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KAM theory in momentum space and quasiperiodic Schrödinger operators

by

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ABSTRACT. — We construct a family of quasiperiodic Schrödinger operators in dimension one and in the tight binding approximation, having purely absolutely continuous spectrum. We work in momentum space and use a superconvergent approximation scheme to construct a unitary transformation that diagonalizes these operators on $L^2(B)$, $B=[-\pi, \pi)$ being the first Brillouin zone of the unperturbed part. The transformed operators are multiplications by a function $E_\infty(k) \in L^\infty(B)$ which might have a dense set of jump discontinuities and is the uniform limit as $n \to \infty$ of functions $E_n(k)$ with a finite number of discontinuities. Our control on the functions $E_n(k)$ and its first two derivatives is good enough to ensure the absence of pure point and singular continuous spectrum.

Key words : Quasiperiodic Schrödinger, small divisors, Spectral analysis.

RÉSUMÉ. — Nous construisons une famille d'opérateurs de Schrödinger en dimension un et avec interaction à courte portée, ayant un spectre purement absolument continu. Nous travaillons dans l'espace de Fourier et utilisons un schéma d'approximation super-convergent pour construire une transformation unitaire qui diagonalise ces opérateurs sur $L^2(B)$, $B=[-\pi, \pi]$ étant la première zone de Brillouin de la partie libre. Les

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opérateurs transformés sont des multiplications par une fonction $E_{\infty}(k) \in L^{\infty}(B)$ qui peut avoir un ensemble dense de sauts et est la limite uniforme quand $n \to \infty$ de fonctions $E_n(k)$ avec un nombre fini de discontinuités. Le contrôle sur les fonctions $E_n(k)$ et leur deux premières dérivées est suffisant pour impliquer l'absence de spectre ponctuel et singulier continu.

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1. INTRODUCTION

In this paper, we construct a class of finite-difference Schrödinger operators in dimension one with a weak quasi-periodic potential, whose spectrum is purely absolutely continuous. The operators we consider act on $l^2(\mathbb{Z})$ and have the form

$$\hat{H}_0 = -\Delta_{E_0} + \varepsilon \sum_{i=1}^{s} v_i^c \cos(\omega_i x) + v_i^s \sin(\omega_i x). \tag{1.1}$$

To explain the notations, let $B=[-\pi, \pi)$ and let $\mathcal{F}: l^2(\mathbb{Z}) \to L^2(B)$ be the Fourier transform operator such that if $u \in l^2(\mathbb{Z})$ we have

$$(\mathcal{F} u)(k) = \frac{1}{\sqrt{2\pi}} \sum_x e^{ikx} u(x) \tag{1.2}$$

The operator Δ_{E_0} is the generalized Laplacian given by

$$\Delta_{E_0} = \mathcal{F}^{-1} \mathbf{E}_0 \mathcal{F} \tag{1.3}$$

where $\mathbf{E}_0: L^2(B) \to L^2(B)$ is the operator of multiplication by the function $E_0(k)$ on B. If $E_0(k) = 1 - \cos k$, then (1.3) is the usual Laplacian operator. We assume that $E_0(k)$ belongs to the space $\mathscr{C}_p^r(B)$ of the functions on B satisfying periodic boundary conditions and having r bounded derivatives, for some fixed $r \geqq 2$. We refer $\mathscr{C}_p^r(B)$ to the topology induced by the norm

$$\| E(k) \|_r = \sup_{m=1,\ldots,r} \sup_{k \in B} \left| \frac{d^m}{dk^m} E(k) \right|. \tag{1.4}$$

The frequencies ω_i, $i=1, \ldots, s$, in (1.1) are numbers in $[-\pi, \pi]$ satisfying the following Diophantine conditions

$$\left| \sum_{i=1}^{s} p_i \omega_i + 2\pi q \right| \geqq D_0^{-1} \left(\sum_{i=1}^{s} |p_i| \right)^{-\theta_0} \tag{1.5}$$

for all $(s+1)$-ple of integers q, p_i, $i=1, \ldots, s$, for some $D_0 > 0$ and some $\theta_0 > s+1$. The set of these frequencies has full Lebesgue measure in $[-\pi, \pi]^s$. In (1.1), v_i^c and v_i^s are numbers in $[-1, 1]$. Finally, the coupling ε is our small perturbation parameter.

Our aim in this paper is to diagonalize the operator $\hat{H}_0$ in momentum space and to prove the following result:

THEOREM 1. – *For all choices of the parameters* ω_i, v_i^c, v_i^s $(i=1, \ldots, s)$ *satisfying the conditions above and for all integers* $r \geqq 3$, *there is a dense subset* $\mathscr{E}$ *of* $\mathscr{C}_p^r(B)$ *such that if* $E_0 \in \mathscr{C}$ *then the operator* $\hat{H}_0$ *in* (1.1) *has purely absolutely continuous spectrum for at least one value of the coupling* ε *and all eigenfunctions are uniformly bounded.*

In the rest of this introductory section we review preceeding results on this class of problems and explain our motivations to study the specific question of the existence of quasiperiodic Schrödinger operators with pure absolutely continuous spectrum with direct diagonalization methods. Then, we give an outline of the method we use to construct a family of such operators.

The first results obtained on one dimensional Schrödinger operators with weak quasi-periodic potentials concern the Floquet theory for operators with standard laplacian in the continuum of the following form

$$\mathbf{H} = -\frac{d^2}{dx^2} + \varepsilon v_\omega(x) \tag{1.6}$$

where

$$v_\omega(x) = \sum_i v_i^c \cos \omega_i x + v_i^s \sin \omega_i x. \tag{1.7}$$

The eigenvalue equation $\mathbf{H}\psi = \mathrm{E}\psi$ can be seen as the ordinary differential equation

$$\mathrm{X}'(t) = \begin{pmatrix} 0 & 1 \\ v_\omega(x) - \mathrm{E} & 0 \end{pmatrix} \mathrm{X}(t) \tag{1.8}$$

where $t = x$ and

$$\mathrm{X}(t) = \begin{pmatrix} \psi'(t) \\ \psi(t) \end{pmatrix}. \tag{1.9}$$

Equation (1.8) is said reducible if there is a matrix $\mathrm{A} \in sl(2, \mathbf{R})$ and an analytic matrix valued function $\mathrm{Y} : \mathbf{T}^{\mathrm{S}} \to \mathrm{GL}(2, \mathbf{R})$, $\mathbf{T}^{\mathrm{S}} = [0, 2\pi]^{\mathrm{S}}$ being the S dimensional torus, such that

$$\mathrm{X}(t) = \mathrm{Y}\left(\frac{\omega t}{2}\right) e^{\mathrm{A}t}. \tag{1.10}$$

If $v_\omega(x)$ is periodic, then the system (1.7) is reducible for all energies, see for instance [RS4]. On the other hand, the quasiperiodic case is not so straightforward. In case the frequencies ω_i, $i = 1, \ldots s$, satisfy a Diophantine condition as (1.5), reducibility depends on the arithmetic properties of the rotation number

$$\rho(\mathrm{E}) = \lim_{|t| \to \infty} \arg \mathrm{X}(t) v, \tag{1.11}$$

where $v \in \mathbf{R}^2$. One can show [JM] that this limit exists, it doesn't depend on v and $\rho(\mathrm{E})$ is a monotone continuous function of E. Moreover, the spectrum of $\mathbf{H}$ is given by

$$\sigma(\mathbf{H}) = \overline{\mathbf{R} \setminus \rho^{-1}(\mathcal{U})}, \tag{1.12}$$

$\mathcal{U}=\left\{\sum_{j=1}^{s} p_i \omega_i,\ p_i \in \mathbb{Z}\right\}$ being the frequency module. Thus, if $\rho(\mathrm{E}) \in \mathcal{U}$ then E is in the closure of a gap. In this case, Moser and Poschel [MP] prove that the system (1.7) is reducible and that $\mathrm{A}=0$ if $\{\mathrm{E}\}$ is a collapsed gap, A is nilpotent and $\neq 0$ if E is the endpoint of a gap and $\det \mathrm{A} \neq 0$ if E is inside a gap. The opposite case in which $\rho(\mathrm{E})$ is Diophantine with respect to the frequency module $\mathcal{U}$, *i.e.*

$$\left|\rho(\mathrm{E})-\frac{1}{2}\sum_{i=1}^{s} p_i \omega_i\right| \geqq \frac{\mathrm{C}}{(\Sigma|p_i|)^{\sigma}} \tag{1.13}$$

for some c, $\sigma>0$ and all $p_i \in \mathbb{Z}$, is studied in the pioneering work [DS] by Dinaburg and Sinai, who show the existence of a subset of $\sigma(\mathbf{H})$ of large but not full Lebesgue measure for which $\rho(\mathrm{E})$ is Diphantine and system (1.8) is reducible. Finally, Eliasson [E] improved these results by showing reducibility for all energies for which $\rho(\mathrm{E})$ is either Diophantine or rational.

Eliasson also proves that if $\rho(\mathrm{E})$ is Liouville, i.e. neither Diophantine with respect to $\mathcal{U}$ nor in $\mathcal{U}$, then we have

$$\inf \lim_{|t| \to \infty} |\mathrm{X}(t)-\mathrm{X}(0)| < \frac{1}{2}|\mathrm{X}(0)| \tag{1.14}$$

and

$$\lim_{|t| \to \infty} \frac{|\mathrm{X}(t)|}{|t|}=0. \tag{1.15}$$

This is far from implying reducibility. On the contrary, Eliasson can also show that for a generic quasiperiodic potential there exists an energy for which $\rho(\mathrm{E})$ is Liouville and $\mathrm{X}(t)$ is unbounded. Finally, by using an idea in the paper by Delyon and Sinai [DS], Eliasson also proves that the spectrum is purely absolutely continuous. Hence, in the one dimensional case Floquet theory gives a complete description of the spectral properties of quasiperiodic operators with small potential or for large energies. However, these methods are genuinely one-dimensional and they cannot be extended to higher dimensions.

A result that may prove more useful to understand the two dimensional case is the one proven by Chulayevskij and Delyon [CD] according to which the Schrödinger operator

$$\mathbf{H}=-\Delta+\varepsilon \cos \omega x \tag{1.16}$$

has purely absolutely continuous spectrum for small ε. Here, Δ is the standard discretized laplacian and ω is a Diophantine number. By means of an Aubry [AA] duality transformation, the problem can be reduced to the localization result in the strong coupling regime for large ε obtained by Sinai [S] and by Fröhlich, Spencer and Wittwer [FSW]. It turns out that the information provided by Sinai's constructive proof of localization suffices to exclude the presence of the point and singular continuous components from the spectrum of **H**. Unfortunately, Aubry's transformation has a quite limited range of validity and applies only to the case of one frequency only. In this paper, we make use of ideas similar to those in Sinai's paper to set up a superconvergent algorithm that allows one to diagonalize the operator in (1.1) with an arbitrary number of frequencies and without using any mapping to the strong coupling regime. The interest of the problem is that, unlike Floquet theory, such constructions can be generalized to higher dimensions. The new difficulty one finds as one tries to extend this construction to dimension two is that the resonances are not isolated points in the Brillouin zone, but they are lines that can generically have pairwise intersections. Very interestingly, this problem resembles very closely the problem of overlapping singularities that Chulayevskij and Sinai solved in [DS] where they extend the proof of localization to the case with two independent frequencies. It is thus conceivable that by combining their ideas with the KAM theory in momentum space developed in this paper one may be able to control the spectrum of two dimensional quasiperiodic Schrödinger operators.

Our method to study the spectrum of the operator (1.1) is based on a superconvergent algorithm of KAM type in momentum space by means of which we diagonalize the Hamiltonian operator. In order the iteration scheme to proceed, a large number of non-resonance conditions have to be fulfilled at each step of the inductive construction. This forces us to play with the dispersion law $E_0(k)$ itself as we try to avoid resonances by excluding a set of function $E_0(k)$ at each step. More precisely, to prove Theorem 1 we fix a small $\varepsilon>0$, an integer $r\geqq 3$ and a dispersion law $E_0(k)\in\mathscr{C}^r$ and we define a family $E_0(z;k)\in\mathscr{C}^r$ indexed by z in a set $Z=\bigotimes_{n=1}^{\infty}[0,1]$. By varying z in Z we can change $E_0(z;k)$ on intervals of arbitrarily small size so that the $\mathscr{C}^r$ norm doesn't vary too much. At each iteration step we eliminate a subset of parameters z in Z for which resonances occur. In this way, we obtain a decreasing sequence of sets

$$Z\supset Z_1(\varepsilon)\supset\ldots\supset Z_m(\varepsilon)\supset\ldots \tag{1.17}$$

The set

$$Z_\infty(\varepsilon)=\bigcap_{m=1}^{\infty} Z_m(\varepsilon) \tag{1.18}$$

containing the values of z for which the diagonalization can be completed, turns out to be a Cantor set depending on ε and of measure $1-0(\varepsilon)$.

