity, we get

$$\operatorname{vrai}\max_{x\in\Omega^{(s)}} w_s(x) \leqslant C\left\{1 + \int_{\Omega^{(s)}} w_s^{\frac{nm}{n-m}}(x)\,dx\right\}^{\frac{n-m}{nm}}.$$

This implies that

$$\operatorname{vrai}\max_{x\in\Omega^{(s)}} \left|u_s(x)\right| \leqslant C\left\{\|u_s - f\|_{L_{\frac{nm}{n-m}}(\Omega,N)} + \|f\|_{W_q^1(\Omega,N)} + 1\right\};$$

here we have used inequality (9). The theorem is proved. □

We now state the conditions on the geometry of the set $\Omega^{(s)}$. Let $d_i^{(s)}$ be the diameter of the set $F_i^{(s)}$ and let $x_i^{(s)}$ be the center of the ball $B(x_i^{(s)}, d_i^{(s)})$ of radius $d_i^{(s)}$ such that $F_i^{(s)} \subset \overline{B(x_i^{(s)}, d_i^{(s)})}$. We denote by $r_i^{(s)}$ the distance from the ball $B(x_i^{(s)}, d_i^{(s)})$ to the set $\bigcup_{j\neq i} B(x_j^{(s)}, d_j^{(s)}) \cup \partial\Omega$. We assume that the following conditions are satisfied: There exist the constants C_0 and C_1 independent of i and s such that

B1. $\lim_{s\to\infty} \max\{r_i^{(s)}\} = 0$, $d_i^{(s)} \leqslant C_1 r_i^{(s)}$.

B2.

$$\sum_{i=1}^{I(s)} \left[\frac{[d_i^{(s)}]^{m(n-m)}}{[r_i^{(s)}]^n} \right]^{\frac{1}{m-1}} \leqslant C_0.$$

For the formulation of another condition we introduce a model problem which will play a central role in our investigations. Let $\vec{k} \in \mathbf{R}^N$ and k the euclidean norm of $\vec{k}$. We denote by $\vec{e}$ the unit N-dimensional vector $\frac{\vec{k}}{k}$, and set $\Omega_i^{(s)} = B(x_i^{(s)}, 1) \setminus F_i^{(s)}$. Let $\psi \in \mathbf{C}_o^\infty(B(x_i^{(s)}, 1))$ with $\psi(x) = 1$ in $B(x_i^{(s)}, \frac{1}{2})$. For any $\vec{k}$, when $d_i < \frac{1}{2}$, we denote by $v_i^{(s)}(x, \vec{k})$, the vector-function from $k\psi(x - x_i^{(s)})\vec{e} + \overset{\circ}{W}{}_m^1(\Omega_i^{(s)}, N)$ which satisfies the integral identity

$$\sum_{j=1}^{N} \sum_{l=1}^{n} \int_{\Omega_i^{(s)}} A_l^j \left(x, \frac{\partial v_i^{(s)}}{\partial x} \right) \frac{\partial \varphi_j}{\partial x_l} \, dx = 0, \tag{20}$$

for any $\varphi(x) = (\varphi_1(x), \ldots, \varphi_N(x)) \in \overset{\circ}{W}{}_m^1(\Omega_i^{(s)}, N)$. Further the functions $v_i^{(s)}(x, \vec{k})$ are extended to Ω by setting $v_i^{(s)}(x, \vec{k}) = k\psi(x - x_i^{(s)})\vec{e}$ for $x \notin \Omega_i^{(s)}$. In other words $v_i^{(s)}(x, \vec{k})$ is a weak solution of the boundary value problem

$$\sum_{l=1}^{n} \frac{\partial}{\partial x_l} A_l^j \left(x, \frac{\partial v}{\partial x} \right) = 0 \quad \text{in } \Omega_i^{(s)}, \ \ j = 1, \ldots, N, \tag{21}$$

$$v(x) = k\psi\left(x - x_i^{(s)}\right)\vec{e} \quad \text{on } \partial\Omega_i^{(s)}. \tag{22}$$

Under the conditions 1) and 2) the existence of a solution to the problem (21)–(22) follows from Leray and Lions [9]. We impose on the sets $F_i^{(s)}$ the following restriction:

C. Let $\vec{k} = (k_1, \ldots, k_N) \in \mathbf{R}^N$. For any $\alpha = 1, \ldots, N$, there exists a continuous function $C^{(\alpha)}(x, \vec{k})$ such that for any subset $D \subset \Omega$, we have

$$\int_D C^{(\alpha)}(x, \vec{k}) \, dx = \begin{cases} \lim\limits_{s\to\infty} \sum\limits_{i \in I_s(D)} \sum\limits_{j=1}^{N} \dfrac{1}{k_\alpha} \displaystyle\int_D A^j \left(x, \frac{\partial v_i^{(s)}(x, \vec{k})}{\partial x} \right) \frac{\partial v_{i\alpha,j}^{(s)}(x, \vec{k})}{\partial x} \, dx & \text{if } k_\alpha \neq 0, \\ 0 & \text{if } k_\alpha = 0, \end{cases} \tag{23}$$

where $v_{i\alpha,j}^{(s)}(x, \vec{k})$ $(j = 1, \ldots, N)$ are the components of the vector-function $v_{i\alpha}^{(s)}(x, \vec{k})$ which is a solution of problem (21)–(22) with $\vec{e} = \vec{e}^{(\alpha)}$; $\vec{e}^{(\alpha)}$ being the N-dimensional

vector whose αth component equals $k_\alpha/|k_\alpha|$ ($k_\alpha \neq 0$) and the remaining components equal zero, $I_s(D) = \{i = 1, \ldots, I(s)\colon\ x_i^{(s)} \in D\}$, and the convergence in the limit in (23) is uniform in $\vec{k}$ with $|\vec{k}| < \infty$. We can look at the vector-function $\vec{C}(x, \vec{k}) = (C^{(1)}(x, \vec{k}), \ldots, C^{(N)}(x, \vec{k}))$ as the vector-valued analog of the additional term of capacity type in the works [17] and [18].

Now we are in a position to formulate our main result.

THEOREM 3. – *Let the conditions* 1), 2), B1, B2 *and* C *be satisfied. Assume that* $f \in W_q^1(\Omega, N)$ *with* $q > n$, *and let* u_s *be the sequence of solutions of problem* (1)–(2) *which satisfies inequality* (7) *and converges weakly to the vector-function* u_0 *in* $W_m^1(\Omega, N)$. *Then the sequence* u_s *converges strongly to* u_0 *in* $W_p^1(\Omega, N)$ *for every* $p \in (1, m)$, *and the function* u_0 *is a solution to the boundary value problem*

$$\sum_{l=1}^{n} \frac{\partial}{\partial x_l} A_l^j\left(x, \frac{\partial u}{\partial x}\right) + C^{(j)}\big(x, f(x) - u(x)\big) = 0 \quad \text{in } \Omega,\ j = 1, \ldots, N, \tag{24}$$

$$u(x) = f(x) \quad \text{on } \partial\Omega. \tag{25}$$

When $N = 1$, this theorem is due to Skrypnik [17, Chap. 9]. The theorem will follow from a suitable asymptotic expansion of the solutions u_s of problem (1)–(2) which plays an important role in the construction of the limit problem (24)–(25). The asymptotic expansion and the derivation of the limit problem rely on the solutions of the model problems (21)–(22) and some of their integral and pointwise a priori estimates. In the next section we deal with these a priori estimates.

3. Integral and pointwise estimates of the solutions of problem (21)–(22)

We start with

LEMMA 4. – *Le the assumptions* 1) *and* 2) *be satisfied. Then for any vector* $\vec{k} \neq 0$, *any solution* $v_i^{(s)}(x, \vec{k}) \in k\psi(x - x_i^{(s)})\vec{e} + \overset{\circ}{W}{}_m^1(\Omega_i^{(s)}, N)$ *of problem* (21)–(22) *satisfies the inequality*

$$\big|v_i^{(s)}(x, \vec{k})\big| \leqslant k. \tag{26}$$

Proof. – In the integral identity (20), let

$$\varphi(x) = \begin{cases} 0 & \text{if } |v_i^{(s)}(x, \vec{k})| \leqslant k, \\ v_i^{(s)}(x, \vec{k}) - k\vec{e} & \text{if } |v_i^{(s)}(x, \vec{k})| > k. \end{cases}$$

Let $E_k = \{x \in \Omega\colon\ |v_i^{(s)}(x, \vec{k})| \leqslant k\}$. Thus we have

$$\int\limits_{B(x_i^{(s)}, 1)\setminus E_k} \left|\frac{\partial v_i^{(s)}}{\partial x}\right|^m dx \leqslant \int\limits_{B(x_i^{(s)}, 1)\setminus E_k} \sum_{j=1}^{N} A^j\left(x, \frac{\partial v_i^{(s)}}{\partial x}\right) \frac{\partial v_{ij}^{(s)}}{\partial x}\, dx = 0.$$

This implies that either meas $B(x_i^{(s)}, 1)\setminus E_k = 0$ or $v_i^{(s)}$ has constant components. In the last case $v_i^{(s)}(x, k) = k\vec{e}$. Hence in both situations we have (26). □

In analogy with [13], we have

THEOREM 5. – *Let the conditions* 1) *and* 2) *be satisfied. Then there exist the constants* K_1, K_2 *and* K_3 *depending only on* N, n, m, ν_1, ν_2, ν_3 *such that for* $i = 1, \dots, I(s)$, $s = 1, 2, \dots$, *for any solution* $v_i^{(s)}(x, \vec{k}) \in k\psi(x - x_i^{(s)})\vec{e} + \overset{\circ}{W}{}_m^1(\Omega_i^{(s)}, N)$ *of problem* (21)–(22) *the following claims hold*:

1.

$$\int\limits_{B(x_i^{(s)},1)} \left| \frac{\partial v_i^{(s)}}{\partial x} \right|^m dx \leqslant K_1 k^m \left[d_i^{(s)} \right]^{n-m}. \tag{27}$$

2. *For* $0 < t < k$ *and* $E_t = \{x \in B(x_i^{(s)}, 1)\colon |v_i^{(s)}(x, \vec{k})| \leqslant t\}$

$$\int\limits_{E_t} \left| \frac{\partial v_i^{(s)}}{\partial x} \right|^m dx \leqslant K_2 t k^{m-1} \left[d_i^{(s)} \right]^{n-m}. \tag{28}$$

3.

$$\left| v_i^{(s)}(x, \vec{k}) \right| \leqslant K_3 k \left[\frac{d_i^{(s)}}{|x - x_i^{(s)}|} \right]^{\frac{n-m}{m-1}}, \tag{29}$$

for any $x \in B(x_i^{(s)}, 1) \backslash B(x_i^{(s)}, d_i^{(s)})$.