The algorithm we use to diagonalize our operator is a KAM type superconvergent algorithm with an infinite number of adjustable parameter by means of which we construct a unitary operator $\mathbb{U}$ that diagonalizes our Hamiltonian. Since we work in the momentum representation, $\mathbb{U}$ is defined on the space $L^2([-\pi, \pi))$, where $[-\pi, \pi)$ is the first Brillouin zone. Our algorithm produces $\mathbb{U}$ as an infinite product

$$\mathbb{U} = \lim_{n \to \infty} \mathbb{U}_1 \dots \mathbb{U}_n \tag{1.19}$$

where $\mathbb{U}_n$ is the unitary operator computed at the n-th iteration step. After n iterations, the renormalized Hamiltonian

$$\mathbf{H}_n = \mathbb{U}_n^{-1} \dots \mathbb{U}_1 \mathbf{H} \mathbb{U}_1 \dots \mathbb{U}_n \tag{1.20}$$

has the form

$$\mathbf{H}_n = \mathrm{E}_n(k) + \varepsilon_n \int dk \sum_{\omega \in \mathcal{U}} v_n(\omega; k) \,|\, t_\omega k \rangle \langle k \,| + O(\varepsilon_n^2) \tag{1.21}$$

where ε_n is the renormalized coupling

$$\varepsilon_n = (\varepsilon)^{(3/2)^n} \tag{1.22}$$

and $v_n(\omega; k)$ are functions such that $v_n(\omega; k) = 0$ if $|\omega| > C^n$, where

$$\left| \sum_{i=1}^{s} p_i \omega_i \right| \equiv \sum_{i=1}^{s} |p_i|. \tag{1.23}$$

At the n-th interation, we eliminate the non diagonal terms of order ε_n by means of a two steps procedure. First, we consider the singular values of k for which

$$\mathrm{E}_n(k) = \mathrm{E}_n(t_\omega k) \tag{1.24}$$

for some ω such that $v_n(\omega; k) \neq 0$. By restricting the set of allowed dispersion laws, we can assume that there are *no overlapping divergences*. This allows us to eliminate the matrix elements $v_n(\omega; k) \,|\, t_\omega k > < k \,|$ corresponding to resonant transitions by means of a unitary operator given by a direct intergral of 2×2 matrices. At this point, one can eliminate the other terms of order ε_n by means of a unitary transformation determined by an homology equation, as is commonly done in KAM theory. During this process, the functions $\mathrm{E}_n(k)$ aquire jump discontinuities each time we eliminate a resonant transition. At the end we obtain a function

$$\mathrm{E}_\infty(k) = \lim_{n \to \infty} \mathrm{E}_n(k) \tag{1.25}$$

with a dense set of jump discontinuities which gives the diagonalization of $\mathbf{H}$. As a corollary of all the information we have to accumulate on $\mathrm{E}_n(k)$ in order to control the iterative procedure, we finally obtain the following result which implies Theorem 1:

THEOREM 2. — *Let us fix the parameters ω_i, v_i^c, v_i^s, $i = 1, \dots, s$, satisfying the conditions above, let r be an integer ≥ 3 and let $\hat{\mathrm{E}}_0 \in \mathscr{C}_p^r(\mathrm{B})$. For all*

$\mu>0$ *there is an* $\varepsilon_0>0$ *and for all* $\varepsilon\in(0, \varepsilon_0]$ *there is a* $E_0\in\mathscr{C}^r_p(B)$ *such that* $\| \hat{E}_0-E_0 \|_r<\mu$, *the operator* (1.1) *has purely absolutely continuous spectrum and all its generalized eigenfunctions are bounded in configuration space. More precisely, in this case there is a unitary operatori* $\mathbb{U}$ *on* $L^2(B)$ *such that* $\mathbb{U}^{-1} H_0 \mathbb{U}$ *is an operator of multiplication by an* $L^\infty(B)$*-function* $E_\infty(k)$ *such that*

$$\sup_{0<\bar{\varepsilon}<1} \int_{-\infty}^{\infty} dx \left(\int_B dk \frac{\bar{\varepsilon}}{(E_\infty(k)-x)^2+\bar{\varepsilon}^2} \right)^{3/2} < \infty. \tag{1.26}$$

The fact that condition (1.7) implies that the operator of multiplication by the function $E_\infty(k)$ has no singular continuous spectrum, is a consequence of Stone's formula (*see* [13] for details).

2. NOTATIONS, STRATEGY OF THE PROOF AND INDUCTION HYPOTHESIS

2.1. Introduction and Basic Notations

To prove the theorems in section 1, we fix an s-uple of frequencies (ω_α), $\alpha=1, \ldots, s$, satisfying the Diophantine conditions (1.5) and, for all integers $r\geqq 2$, all $\varepsilon>0$ small enough and all $\hat{E}_0\in\mathscr{C}^r_p(B)$, we give a fairly explicit construction of a dispersion law $E_0\in\mathscr{C}^r_p(B)$ which is close to $\hat{E}_0$ in $\mathscr{C}^r_p(B)$-norm and of a unitary operator U on $L^2(B)$ which diagonalizes the Hamiltonian (1.6). Our construction is iterative and at each step we establish several induction hypotheses. The purpose of this section is to introduce some basic notations, to illustrate the strategy of the construction and to state the induction hypothesis.

Notations. – In the following we introduce several positive constants denoted with D_i, θ_i where i is an integer $\geqq 1$. There is also a constant $\alpha>0$. The value of these parameters is given in subsection 2.8.

Let

$$B=[\pi, -\pi) \tag{2.1.1}$$

be our choice for the first Brillouin zone. If $\omega\in\mathbf{R}$, let $t_\omega: B\to B$ be the map such that

$$t_\omega k=k+\omega \bmod 2\pi \tag{2.1.2}$$

for all $k\in B$. It is convenient to think of B as of the circle S^1. If a, $b\in B\approx S^1$ are not antipodal points, let $[a, b]$ denote the shortest closed arc joining a to b and let $d(a, b)$ denote its length.

Let us introduce the set

$$\mathcal{U}=\left\{\omega\in\mathbf{R}\,\middle|\,\omega=\sum_{\alpha=1}^{s}p_\alpha\omega_\alpha \text{ with } p_\alpha\in\mathbb{Z}\right\}. \tag{2.1.3}$$

If

$$\omega=\sum_{\alpha=1}^{s}p_\alpha\omega_\alpha\in\mathcal{U}\setminus\{0\}, \tag{2.1.4}$$

let

$$\|\omega\|=\sum_{\alpha=1}^{s}|p_\alpha| \tag{2.1.5}$$

denote its order. By convention, we also set

$$\|0\|=1. \tag{2.1.6}$$

If $\omega\in\mathbf{R}$, let $|\omega|$ denote the modulus of the number $\omega'\in\mathrm{B}$ such that

$$\omega'=\omega \bmod 2\pi. \tag{2.1.7}$$

Finally, if $n\geqq 0$ let us define the set

$$\mathcal{U}_n=\{\omega\in\mathcal{U}\setminus\{0\}\mid \|\omega\|\leqq 2^{\theta_1 n}\}. \tag{2.1.8}$$

2.2. The unperturbed dispersion law

Let us fix a function $\hat{\mathrm{E}}_0(k)\in\mathscr{C}_p^{r+1}(\mathrm{B})$. Without restricting the generality, we can assume that $\hat{\mathrm{E}}_0(k)$ is a Morse function (*see* [M]) and that

$$|\hat{\mathrm{E}}_0(k)|,\ \left|\frac{d}{dk}\hat{\mathrm{E}}_0(k)\right|,\ \left|\frac{d^2}{dk^2}\hat{\mathrm{E}}_0(k)\right|\leqq\left|\frac{d^3}{dk^3}\hat{\mathrm{E}}_0(k)\right|\leqq\frac{1}{2} \tag{2.2.1}$$

for all $k\in\mathrm{B}$. We consider a family $\mathrm{E}_0(z,k)$, $z\in\mathrm{Q}$, of functions in $\mathscr{C}_p^{r+1}(\mathrm{B})$ close to $\hat{\mathrm{E}}_0(k)$ with respect to the $\mathscr{C}_p^r(\mathrm{B})$ topology. The index set Q has the following form

$$\mathrm{Q}=\mathop{\times}_{p=1}^{\infty}\mathrm{Q}_p \tag{2.2.2}$$

where

$$\mathrm{Q}_p=\mathrm{Q}_p^{\mathrm{K}}\times\mathrm{Q}_p^{\mathrm{H}}, \tag{2.2.3}$$

$$\mathrm{Q}_p^{\mathrm{K}}=\mathop{\times}_{\nu=1}^{\nu_p}[-1,1],\qquad \mathrm{Q}_p^{\mathrm{H}}=\mathop{\times}_{\beta=1}^{\beta_p}[-1,1] \tag{2.2.4}$$

and ν_p, β_p are given by

$$\nu_p=\{2\pi\mathrm{D}_2 2^{\theta_2 p}\},\qquad \beta_p=\{4\pi\mathrm{D}_3 2^{\theta_3 p}\} \tag{2.2.5}$$

where $\{\,.\,\}$ denotes the integer part plus one. A typical element of Q is a double sequence

$$\left.\begin{array}{c} z=(z_p^k(\nu),\ z_p^{\mathrm{H}}(\beta)), \\ p=0,1,\ldots, \qquad \nu=1,\ldots,\nu_p, \qquad \beta=1,\ldots\beta_p \end{array}\right\} \qquad (2.2.6)$$

of numbers $z_p^k(\nu)$, $z_p^{\mathrm{H}}(\beta)\in[-1,1]$. The function $\mathrm{E}_0(z;k)$ has the following form:

$$\mathrm{E}_0(z;k)=\hat{\mathrm{E}}_0(k)+\sum_{p=1}^{\infty}\left(\sum_{\nu=1}^{\nu_p} z_p^k(\nu)\,\varphi_p^k(\nu;k)\right.$$
$$\left.+\sum_{\beta=1}^{\beta_p(z)} z_p^{\mathrm{H}}(\beta)\,\varphi_p^{\mathrm{H}}(z,\beta;k)\right) \qquad (2.2.7)$$

Here, $\beta_p(z)$ is an integer $\leqq\beta_p$ depending on z and $\varphi_p^k(\nu;k)$ and $\varphi_p^{\mathrm{H}}(z,\beta;k)$ are functions in $\mathscr{C}_p^\infty(\mathrm{B})$ whose definition requires some new notations.

Notations. – If $t\in\mathbf{R}$, let $k_p(t)\in\mathrm{B}$ be the point such that

$$k_p(t)=t\,\mathrm{D}_2^{-1}z^{-\theta_2 p}\bmod 2\pi. \qquad (2.2.8)$$

Let us introduce the following arcs of B

$$\mathrm{K}_p(\nu)=\left[k_p\left(\nu-\frac{3}{4}\right),\,k_p\left(\nu+\frac{3}{4}\right)\right] \qquad (2.2.9)$$

and

$$\bar{\mathrm{K}}_p(\nu)=[k_p(\nu-1),\,k_p(\nu+1)]. \qquad (2.2.10)$$

Finally, let $f\in\mathscr{C}^\infty(\mathbf{R})$ be a nonincreasing function such that $f(x)=1$ if $\alpha\leqq 0$, $f(x)=0$ if $x\geqq 1/4$ and

$$|f'(x)|\leqq 2^4, \qquad |f''(x)|\leqq 2^8 \qquad (2.2.11)$$

for all $x\in\mathbf{R}$. An example of such a function is given by the convolution

$$f(x)=\int dy\,f_1(y-x)\,f_2(y) \qquad (2.2.12)$$

where

$$f_1(x)=\begin{cases} 1 & \text{if } t\leqq 2^{-4} \\ 1-2^7(t-2^{-4})^2 & \text{if } 2^{-4}\leqq t\leqq 2^{-3} \\ 2^7(t-3.2^{-4})^2 & \text{if } 2^{-3}\leqq t\leqq 3.2^{-4} \\ 0 & \text{if } 3.2^{-4}\leqq t \end{cases} \qquad (2.2.13)$$

and

$$f_2(x)=\left\{\begin{array}{ll} \mathrm{N}\exp((x+2^{-4})^{-1}(x-2^{-4})^{-1}) & \text{if } -2^{-4}\leqq x\leqq 2^{-4} \\ 0 & \text{otherwise.} \end{array}\right\} \qquad (2.2.14)$$

Here, N is a normalization factor such that

$$\int dx f_2(x)=1. \tag{2.2.15}$$

The functions $\varphi_p^k(\nu;k)$ have the form

$$\varphi_p^k(\nu;k)=D_2^{-(r+1)}2^{-(r+1)\theta_2 p}f\left(-D_2 2^{\theta_2 p}\left(k-k_p\left(\nu+\frac{3}{4}\right)\right)\right)\times f\left(D_2 2^{\theta_2 p}\left(k-k_p\left(\nu-\frac{3}{4}\right)\right)\right) \tag{2.2.16}$$

In (2.2.16) and in the following, the arithmetic operations on B are defined mod 2π. Let us remark that we have

$$0\leqq\varphi_p^K(\nu;k)\leqq D_2^{-(r+1)}2^{-(r+1)\theta_2 p} \tag{2.2.17}$$

for all $k\in B$. The first inequality in (2.2.17) is saturated for $k\in B\setminus\bar{K}_p(\nu)$ and the second one for $k\in K_p(\nu)$. Moreover, we have

$$\sup_{k\in B}\left|\frac{d^m}{dk^m}\varphi_p^k(\nu;k)\right|\leqq\|f_2\|_r D_2^{-(r+1-m)}2^{-(r+1-m)\theta_2 p} \tag{2.2.18}$$

for all m such that $1\leqq m\leqq r$.

The functions $\varphi_p^H(z,\nu;k)$ have the form

$$\varphi_p^H(z,\nu;k)=D_3^{-(r+1)}2^{-(r+1)\theta_3 p}f\left(-D_3 2^{\theta_3 p}(k-h_p(z,\beta))-\frac{3}{4}\right)\times f\left(D_3 2^{\theta_3 p}(k-h_p(z,\beta))+\frac{2}{3}\right) \tag{2.2.19}$$

The definition of the points $h_p(z,\beta)$ and of the integer $\beta_p(z)$ in (2.2.7) is given in Section 3. This definition is such that the distance among two points $h_p(z,\beta)$, $h_p(z,\beta')$ with $\beta\neq\beta'$ and $\beta,\beta'\leqq\beta_p(z)$, is not less than $D_3^{-1}2^{-\theta_3 p}$. Moreover, we have

$$\sup_{k\in B}\left|\frac{d^m}{dk^m}\varphi_p^H(z,\nu;k)\right|\leqq\|f_2\|_r D_3^{-(r+1-m)}2^{-(r+1-m)\theta_3 p} \tag{2.2.20}$$

for all m such that $1\leqq m\leqq r$.

This proves the first part of the following lemma:

LEMMA 2.1. – *The function* $E_0(z;k)$ *belongs to* $\mathscr{C}_p^r(B)$ *for all* $z\in Q$ *and for all* $\mu>0$ *there are constants* $D_2(\mu)$, $D_3(\mu)<\infty$ *such that if* $D_2\geqq D_3(\mu)$ *and* $D_3\geqq D_3(\mu)$, *then we have*

(i) $$\|E_0(z;\,.)-\hat{E}_0(.)\|_r<\mu \tag{2.2.21}$$

(ii) *Let* $k_i(z)$, $i=1,\ldots,c_0$, *be the critical points of* $E_0(z;\,.)$. *If* $\hat{E}_0(k)$ *is generic and* $\mu>0$ *is small enough, then* c_0 *does not depend on* z *and there* *are two constants* $\lambda_1, \lambda_2>0$ *such that for all* $z\in Q$ *we have*

$$\left|\frac{d^2}{dk^2}E_0(z;k)\right|\geqq\frac{1}{2}\lambda_1 \tag{2.2.22}$$

for all k with $|k-k_i(z)|\leqq\frac{1}{8}\lambda_1$ *for some* $i=1,\dots,c_0$ *and*

$$\left|\frac{d}{dk}E_0(z;k)\right|\geqq\frac{1}{2}\lambda_2 \tag{2.2.23}$$

otherwise.

(iii) *Under the hypothesis in* (ii), *we have*

$$\sup_{0<\bar{\varepsilon}<1}\int dx\left[\int_B dk\frac{\bar{\varepsilon}}{(E_0(z;k)-x)^2+\bar{\varepsilon}^2}\right]^{3/2} \leqq 2^{7/2}\rho_0^{-4}+\frac{(2c_0)^{3/2}\rho_0}{\pi\lambda_2}+\frac{(2c_0)^{3/2}\rho_0^{1/2}}{\pi\sqrt{\lambda_1}} \tag{2.2.24}$$

where

$$\rho_0=\min(2^{-5}\lambda_1\lambda_2,\ 2^{-11}\lambda_1^3). \tag{2.2.25}$$

Proof (The following proof can be skipped in a first reading).

(ii) A generic $\hat{E}_0\in\mathscr{C}_p^{r+1}(B)$ is a Morse function. In particular, it has a finite number c_0 of critical points k_i, $i=1,\dots,c_0$, and we have

$$\left|\frac{d^2}{dk^2}\hat{E}_0(k_i(z))\right|\geqq\lambda_1 \tag{2.2.26}$$

for all $i=1,\dots,c_0$ and some $\lambda_1>0$. Thanks to (2.2.1), if $|k-k_i(z)|\leqq\lambda_1$ for some $i=1,\dots,c_0$, we have

$$\left|\frac{d^2}{dk^2}\hat{E}_0(k)\right|\geqq\frac{3}{4}\lambda_1 \tag{2.2.27}$$

for all k such that $|k-k_i(z)|\leqq\frac{1}{4}\lambda_1$. Let us define λ_2 as follows:

$$\lambda_2=\min\left\{\left|\frac{d}{dk}E_0(k)\right| \text{ for } k \text{ such that } |k-k_i(z)|\geqq\frac{1}{4}\lambda_1 \text{ for all } i=1,\dots,c_0\right\}. \tag{2.2.28}$$

If

$$\mu<\min\left(\frac{1}{2}\lambda_2,\ 3.2^{-8}\lambda_1^3,\frac{1}{2}\lambda_1\right), \tag{2.2.29}$$

then for all $z \in Q$, all critical points $k_i(z)$ of the function $E_0(z; k)$ are at a distance $< \frac{1}{8}\lambda_1$ from a critical point of $\hat{E}_0(k)$ and (2.2.22) and (2.2.23) hold.

(iii) Let us define the set

$$\mathscr{A}(z; x) = \{ k : |E_0(z; k) - x| \leqq \rho_0 \} \qquad (2.2.30)$$

for all $z \in Q$ and $x \in \mathbf{R}$ $\mathscr{A}(z; x)$ is the union of a family of disjoint intervals $\mathscr{A}_\alpha(z; x)$ of length $\leqq \frac{1}{8}\lambda_1$. If α is such that the distance between $\mathscr{A}_\alpha(z; x)$ and the closest critical point of $E_0(z; k)$ is $\geqq \frac{1}{8}\lambda_1$, then the function $E_0(z; k)$ takes the value α in $\mathscr{A}_\alpha(z; x)$ and, if $\bar{\varepsilon} \in (0, 1)$, we have

$$\int_{\mathscr{A}_\alpha(z; x)} dk \frac{\bar{\varepsilon}}{(E_0(z; k) - x)^2 + \bar{\varepsilon}^2} \leqq \int \frac{\bar{\varepsilon}\, dk}{(1/2\lambda_2 k)^2 + \bar{\varepsilon}^2} = \frac{1}{\pi\lambda_2}. \qquad (2.2.31)$$

Otherwise, if $\mathscr{A}_\alpha(z; x)$ is at distance $\leqq \frac{1}{8}\lambda_1$ from a critical point $k_i(z)$ of $E_0(z; k)$ and $y = E_0(z; k_i(z)) - x$, we have

$$\int_{\mathscr{A}_\alpha(z; x)} dk \frac{\bar{\varepsilon}}{(E_0(z; k) - x)^2 + \bar{\varepsilon}^2} \leqq \int \frac{\bar{\varepsilon}\, dk}{(1/4\lambda_1 k^2 - y)^2 + \bar{\varepsilon}^2}$$

$$= \frac{1}{2\pi\sqrt{\lambda_1}} \frac{|y|}{(y^2 + \bar{\varepsilon}^2)^{3/4}} \leqq \frac{1}{2\pi}(\lambda_1 |y|)^{-1/2} \qquad (2.2.32)$$

Hence we have

$$\sup_{0 < \bar{\varepsilon} < 1} \int dx \left[\int dk \frac{\bar{\varepsilon}}{(E_0(z; k) - x)^2 + \bar{\varepsilon}^2} \right]^{3/2}$$

$$\leqq \sqrt{2}\,.\,8\,\rho_0^{-4} + \sqrt{2}\, c_0^{3/2} \int_{-\rho_0}^{\rho_0} \left(\frac{1}{\pi\lambda_2} + \frac{1}{2\pi}(\lambda_1 |y|)^{-1/2} \right) dy$$

$$= 2^{7/2} \rho_0^{-4} + \frac{(2c_0)^{3/2}\rho_0}{\pi\lambda_2} + \frac{(2c_0)^{3/2}}{\pi\sqrt{\lambda_1}} \rho_0^{1/2}. \text{Q.E.D.} \qquad (2.2.34)$$

2.3. An Outline of the Strategy

If $n \geqq 0$, the n times renormalized Hamiltonian operator $\mathbb{H}_n(z)$ is defined for z restricted to a subset Z_n of Q. If $n = 0$, we have $Z_n = Q$. At the $(n+1)$ st iteration step we construct a set

$$Z_{n+1} \subset Z_n \qquad (2.3.1)$$

and, for all $z \in Z_{n+1}$, we define a unitary transformation $\mathbb{U}_n(z)$ on $L^2(B)$ which gives $\mathbf{H}_{n+1}(z)$, *i.e.*

$$\mathbf{H}_{n+1}(z) = \mathbb{U}_n(z)^{-1} \mathbf{H}_n(z) \mathbb{U}_n(z) \tag{2.3.2}$$

The operator $\mathbf{H}_n(z)$ has the form

$$\mathbf{H}_n(z) = \mathbf{E}_n(z) + \varepsilon_n \mathbb{V}_n(z) + \sum_{N=2}^{\infty} \varepsilon_n^N \mathbf{H}_n(z) \tag{2.3.3}$$

where

$$\varepsilon_n = \varepsilon^{(3/2)^n} \tag{2.3.4}$$

is the renormalized coupling, $\mathbf{E}_n(z)$ is a multiplication operator by a function $E_n(z; k)$ that we call the n-th renormalized dispersion law and $\mathbb{V}_n(z)$, $\mathbf{H}_{nN}(z)$, $N \geqq 2$, are operators of the form

$$\mathbb{V}_n(z) = \sum_{\omega \in \mathcal{U}_n} \int_B dk\, v_n(z, \omega; k) \,|\, t_\omega k \rangle \langle k \,| \tag{2.3.5}$$

$$\mathbf{H}_{nN}(z) = \sum_{\omega \in \mathcal{U}} \int_B dk\, h_{nN}(z, \omega; k) \,|\, t_\omega k \rangle \langle k \,|. \tag{2.3.6}$$

If $n=0$, we have

$$v_0(z, \omega; k) = \sum_{\alpha=1}^{s} \frac{1}{2} \delta_{\omega, \omega_\alpha} (v_{\omega\alpha}^c + v_{\omega_\alpha}^s) + \frac{1}{2} \delta_{\omega, -\omega_\alpha} (v_{-\omega_\alpha}^c - v_{-\omega_\alpha}^s) \tag{2.3.7}$$

Let us remark that at each renormalization step we perform a *resummation* in ε. Consequently, the functions $E_n(z; k)$, $v_n(z, \omega; k)$ and $h_{nN}(z, \omega; k)$ depend on ε. Also the set $Z_n \subset Q$ depends on ε. However, we are going to neglect this dependency in our notations.

The unitary operator $\mathbb{U}_n(z)$ is defined in such a way as to diagonalize the truncated Hamiltonian

$$\mathbf{E}_n(z) + \varepsilon_n \mathbb{V}_n(z) \tag{2.3.7'}$$

up to terms of order ε_n^2. If $\omega \in \mathcal{U}_n$, let us consider the function

$$\mathcal{E}_n(z, \omega; k) = E_n(z; t_\omega k) - E_n(z; k) \tag{2.3.8}$$

and the corresponding zero set

$$\Lambda_n(z, \omega) = \{ k \in B \,|\, \mathcal{E}_n(z, \omega; k) = 0 \} \tag{2.3.9}$$

The regions of B close to $\Lambda_n(z, \omega)$ require a special consideration because there the non-diagonal matrix elements in $\varepsilon_n \mathbb{V}_n(z)$ can dominate. Let us remark that the resonances we have to consider at the n-th step correspond to frequencies ω in the set $\mathcal{U}_n$ defined in (2.1.8). In fact, these are the only frequencies entering in the expansion (2.3.5) for $\mathbb{V}_n(z)$. This is one of the basic features of our construction.