We prove the following

THEOREM 6. – *Let* $\bar{k} = (\bar{k}_1, \dots, \bar{k}_N)$ *and* $\hat{k} = (\hat{k}_1, \dots, \hat{k}_N)$ *be some* N*-dimensional vectors with* $|\bar{k}|$, $|\hat{k}| < \infty$ *and let* $v_i^{(s)}(x, \bar{k}) \in \bar{k}\psi(x - x_i^{(s)}) + \overset{\circ}{W}{}_m^1(\Omega_i^{(s)}, N)$, $v_i^{(s)}(x, \hat{k}) \in \hat{k}\psi(x - x_i^{(s)}) + \overset{\circ}{W}{}_m^1(\Omega_i^{(s)}, N)$ *be solutions of problem* (21)–(22). *If the conditions* 1) *and* 2) *are satisfied, then the following inequalities hold*:

$$\int\limits_{B(x_i^{(s)},1)} \left| \frac{\partial}{\partial x} \left[v_i^{(s)}(x, \bar{k}) - v_i^{(s)}(x, \hat{k}) \right] \right|^m \leqslant K_4 |\bar{k} - \hat{k}|^{\frac{m}{m-1}} \left[d_i^{(s)} \right]^{n-m}, \tag{30}$$

and for $\bar{k}_\alpha$, $\hat{k}_\alpha \neq 0$,

$$\left| \sum_{j=1}^{N} \int\limits_{B(x_i^{(s)},1)} \left\{ A^j \left(x, \frac{\partial v_i^{(s)}(x, \bar{k})}{\partial x} \right) \frac{1}{\bar{k}_\alpha} \frac{\partial v_{i\alpha}^{(s)}(x, \bar{k})}{\partial x} \right.\right.$$

$$\left.\left. - A^j \left(x, \frac{\partial v_i^{(s)}(x, \hat{k})}{\partial x} \right) \frac{1}{\hat{k}_\alpha} \frac{\partial v_{i\alpha}^{(s)}(x, \hat{k})}{\partial x} \right\} dx \right| \leqslant K_5 |\bar{k} - \hat{k}|^{\frac{1}{m-1}} \left[d_i^{(s)} \right]^{n-m}; \tag{31}$$

the constants K_4 *and* K_5 *depend only on* N, n, m, ν_1, ν_2, ν_3.

Proof. – For simplicity, we write $\bar{v}(x) = v_i^{(s)}(x, \bar{k})$, $\hat{v}(x) = v_i^{(s)}(x, \hat{k})$. Substituting the test function

$$\varphi(x) = \bar{v}(x) - \hat{v}(x) - |\bar{k} - \hat{k}|\psi\left(\frac{x - x_i^{(s)}}{d_i^{(s)}}\right)\vec{q},$$

with $\vec{q} = \frac{\bar{k}-\hat{k}}{|\bar{k}-\hat{k}|}$, $\psi \in \mathbf{C}_o^\infty(B(x_i^{(s)}, 1))$, $\psi(y) = 1$ in $B(x_i^{(s)}, \frac{1}{2})$, in the integral identity (20) for $\bar{v}$ and $\hat{v}$, and subtracting the resulting equations from each other, and using the condition 2), we get

$$\int\limits_{B(x_i^{(s)},1)} \left|\frac{\partial(\bar{v} - \hat{v})}{\partial x}\right|^m dx \leqslant C\{I_1 + I_2 + I_3\}, \tag{32}$$

where

$$I_1 = \frac{|\bar{k} - \hat{k}|}{d_i^{(s)}} \int\limits_{B(x_i^{(s)},1)} |\psi'| \left|\frac{\partial(\bar{v} - \hat{v})}{\partial x}\right| dx,$$

$$I_2 = \frac{|\bar{k} - \hat{k}|}{d_i^{(s)}} \int\limits_{B(x_i^{(s)},1)} |\psi'| \left|\frac{\partial \bar{v}}{\partial x}\right|^{m-2} \left|\frac{\partial(\bar{v} - \hat{v})}{\partial x}\right| dx,$$

$$I_3 = \frac{|\bar{k} - \hat{k}|}{d_i^{(s)}} \int\limits_{B(x_i^{(s)},1)} |\psi'| \left|\frac{\partial \hat{v}}{\partial x}\right|^{m-2} \left|\frac{\partial(\bar{v} - \hat{v})}{\partial x}\right| dx.$$

By Young's inequality, we have

$$I_1 \leqslant \varepsilon \int\limits_{B(x_i^{(s)},1)} \left|\frac{\partial(\bar{v} - \hat{v})}{\partial x}\right|^m dx + C_\varepsilon |\hat{k} - \bar{k}|^{\frac{m}{m-1}} \left[d_i^{(s)}\right]^{n-m}, \tag{33}$$

$$I_2 \leqslant \varepsilon \int\limits_{B(x_i^{(s)},1)} \left|\frac{\partial(\bar{v} - \hat{v})}{\partial x}\right|^m dx + C_\varepsilon \left[\frac{|\bar{k} - \hat{k}|}{d_i^{(s)}}\right]^{\frac{m}{m-1}} \int\limits_{B(x_i^{(s)},1)} \left[|\psi'| \left|\frac{\partial \bar{v}}{\partial x}\right|^{m-2}\right]^{\frac{m}{m-1}} dx.$$

Applying Hölder's inequality to the second integral in the right-hand side of this inequality and using inequality (27), we get

$$I_2 \leqslant \varepsilon \int\limits_{B(x_i^{(s)},1)} \left|\frac{\partial(\bar{v} - \hat{v})}{\partial x}\right|^m dx + C|\bar{k} - \hat{k}|^{\frac{m}{m-1}} |\bar{k}|^{\frac{m(m-2)}{m-1}} \left[d_i^{(s)}\right]^{n-m}. \tag{34}$$

We estimate I_3 analogously. Thus from (32)–(34), with ε sufficiently small, we obtain (30).

Next we prove (31). We substitute the test functions

$$\varphi(x) = \frac{1}{\bar{k}_\alpha} v_{i\alpha}^{(s)}(x, \bar{k}) - \psi\left(\frac{x - x_i^{(s)}}{d_i^{(s)}}\right)\vec{e}^{(\alpha)}$$

and

$$\varphi(x) = \frac{1}{\hat{k}_\alpha} v_{i\alpha}^{(s)}(x, \hat{k}) - \psi\left(\frac{x - x_i^{(s)}}{d_i^{(s)}}\right)\vec{e}^{(\alpha)},$$

in the integral identity (20) corresponding to $v_i^{(s)}(x, \bar{k})$ and $v_i^{(s)}(x, \hat{k})$ respectively, the function $v_{i\alpha}^{(s)}(x, \vec{k})$ are defined as in the formulation of condition C. Subtracting the resulting equations from each other, we get

$$\sum_{j=1}^{N} \int\limits_{B(x_i^{(s)},1)} \left\{ A^j\left(x, \frac{\partial \bar{v}}{\partial x}\right)\frac{1}{\bar{k}_\alpha}\frac{\partial v_{i\alpha}^{(s)}(x, \bar{k})}{\partial x} - A^j\left(x, \frac{\partial \hat{v}}{\partial x}\right)\frac{1}{\hat{k}_\alpha}\frac{\partial v_{i\alpha}^{(s)}(x, \hat{k})}{\partial x} \right\} dx$$

$$\leqslant \sum_{j=1}^{N} \int\limits_{B(x_i^{(s)},1)} \left| A^j\left(x, \frac{\partial \bar{v}}{\partial x}\right) - A^j\left(x, \frac{\partial \hat{v}}{\partial x}\right)\right| \left|\frac{\partial \psi}{\partial x}\right| dx = J_1. \tag{35}$$

Let us estimate J_1. From condition 2) and Hölder's inequality, we have

$$J_1 \leqslant C\left\{ \int\limits_{B(x_i^{(s)},1)} \left|\frac{\partial(\bar{v} - \hat{v})}{\partial x}\right|^m dx \right\}^{\frac{1}{m}} \left[d_i^{(s)}\right]^{\frac{n(m-1)}{m} - 1}$$

$$+ C\left[\left\{ \int\limits_{B(x_i^{(s)},1)} \left|\frac{\partial \bar{v}}{\partial x}\right|^m dx \right\}^{\frac{m-2}{m}} + \left\{ \int\limits_{B(x_i^{(s)},1)} \left|\frac{\partial \hat{v}}{\partial x}\right|^m dx \right\}^{\frac{m-2}{m}} \right]$$

$$\times \left\{ \int\limits_{B(x_i^{(s)},1)} \left|\frac{\partial(\bar{v} - \hat{v})}{\partial x}\right|^m dx \right\}^{\frac{1}{m}} \left[d_i^{(s)}\right]^{\frac{n-m}{m}}.$$

Thus from (27) and (30), we get

$$J_1 \leqslant C|\bar{k} - \hat{k}|^{\frac{1}{m-1}} \left[d_i^{(s)}\right]^{n-m}.$$

(31) immediately follows from this inequality and inequality (35). The theorem is proved. □

4. Asymptotic expansion of the sequence of solutions

In this section we construct an asymptotic expansion for the sequence of solutions u_s of the problem (1)–(2) with a remainder. We prove the convergence of the remainder term to zero in appropriate topologies. Let us introduce the sequence

$$\rho_i^{(s)} = \max\left\{ \left(1 + \frac{1}{2C_1}\right) d_i^{(s)}, \frac{1}{2C_3}\left[r_i^{(s)}\right]^{\frac{n}{n-m}} \ln^2 r_i^{(s)} \right\},$$

where C_1 is the constant from hypothesis B1 and

$$C_3 = \max_{0 \leqslant t \leqslant \operatorname{diam}\Omega} \left\{ t^{\frac{m}{n-m}} \ln^2 t \right\}.$$