As a consequence of the resonances, the function $E_n(z; k)$ is piecewise differentiable and has a finite number of jump discontinuities located on the "jump set" $J_n(z) \subset B$. Such a set has the form

$$J_n(z) = J_n^p(z) \cup J_n^s(z), \tag{2.3.10}$$

where "p" stands for principal and "s" for secondary. We have

$$J_n^p(z) = \bigcup_{m=0}^{n-1} \bigcup_{\omega \in \mathcal{U}_m \setminus \mathcal{U}_{m-1}} \Lambda_m(z, \omega) \tag{2.3.11}$$

where $\mathcal{U}_{-1} = \varnothing$. The nonresonance conditions we impose to define Z_n imply that

$$\Lambda_{m_1}(z, \omega_1) \cap t_{\omega_0} \Lambda_{m_2}(z, \omega_2) = \varnothing \tag{2.3.12}$$

for $m_1, m_2 \leqq m$, $\omega_1 \in \mathcal{U}_{m_1} \setminus \mathcal{U}_{m_1 - 1}$, $\omega_2 \in \mathcal{U}_{m_2} \setminus \mathcal{U}_{m_2 - 1}$, $\omega_1 \neq \omega_2$ and $\omega_0 \in \mathcal{U}$ or for $\omega_1 = \omega_2$ and $\omega_0 \in \mathcal{U} \setminus \{0\}$. In particular, we see that if $j(z) \in J_n^p(z)$, then there exists one and only one frequency

$$\omega_n(z; j(z)) \in \mathcal{U}_n \tag{2.3.13}$$

such that

$$j(z) \in \Lambda_m(z, \omega_n(z; j(z))) \tag{2.3.14}$$

for some $m \leqq n$. Let us remark that in this case we have

$$t_{\omega_n(z; j(z))} j(z) \in \Lambda_m(z, -\omega_n(z; j(z))). \tag{2.3.15}$$

The two jumps $j(z)$ and $t_{\omega_n(z; j(z))} j(z)$ of $J_n^p(z)$ are said to be mutually conjugated. Finally, the secondary jump set is defined as follows:

$$J_n^s(z) = \bigcup_{\omega \in \mathcal{U}_n} t_\omega J_n^p(z). \tag{2.3.16}$$

Also the functions $v_n(z, \omega; k)$ have only jump discontinuities which are located inside $J_n(z)$. If $N \geqq 2$, the functions $h_{nN}(z, \omega; k)$ have two derivatives for $k \in B \setminus J_n^N(z)$ and jump discontinuities in the set

$$J_n^N(z) = \bigcup_{\|\omega\| \leqq N\, 2^{\theta_1 n}} t_\omega J_n^p(z). \tag{2.3.17}$$

2.4. The Nonresonance Conditions

Let us introduce some notations

Notations. – If p is an integer $\geqq 1$, let $\delta(p)$ be the least integer $\geqq 0$ for which we have

$$\varepsilon_{\delta(p)} \leqq D_4^{-1} 2^{-\theta_4 p - \delta(p) - 1}. \tag{2.4.1}$$

If n is an integer $\geqq 0$, let $\delta_0^{-1}(n)$ be defined as the least integer such that

$$\delta(\delta_0^{-1}(n)) = n. \tag{2.4.2}$$

Let us remark that we have

$$\varepsilon_{n-1} \leqq D_4^{-1} 2^{-\theta_4 \delta_0^{-1}(n-1)-n} \tag{2.4.3}$$

Let

$$\Pi_{<p}: \quad Q \to \underset{1 \leqq p' \leqq p}{\times} Q_{p'} \tag{2.4.4}$$

be the map such that if $(z_{p'})_{p' \geqq 1} \in Q$, then

$$\Pi_{<p} z = (z_{p'})_{1 \leqq p' \leqq p}. \tag{2.4.5}$$

Finally, let $F_p : \mathscr{P}(Q) \to \mathscr{P}(Q)$ be the map such that if $A \subset Q$ is a subset, then

$$F_p A = \{ z \in Q \mid \exists z' \in A \text{ for which } \Pi_{<\delta_0^{-1}(p-1)} z = \Pi_{<\delta_0^{-1}(p-1)} z' \}. \tag{2.4.6}$$

At the $(n+1)$-st iteration step, $n \geqq 0$, we define the sets $\Delta_{np} \subset Q$ for all $p \in \mathscr{P}_n$, where

$$\mathscr{P}_n = \{ p \in N \mid n+1 \leqq p < \delta_0^{-1}(n) \}. \tag{2.4.7}$$

Let us set

$$Z_{np} = \bigcap_{\substack{q \leqq p \\ m \leqq n}} \Delta_{mq} \tag{2.4.8}$$

The set Z_{n+1} is defined as follows:

$$Z_{n+1} = F_{n+1} Z_{n,\, n+1}. \tag{2.4.9}$$

In Section 3 we prove that it is possible to define the sets Δ_{np}, $p \in \mathscr{P}_n$, so that the following induction hypothesis holds:

$\mathscr{I}_1(n+1)$. Properties of the sets Δ_{np}, $p \in \mathscr{P}_n$.

For all $p \in \mathscr{P}_n$, we have

(i) $$l(Q \setminus \Delta_{np}) \leqq 2^{-\bar{p}} \tag{2.4.10}$$

where

$$\bar{p} = \begin{cases} p & \text{if } p \geqq \delta_0^{-1}(n-1) \\ \theta_{10}\, \delta_0^{-1}(n-1) & \text{otherwise;} \end{cases} \tag{2.4.11}$$

(ii) *If $z \in F_p Z_{np}$, we have*

$$|\mathscr{E}_n(z, \omega; j(z))| \geqq D_5^{-1} 2^{-\theta_5 p - n} \tag{2.4.12}$$

for all $j(z) \in J_n(z)$ and all

$$\omega \in \mathscr{U}_p \setminus \{ \omega_n(z; j(z)) \}; \tag{2.4.13}$$

(iii) *If $z \in F_p Z_{np}$, we have*

$$\left| \frac{d}{dk}\bigg|_{k=k_0} E_n(z; k) \right| \geqq D_6^{-1} 2^{-\theta_6 p - n} \tag{2.4.14}$$

and

$$\left|\frac{d}{dk}\right|_{k=k_0} \mathscr{E}_n(z, \omega; k)\left| \geqq D_6^{-1} 2^{-\theta_6 p-n} \right. \tag{2.4.15}$$

for all $\omega \in \mathscr{U}_p \setminus \mathscr{U}_{p-1}$, *and all points* $k_0 \in B$ *such that*

$$|\mathscr{E}_n(z, \omega; k_0)| \leqq D_7^{-1} 2^{-\theta_7 p-n} \tag{2.4.16}$$

(iv) *If* $z \in F_p Z_{np}$, $\omega \in \mathscr{U}_p \setminus \mathscr{U}_{p-1}$, *we have*

$$|\mathscr{E}_n(z, \omega; \lambda(z))| \geqq D_5^{-1} 2^{-\theta_5 p-n} \tag{2.4.17}$$

for all $\lambda(z) \in \Lambda_n(z, \omega_0)$ *with* $\omega_0 \in \mathscr{U}_{p-1} \setminus \mathscr{U}_n$.

(v) *If* $p \geqq \delta_0^{-1}(n-1)$, $z \in F_p Z_{np}$, $\omega_1 \in \mathscr{U}_p \setminus \mathscr{U}_{p-1}$, $\omega_2 \in \mathscr{U}_p \setminus \mathscr{U}_n$ *and* $\lambda_1(z) \in \Lambda_n(z, \omega_1)$, $\lambda_2(z) \in \Lambda_n(z, \omega_2)$ *are two distinct and not mutually conjugated zeros, then we have*

$$d(\lambda_1(z), t_{\omega_0} \lambda_2(z)) \geqq 2 D_3^{-1} 2^{-\theta_3 p} \tag{2.4.18}$$

for all $\omega_0 \in \mathscr{U}_n \cup \{0\}$.

(vi) *If* $z \in Z_{np}$, $\omega_1 \in \mathscr{U}_p \setminus \mathscr{U}_{p-1}$, $\omega_2 \in \mathscr{U}_p \setminus \mathscr{U}_n$ *and* $\lambda_1(z) \in \Lambda_n(z, \omega_1)$, $\lambda_2(z) \in \Lambda_n(z, \omega_2)$ *are two distinct and not mutually conjugated zeros, then we have*

$$d(\lambda_1(z), t_{\omega_0} J_n(z)) \geqq D_8^{-1} 2^{-\theta_8 p-n} \|\omega_0\|^{-\theta_0} \tag{2.4.19}$$

and

$$d(\lambda_1(z), t_{\omega_0} \lambda_2(z)) \geqq D_8^{-1} 2^{-\theta_8 p-n} \|\omega_0\|^{-\theta_0} \tag{2.4.20}$$

for all $\omega_0 \in \mathscr{U}$.

(vii) *If* $z \in F_n Z_{n+1}$ *and* $j_1(z), j_2(z)$ *are two distinct and not mutually conjugated jump points in* $J_{n+1}(z)$, *then we have*

$$d(j_1(z), t_{\omega_0} j_2(z)) \geqq \frac{1}{2} D_8^{-1} 2^{-\theta_8 (n+1)-n} \|\omega_0\|^{-\theta_0} \tag{2.4.21}$$

for all $\omega_0 \in \mathscr{U}$ *such that*

$$\|\omega_0\| \leqq 2^{\theta_{11}(n+1)}. \tag{2.4.22}$$

2.5. The Singular Sets

Notation. – In the rest of this section and in sections 4, 5 and 6, we fix a $z \in Z_{n+1}$ and drop it from our notations, unless otherwise stated.

Let us introduce the singular sets of order n

$$\mathscr{S}_n = \bigcup_{j \in J_{n+1}^p} C_n(j), \qquad \bar{\mathscr{S}}_n = \bigcup_{j \in J_{n+1}^p} \bar{C}_n(j), \tag{2.5.1}$$

where $C_n(j)$ and $\bar{C}_n(j)$ are open arcs centered at $j \in J_{n+1}^p$ and of length c_n and $2c_n$, respectively, where

$$c_n = D_{12}^{-1} 2^{-\theta_{12} n}. \tag{2.5.2}$$

The formalism of singular sets we use reminds quite closely the one developed by Fröhlich and Spencer to prove localization for large disorder for the Anderson model, *see* [FS]. We introduce also a family of "semi-distance" functions $d_n(k, k')$ on B, where $n=0, 1, \ldots$ If $k' \neq t_\omega k$ for all $\omega \in \mathcal{U}$, we set $d_n(k, k') = \infty$, while if $k=k'$ we set $d_n(k, k)=0$. Otherwise, if $n=0$ we define

$$d_0(k, t_\omega k) = \|\omega\| \tag{2.5.3}$$

for all $\omega \in \mathcal{U} \setminus \{0\}$. If $n \geqq 1$, $d_n(k, t_\omega k)$ is defined inductively in n so that

$$d_{n+1}(k, t_\omega k) = \inf \sum_{l=0}^{m-1} d_n(t_{\omega_l} k, t_{\omega_{l+1}} k) g_n(t_{\omega_l} k, t_{\omega_{l+1}} k) \tag{2.5.4}$$

where the infimum is taken over all m-tuples $\omega_1, \ldots, \omega_m$ such that

$$\omega_1 + \ldots + \omega_m = \omega. \tag{2.5.5}$$

In (2.5.4), we set $\omega_0 = 0$ and define g_n as the function such that $g_n(t_{\omega_l} k, t_{\omega_{l+1}} k) = 0$ in case $\omega_{l+1} - \omega_l \notin \mathcal{U}_{n+1}$ and in case $\omega_{l+1} - \omega_l \in \mathcal{U}_{n+1}$ and both $t_{\omega_l} k$ and $t_{\omega_{l+1}} k \in \bar{\mathcal{S}}_n$. Otherwise, we get $g_n(t_{\omega_l} k, t_{\omega_{l+1}}) = 1$.

Let us notice that the function $d_n(k, k')$ presently defined is not a distance on B because $d_n(k, k')$ can be zero $k \neq k'$. Moreover, let us remark that $d_n(k, k')$ must not be confused with the euclidean distance $d(k, k')$ introduced in subsection 2.1. In section 4 prove that the following induction hypothesis holds:

$\mathcal{I}_2(n+1)$. *Properties of the Singular Sets* $\mathcal{S}_n$ *and* $\bar{\mathcal{S}}_n$.

For all $z \in \mathrm{Z}_{n+1}$, *we have*

(i) $$\bar{\mathrm{C}}_n(j) \cap \mathrm{J}_{n+1}^p = \{j\} \tag{2.5.6}$$

for all $j \in \mathrm{J}_{n+1}^p$.

(ii) $$\bar{\mathrm{C}}_n(j) \cap t_{\omega_0} \bar{\mathrm{C}}_n(j') = \varnothing \tag{2.5.7}$$

for all pairs of distinct, non-conjugated jump points $j, j' \in \mathrm{J}_{n+1}^p$ *and all* $\omega_0 \in \mathcal{U}$ *such that*

$$\|\omega_0\| \leqq 2^{\theta_{11}(n+1)}. \tag{2.5.8}$$

(iii) $$|\mathcal{E}_n(\omega; k)| \geqq \mathrm{D}_{13}^{-1} 2^{-\theta_{13} n} \tag{2.5.9}$$

in case $k \in \mathrm{B} \setminus \mathcal{S}_n$ *and* $\omega \in \mathcal{U}_{n+1}$ *and in case* $k \in \mathcal{S}_n$ *and* $\omega \in \mathcal{U}_{n+1} \setminus \{\omega_n(k)\}$.

(iv) *If* $j \in \mathrm{J}_{n+1}^p$ *is of order* m [*see Definition after* (2.6.6)] *and* $k \in \bar{\mathrm{C}}_n(j)$, *then we have*

$$|\mathcal{E}_n(\omega_n(j); k)| \geqq (1+n^{-1}) \mathrm{D}_{14}^{-1} 2^{-\theta_{14} m} d(k, j). \tag{2.5.10}$$

(v) *For all* $\mathrm{N} \geqq 1$, *we have*

$$\sup\{d_0(k, k') \mid k, k' \in \mathrm{B} \quad \text{and} \quad d_{n+1}(k, k') \leqq \mathrm{N}\} \leqq 2^{\mathrm{N}\theta_1(n+1)} \tag{2.5.11}$$

Before concluding this subsection, let us introduce some notations concerning singular sets.

Let

$$\mathrm{J}_{n+1}^{p+}=\{j\in \mathrm{J}_{n+1}^{p}\,|\,\omega_n(j)>0\} \tag{2.5.13}$$

$$\bar{\mathscr{S}}_n^{+}=\bigcup_{j\in \mathrm{J}_{n+1}^{p+}} \bar{\mathrm{C}}_n(j) \tag{2.5.14}$$

and

$$\bar{\mathscr{S}}_n^{-}=\bigcup_{j\in \mathrm{J}_{n+1}^{p+}} t_{\omega_n(j)}\bar{\mathrm{C}}_n(j). \tag{2.5.15}$$

Let

$$\omega_n:\quad \bar{\mathscr{S}}_n\to \mathscr{U}_n \tag{2.5.16}$$

be the function such that

$$\omega_n(k)=\omega_n(j) \tag{2.5.17}$$

for all $k\in\bar{\mathrm{C}}_n(j)$ with $j\in\mathrm{J}_{n+1}^{p}$. Let

$$t_n:\quad \bar{\mathscr{S}}_n\to \bar{\mathscr{S}}_n \tag{2.5.18}$$

be the map such that

$$t_n k=t_{\omega_n(k)}k \tag{2.5.19}$$

for all $k\in\bar{\mathscr{S}}_n$.

If $j\in\mathrm{J}_{n+1}^{p}$, let $\psi_n(j;k)\in\mathscr{C}_p^2(\mathrm{B})$ be the function such that

$$\psi_n(j;k)=f\left(\frac{1}{2}c_n^{-1}\left(k-j-\frac{1}{2}c_n\right)\right)f\left(-\frac{1}{2}c_n^{-1}\left(k-j+\frac{1}{2}c_n\right)\right) \tag{2.5.20}$$

where c_m is defined in (2.5.2) and the function f in (2.2.12). We have

$$0\leqq\psi_n(j;k)\leqq 1 \tag{2.5.21}$$

for all $k\in\mathrm{B}$. The first inequality is saturated for $k\in\mathrm{B}\setminus\bar{\mathrm{C}}_n(j)$ and the second one for $k\in\mathrm{C}_n(j)$. Moreover, we have

$$\sup_{k\in\mathrm{B}}\left|\frac{d}{dk}\psi_n(j;k)\right|\leqq 2^3 c_n^{-1} \tag{2.5.22}$$

and

$$\sup_{k\in\mathrm{B}}\left|\frac{d^2}{dk^2}\psi_n(j;k)\right|\leqq 2^6 c_n^{-2} \tag{2.5.23}$$

2.6. The Singular Part of $\mathbb{U}_n$

The unitary transformation in (2.3.2) that we construct at the $(n+1)$-st iteration step, has the form of a product of two unitary operators

$$\mathbb{U}_n=\mathbb{S}_n\exp(\varepsilon\mathbf{R}_n). \tag{2.6.1}$$

$\mathbb{S}_n$ is named the "singular" part of $\mathbb{U}_n$, $\exp(\varepsilon_n \mathbf{R}_n)$ is the "regular" part of $\mathbb{U}_n$. In this and the next subsection we state the induction hypothesis fulfilled by these two operators.

The singular part $\mathbb{S}_n$ is defined in such a way as to kill most of the matrix elements in $\mathbf{H}_n$ corresponding to transitions $|t_\omega k\rangle\langle k|$ with both k and $t_\omega k$ in the singular set $\mathscr{S}_n$. More precisely, for all jump points $j \in \mathrm{J}^p_{n+1}$ let us introduce the operator

$$\mathbb{W}_n(j) = \int_{\mathrm{B}} dk\, w_n(j; k)\,|t_n k\rangle\langle k| \tag{2.6.2}$$

where

$$w_n(j; k) = \psi_n(j; k)\, v_n(\omega_n(k); k) \tag{2.6.3}$$

and $\psi_n(j; k)$ is defined in (2.5.20). Since

$$\operatorname{supp} w_n(j; .) \subset \bar{\mathrm{C}}_n(j), \tag{2.6.4}$$

the right hand side of (2.6.3) is well defined. Thanks to $\mathscr{I}_2(n+1)$ (*i*), any two operators $\mathbb{W}_n(j)$ and $\mathbb{W}_n(j')$ with $j \neq j' \in \mathrm{J}^p_{n+1}$, commute. Moreover, each operator $\mathbb{W}_n(j)$ is the direct integral over B of 2×2 matrices. Hence, it is possible to find a rather explicit expression for a unitary operator diagonalizing $\mathbb{W}_n(j)$ on $\mathrm{L}^2(\mathrm{B})$. We introduce a modification $\mathbb{S}_n(j)$ of such an operator and define $\mathbb{S}_n$ as the product

$$\mathbb{S}_n = \prod_{j \in \mathrm{J}^{p+}_{n+1}} \mathbb{S}_n(j). \tag{2.6.5}$$

Definitions. – The jump point $j \in \mathrm{J}^p_{n+1}$ is said to be $(n+1)$-*regular* if $n \geqq 1$, $j \in \mathrm{J}^p_n$ and j is n-regular or if $n \geqq 0$ and we have

$$\varepsilon_n |w_n(j; j)| \geqq 2\, \varepsilon_n^{7/4}. \tag{2.6.6}$$

Otherwise we say that j is $(n+1)$-*degenerate*. Furthermore, we say that the *order* of j is the least integer m such that $j \in \mathrm{J}^p_{m+1}$ and the *height* of j is the least integer n' such that $j \in \mathrm{J}^p_{n'+1}$ and it is $(n'+1)$-regular.

Remark. – This definition makes sense because, as we discuss below, if $j \in \mathrm{J}^p_{n+1}$ is not an n-regular jump point of J^p_n, then $w_n(j; k)$ is continuous at $k = j$.

Notation. – If $j \in \mathrm{J}^p_{n+1}$, let $\mathring{\mathrm{C}}_n(j)$ and $\mathring{\bar{\mathrm{C}}}_n(j)$ be the subsets of $\bar{\mathrm{C}}_n(j)$ such that

$$\mathring{\mathrm{C}}_n(j) = \mathring{\bar{\mathrm{C}}}_n(j) = \bar{\mathrm{C}}_n(j) \setminus \{j\} \tag{2.6.7}$$

in case j is $(n+1)$-regular and

$$\mathring{\mathrm{C}}_n(j) = \bar{\mathrm{C}}_n(j) \setminus \{k \mid d(j; k) \leqq \varepsilon_m^{-1} \varepsilon_n^{7/4}\} \tag{2.6.8}$$

$$\mathring{\bar{\mathrm{C}}}_n(j) = \bar{\mathrm{C}}_n(j) \setminus \left\{k \mid d(j; k) \leqq \frac{1}{2} \varepsilon_m^{-1} \varepsilon_n^{7/4}\right\} \tag{2.6.9}$$

in case j is $(n+1)$-degenerate of order $m \leqq n$. We also introduce the sets

$$\mathscr{S}_n(j)=\bigcup_{j \in J^p_{n+1}} \mathring{C}_n(j), \qquad \bar{\mathscr{S}}_n(j)=\bigcup_{j \in J^p_{n+1}} \bar{C}_n(j). \tag{2.6.10}$$

If $j \in J^{p+}_{n+1}$, the operator $\mathbb{S}_n(j)$ has the form

$$\mathbb{S}_n(j)=\int_{B \setminus (\bar{C}_n(j) \cup \bar{C}_n(t_n j))} dk\,|k\rangle\langle k| + \int_{\bar{C}_n(j)} dk\, S_n(j;k), \tag{2.6.11}$$

where $S_n(j;k)$ is an O(2) operator of the form

$$\begin{aligned} S_n(j;k)&=\cos\theta_n(j;k)\,|k\rangle\langle k| - \sin\theta_n(j;k)\,|k\rangle\langle t_n k| \\ &\quad + \sin\theta_n(j;k)\,|t_n k\rangle\langle k| + \cos\theta_n(j;k)\,|t_n k\rangle\langle t_n k|. \end{aligned} \tag{2.6.12}$$

The function $\theta_n(j;\,.): \bar{C}_n(j) \to S^1$ is a real valued function defined mod 2π. It is twice differentiable for $k \neq j$ in case j is regular, and for all $k \in \bar{C}_n(j)$ if j is degenerate. If $j \in J^{p+}_{n+1}$ and $k \in \bar{C}_n(j)$, let $F_n(j;k)$ be the symmetric operator

$$\begin{aligned} F_n(j;k)&=E_n(k)\,|k\rangle\langle k| + w_n(j;k)\,|t_n k\rangle\langle k| \\ &\quad + w_n(j;k)\,|k\rangle\langle t_n k| + E_n(t_n k)\,|t_n k\rangle\langle t_n k|. \end{aligned} \tag{2.6.13}$$

If j is $(n+1)$-regular, then we define $\theta_n(j;k)$ so that $S_n(j;k)$ diagonalizes $F_n(j;k)$ for all $k \in \bar{C}_n(j)$. Otherwise, if j is $(n+1)$-degenerate we define $\theta_n(j;k)$ so that $S_n(j;k)$ diagonalizes $F_n(j;k)$ for all $k \in \mathring{C}_n(j)$ and we extend this definition to the rest of $\bar{C}_n(j)$ in such a way that

$$\theta_n(j;k)=0 \tag{2.6.14}$$

if

$$d(k,j) \leqq \frac{1}{2}\varepsilon_m^{-1}\varepsilon_n^{7/4} \tag{2.6.15}$$

and the induction hypothesis below holds. *See* section 5, Lemma 5.7 for the details.