A simple calculation shows that

$$\rho_i^{(s)} \leqslant d_i^{(s)} + \frac{r_i^{(s)}}{2},$$

and from the definition of the numbers $d_i^{(s)}$ and $r_i^{(s)}$, it follows that the balls $B(x_i^{(s)}, d_i^{(s)} + r_i^{(s)}/2)$ $(i = 1, \ldots, I(s))$ are pairwise disjoint. Let θ_1, θ_2 and θ_3 be some numbers such that $0 < \theta_3 < \theta_2 < \theta_1 < 1$. We consider the infinitely differentiable functions χ_j, $j = 1, 2$, satisfying the conditions: $0 \leqslant \chi_j(t) \leqslant 1$, $\chi_j(t) = 1$ if $t < \theta_{j+1}$, $\chi_j(t) = 0$ if $t > \theta_j$, $\chi_j'(t) < \infty$ $(j = 1, 2)$. We introduce the functions

$$\psi_i^{(s)}(x) = \chi_1\left(\frac{|x - x_i^{(s)}|}{\rho_i^{(s)}}\right),\ \varphi_i^{(s)}(x) = \chi_2\left(\frac{|x - x_i^{(s)}|}{\rho_i^{(s)}}\right);\quad i = 1, \ldots, I(s).$$

We set

$$I_s' = \left\{i = 1, \ldots, I(s):\ \theta_1 d_i^{(s)} \geqslant \frac{1}{2C_3}\left[r_i^{(s)}\right]^{\frac{n}{n-m}} \ln^2 r_i^{(s)}\right\},$$

$$I_s'' = \left\{i = 1, \ldots, I(s):\ \theta_1 d_i^{(s)} < \frac{1}{2C_3}\left[r_i^{(s)}\right]^{\frac{n}{n-m}} \ln^2 r_i^{(s)}\right\}.$$

We have

LEMMA 7. – *If conditions* B1 *and* B2 *are satisfied, then*

$$\lim_{s\to\infty} \sum_{i\in I_s'} \left[d_i^{(s)}\right]^{n-m} = 0, \tag{36}$$

$$\lim_{s\to\infty} \sum_{i\in I_s''} \left[\rho_i^{(s)}\right]^{n} = 0. \tag{37}$$

Proof. – We have

$$\sum_{i\in I_s'} \left[d_i^{(s)}\right]^{n-m} \leqslant C \max_{1\leqslant i\leqslant I(s)} \left\{\frac{1}{[\ln r_i^{(s)}]^{2\frac{n-m}{m-1}}}\right\} \sum_{i\in I_s'} \left[\frac{[d_i^{(s)}]^{m(n-m)}}{[r_i^{(s)}]^n}\right]^{\frac{1}{m-1}}.$$

By condition B2, (36) follows from a passage to the limit as $s \to \infty$ in the above inequality; here we have made use of the definition of the set I_s'.

For the proof of the relation (37), we note that since the balls $B(x_i^{(s)}, d_i^{(s)} + r_i^{(s)}/2)$, $i = 1, \ldots, I(s)$ are pairwise disjoint and Ω is bounded, it follows that

$$\sum_{i=1}^{I(s)} \left[r_i^{(s)}\right]^n \leqslant C; \tag{38}$$

C is a constant independent of s. We have, by the definition of the set I_s'',

$$\sum_{i\in I_s''} \left[\rho_i^{(s)}\right]^n \leqslant \left[\frac{1}{2C_3}\right]^n \max_{1\leqslant i\leqslant I(s)} \left\{\left[r_i^{(s)}\right]^{\frac{nm}{n-m}} \ln^{2n} r_i^{(s)}\right\} \sum_{i\in I_s''} \left[r_i^{(s)}\right]^n.$$

A passage to the limit in both sides of this inequality and the relation (38) yield (37). The lemma is proved. □

As an immediate consequence of Lemma 7, we have

$$\lim_{s\to\infty}\sum_{i=1}^{I(s)}\bigl[\rho_i^{(s)}\bigr]^n = 0. \tag{39}$$

Further we see that

$$\sum_{i=1}^{I(s)}\bigl[d_i^{(s)}\bigr]^{n-m} < C; \tag{40}$$

C is independent of s. Indeed by Hölder's inequality, we have

$$\sum_{i=1}^{I(s)}\bigl[d_i^{(s)}\bigr]^{n-m} \leqslant \left\{\sum_{i=1}^{I(s)}\left[\frac{[d_i^{(s)}]^{n-m}}{[r_i^{(s)}]^{\frac{n}{m}}}\right]^{\frac{m}{m-1}}\right\}^{\frac{m-1}{m}}\left\{\sum_{i=1}^{I(s)}\bigl[r_i^{(s)}\bigr]^n\right\}^{\frac{1}{m}}.$$

Thus, (40) is an immediate consequence of condition B2 and relation (39). Inequality (40) implies in particular that

$$\lim_{s\to\infty}\sum_{i=1}^{I(s)}\bigl[d_i^{(s)}\bigr]^n = 0;$$

i.e., the set $\bigcup_{i=1}^{I(s)} F_i^{(s)}$ vanishes as $s\to\infty$.

Now we are in a position to construct the asymptotic expansion of the sequence of solutions u_s of the problem (1)–(2). We assume that u_s satisfies the inequalities (7) and (8) and converges weakly to the vector-function $u_0 \in f + \overset{\circ}{W}{}^1_m(\Omega, N)$ with $f \in W^1_q(\Omega, N)$ and $q > n$. We denote the means of the functions u_0 and f over the balls $B_i^{(s)} =: B(x_i^{(s)}, 1)$ by

$$u_i^{(s)} = \fint_{B_i^s} u_0(x)\,dx =: \frac{1}{\operatorname{meas} B_i^{(s)}}\int_{B_i^{(s)}} u_0(x)\,dx,$$

$$f_i^{(s)} = \frac{1}{\operatorname{meas} B_i^{(s)}}\int_{B_i^{(s)}} f(x)\,dx,$$

respectively. In the sequel we set $D_i^{(s)} =: B(x_i^{(s)}, \theta_2\rho_i^{(s)})$.

We seek a solution of problem (1)–(2) in the form

$$u_s(x) = u_0(x) + \sum_{k=1}^{4} R_s^{(k)}(x) + w_s(x), \tag{41}$$

where

$$R_s^{(1)}(x) = \sum_{i\in I_s'} \{[u_i^{(s)} - u_0(x)] + [f(x) - f_i^{(s)}]\}\psi_i^{(s)}(x),$$

$$R_s^{(2)} = \sum_{i\in I_s''} \{[u_i^{(s)} - u_0(x)] + [f(x) - f_i^{(s)}]\}\varphi_i^{(s)}(x),$$

$$R_s^{(3)} = \sum_{i\in I_s'} v_i^{(s)}\big(x, f_i^{(s)} - u_i^{(s)}\big)\varphi_i^{(s)}(x),$$

$$R_s^{(4)} = \sum_{i\in I_s''} v_i^{(s)}\big(x, f_i^{(s)} - u_i^{(s)}\big)\varphi_i^{(s)}(x),$$

where $v_i^{(s)}(x, f_i^{(s)} - u_i^{(s)})$ is a solution of problem (21)–(22) with $k = |f_i^{(s)} - u_i^{(s)}|$ and $\vec{e} = \frac{f_i^{(s)} - u_i^{(s)}}{|f_i^{(s)} - u_i^{(s)}|}$, w_s is the remainder term whose behavior will be dealt with in the following corrector result which gives a justification of the expansion (41). For simplicity we shall write $v_i^{(s)}(x, f_i^{(s)} - u_i)$ as $v_i^{(s)}$ and indicate "converge to" by $\to$.

THEOREM 8. – *Let the conditions of Theorem* 3 *be satisfied. Then*

1. $R_s^{(k)} \to 0$ *strongly in* $W_m^1(\Omega, N)$ $(k = 1, 2, 3)$.
2. $R_s^{(4)} \to 0$ *weakly in* $W_m^1(\Omega, N)$ *and strongly in* $W_p^1(\Omega, N)$ *for all* $p \in (1, m)$.
3. $w_s \to 0$ *strongly in* $W_m^1(\Omega, N)$.

Proof. –

Proof of 1. We have

$$\big\|R_s^{(1)}\big\|_{L_m(\Omega,N)} \leqslant \sum_{i\in I_s'} \int_\Omega \big[|u_i^{(s)} - u_0(x)|^m + |f_i^{(s)} - f(x)|^m\big]|\psi_i^{(s)}|^m\, dx.$$

Theorem 2 implies that u_0 is uniformly bounded in Ω. Thus using inequality (9), Poincare inequality and the definition of I_s', we get

$$\big\|R_s^{(1)}\big\|_{L_m(\Omega,N)} \leqslant C \sum_{i\in I_s'} \big[d_i^{(s)}\big]^{n-m}. \tag{42}$$

Next

$$\begin{aligned}\left\|\frac{\partial R_s^{(1)}}{\partial x}\right\|_{L_m(\Omega,N)} &\leqslant C \sum_{i\in I_s'} \int_\Omega \left\{\big[|u_i^{(s)} - u_0(x)|^m + |f_i^{(s)} - f(x)|^m\big]\left|\frac{\partial \psi_i^{(s)}}{\partial x}\right|^m \right.\\ &\quad \left. + \left[\left|\frac{\partial u_0}{\partial x}\right|^m + \left|\frac{\partial f}{\partial x}\right|^m\right]|\psi_i^{(s)}(x)|^m\right\} dx\\ &\leqslant C\left\{\sum_{i\in I_s'} \big[d_i^{(s)}\big]^{n-m} + \int_\Omega \left[\left|\frac{\partial u_0}{\partial x}\right|^m + \left|\frac{\partial f}{\partial x}\right|^m\right] \sum_{i\in I_s'} |\psi_i^{(s)}(x)|^m\, dx\right\},\end{aligned}$$

where as above, we have used Theorem 2, inequality (9) and the definition of I'_s. The function $\Phi_s(x) = \sum_{i\in I'_s} |\psi_i^{(s)}(x)|^m$ is bounded and by Poincare's inequality, we have

$$\int_\Omega \Phi_s(x)\,dx \leqslant C \sum_{i\in I'_s} \int_{D_i^{(s)}} \left|\frac{\partial \psi_i^{(s)}}{\partial x}\right|^m dx \leqslant C \sum_{i\in I'_s} \left[d_i^{(s)}\right]^{n-m}.$$

Thus, we get

$$\lim_{s\to\infty} \left\| \frac{\partial R_s^{(1)}}{\partial x} \right\|_{L_m(\Omega,N)} \leqslant C \sum_{i\in I'_s} \left[d_i^{(s)}\right]^{n-m}. \tag{43}$$

Passing to the limit in (42) and (43) and using (36), we obtain that $R_1^{(s)}$ converges to zero strongly in $W_m^1(\Omega, N)$.