The renormalized Hamiltonian

$$\mathbf{H}^s_n=\mathbb{S}_n^{-1}\,\mathbf{H}_n\,\mathbb{S}_n \tag{2.6.16}$$

can be split as follows:

$$\mathbf{H}^s_n=\mathbf{E}^s_n+\varepsilon_n\,\mathbb{V}^s_n+\sum_{N=2}^{\infty}\varepsilon_n^N\,\mathbf{H}^s_{n\,N}. \tag{2.6.17}$$

Here, E^s_n is the operator of multiplication by the function

$$E^s_n(k)=\langle k|\,\mathbb{S}_n^{-1}(\mathbf{E}_n+\varepsilon_n\mathbb{V}_n)\,\mathbb{S}_n\,|k\rangle. \tag{2.6.18}$$

$\mathbb{V}^s_n$ is the operator

$$\mathbb{V}^s_n=\sum_{\omega \in \mathscr{U}\setminus\{0\}}\int_B dk\, v^s_n(\omega;k)\,|t_\omega k\rangle\langle k| \tag{2.6.19}$$

where, if $k \in \mathrm{B} \setminus \mathcal{S}_n$ and if $k \in \mathcal{S}_n$ and $\omega \neq \omega_n(k)$ we have

$$v_n^s(\omega; k) = \langle t_\omega k \,|\, \mathbb{S}_n^{-1} (\mathbf{E}_n + \varepsilon_n \mathbb{V}_n) \mathbb{S}_n \,|\, k \rangle \tag{2.6.20}$$

while if $k \in \mathcal{S}_n$ and $\omega = \omega_n(k)$ we set

$$v_n^s(\omega_n(k); k) = 0. \tag{2.6.21}$$

Finally, the operators $\mathbf{H}_{n\,\mathrm{N}}^s$, $\mathrm{N} \geqq 2$, have the form

$$\mathbf{H}_{n\,\mathrm{N}} = \sum_{\omega \in \mathcal{U}} \int_{\mathrm{B}} dk\, h_{n\,\mathrm{N}}^s(\omega; k) \,|\, t_\omega k \rangle \langle k \,|. \tag{2.6.22}$$

If $\mathrm{N} \geqq 3$, $\mathrm{N}=2$ and $k \in \mathrm{B} \setminus \bar{\mathcal{S}}_n$ or if $\mathrm{N}=2$, $k \in \bar{\mathcal{S}}_n$ and $\omega \neq \omega_n(k)$, we define

$$h_{n\,\mathrm{N}}^s(\omega; k) = \langle t_\omega k \,|\, \mathbb{S}_n^{-1} \mathbf{H}_{n\,\mathrm{N}} \mathbb{S}_n \,|\, k \rangle. \tag{2.6.23}$$

Otherwise, if $k \in \bar{\mathcal{S}}_n$ we set

$$h_{n\,2}^s(\omega_n(k); k) = \langle t_n k \,|\, \mathbb{S}_n^{-1} (\mathbf{H}_{n\,2} + \varepsilon_n^{-2} \mathrm{E}_n + \varepsilon_n^{-1} \mathbb{V}_n) \mathbb{S}_n \,|\, k \rangle. \tag{2.6.24}$$

In Section 5, we prove that the following three induction hypotheses are fulfilled:

$\mathcal{I}_3(n+1)$ Properties of $\mathrm{E}_n^s(k)$.

For all $z \in \mathbf{Z}_{n+1}$, the following are true

(i)
$$\mathrm{E}_n^s(k) = \mathrm{E}_n(k) \tag{2.6.25}$$

for all $k \in \mathrm{B} \setminus \bar{\mathcal{S}}_n$ and all k such that $d(k, j) < \frac{1}{2} \varepsilon_n^{7/4} \mathrm{D}_{32} 2^{\theta_{32} m + \alpha m^2}$ *for some $(n+1)$-degenerate jump point $j \in \mathrm{J}_{n+1}^p$.*

For all $k \in \bar{\mathcal{S}}_n \setminus \mathrm{J}_{n+1}^p$ we have

(ii)
$$|\mathrm{E}_n^s(k) - \mathrm{E}_n(k)| \leqq \min(\varepsilon_n, \varepsilon_n^2 \mathrm{D}_{18} 2^{\theta_{18} n} d(k, j)^{-1}) \tag{2.6.26}$$

(iii)
$$\left| \frac{d}{dk} (\mathrm{E}_n^s(k) - \mathrm{E}_n(k)) \right| \leqq \mathrm{D}_{22} 2^{\alpha n^2 + \theta_{22} n} \min(1, \varepsilon_n d(k, j)^{-1}) \tag{2.6.27}$$

(iv)
$$\left| \frac{d^2}{dk^2} (\mathrm{E}_n^s(k)) - \mathrm{E}_n(k)) \right| \leqq \mathrm{D}_{23} 2^{2\alpha n^2 + \theta_{23} n} \min(\varepsilon_n^{-7/4}, d(k, j)^{-1}) \tag{2.6.28}$$

(v) *For all $(n+1)$-regular jumps $j \in \mathrm{J}_{n+1}^p$ of height $n' \leqq n$ and all $k \in \mathrm{C}_n(j)$, we have*

$$|\mathrm{E}_n^s(t_n k) - \mathrm{E}_n^s(k)| \geqq \frac{1}{2} \mathrm{D}_{14}^{-1} 2^{-\theta_{14} n} \max(\varepsilon_{n'} \omega_{n'}(j), d(k, j)). \tag{2.6.29}$$

(vi) *For all $(n+1)$-degenerate jumps of order $m \leqq n$, $j \in \mathrm{J}_{n+1}^p$, the function $\mathrm{E}_n^s(k)$ is continuous at $k=j$ and we have*

$$|\mathrm{E}_n^s(t_n k) - \mathrm{E}_n^s(k)| \geqq |\mathrm{E}_n(t_n k) - \mathrm{E}_n(k)| \tag{2.6.30}$$

for all $k \in C_n(j)$ *and*

$$E_n^s(k) = E_n(k) \tag{2.6.31}$$

for all k *such that* $d(k, j) \leqq \frac{1}{2} \varepsilon_n^{7/4} \varepsilon_m^{-1}$.

(vii) *Let* $j \in J_{n+1}^p$ *be* $(n+1)$*-regular of order* m *and height* n' *and let* $m' = m$ *if* $n' = n$, $m' = n$ *otherwise. If* $k \in \bar{C}_n(j)$ *is such that*

$$d(j, k) \geqq D_{32} 2^{\alpha m'^2 + \theta_{32} m'} \varepsilon_n |w_n(j)|, \tag{2.6.32}$$

we have

$$\left| \frac{d}{dk} (E_n^s(k) - E_n(k)) \right| \leqq D_{33}^{-1} 2^{-\theta_{33} m'}. \tag{2.6.33}$$

Otherwise, if (2.6.32) *fails, we have*

$$\operatorname{sgn}\left(\frac{d^2}{dk^2} E_n^s(k) \right) = \operatorname{sgn} \mathscr{E}_n(k) \tag{2.6.34}$$

and

$$\left| \frac{d^2}{dk^2} E_n^s(k) \right| \geqq \left(n + \frac{1}{2} \right)^{-1} D_{34}^{-1} 2^{-\theta_{34} m - 6 \alpha m^2} \varepsilon_{n'}^{-1} |w_{n'}(j)|^{-1} \tag{2.6.35}$$

(viii) *For all* $(n+1)$*-degenerate jumps* $j \in J_{n+1}^p$ *of order* m, *we have*

$$\left| \frac{d}{dk} (E_n^s(k) - E_n(k)) \right| \leqq \varepsilon_m \tag{2.6.36}$$

for all $k \in \bar{C}_n(j)$.

$\mathscr{I}_4(n+1)$ Properties of $\mathbb{V}_n^s$.

For all $z \in Z_{n+1}$, *the following are true*

(i)
$$v_n^s(\omega; k) = 0 \tag{2.6.37}$$

for all $\omega \in \mathscr{U}$, $k \in B$ *such that*

$$d_{n+1}(k, t_\omega k) > 1; \tag{2.6.38}$$

(ii)
$$v_n^s(\omega; k) = v_n(\omega; k) \tag{2.6.39}$$

for all $\omega \in \mathscr{U}$ *and* $k \in B \setminus (\mathscr{S}_n \cup t_{-\omega} \mathscr{S}_n)$ *and for all* $k \in \mathscr{S}_n \setminus \mathring{\bar{\mathscr{S}}}_n$ *and* $\omega \neq \omega_n(k)$.

(iii)
$$v_n^s(\omega_n(k); k) = 0 \tag{2.6.40}$$

for all $k \in \mathscr{S}_n$.

For all $j \in J_{n+1}^p$ *and all* $k \in \mathring{\bar{\mathscr{S}}}_n(j)$, *we have*

(iv)
$$\sum_{\omega \in \mathscr{U}} |v_n^s(\omega; k)| \leqq 3 \tag{2.6.41}$$

(v) $\sum_{\omega \in \mathscr{U}} \left| \frac{d}{dk} v_n^s(\omega; k) \right|$

$$\leqq D_{14} 2^{\alpha n^2 + \theta_{14} n + 6} \min(\varepsilon_n^{-7/4}, d(k, j)^{-1}) \quad (2.6.42)$$

(vi) $\sum_{\omega \in \mathscr{U}} \left| \frac{d^2}{dk^2} v_{n'}^s(\omega; k) \right|$

$$\leqq D_{14}^2 2^{2\alpha n^2 + 2\theta_{14} n + 12} \min(\varepsilon_n^{-7/2} d(k, j)^{-2}) \quad (2.6.43)$$

$\mathscr{I}_5(n+1)$ Properties of $\mathbf{H}_{n\,N}^s$, $N \geqq 2$.

For all $z \in Z_{n+1}$ and all $N \geqq 2$, we have

(i) $$h_{n\,N}^s(\omega; k) = 0 \quad (2.6.44)$$

for all $\omega \in \mathscr{U}$, $k \in B$ such that

$$d_{n+1}(k, t_\omega k) > N. \quad (2.6.45)$$

(ii) *Let $\omega \in \mathscr{U}$. If $N \geqq 3$, if $N = 2$ and $k \in B \setminus (\mathscr{S}_n \cup t_{-\omega} \mathscr{S}_n)$ or if $N = 2$, $k \in B \setminus (\mathring{\mathscr{S}}_n \cup t_{-\omega} \mathring{\mathscr{S}}_n)$ and $\omega \neq \omega_n(k)$, we have*

$$h_{n\,N}^s(\omega; k) = h_{n\,N}(\omega; k). \quad (2.6.46)$$

For all $j \in J_{n+1}^p$, we have

(iii) $$\sum_{\omega \in \mathscr{U}'} |h_{n\,N}^2(\omega; k)| \leqq 5 \quad (2.6.43)$$

(iv) $\sum_{\omega \in \mathscr{U}'} \left| \frac{d}{dk} h_{n\,N}^s(\omega; k) \right|$

$$\leqq D_{14} 2^{\alpha n^2 + \theta_{14} n + 6} \min(\varepsilon_n^{-7/4}, d(k, j)^{-1}) \quad (2.6.44)$$

(v) $\sum_{\omega \in \mathscr{U}'} \left| \frac{d^2}{dk^2} h_{n\,N}^s(\omega; k) \right|$

$$\leqq D_{14}^2 2^{2\alpha n^2 + 2\theta_{14} n + 12} \min(\varepsilon_n^{-7/2}, d(k, j)^{-2}) \quad (2.6.45)$$

where $\mathscr{U}' = \mathscr{U}$ if $N \geqq 3$ or if $N = 0$ and $k \in \mathring{C}_n(j)$; otherwise $\mathscr{U}' = \mathscr{U} \setminus \{\omega_n(k)\}$.

If $j \in J_{n+1}^p$ is $(n+1)$-degenerate of order m and $0 < d(k, j) \leqq \frac{1}{2} \varepsilon_m^{-1} \varepsilon_n^{7/4}$, we have

(vi) $$|\varepsilon_n^2 h_{n\,2}^s(\omega_n(k); k)| \leqq 2 \varepsilon_{n-1}^{7/4} + \varepsilon_m d(k, j) \quad (2.6.46)$$

(vii) $$\left| \frac{d}{dk} \varepsilon_n^2 h_{n\,2}^s(\omega_n(k); k) \right| \leqq \varepsilon_m \left(2 - \frac{1}{n + 1/2} \right) \quad (2.6.47)$$

(viii) $$\left| \frac{d^2}{dk^2} \varepsilon_n^2 h_{n\,2}^s(\omega_k(k); k) \right| \leqq \varepsilon_m \left(2 - \frac{1}{n + (1/2)} \right) \quad (2.6.48)$$

If $j \in J_{n+1}^p$ is $(n+1)$-degenerate of order m and

$$\frac{1}{2} \varepsilon_m^{-1} \varepsilon_n^{7/4} \leqq d(k, j) \leqq \varepsilon_m^{-1} \varepsilon_n^{7/4},$$

we have

(ix) $$\left|\varepsilon_n^2 h_{n\,2}^s(\omega_n(k); k)\right| \leqq D_{14} D_{31} 2^{(\theta_{14}+\theta_{31})m+5} \varepsilon_n^{7/4} \qquad (2.6.49)$$

(x) $$\left|\frac{d}{dk} \varepsilon_n^2 h_{n\,2}^s(\omega_n(k); k)\right| \leqq D_{14}^2 D_{31} 2^{(2\theta_{14}+\theta_{31})m+2} \varepsilon_m \qquad (2.6.50)$$

(xi) $$\left|\frac{d}{dk^2} \varepsilon_n^2 h_{n\,2}^s(\omega_n(k); k)\right| \leqq D_{14}^2 D_{31} 2^{(2\theta_{14}+\theta_{31})m} \varepsilon_n^{-7/4} \qquad (2.6.51)$$

In order to formulate the next family of induction hypotheses, we need to introduce some notations.