In a similar fashion we show that

$$\lim_{s\to\infty} \left\| R_s^{(2)} \right\|_{W_m^1(\Omega,N)} = 0. \tag{44}$$

Next we have

$$\begin{aligned}\left\| R_s^{(3)} \right\|_{L_m(\Omega,N)} &\leqslant \sum_{i\in I'_s} \int_\Omega \left| v_i^{(s)}\left(x, f_i^{(s)} - u_i^{(s)}\right)\right|^m \left|\varphi_i^{(s)}\right|^m dx \\ &\leqslant C \sum_{i\in I'_s} \int_\Omega \left|\frac{\partial v_i^{(s)}}{\partial x}\right|^m dx \leqslant C \sum_{i\in I'_s} \left[d_i^{(s)}\right]^{n-m},\end{aligned}$$

where we have used Poincare inequality and inequality (27) from Theorem 5.

Next we have

$$\begin{aligned}\left\| \frac{\partial R_s^{(3)}}{\partial x} \right\|_{L_m(\Omega,N)} &\leqslant C \sum_{i\in I'_s} \int_\Omega \left[\left|v_i^{(s)}(x)\right|^m \left|\frac{\partial \varphi_i^{(s)}}{\partial x}\right|^m + \left|\frac{\partial v_i^{(s)}}{\partial x}\right| \left|\varphi_i^{(s)}(x)\right|^m \right] dx \\ &\leqslant C \sum_{i\in I'_s} \left[d_i^{(s)}\right]^{n-m}.\end{aligned}$$

Here we have used the boundedness of $v_i^{(s)}$, the definition of $\varphi_i^{(s)}$ and the inequality (27) from Theorem 5. Combining this inequality and the previous one and taking account of (36), we get

$$\lim_{s\to\infty} \left\| R_s^{(3)} \right\|_{W_m^1(\Omega,N)} = 0. \tag{45}$$

Part 1 of the theorem is proved.

Proof of 2. By Theorem 5 (inequality (27)) and the boundedness of $v_i^{(s)}$ we have

$$\left\| R_s^{(4)} \right\|_{W_m^1(\Omega,N)}^m \leqslant C \sum_{i\in I''_s} \left\{ \left[d_i^{(s)}\right]^{n-m} + \left[\rho_i^{(s)}\right]^n \right\}.$$

Thus the relation (40) and (39) imply

$$\lim_{s\to\infty} \left\| R_s^{(4)} \right\|_{W_m^1(\Omega,N)} < \infty. \tag{46}$$

We deduce from here that $R_s^{(4)}$ converges weakly in $W_m^1(\Omega, N)$. Next by Hölder's inequality and (46), we have for $1 < p < m$

$$\|R_s^{(4)}\|_{W_p^1(\Omega,N)} \leqslant C\|R_s^{(4)}\|_{W_m^1(\Omega,N)} \left\{\sum_{i\in I_s''} \operatorname{meas} D_i^{(s)}\right\}^{\frac{1}{p}-\frac{1}{m}} \leqslant C\left\{\sum_{i\in I_s''} [\rho_i^{(s)}]^n\right\}^{\frac{1}{p}-\frac{1}{m}};$$

we have used the fact that $\varphi_i^{(s)}(x) = 0$ outside $\bigcup_{i=1}^{I(s)} D_i^{(s)}$. Hence by relation (37), we get

$$\lim_{s\to\infty} \|R_s^{(4)}\|_{W_p^1(\Omega,N)} = 0. \tag{47}$$

This implies the claimed strong convergence of $R_s^{(4)}$ to zero in $W_p^1(\Omega, N)$.

Proof of 3. From the assertions 1 and 2 of the theorem and the weak convergence of u_s to u in $W_m^1(\Omega, N)$, it clearly follows that w_s weakly converges to zero in $W_m^1(\Omega, N)$. Furthermore, since the sequence w_s is uniformly bounded, we also have that w_s strongly converges to zero in any $L_r(\Omega, N)$ for all $r < \infty$ by Sobolev embedding's theorem.

We have $w_s \in \overset{\circ}{W}{}_m^1(\Omega^{(s)}, N)$; in particular $w_s(x) = 0$ in $\bigcup_i^{I(s)} F_i^{(s)}$. Hence we can substitute $\varphi(x) = w_s(x) = (w_{s1}(x), \ldots, w_{sN}(x))$ in the integral identity (6) and get

$$\sum_{j=1}^{N} \int_\Omega A^j\left(x, \frac{\partial u_s}{\partial x}\right) \frac{\partial w_{sj}(x)}{\partial x}\, dx = 0. \tag{48}$$

We rewrite the left-hand side of (48) as

$$\sum_{j=1}^{N} \int_{\Omega^{(s)}} A^j\left(x, \frac{\partial u_s}{\partial x}\right) \frac{\partial w_{sj}(x)}{\partial x}\, dx = I_1^{(s)} + I_2^{(s)} + I_3^{(s)} + I_4^{(s)},$$

where

$$I_1^{(s)} = \int_\Omega \sum_{j=1}^{N} \left[A^j\left(x, \frac{\partial u_s}{\partial x}\right) - A^j\left(x, \frac{\partial u_s}{\partial x} - \frac{\partial w_s}{\partial x}\right)\right] \frac{\partial w_{sj}(x)}{\partial x}\, dx,$$

$$I_2^{(s)} = \int_\Omega \sum_{j=1}^{N} \left[A^j\left(x, \frac{\partial u_s}{\partial x} - \frac{\partial w_s}{\partial x}\right) - A^j\left(x, \frac{\partial u_0}{\partial x} + \frac{\partial R_s^{(4)}}{\partial x}\right)\right] \frac{\partial w_{sj}(x)}{\partial x}\, dx,$$

$$I_3^{(s)} = \int_\Omega \sum_{j=1}^{N} \left[A^j\left(x, \frac{\partial u_0}{\partial x} + \frac{\partial R_s^{(4)}}{\partial x}\right) - A^j\left(x, \frac{\partial R_s^{(4)}}{\partial x}\right)\right] \frac{\partial w_{sj}(x)}{\partial x}\, dx,$$

$$I_4^{(s)} = \int_\Omega \sum_{j=1}^{N} A^j\left(x, \frac{\partial R_s^{(4)}}{\partial x}\right) \frac{\partial w_{sj}(x)}{\partial x}\, dx.$$

By condition (4) on the A^j's, we obtain the estimate

$$I_1^{(s)} \geqslant \nu_2 \int_\Omega \left|\frac{\partial w_s}{\partial x}\right|^m dx. \tag{49}$$

Inequality (5) and Hölder's inequality yield

$$I_2^{(s)} \leqslant C\left\{\int_\Omega \left[1+\left|\frac{\partial u_s}{\partial x}\right|+\left|\frac{\partial u_0}{\partial x}\right|+\left|\frac{\partial R_s^{(4)}}{\partial x}\right|+\left|\frac{\partial w_s}{\partial x}\right|\right]^m dx\right\}^{\frac{m-2}{m}}$$
$$\times\left\{\int_\Omega \left[\sum_{k=1}^{3}\left|\frac{\partial R_s^{(k)}}{\partial x}\right|\right]^m dx\right\}^{\frac{1}{m}}\left\{\int_\Omega \left|\frac{\partial w_s}{\partial x}\right|^m\right\}^{\frac{1}{m}}.$$

From relations (43)–(45) we deduce from this inequality that

$$\lim_{s\to\infty} I_2^{(s)} = 0. \tag{50}$$

For the estimation of $I_3^{(s)}$, we note that since $R_s^{(4)}(x) = 0$ in $K^{(s)} = \Omega\setminus\bigcup_{i=1}^{I''(s)} D_i^{(s)}$, we have the following representation

$$I_3^{(s)} = \int_{K^{(s)}} \sum_{j=1}^{N} A^j\left(x, \frac{\partial u_0}{\partial x}\right)\frac{\partial w_{sj}}{\partial x}\,dx$$
$$+\sum_{j=1}^{N}\sum_{i\in I_s''}\int_{D_i^{(s)}}\left[A^j\left(x, \frac{\partial u_0}{\partial x}+\frac{\partial R_s^{(4)}}{\partial x}\right)-A^j\left(x, \frac{\partial R_s^{(4)}}{\partial x}\right)\right]\frac{\partial w_{sj}}{\partial x}\,dx$$
$$\equiv I_{31}^{(s)} + I_{32}^{(s)}. \tag{51}$$

By inequality (5) and Young's inequality, we have

$$\int_{K^{(s)}}\left|A^j\left(x, \frac{\partial u_0}{\partial x}\right)\right|^{\frac{m}{m-1}} dx \leqslant C\int_\Omega \left|\frac{\partial u_0}{\partial x}\right|^m dx.$$

This implies that $|A^j(x, \frac{\partial u_0}{\partial x})| \in L_{\frac{m}{m-1}}(\Omega)$. Hence from the weak convergence of w_s to zero in $W_m^1(\Omega, N)$, we obtain that $I_{31}^{(s)}$ converges to zero as $s\to\infty$. Further by inequality (5) and Hölder's inequality, we have

$$I_{32}^{(s)} \leqslant C\left\{\int_\Omega \left[1+\left|\frac{\partial u_0}{\partial x}\right|+\left|\frac{\partial R_s^{(4)}}{\partial x}\right|\right]^m dx\right\}^{\frac{m-2}{m}}\left\{\int_\Omega \left|\frac{\partial w_s}{\partial x}\right|^m dx\right\}^{\frac{1}{m}}\left\{\sum_{i\in I_s''}\int_{D_i^{(s)}}\left|\frac{\partial u_0}{\partial x}\right|^m dx\right\}^{\frac{1}{m}}.$$

In view of (39) and the absolute continuity of integrals, the last integral converges to zero as $s\to\infty$; the two previous are bounded. Thus $I_{32}^{(s)}$ converges to zero. Hence we obtain

$$\lim_{s\to\infty} I_3^{(s)} = 0. \tag{52}$$

Now we proceed to the estimation of $I_4^{(s)}$. It is technically the most complicated part of the proof of the theorem. But firstly, let us introduce a new test function. Let $\lambda \in (0, \theta_3)$ and let us consider the functions $\chi_i^{(s)} \in \mathbf{C}_o^\infty(\Omega)$ such that $0 \leqslant \chi_i^{(s)}(x) \leqslant 1$,

$\chi_i^{(s)}(x) = 1$ in $B(x_i^{(s)}, \lambda\rho_i^{(s)})$, $\chi_i^{(s)}(x) = 0$ outside $B(x_i^{(s)}, \theta_3\rho_i^{(s)})$ and $|\frac{\partial\chi_i^{(s)}}{\partial x}| \leqslant \frac{C}{\rho_i^{(s)}}$. Using these functions, we rewrite $I_4^{(s)}$ as follows

$$I_4^{(s)} = I_{41}^{(s)} + I_{42}^{(s)}, \tag{53}$$

where

$$I_{41}^{(s)} = \sum_{i\in I_s''} \int_{D_i^{(s)}} \sum_{j=1}^{N} A^j\left(x, \frac{\partial v_i^{(s)}}{\partial x}\right) \frac{\partial[w_{sj}\chi_i^{(s)}]}{\partial x}\, dx$$

and

$$I_{42}^{(s)} = \sum_{i\in I_s''} \int_{D_i^{(s)}} \sum_{j=1}^{N} A^j\left(x, \frac{\partial[\varphi_i^{(s)} v_i^{(s)}]}{\partial x}\right) \frac{\partial[w_{sj}(1-\chi_i^{(s)})]}{\partial x}\, dx.$$

Let s be sufficiently large so that $\theta_3\rho_i^{(s)} < 1$. Then $I_{41}^{(s)} = 0$ by the definition of the functions $v_i^{(s)}(x, f_i^{(s)} - u_i^{(s)})$. For simplicity let us denote the ball $B(x_i^{(s)}, \lambda\rho_i^{(s)})$ by $G_i^{(s)}$. Appealing to Hölder's and Young's inequalities and using inequality (5) we get

$$\begin{aligned} I_{42}^{(s)} \leqslant C \sum_{i\in I_s''} \Bigg\{ \int_{D_i^{(s)}} \left|\frac{\partial[w_{sj}(1-\chi_i^{(s)})]}{\partial x}\right|^m dx \Bigg\}^{\frac{1}{m}} \\ \times \left\{ [\rho_i^{(s)}]^n + \int_{D_i^{(s)}\setminus G_i^{(s)}} \left|\frac{\partial[\varphi_i^{(s)} v_i^{(s)}]}{\partial x}\right|^m dx \right\}. \end{aligned} \tag{54}$$

Let $m_i^{(s)} = \max_{x\in D_i^{(s)}\setminus G_i^{(s)}} |v_i^{(s)}(x)|$. By Theorem 5 (inequality (29)), we have

$$m_i^{(s)} \leqslant C\left[\frac{d_i^{(s)}}{\rho_i^{(s)}}\right]^{\frac{n-m}{m-1}}.$$

Thus using the properties of the functions $\varphi_i^{(s)}$ and the inequality (28) in Theorem 5, we get

$$\begin{aligned} \int_{D_i^{(s)}\setminus G_i^{(s)}} \left|\frac{\partial[\varphi_i^{(s)} v_i^{(s)}]}{\partial x}\right|^m dx &\leqslant C\left\{ \int_{E_{m_i^{(s)}}} \left|\frac{\partial v_i^{(s)}}{\partial x}\right|^m dx + \int_{D_i^{(s)}\setminus G_i^{(s)}} |v_i^{(s)}(x)|^m \left|\frac{\partial\varphi_i^{(s)}}{\partial x}\right|^m dx \right\} \\ &\leqslant C\{m_i^{(s)}[d_i^{(s)}]^{n-m} + [m_i^{(s)}]^m[\rho_i^{(s)}]^{n-m}\} \\ &\leqslant C\left[\frac{[d_i^{(s)}]^m}{\rho_i^{(s)}}\right]^{\frac{n-m}{m-1}}. \end{aligned} \tag{55}$$

For the estimation of the first factor in the last inequality in (54), we need the following Poincare's inequality from [17, Lemma 1.4, Chap. 8]. Let $1 < p < n$, then for any

function $u \in W_p^1(B(0,r))$ and any numbers ρ_1 and ρ_2 such that $0 \leqslant \rho_1 < \rho_2 \leqslant \frac{r}{2}$, the inequality

$$\int\limits_{K(\rho_1,\rho_2)} |u(x)|^p\,dx \leqslant C\left\{(\rho_2^p - \rho_1^p)\int\limits_{K(\rho_1,r)} \left|\frac{\partial u}{\partial x}\right|^p dx + \frac{\rho_2^n - \rho_1^n}{r^n}\int\limits_{K(\frac{r}{2},r)} |u(x)|^p\,dx\right\}, \quad (56)$$

holds with the constant C depending only on n and p; $K(a,b) = \{x\colon a < |x| < b\}$.

We have

$$\int\limits_{D_i^{(s)}} \left|\frac{\partial[w_{sj}(1-\chi_i^{(s)})]}{\partial x}\right|^m dx \leqslant C\left\{\int\limits_{D_i^{(s)}} \left|\frac{\partial w_{sj}}{\partial x}\right|^m dx + \left[\rho_i^{(s)}\right]^{-m}\int\limits_{D_i^{(s)}} |w_{sj}|^m\,dx\right\}.$$

Let $\bar r_i^{(s)} = d_i^{(s)} + \frac{r_i^{(s)}}{2}$. Thus $D_i^{(s)} \subset B(x_i^{(s)}, \bar r_i^{(s)}) \subset B(x_i^{(s)}, 2\bar r_i^{(s)})$. By inequality (56) it follows that

$$\int\limits_{D_i^{(s)}} |w_{sj}(x)|^m\,dx \leqslant C\left\{[\theta_2\rho_i^{(s)}]^m \int\limits_{B(x_i^{(s)},\bar r_i^{(s)})} \left|\frac{\partial w_{sj}}{\partial x}\right|^m dx \frac{[\theta_2\rho_i^{(s)}]^n}{[r_i^{(s)} + 2d_i^{(s)}]^n} \times \int\limits_{B(x_i^{(s)},\bar r_i^{(s)})} |w_{sj}(x)|^m\,dx\right\}.$$

Thus we obtain

$$\int\limits_{D_i^{(s)}} \left|\frac{\partial[w_{sj}(1-\chi_i^{(s)})]}{\partial x}\right|^m dx \leqslant C\left\{\int\limits_{B(x_i^{(s)},\bar r_i^{(s)})} \left|\frac{\partial w_{sj}}{\partial x}\right|^m dx + \frac{[\rho_i^{(s)}]^{n-m}}{[r_i^{(s)}]^n}\int\limits_{B(x_i^{(s)},\bar r_i^{(s)})} |w_{sj}(x)|^m\,dx\right\}.$$

From this inequality and (54)–(55), we get

$$\begin{aligned} I_{42}^{(s)} \leqslant{}& C_1\left\{\sum_{i\in I_s''}\left[\frac{[d_i^{(s)}]^m}{\rho_i^{(s)}}\right]^{\frac{n-m}{m-1}} + \sum_{i\in I_s''}\left[\rho_i^{(s)}\right]^n\right\}^{\frac{m-1}{m}}\left\{\int\limits_\Omega \left|\frac{\partial w_s}{\partial x}\right|^m dx\right\}^{\frac{1}{m}} \\ &+ C_2\left\{\sum_{i\in I_s''}\left[\frac{[d_i^{(s)}]^{m(n-m)}}{[r_i^{(s)}]^n}\right]^{\frac{1}{m-1}} + \sum_{i\in I_s''}\left[\rho_i^{(s)}\right]^n\left[\frac{[\rho_i^{(s)}]^{n-m}}{[r_i^{(s)}]^n}\right]^{\frac{1}{m-1}}\right\}^{\frac{m-1}{m}} \\ &\times\left\{\int\limits_\Omega |w_{sj}(x)|^m\,dx\right\}. \end{aligned} \quad (57)$$

By the definition of the set I_s'', we get

$$A_s = \sum_{i\in I_s''}\left[\frac{[d_i^{(s)}]^m}{\rho_i^s}\right]^{\frac{n-m}{m-1}} \leqslant C\max_{1\leqslant i\leqslant I(s)}\frac{1}{\ln^{2\frac{n-m}{m-1}} r_i^{(s)}}\sum_{i\in I_s''}\left[\frac{[d_i^{(s)}]^{m(n-m)}}{[r_i^{(s)}]^n}\right]^{\frac{1}{m-1}}.$$

Passing to the limit in this inequality we get from the conditions B1 and B2 that A_s converges to zero as $s \to \infty$. Thus in view of (37), we obtain that the first term in the right-hand side of (57) converges to zero as $s \to \infty$.

Further, we have

$$E_s = \sum_{i \in I_s''} \bigl[\rho_i^{(s)}\bigr]^n \left[\frac{[\rho_i^{(s)}]^{n-m}}{[r_i^{(s)}]^n}\right]^{\frac{1}{m-1}} \leqslant \max_{1 \leqslant i \leqslant I(s)} \bigl\{\bigl[r_i^{(s)}\bigr]^{\frac{nm}{n-m}} \bigl[\ln r_i^{(s)}\bigr]^{2[n+\frac{n-m}{m-1}]}\bigr\} \sum_{1 \leqslant i \leqslant I(s)} \bigl[r_i^{(s)}\bigr]^n.$$

The first factor in the right-hand side of this inequality converges to zero as $s \to \infty$ while the second factor is bounded by (38). Thus by condition B2 and the strong convergence of w_s to zero in $L_m(\Omega, N)$ it follows that the second term in (57) converges to zero as $s \to \infty$. Hence we obtain that $\lim_{s\to\infty} I_{42}^{(s)} = 0$ and this shows that

$$\lim_{s \to \infty} I_4^{(s)} = 0, \tag{58}$$

since $I_{42}^{(s)} = 0$. Relations (48), (49), (50), (52) and (58) imply that w_s strongly converges to zero in $W_m^1(\Omega, N)$. The theorem is proved. □

The results of this section establish the first claim in the main Theorem 3. The remaining of our work is devoted to the proof of the second claim of the theorem. We deal with this question in the next section.