Notations. – If $g(k)$ is a finite sum of δ-functions, i. e.

$$g(k) = \sum_{i=1,\ldots,M} g_i \delta(k-k_i) \qquad (2.6.51')$$

with $k_i \in B$, we define

$$\|g\|_\infty = \sum_{i=1,\ldots,M} |g_i| \qquad (2.6.52)$$

$$\operatorname{supp} g = \{k_i, i=1, \ldots, M\}. \qquad (2.6.53)$$

$\mathscr{I}_6(n+1)$ *Dependency of* E_n^s, $\mathbb{V}_n^s$ *and* $H_{n\,N}^s$ *on* $E_0(k)$.

For all z belonging to the interior $\mathring{Z}_{n+1}$ *of* Z_{n+1}, *all* $\omega \in \mathscr{U}$, $N \geqq 2$ *and* $k \in B$, *the distributions*

$$\frac{\delta}{\delta E_0} E_n^s(k), \qquad \frac{\delta}{\delta E_0} v_n^s(\omega; k), \qquad \frac{\delta}{\delta E_0} h_{n\,N}^s(\omega; k) \qquad (2.6.54)$$

are finite sums of delta functions and we have

(i) $$\operatorname{supp} \frac{\delta}{\delta E_0} E_n^s(k) \subset \{k' \in B \mid d_{n+1}(k', k) \leqq 1\} \qquad (2.6.55)$$

(ii) $$\operatorname{supp} \frac{\delta}{\delta E_0} v_n^s(\omega; k) \subset \{k' \in B \mid d_{n+1}(k', k) \leqq 1\} \qquad (2.6.56)$$

(iii) $$\operatorname{supp} \frac{\delta}{\delta E_0} h_{n\,N}^s(\omega; k) \subset \{k' \in B \mid d_{n\,N}(k', k) \leqq N\} \qquad (2.6.57)$$

For all $k \in \bar{\mathscr{S}}_n$ *and* $N \geqq 2$, *we have*

(iv) $$\left\|\frac{\delta}{\delta E_0}(E_n^s(k) - E_n(k))\right\|_\infty \leqq D_{22} 2^{\alpha n^2 + \theta_{22} n} \min(1, \varepsilon_n d(k, j)^{-1}) \qquad (2.6.58)$$

(v) $$\sum_{\omega \in \mathscr{U}} \left\|\frac{\delta}{\delta E_0} v_n^s(\omega; k)\right\|_\infty \leqq D_{14} 2^{\alpha n^2 + \theta_{14} n + 6} \min(\varepsilon_n^{-7/4}, d(k, j)^{-1}) \qquad (2.6.59)$$

(vi) $$\sum_{\omega \in \mathcal{U}} \left\| \frac{\delta}{\delta E_0} h^s_{n\,N}(\omega; k) \right\|_\infty \leqq D_{14}\, 2^{\alpha n^2 + \theta_{14} n + 7} \min(\varepsilon_n^{-7/4}, d(k, j)^{-1}) \quad (2.6.60)$$

(vii) $$\left\| \frac{\delta}{\delta E_0} \frac{d}{dk} (E^s_n(k) - E_n(k)) \right\|_\infty \leqq D_{23}\, 2^{2\alpha n^2 + \theta_{23} n} \min(\varepsilon_n^{-7/4}, d(k, j)^{-1}) \quad (2.6.61)$$

(viii) $$\sum_{\omega \in \mathcal{U}} \left\| \frac{\delta}{\delta E_0} \frac{d}{dk} v^s_n(\omega; k) \right\|_\infty \leqq D_{14}^2\, 2^{2\alpha n^2 + 2\theta_{14} n + 12} \min(\varepsilon_n^{-7/4}, d(k, j)^{-2}) \quad (2.6.62)$$

(ix) $$\sum_{\omega \in \mathcal{U}} \left\| \frac{\delta}{\delta E_0} \frac{d}{dk} h^s_{n\,N}(\omega; k) \right\|_\infty \leqq D_{14}^2\, 2^{2\alpha n^2 + 2\theta_{14} n + 12} \min(\varepsilon_n^{-7/2}, d(k, j)^{-2}) \quad (2.6.63)$$

2.7. The Regular Part of $\mathbb{U}_n$

In this subsection, we discuss the regular part $\exp(\varepsilon_n \mathbf{R}_n)$ of the unitary transformation $\mathbb{U}_n$ in (2.6.1). $\mathbf{R}_n$ is a skewsymmetric operator of the form

$$\mathbf{R}_n = \sum_{\omega \in \mathcal{U} \setminus \{0\}} \int_B dk\, r_n(\omega; k) \,|\, t_\omega k \rangle \langle k \,|. \quad (2.7.1)$$

The functions $r_n(\omega; k)$ are computed in such a way that $\mathbf{R}_n$ solves the following homology equation:

$$[\mathbf{E}^s_n, \mathbf{R}_n] = -\mathbb{V}^s_n. \quad (2.7.2)$$

To this end, we have to choose $r_n(\omega; k)$ as follows:

$$r_n(\omega; k) = -\mathcal{E}^s_n(\omega; k)^{-1} v^s_n(\omega; k) \quad (2.7.3)$$

where

$$\mathcal{E}^s_n(\omega; k) = E^s_n(t_\omega k) - E^s_n(k). \quad (2.7.4)$$

The $(n+1)$-st renormalized Hamiltonian is given by

$$\begin{aligned} \mathbf{H}_{n+1} &= \exp(-\varepsilon_n \mathbf{R}_n)\, \mathbf{H}^s_n \exp(\varepsilon_n \mathbf{R}_n) \\ &= \mathbf{E}_{n+1} + \varepsilon_{n+1} \mathbb{V}_{n+1} + \sum_{N=2}^{\infty} \varepsilon^N_{n+1} \mathbf{H}_{n+1\,N} \end{aligned} \quad (2.7.5)$$

where $\varepsilon_{n+1}=\varepsilon_n^{3/2}$ is the renormalized coupling. $\mathbf{E}_{n+1}$ is the operator of multiplication by the function

$$\mathrm{E}_{n+1}(k)=\mathrm{E}_n^s(k)+\Big\langle k\Big|\frac{1}{2}\varepsilon_n^2\,\mathrm{ad}(\mathbf{R}_n)^2\,\mathbf{E}_n^s+\frac{1}{6}\varepsilon_n^3\,\mathrm{ad}(\mathbf{R}_n)^3\,\mathbf{E}_n^s$$
$$+\varepsilon_n^2\,\mathrm{ad}(\mathbf{R}_n)\,\mathbb{V}_n^s+\frac{1}{2}\varepsilon_n^3\,\mathrm{ad}(\mathbf{R}_n)^2\,\mathbb{V}_n^s$$
$$+\varepsilon_n^2\,\mathbf{H}_{n\,2}^s+\varepsilon_n^3\,\mathrm{ad}(\mathbf{R}_n)\,\mathbf{H}_{n\,2}^s+\varepsilon_n^3\,\mathbf{H}_{n\,3}^s\Big|k\Big\rangle \qquad (2.7.6)$$

where we use the following notation:

Notation. – If **A**, **B** are two operators and p is an integer $\geqq 1$, then $\mathrm{ad}(\mathbf{A})^p\,\mathbf{B}$ is the operator $[\mathbf{B}, \mathbf{A}]$ in case $p=1$ and $[\mathrm{ad}(\mathbf{A})^{p-1}\,\mathbf{B}, \mathbf{A}]$ if $p\geqq 1$.

The operator $\mathbb{V}_{n+1}$ has the form

$$\mathbb{V}_{n+1}=\sum_{\omega\in\mathscr{U}\setminus\{0\}}\int_{\mathrm{B}} dk\, v_{n+1}(\omega;k)\,|\,t_\omega k\rangle\langle k\,| \qquad (2.7.7)$$

where

$$v_{n+1}(\omega;k)=\sqrt{\varepsilon_n}\Big\langle t_\omega k\Big|\frac{1}{2}\mathrm{ad}(\mathbf{R}_n)^2\,\mathbf{E}_n^s+\frac{1}{6}\varepsilon_n\,\mathrm{ad}(\mathbf{R}_n)^3\,\mathbf{E}_n^s$$
$$+\mathrm{ad}(\mathbf{R}_n)\,\mathbb{V}_n^s+\frac{1}{2}\varepsilon_n\,\mathrm{ad}(\mathbf{R}_n)^2\,\mathbb{V}_n^s$$
$$+\mathbf{H}_{n\,2}^s+\varepsilon_n\,\mathrm{ad}(\mathbf{R}_n)\,\mathbf{H}_{n\,2}^s+\varepsilon_n\,\mathbf{H}_{n\,3}^s\Big|dk\Big\rangle. \qquad (2.7.8)$$

Finally, the operators $\mathbf{H}_{n+1,\mathrm{N}}$, $\mathrm{N}\geqq 2$ are given by

$$\mathbf{H}_{n+1,\mathrm{N}}=\sum_{\omega\in\mathscr{U}}\int_{\mathrm{B}} dk\, h_{n+1,\mathrm{N}}(\omega;k)\,|\,t_\omega k\rangle\langle k\,| \qquad (2.7.9)$$

where

$$h_{n+1,\mathrm{N}}(\omega;k)=\sqrt{\varepsilon_n^{\mathrm{N}}}\Big\langle t_\omega k\Big|\Big(\frac{1}{(2\mathrm{N})!}\mathrm{ad}(\mathbf{R}_n)^{2\mathrm{N}}\,\mathbf{E}_n^s$$
$$+\frac{1}{(2\mathrm{N}+1)!}\varepsilon_n\,\mathrm{ad}(\mathbf{R}_n)^{2\mathrm{N}+1}\,\mathbf{E}_n^s$$
$$+\frac{1}{(2\mathrm{N}-1)!}\mathrm{ad}(\mathbf{R}_n)^{2\mathrm{N}-1}\,\mathbb{V}_n^s+\frac{1}{(2\mathrm{N})!}\varepsilon_n\,\mathrm{ad}(\mathbf{R}_n)^{2\mathrm{N}}\,\mathbb{V}_n^s$$
$$+\sum_{\mathrm{N}'=2}^{2\mathrm{N}}\frac{1}{(2\mathrm{N}-\mathrm{N}')!}\mathrm{ad}(\mathbf{R}_n)^{2\mathrm{N}-\mathrm{N}'}\,\mathbf{H}_{n\,\mathrm{N}'}^s$$
$$+\varepsilon_n\sum_{\mathrm{N}'=2}^{2\mathrm{N}+1}\frac{1}{(2\mathrm{N}+1-\mathrm{N}')!}\mathrm{ad}(\mathbf{R}_n)^{2\mathrm{N}+1-\mathrm{N}'}\,\mathbf{H}_{n\,\mathrm{N}'}^s\Big)\Big|k\Big\rangle. \qquad (2.7.10)$$

In section 6 we prove that the following four families of induction hypotheses are fulfilled.

$\mathcal{I}_7(n+1)$ *Properties of* $E_{n+1}(k)$.