5. Derivation of the limit problem

In this section we shall prove that the vector-function u_0, the weak limit of the sequence of solutions of problem (1)–(2) is a solution of problem (24)–(25). Let g be an arbitrary vector-function in $\mathbf{C}_o^\infty(\Omega, N)$. We consider the sequence of functions

$$g_s(x) = g(x) + \sum_{k=1}^{3} \rho_s^{(k)}(x), \tag{59}$$

where

$$\rho_s^{(1)}(x) = \sum_{i=1}^{I(s)} \bigl[g_i^{(s)} - g(x)\bigr] \psi_i^{(s)}(x),$$

$$\rho_s^{(2)}(x) = -\sum_{\alpha=1}^{N} \sum_{i \in I'_{s\alpha}} w_{i\alpha}^{(s)}(x) g_{i\alpha}^{(s)} \varphi_i^{(s)}(x),$$

$$\rho_s^{(3)}(x) = -\sum_{\alpha=1}^{N} \sum_{i \in I''_{s\alpha}} v_{i\alpha}^{(s)}(x, 1) g_{i\alpha}^{(s)} \varphi_i^{(s)}(x),$$

where

$$w_{i\alpha}^{(s)}(x) = \begin{cases} \frac{1}{(f_i^{(s)} - u_i^{(s)})_\alpha} v_{i\alpha}^{(s)}(x, f_i^{(s)} - u_i^{(s)}), & \text{if } (f_i^{(s)} - u_i^{(s)})_\alpha \neq 0, \\ 0, & \text{if } (f_i^{(s)} - u_i^{(s)})_\alpha = 0; \end{cases}$$

$(\cdot)_\alpha$ denotes the αth component of the vector $\cdot$, $g_i^{(s)}$ is the mean of the vector-function g over the ball $B(x_i^{(s)}, \theta_2\rho_i^{(s)})$, $v_{i\alpha}^{(s)}(x, f_i^{(s)} - u_i^{(s)})$ is a solution of problem (21)–(22) with $k = |(f_i^{(s)} - u_i^{(s)})_\alpha|$ and $\vec{e}$ the N-dimensional vector whose αth component is $(f_i^{(s)} - u_i^{(s)})_\alpha / |(f_i^{(s)} - u_i^{(s)})_\alpha|$ and the remaining components are zero, $v_{i\alpha}^{(s)}(x, 1)$ is a solution of problem (21)–(22) with $k = 1$ and the vector $\vec{e}$ has its αth component equal to 1 and the remaining components zero, $\psi_i^{(s)}$ and $\varphi_i^{(s)}$ are the test functions defined in the previous section. By the symbols $I'_{s\alpha}$ and $I''_{s\alpha}$ we mean the sets

$$I'_{s\alpha} = \{i = 1, \dots, I(s): \ |(f_i^{(s)} - u_i^{(s)})_\alpha| \geqslant d_i^{(s)}\},$$
$$I''_{s\alpha} = \{i = 1, \dots, I(s): \ |(f_i^{(s)} - u_i^{(s)})_\alpha| < d_i^{(s)}\}.$$

Using the same arguments as in the proof of Theorem 7, we can show that the sequences $\rho_1^{(s)}$ and $\rho_3^{(s)}$ strongly converge to zero in $W_m^1(\Omega, N)$, i.e.,

$$\lim_{s\to\infty} \{\|\rho_1^{(s)}\|_{W_m^1(\Omega,N)} + \|\rho_3^{(s)}\|_{W_m^1(\Omega,N)}\} = 0. \tag{60}$$

We also get that

$$\|\rho_2^{(s)}\|_{W_m^1(\Omega,N)} < \infty, \tag{61}$$

and

$$\lim_{s\to\infty} \|\rho_2^{(s)}\|_{W_p^1(\Omega,N)} = 0, \tag{62}$$

for all $p \in (1, m)$.

A simple verification shows that $g_s \in \overset{\circ}{W}{}_m^1(\Omega^{(s)}, N)$. Hence we can substitute $\varphi(x) = g_s(x)$ in the integral identity (6). By doing so, we get

$$J_1^{(s)} + J_2^{(s)} + J_3^{(s)} + J_4^{(s)} = 0, \tag{63}$$

where

$$J_1^{(s)} = \sum_{j=1}^{N}\sum_{l=1}^{n} \int_\Omega A_l^j\left(x, \frac{\partial u_s}{\partial x}\right)\frac{\partial g_j}{\partial x_l}\,dx,$$
$$J_2^{(s)} = \sum_{j=1}^{N}\sum_{l=1}^{n} \int_\Omega A_l^j\left(x, \frac{\partial u_s}{\partial x}\right)\left(\frac{\partial \rho_{sj}^{(1)}}{\partial x_l} + \frac{\partial \rho_{sj}^{(3)}}{\partial x_l}\right)dx,$$
$$J_3^{(s)} = \sum_{j=1}^{N}\sum_{l=1}^{n} \int_\Omega A_l^j\left(x, \frac{\partial u_s}{\partial x}\right)\frac{\partial \rho_{sj}^{(2)}}{\partial x_l}\,dx.$$

We investigate the behavior of each $J_k^{(s)}$ $(k = 1, 2, 3)$ as $s \to \infty$.

We have

$$J_1^{(s)} = \sum_{j=1}^{N}\sum_{l=1}^{n} \int_\Omega A_l^j\left(x, \frac{\partial u_0}{\partial x}\right)\frac{\partial g_j}{\partial x_l}\,dx + J_{11}^{(s)},$$

where

$$J_{11}^{(s)} = \sum_{j=1}^{N}\sum_{l=1}^{n}\int_{\Omega}\left[A_l^j\left(x,\frac{\partial u_s}{\partial x}\right) - A_l^j\left(x,\frac{\partial u_0}{\partial x}\right)\right]\frac{\partial g_j}{\partial x_l}\,dx.$$

From condition 2) and Hölder's inequality we have

$$J_{11}^{(s)} \leqslant C'\left\{\int_{\Omega}\left[\left|\frac{\partial u_0}{\partial x}\right|^m + \left|\frac{\partial u_s}{\partial x}\right|^m\right]dx\right\}^{\frac{m-2}{m}}\left\{\int_{\Omega}\left|\frac{\partial(u_s-u_0)}{\partial x}\right|^{m-1}dx\right\}^{\frac{1}{m-1}}$$
$$+ C''\left\{\int_{\Omega}\left|\frac{\partial(u_s-u_0)}{\partial x}\right|^{m-\varepsilon}dx\right\}^{\frac{1}{m-\varepsilon}}, \tag{64}$$

with $\varepsilon \in (0,1)$. Since u_s strongly converges to u_0 in $W_p^1(\Omega,N)$ for any $p \in (1,m)$, and the first factor in the second inequality in (64) is bounded, we see that $J_{11}^{(s)}$ converges to zero as $s \to \infty$. Thus

$$\lim_{s\to\infty} J_1^{(s)} = \sum_{j=1}^{N}\sum_{l=1}^{n}\int_{\Omega} A_l^j\left(x,\frac{\partial u_0}{\partial x}\right)\frac{\partial g_j}{\partial x_l}\,dx. \tag{65}$$

Since $\rho_s^{(1)}$ and $\rho_s^{(3)}$ strongly converge to zero in $W_m^1(\Omega,N)$ then by condition 2) and Hölder's inequality, it follows that

$$\lim_{s\to\infty} J_2^{(s)} = 0. \tag{66}$$

Using expansion (41), let us rewrite $J_3^{(s)}$ as follows

$$J_3^{(s)} = J_{31}^{(s)} + J_{32}^{(s)} \equiv \sum_{j=1}^{N}\sum_{l=1}^{n}\int_{\Omega}\left[A_l^j\left(x,\frac{\partial u_s}{\partial x}\right) - A_l^j\left(x,\frac{\partial R_s^{(4)}}{\partial x}\right)\right]\frac{\partial \rho_{sj}^{(2)}}{\partial x_l}\,dx$$
$$+ \sum_{j=1}^{N}\sum_{l=1}^{n}\int_{\Omega} A_l^j\left(x,\frac{\partial R_s^{(4)}}{\partial x}\right)\frac{\partial \rho_{sj}^{(2)}}{\partial x_l}\,dx.$$

The inequality (5) from condition 2) applied to $J_{31}^{(s)}$ yields

$$J_{31}^{(s)} \leqslant C\{J_{31}^{(s)1} + J_{32}^{(s)2}\} \equiv C\Bigg\{\int_{\Omega}\left|\frac{\partial \rho_s^{(2)}}{\partial x}\right|\left(\sum_{k=1}^{3}\left|\frac{\partial R_s^{(k)}}{\partial x}\right| + \left|\frac{\partial w_s}{\partial x}\right|\right)$$
$$\times\left(1 + \left[\left|\frac{\partial u_s}{\partial x}\right| + \left|\frac{\partial R_s^{(4)}}{\partial x}\right|\right]^{m-2}\right)dx$$
$$+ \int_{\Omega}\left|\frac{\partial \rho_s^{(2)}}{\partial x}\right|\left|\frac{\partial u_0}{\partial x}\right|\left[1 + \left|\frac{\partial R_s^{(4)}}{\partial x}\right|^{m-2}\right]dx\Bigg\}. \tag{67}$$

By Hölder's inequality, we have

$$J_{31}^{(s)1} \leqslant \left\{ \int_\Omega \left| \frac{\partial \rho_s^{(2)}}{\partial x} \right|^m dx \right\}^{\frac{1}{m}} \left\{ \int_\Omega \left(\sum_{k=1}^{3} \left| \frac{\partial R_s^{(k)}}{\partial x} \right| + \left| \frac{\partial w_s}{\partial x} \right| \right)^m dx \right\}^{\frac{1}{m}} \times \left\{ \int_\Omega \left(1 + \left| \frac{\partial u_s}{\partial x} \right| + \left| \frac{\partial R_s^{(4)}}{\partial x} \right| \right)^m dx \right\}^{\frac{m-2}{m}}.$$

The second factor in the right-hand side of this inequality converges to zero by Theorem 8, while the other factors are bounded. Thus

$$\lim_{s\to\infty} J_{31}^{(s)1} = 0. \tag{68}$$

By Hölder's inequality, we have

$$J_{31}^{(s)2} \leqslant \left\{ \int_\Omega \left[1 + \left| \frac{\partial u_s}{\partial x} \right| + \left| \frac{\partial R_s^{(4)}}{\partial x} \right| \right]^m dx \right\}^{\frac{m-2}{m}} \left\{ \int_\Omega \left[\left| \frac{\partial \rho_s^{(2)}}{\partial x} \right| \left| \frac{\partial u_0}{\partial x} \right| \right]^{\frac{m}{2}} dx \right\}^{\frac{2}{m}}. \tag{69}$$

Let us show that the second factor (that we denote by $H_1^{(s)}$) in this inequality converges to zero. Indeed applying Hölder's inequality, taking account of the properties of the functions $v_{i\alpha}^{(s)}$ and $\varphi_i^{(s)}$, and appealing to the estimate (27) in Theorem 5, we get

$$\begin{aligned} H_1^{(s)} &= \int_\Omega \left[\left| \frac{\partial \rho_s^{(2)}}{\partial x} \right| \left| \frac{\partial u_0}{\partial x} \right| \right]^{\frac{m}{2}} dx \\ &\leqslant C \left\{ \sum_{i\in I_s'} \left[d_i^{(s)} \right]^{n-m} \right\}^{\frac{1}{2}} \left\{ \int_\Omega \left| \frac{\partial u_0}{\partial x} \right|^m dx \right\}^{\frac{1}{2}} \\ &\quad + C \left\{ \sum_{i\in I_s''} \left[d_i^{(s)} \right]^{n-m} \right\}^{\frac{1}{2}} \left\{ \sum_{i\in I_s''} \int_{D_i^{(s)}} \left| \frac{\partial u_0}{\partial x} \right|^m dx \right\}^{\frac{1}{2}}. \end{aligned}$$

The first term in the right-hand side of this inequality converges to zero by (36), the second term also converges to zero by (39), (40) and the absolute continuity of an integral. Hence $H_1^{(s)}$ converges to zero as $s \to \infty$. Thus from (69), we get $\lim_{s\to\infty} J_{31}^{(s)2} = 0$. This together with (68) and (67) shows that $\lim_{s\to\infty} J_{31}^{(s)} = 0$.

Now we are left with the investigation of $J_{32}^{(s)}$. Using the definition of $R_s^{(4)}$, we rewrite $J_{32}^{(s)}$ as follows.

$$J_{32}^{(s)} = -\sum_{j=1}^{N} \sum_{l=1}^{n} \sum_{i=1}^{I(s)} g_{i\alpha}^{(s)} \int_\Omega A_l^j \left(x, \frac{\partial v_i^{(s)}}{\partial x} \right) \frac{\partial w_{i\alpha,j}^{(s)}}{\partial x_l} dx + J_{32}^{(s)1} + J_{32}^{(s)2}, \tag{70}$$

where

$$J_{32}^{(s)1} = \sum_{\alpha,j=1}^{N} \sum_{l=1}^{n} \sum_{i\notin I_s''\cap I_{s\alpha}'} g_{i\alpha}^{(s)} \int_\Omega A_l^j \left(x, \frac{\partial v_i^{(s)}}{\partial x} \right) \frac{\partial w_{i\alpha,j}^{(s)}}{\partial x} dx,$$

$$J_{32}^{(s)2} = \sum_{\alpha,j=1}^{N} \sum_{l=1}^{n} \sum_{i \in I_s'' \cap I_{s\alpha}'} g_{i\alpha}^{(s)} \int_{\Omega} \left\{ A_l^j \left(x, \frac{\partial [v_i^{(s)} \varphi_i^{(s)}]}{\partial x} \right) \frac{\partial [\varphi_i^{(s)} w_{i\alpha,j}^{(s)}]}{\partial x_l} - A_l^j \left(x, \frac{\partial v_i^{(s)}}{\partial x} \right) \frac{\partial w_{i\alpha,j}^{(s)}}{\partial x_l} \right\} dx,$$

where $w_{i\alpha,j}^{(s)}$ is the jth component of $w_{i\alpha}^{(s)}$. By condition 2) and Hölder's inequality, we have

$$J_{32}^{(s)1} \leqslant C \{ H_{31}^{(s)} \times H_{32}^{(s)} \},$$

where

$$H_{31}^{(s)} = \sum_{i \notin I_s'' \cap I_{s\alpha}'} \left\{ \int_{D_i^{(s)}} \left(1 + \left| \frac{\partial v_i^{(s)}}{\partial x} \right| \right)^m dx \right\}^{\frac{m-2}{m}} \left\{ \int_{D_i^{(s)}} \left| \frac{\partial v_i^{(s)}}{\partial x} \right|^m dx \right\}^{\frac{1}{m}},$$

and

$$H_{32}^{(s)} = \sum_{\alpha=1}^{N} \sum_{i \notin I_s'' \cap I_{s\alpha}'} \left\{ \int_{D_i^{(s)}} \left| \frac{\partial w_{i\alpha}^{(s)}}{\partial x} \right|^m \right\}^{\frac{1}{m}}.$$

Noticing that $\{i\colon i = 1, \ldots, I(s)\} \backslash [I_s'' \cap I_{s\alpha}'] = I_s' \cup [I_s'' \cap I_{s\alpha}'']$, by Theorem 5, we get

$$\begin{aligned} H_{31}^{(s)} &\leqslant C \sum_{\alpha=1}^{N} \sum_{i \notin I_s'' \cap I_{s\alpha}'} [d_i^{(s)}]^{n-m} \left| \left(f_i^{(s)} - u_i^{(s)} \right)_\alpha \right|^m \\ &\leqslant C \left\{ \sum_{i \in I_s'} [d_i^{(s)}]^{n-m} + \sum_{i \in I_s'' \cap I_{s\alpha}''} [\rho_i^{(s)}]^n \right\}. \end{aligned}$$

Here we have used the definition of the sets I_s'' and $I_{s\alpha}''$. By (36) and (37), we conclude from this inequality that $\lim_{s\to\infty} H_{31}^{(s)} = 0$. Analogous arguments show that $\lim_{s\to\infty} H_{32}^{(s)} = 0$. Thus

$$\lim_{s\to\infty} J_{32}^{(s)1} = 0. \tag{71}$$

Next, we write

$$J_{32}^{(s)2} = L_1^{(s)} + L_2^{(s)} + L_3^{(s)}, \tag{72}$$

where

$$\begin{aligned} L_1^{(s)} = &- \sum_{\alpha,j=1}^{N} \sum_{l=1}^{n} \sum_{i \in I_s'' \cap I_{s\alpha}'} g_{i\alpha}^{(s)} \int_{\Omega} \left\{ A_l^j \left(x, \frac{\partial [v_i^{(s)} \varphi_i^{(s)}]}{\partial x} \right) \frac{\partial [\varphi_i^{(s)} w_{i\alpha,j}^{(s)}]}{\partial x_l} \right. \\ &\left. - A_l^j \left(x, \frac{\partial v_i^{(s)}}{\partial x} \right) \varphi_i^{(s)} \frac{\partial w_{i\alpha,j}^{(s)}}{\partial x_l} \right\} dx, \\ L_2^{(s)} = &\sum_{\alpha,j=1}^{N} \sum_{l=1}^{n} \sum_{i \in I_s'' \cap I_{s\alpha}'} g_{i\alpha}^{(s)} \int_{\Omega} (1 - \varphi_i^{(s)}) A_l^j \left(x, \frac{\partial v_i^{(s)}}{\partial x} \right) \frac{\partial w_{i\alpha,j}^{(s)}}{\partial x_l} dx, \end{aligned}$$

$$L_3^{(s)} = -\sum_{\alpha,j=1}^{N}\sum_{l=1}^{n}\sum_{i\in I_s''\cap I_{s\alpha}'} g_{i\alpha}^{(s)}\int_\Omega w_{i\alpha,j}^{(s)} A_l^j\left(x, \frac{\partial[v_i^{(s)}\varphi_i^{(s)}]}{\partial x}\right)\frac{\partial\varphi_i^{(s)}}{\partial x_l}\,dx.$$

From condition 2), we have

$$L_1^{(s)} \leqslant C\{L_{11}^{(s)} + L_{12}^{(s)}\}, \tag{73}$$

where

$$L_{11}^{(s)} = \sum_{\alpha=1}^{N}\sum_{i\in I_s''\cap I_{s\alpha}'} |g_{i\alpha}^{(s)}|\int_\Omega \left[\left|\frac{\partial[\varphi_i^{(s)}v_i^{(s)}]}{\partial x}\right| + \left|\frac{\partial v_i^{(s)}}{\partial x}\right|\right]^{m-2} \times \left|\frac{\partial[(1-\varphi_i^{(s)})v_i^{(s)}]}{\partial x}\right|\left|\frac{\partial w_{i\alpha}^{(s)}}{\partial x}\right| dx$$

and

$$L_{12}^{(s)} = \sum_{\alpha=1}^{N}\sum_{i\in I_s''\cap I_{s\alpha}'} |g_{i\alpha}^{(s)}|\int_\Omega \left|\frac{\partial[(1-\varphi_i^{(s)})v_i^{(s)}]}{\partial x}\right|\left|\frac{\partial w_{i\alpha}^{(s)}}{\partial x}\right| dx.$$

Let us estimate $L_{11}^{(s)}$. Denote $B(x_i^{(s)}, \theta_3\rho_i^{(s)})$ by $E_i^{(s)}$. From the definition of $\varphi_i^{(s)}$ and Young's inequality, we have

$$L_{11}^{(s)} \leqslant C\sum_{\alpha=1}^{N}\sum_{i\in I_s''\cap I_{s\alpha}'}\int_{D_i^{(s)}\setminus E_i^{(s)}} \left\{\left|\frac{\partial v_i^{(s)}}{\partial x}\right|^m + |v_i^{(s)}|^m\left|\frac{\partial\varphi_i^{(s)}}{\partial x}\right|^m + \left|\frac{\partial w_{i\alpha}^{(s)}}{\partial x}\right|^m\right\} dx.$$

Let

$$m_i^{(s)} = \max_{x\in D_i^{(s)}\setminus E_i^{(s)}} |v_i^{(s)}(x, f_i^{(s)} - u_i^{(s)})|, \qquad m_{i\alpha}^{(s)} = \max_{x\in D_i^{(s)}\setminus E_i^{(s)}} |v_{i\alpha}^{(s)}(x, f_i^{(s)} - u_i^{(s)})|,$$

$$E_{m_i^{(s)}} = \{x\in D_i^{(s)}\colon |v_i^{(s)}| \leqslant m_i^{(s)}\}, \qquad E_{m_{i\alpha}^{(s)}} = \{x\in D_i^{(s)}\colon |v_{i\alpha}^{(s)}| \leqslant m_{i\alpha}^{(s)}\}.$$

By Theorem 5 and the definition of the set I_s'', we get

$$L_{11}^{(s)} \leqslant C\sum_{\alpha=1}^{N}\sum_{i\in I_s''\cap I_{s\alpha}'}\left\{[m_i^{(s)}]^m[\rho_i^{(s)}]^{n-m} + \int_{E_{m_i^{(s)}}}\left|\frac{\partial v_i^{(s)}}{\partial x}\right|^m dx + \int_{E_{m_{i\alpha}^{(s)}}}\left|\frac{\partial v_{i\alpha}^{(s)}}{\partial x}\right|^m dx\right\}$$

$$\leqslant C\max_{1\leqslant i\leqslant I(s)}\frac{1}{\ln^{2\frac{n-m}{m-1}} r_i^{(s)}}\sum_{i=1}^{I(s)}\left[\frac{[d_i^{(s)}]^{m(n-m)}}{[r_i^{(s)}]^n}\right]^{\frac{1}{m-1}}.$$

A passage to the limit in this inequality and condition B2 yield that

$$\lim_{s\to\infty} L_{11}^{(s)} = 0. \tag{74}$$

For the estimation of $L_{12}^{(s)}$ we use Hölder's and Young's inequalities to get

$$L_{12}^{(s)} \leqslant C\Bigg\{\sum_{\alpha=1}^{N}\sum_{i\in I_s''\cap I_{s\alpha}'}\int\limits_{D_i^{(s)}\setminus E_i^{(s)}}\left(\left|\frac{\partial[(1-\varphi_i^{(s)})v_i^{(s)}]}{\partial x}\right|^m+\left|\frac{\partial w_{i\alpha}^{(s)}}{\partial x}\right|^m\right)dx\Bigg\}^{\frac{2}{m}}.$$

Arguing as in the estimation of $L_{11}^{(s)}$, we see that $\lim_{s\to\infty} L_{12}^{(s)}=0$. This fact together with (74) and (73) imply that

$$\lim_{s\to\infty} L_1^{(s)}=0.$$

The convergence of $L_2^{(s)}$ and $L_3^{(s)}$ to zero as $s\to\infty$ is established analogously. Hence we have shown that

$$\lim_{s\to\infty} J_{32}^{(s)2}=0. \tag{75}$$

We still have to investigate the first term in the right-hand side of (70). In particular we shall show that

$$\begin{aligned}&\lim_{s\to\infty}\sum_{\alpha,j=1}^{N}\sum_{l=1}^{n}\sum_{i=1}^{I(s)} g_{i\alpha}^{(s)}\int\limits_{\Omega} A_l^j\left(x,\frac{\partial v_i^{(s)}}{\partial x}\right)\frac{\partial w_{i\alpha,j}^{(s)}}{\partial x_l}\,dx\\ &\quad=\sum_{\alpha,j=1}^{N}\int\limits_{\Omega} g_\alpha(x)C^{(j)}(x,f-u_0)\delta_\alpha^j\,dx,\end{aligned} \tag{76}$$

where $\delta_\alpha^j=1$ if $\alpha=j$, and $\delta_\alpha^j=0$ if $\alpha\neq j$; the function $C^{(j)}(x,f-u_0)$ is defined as in (23).

Since $u_0\in W_m^1(\Omega,N)$, there exists a uniformly bounded sequence of infinitely differentiable functions $u_k(x)$, $k=1,2,\ldots$, which converges to u_0 in $W_m^1(\Omega,N)$ as $s\to\infty$. For a given k let there be defined a number $d=d(k)>0$ such that for any set $E\subset\Omega$ with its diameter less than $2d$,

$$\max_{x,y}\big\{\big|u_k(x)-u_k(y)\big|+\big|f(x)-f(y)\big|+\big|g(x)-g(y)\big|\big\}<k^{-(m-1)}. \tag{77}$$

Let us represent $\bar\Omega=\bigcup_t\Omega_t$, $t=1,\ldots,T(k)$, $\Omega_t\cap\Omega_{t'}=\emptyset$ for $t\neq t'$, where the Ω_t are some sets with piecewise smooth boundaries such that their diameters is less than d for all t. Let $s_1=s_1(k)$ be such that for $s\geqslant s_1(k)$ and $i=1,\ldots,I(s)$, the inequality $d_i^{(s)}+r_i^{(s)}<d$ holds. Let $I_s(\Omega_t)=\{i\in\{1,\ldots,I(s)\}\colon x_i^{(s)}\in\Omega_t\}$. We have the following representation

$$\begin{aligned}&\sum_{\alpha,j=1}^{N}\sum_{l=1}^{n}\sum_{i=1}^{I(s)} g_{i\alpha}^{(s)}\int\limits_{\Omega} A_l^j\left(x,\frac{\partial v_i^{(s)}}{\partial x}\right)\frac{\partial w_{i\alpha,j}^{(s)}}{\partial x_l}\,dx\\ &\quad=\sum_{t=1}^{T}\sum_{\alpha,j=1}^{N}\sum_{l=1}^{n}\sum_{i\in I_s(\Omega_t)} g_{i\alpha}^{(s)}\int\limits_{\Omega_t} A_l^j\left(x,\frac{\partial v_i^{(s)}}{\partial x}\right)\frac{\partial w_{i\alpha,j}^{(s)}}{\partial x_l}\,dx\\ &\quad=M_1^{(s)}+M_2^{(s)}+M_3^{(s)},\end{aligned} \tag{78}$$

where

$$M_1^{(s)} = \sum_{t=1}^{T} \sum_{\alpha,j=1}^{N} \sum_{i\in I_s(\Omega_t)} g_{i\alpha}^{(s)} \Bigg\{ \sum_{l=1}^{n} \int_{\Omega_t} A_l^j \left(x, \frac{\partial v_i^{(s)}}{\partial x} \right) \frac{\partial w_{i\alpha,j}^{(s)}}{\partial x_l}\, dx - g_{i\alpha}^{(s)} \int_{\Omega_t} C^{(j)}\big(x, f_i^{(s)} - u_i^{(s)}\big)\delta_\alpha^j\, dx \Bigg\},$$

$$M_2^{(s)} = \sum_{t=1}^{T} \sum_{\alpha,j=1}^{N} \sum_{i\in I_s(\Omega_t)} \Bigg\{ g_{i\alpha}^{(s)} \int_{\Omega_t} C^{(j)}\big(x, f_i^{(s)} - u_i^{(s)}\big)\delta_\alpha^j\, dx - \int_{\Omega_t} g_\alpha(x) C^{(j)}(x, f - u_0)\delta_\alpha^j\, dx \Bigg\},$$

$$M_3^{(s)} = \sum_{j,\alpha=1}^{N} \int_{\Omega} g_\alpha(x) C^{(j)}\big(x, f_i^{(s)} - u_i^{(s)}\big)\delta_\alpha^j\, dx.$$

From the assumption C, we see that for any Ω_t in the partition of Ω and s sufficiently large

$$\sum_{j=1}^{N} \sum_{i\in I_s(\Omega_t)} \left[\int_{\Omega_t} A_l^j \left(x, \frac{\partial v_i^{(s)}}{\partial x} \right) \frac{\partial w_{i\alpha,j}^{(s)}}{\partial x}\, dx - \int_{\Omega_t} C^{(\alpha)}\big(x, f_i^{(s)} - u_i^{(s)}\big)\, dx \right] \leqslant \alpha_{1s},$$

with $\lim_{s\to\infty} \alpha_{1s} = 0$. This implies that

$$\lim_{s\to\infty} M_{11}^{(s)} = 0. \tag{79}$$

From the definition of $C^{(j)}(x,\cdot)$ and Theorem 6, we have for $\bar{k} = (k_1, \ldots, k_N)$ and $\hat{k} = (\hat{k}_1, \ldots, \hat{k}_N)$,

$$\big|C^{(j)}(x,\bar{k}) - C^{(j)}(x,\hat{k})\big| \leqslant C|\bar{k} - \hat{k}|^{\frac{1}{m-1}} \big[d_i^{(s)}\big]^{n-m},$$

for $|\bar{k}|, |\hat{k}| < \infty$. This inequality together with (77) and Theorem 5 imply

$$\begin{aligned} M_2^{(s)} &= \sum_{t=1}^{T} \sum_{\alpha,j=1}^{N} \sum_{i\in I_s(\Omega_t)} \Bigg\{ \int_{\Omega_t} \big(g_{i\alpha}^{(s)} - g_\alpha(x)\big) C^{(j)}\big(x, f_i^{(s)} - u_i^{(s)}\big)\delta_\alpha^j\, dx \\ &\quad - \int_{\Omega_t} g_\alpha(x)\big(C^{(j)}\big(x, f_i^{(s)} - u_i^{(s)}\big) - C^{(j)}(x, f - u_0)\big)\delta_\alpha^j\, dx \Bigg\} \\ &\leqslant C \sum_{t=1}^{T} \sum_{\alpha,j=1}^{N} \sum_{i\in I_s(\Omega_t)} \big\{ \big|g_{i\alpha}^{(s)} - g_\alpha(x)\big| + \big|f_i^{(s)} - f(x)\big|^{\frac{1}{m-1}} \\ &\quad + \big|u_i^{(s)} - u_0(x)\big|^{\frac{1}{m-1}} \big\} \times \big[d_i^{(s)}\big]^{n-m} \end{aligned}$$

$$\leqslant \frac{C}{k}\sum_{i=1}^{I(s)}\left[d_i^{(s)}\right]^{n-m} = \alpha_k. \tag{80}$$

By (40), it is clear that $\lim_{k\to\infty}\alpha_k = 0$.

Recapitulating all that has been done so far in this section, we conclude that

$$\sum_{j,\alpha=1}^{N}\int_{\Omega}\left\{\sum_{l=1}^{n}A_l^j\left(x,\frac{\partial u_0}{\partial x}\right)\frac{\partial g_j}{\partial x_l} + g_\alpha(x)C^{(j)}\big(x, f(x)-u_0(x)\big)\delta_\alpha^j\right\}dx = \beta_s + \alpha_k, \tag{81}$$

where α_k is from (80) and $\beta_s \to 0$, as $s \to \infty$. Since the left-hand side in this equation is independent of k and s, then we can make α_k and β_s as small as we wish for k and s, respectively large. This implies that u_0 satisfies the integral identity

$$\sum_{j,\alpha=1}^{N}\int_{\Omega}\left\{\sum_{l=1}^{n}A_l^j\left(x,\frac{\partial u_0}{\partial x}\right)\frac{\partial g_j}{\partial x_l} + g_\alpha(x)C^{(j)}\big(x, f(x)-u_0(x)\big)\delta_\alpha^j\right\}dx = 0,$$

for all $g \in \mathbf{C}_o^\infty(\Omega, N)$. This completes the proof of Theorem 3.

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