For all $z \in Z_{n+1}$, *the following are true*:

(i) *The function* $E_{n+1}(k)$ *has two bounded derivatives for all* $k \in B \setminus J_{n+1}$ *and all its discontinuities are jumps.*

For all $k \in B \setminus J_{n+1}$ *we have*

(ii) $$|E_{n+1}(k) - E_n(k)| \leqq \min(2\varepsilon_n, \varepsilon_n^2 D_{30} 2^{\theta_{30}(n+1)} d(k, J_{n+1}^p)^{-1}) \quad (2.7.11)$$

(iii) $$\left|\frac{d}{dk} E_{n+1}(k)\right| \leqq 2^{\alpha(n+1)^2}; \quad (2.7.12)$$

(iv) $$\left|\frac{d}{dk}(E_{n+1}(k) - E_n(k))\right| \leqq 2^{\alpha(n+1)^2 - 1} \min(1, \varepsilon_n d(k, J_{n+1})^{-1}) \quad (2.7.13)$$

(v) $$\left|\frac{d^2}{dk^2}(E_{n+1}(k) - E_n(k))\right| \leqq 2^{2\alpha(n+1)^2} \min(\varepsilon_n^{-7/4}, d(k, J_{n+1})^{-1}) \quad (2.7.14)$$

(vi) *For all* $(n+1)$*-regular jumps* $j \in J_{n+1}^p$ *and all* $k \in \bar{C}_n(j)$, *we have*

$$|E_{n+1}(t_n k) - E_{n+1}(k)| \geqq D_{14}^{-1} 2^{-\theta_{14}(n+1)} \max(\varepsilon_n^{7/4}, d(k, j)) \quad (2.7.15)$$

(vii) *For* $(n+1)$*-degenerate jumps* $j \in J_{n+1}^p$ *and all* $k \in C_n(j)$, *we have*

$$|E_{n+1}(t_n k) - E_{n+1}(k)| \geqq D_{14}^{-1} 2^{-\theta_{14}(n+1)} d(k, j) \quad (2.7.16)$$

If $j \in J_{n+1}^p$ *is* $(n+1)$*-degenerate of order* m *and* $k \in B$ *is such that* $d_{n+1}\left(k, \left[j - (1/2)\varepsilon_n^{7/4}, j + \frac{1}{2}\varepsilon_n^{7/4}\right]\right) \leqq 1$, *then we have*

(viii) $$\left|\frac{d}{dk} E_{n+1}(k)\right| \leqq D_{31} 2^{\theta_{31}(m+1)}\left(1 - \frac{1}{n+1}\right) \quad (2.7.17)$$

(ix) $$\left|\frac{d^2}{dk^2} E_{n+1}(k)\right| \leqq 2^{2\alpha(n+1)^2} \quad (2.7.18)$$

(x) *If* $k \notin J_{n+1}$ *and we have*

$$\left|\frac{d}{dk} E_{n+1}(k)\right| \leqq \frac{1}{2} D_6^{-1} 2^{-\theta_6(n+1)}, \quad (2.7.19)$$

then either $k \in \bigcup_{m=0}^{n} \mathcal{S}_n$ *and we have*

$$\left|\frac{d^2}{dk^2} E_{n+1}(k)\right| \geqq \frac{1}{16}[1 + (n+1)^{-1}]\lambda_1, \quad (2.7.20)$$

or there is a $(n+1)$-regular jump $j \in \mathrm{J}^p_{n+1}$ of height n' and order m, such that

$$d(k, j) \leqq \mathrm{D}_{32}\, 2^{\alpha m^2 + \theta_{32} m}\, \varepsilon_{n'} \,|\, w_{n'}(j) \,| \tag{2.7.21}$$

and, in this case, we have

$$\left| \frac{d^2}{dk^2} \mathrm{E}_{n+1}(k) \right| \geqq \frac{1}{4} [1 + (n+1)^{-1}]\, \mathrm{D}_{34}^{-1}\, 2^{-\theta_{34} m - 6 \alpha m^2}\, \varepsilon_{n'}^{-1} \,|\, w_{n'}(j) \,|^{-1} \tag{2.7.22}$$

(xi) *If $\bar{\varepsilon} \geqq \varepsilon_n^{1/4}$, we have*

$$\int_{\mathrm{B}} dk \left| \frac{\bar{\varepsilon}}{(\mathrm{E}_{n+1}(k) - x)^2 + \bar{\varepsilon}^2} - \frac{\bar{\varepsilon}}{(\mathrm{E}_n(k) - x)^2 + \bar{\varepsilon}^2} \right| \leqq 12\, \pi \varepsilon_n^{1/4} \tag{2.7.23}$$

for all $x \in \mathbf{R}$.

(xii) *We have*

$$\sup_{0 < \bar{\varepsilon} < 1} \int dx \left[\int_{\mathrm{B}'} dk \frac{\bar{\varepsilon}}{(\mathrm{E}_{n+1}(k) - x)^2 + \bar{\varepsilon}^2} \right]^{3/2} \leqq 2^7 \rho_0^{-4} + 2^6 \pi^{-1} c_0^{3/2} (\rho_0 \lambda_2^{-1} + \rho_0^{1/2} \lambda_1^{-1/2}) \tag{2.7.24}$$

where

$$\mathrm{B}' = \mathrm{B} \bigcup_{m=0}^{n+1} \bigcup_{j \in \mathrm{J}^p_{n+1,m}} \bar{\mathrm{C}}_m, \tag{2.7.25}$$

$$\mathrm{J}^p_{n+1,m} = \{ j \in \mathrm{J}^p_{n+1} \,|\, j \textit{ is regular of order } m \} \tag{2.7.26}$$

and ρ_0, c_0, λ_1 and λ_2 are defined as in Lemma 2.1.

(xiii) *If $j \in \mathrm{J}^p_{n+1}$ is a $(n+1)$-regular jump of order m and $0 < \bar{\varepsilon} < \varepsilon_n^{1/4}$, we have*

$$\int dx \left[\int_{\bar{\mathrm{C}}_m(j)'} dk \frac{\bar{\varepsilon}}{(\mathrm{E}_{n+1}(k) - x)^2 + \bar{\varepsilon}^2} \right]^{3/2} \leqq 7\, \mathrm{D}_6^2\, \mathrm{D}_{12}^{-1}\, 2^{(2\theta_6 - \theta_{12}) m} \tag{2.7.28}$$

where

$$\bar{\mathrm{C}}_m(j)' = \bar{\mathrm{C}}_m(j) \bigcup_{m'm}^{n+1} \bigcup_{\substack{j' \in \mathrm{J}^p_{n+1,m} \\ j' \neq j}} \bar{\mathrm{C}}_m(j') \tag{2.7.29}$$

(xiv) *If $0 < \bar{\varepsilon} < \varepsilon_n^{1/4}$, we have*

$$\int dx \left[\int_{\mathrm{B}} dk \frac{\bar{\varepsilon}}{(\mathrm{E}_{n+1}(k) - x)^2 + \bar{\varepsilon}^2} \right]^{3/2} \leqq 2^7 \rho_0^{-4} + 2^6 \pi^{-1} c_0^{3/2} (\rho_0 \lambda_2^{-1} + \rho_0^{1/2} \lambda_1^{-1/2}) + 8 \tag{2.7.30}$$

$\mathscr{I}_8(n+1)$ Properties of $\mathbb{V}_{n+1}$.

For all $z \in \mathrm{Z}_{n+1}$, we have

(i) $$v_{n+1}(\omega; k) = 0 \tag{2.7.19}$$

for all $\omega \in \mathscr{U} \setminus \mathscr{U}_{n+2}$;

For all $k \in \mathrm{B} \setminus \mathrm{J}_{n+1}$, *we have*

(ii) $$\sum_{\omega \in \mathcal{U}'} |v_{n+1}(\omega; k)| \leqq 1$$

(iii) $$\sum_{\omega \in \mathcal{U}'} \left| \frac{d}{dk} v_{n+1}(\omega; k) \right| \leqq 2^{\alpha (n+1)^2} \sqrt{\varepsilon_n} \min (\varepsilon_n^{-7/4}, d(k, \mathrm{J}_{n+1})^{-1})$$

(iv) $$\sum_{\omega \in \mathcal{U}'} \left| \frac{d^2}{dk^2} v_{n+1}(\omega; k) \right| \leqq 2^{2\alpha (n+1)^2} \sqrt{\varepsilon_n} \min (\varepsilon_n^{-7/2}, d(k, \mathrm{J}_{n+1})^{-2})$$

where $\mathcal{U}' = \mathcal{U}$ *in case there is no* $(n+1)$*-degenerate jump* $j \in \mathrm{J}_{n+1}^p$ *such that* $d_{n+1}(k, [j - \varepsilon_m^{-1} \varepsilon_n^{7/4}, j + \varepsilon_m^{-1} \varepsilon_n^{7/4}]) \leqq 1$. *Otherwise,* $\mathcal{U}' = \mathcal{U} \setminus \{ \omega_n(k) \}$.

If $j \in \mathrm{J}_{n+1}^p$ *is* $(n+1)$*-degenerate of order* m *and* k *is such that* $0 < d(k, j) \leqq \frac{1}{2} \varepsilon_m^{-1} \varepsilon_n^{7/4}$, *we have*

(v) $$\left| \frac{d}{dk} \varepsilon_{n+1} v_{n+1}(\omega_n(k), k) \right| \leqq (2 - (n+1)^{-1}) \varepsilon_m$$

(vi) $$\left| \frac{d}{dk^2} \varepsilon_{n+1} v_{n+1}(\omega_n(k); k) \right| \leqq (2 - (n+1)^{-1}) \varepsilon_m.$$

If $j \in \mathrm{J}_{n+1}^p$ *is* $(n+1)$*-degenerate of order* m *and* k *is such that* $\frac{1}{2} \varepsilon_m^{-1} \varepsilon_n^{7/4} \leqq d(k, j) \leqq \varepsilon_m^{-1} \varepsilon_n^{7/4}$, *we have*

(vii) $$|\varepsilon_{n+1} v_{n+1}(\omega_n(k); k)| \leqq \mathrm{D}_{14} \mathrm{D}_{31} 2^{(\theta_{14} + \theta_{31}) m + 6} \varepsilon_n^{7/4}$$

(viii) $$\left| \frac{d}{dk} \varepsilon_{n+1} v_{n+1}(\omega_n(k); k) \right| \leqq \mathrm{D}_{14}^2 \mathrm{D}_{31} 2^{(2\theta_{14} + \theta_{31}) m + 3} \varepsilon_m$$

(ix) $$\left| \frac{d^2}{dk^2} \varepsilon_{n+1} v_{n+1}(\omega_n(k); k) \right| \leqq \mathrm{D}_{14}^2 \mathrm{D}_{31} 2^{(2\theta_{14} + \theta_{31}) m + 1} \varepsilon_n^{-7/4}$$

$\mathcal{I}_9(n+1)$ *Properties of* $\mathbf{H}_{n+1, \mathrm{N}}$, $\mathrm{N} \geqq 2$.

For all $z \in \mathrm{Z}_{n+1}$ *and all* $\mathrm{N} \geqq 2$, *we have*

(i) $$h_{n+1, \mathrm{N}}(\omega; k) = 0 \tag{2.7.27}$$

for all $\omega \in \mathcal{U}$, $k \in \mathrm{B}$ *such that*

$$d_{n+1}(k, t_\omega k) > \mathrm{N} \tag{2.7.28}$$

(ii) *For all* $k \in \mathrm{B}$, *we have*

$$\sum_{\omega} |h_{n+1, \mathrm{N}}(\omega; k)| \leqq 1 \tag{2.7.29}$$

For all $k \in \mathrm{B} \setminus \mathrm{J}_{n+1}^{\mathrm{N}}$, *we have*

(iii) $$\sum_{\omega} \left| \frac{d}{dk} h_{n+1, \mathrm{N}}(\omega; k) \right| \leqq \sqrt[4]{\varepsilon_n^{\mathrm{N}}} \min (\varepsilon_n^{-7/4}, d(k, \mathrm{J}_{n+1}^{\mathrm{N}})^{-1}) \tag{2.7.30}$$

(iv) $$\sum_{\omega}\left|\frac{d^2}{dk^2}h_{n+1,\mathrm{N}}(\omega;k)\right|\leqq\sqrt[4]{\varepsilon_n^{\mathrm{N}}}\min(\varepsilon_n^{-7/2},\,d(k,\mathrm{J}_{n+1}^{\mathrm{N}})^{-2})\qquad(2.7.31)$$

If $j\in \mathrm{J}_{n+1}^{p}$ *is* $(n+1)$*-degenerate and* $k\in\mathrm{B}$ *is such that* $d_{n+1}\left(k,\,d_{n+1}\left(k,\left[j-\frac{1}{2}\varepsilon_m^{-1}\varepsilon_n^{7/4},j+\frac{1}{2}\varepsilon_m^{-1}\varepsilon_n^{7/4}\right]\right)\right)\leqq\mathrm{N}$, *then we have*

(v) $$\sum_{\omega}\left|\frac{d}{dk}h_{n+1,\mathrm{N}}(\omega;k)\right|\leqq 1\qquad(2.7.32)$$

(vi) $$\sum_{\omega}\left|\frac{d^2}{dk^2}h_{n+1,\mathrm{N}}(\omega;k)\right|\leqq 1\qquad(2.7.33)$$

$\mathscr{I}_{10}(n+1)$ <u>*Dependency of*</u> $\mathbf{E}_{n+1}$, $\mathbb{V}_{n+1}$ <u>*and*</u> $\mathbf{H}_{n+1,\mathrm{N}}$ <u>*on*</u> $\mathrm{E}_0(k)$.

For all z belonging to the interior $\mathring{\mathrm{Z}}_{n+1}$ *of* Z_{n+1}, *all* $\omega\in\mathscr{U}$, $\mathrm{N}\geqq 2$ *and* $k\in\mathrm{B}$, *the distributions*

$$\frac{\delta}{\delta\mathrm{E}_0}\mathrm{E}_{n+1}(k),\qquad\frac{\delta}{\delta\mathrm{E}_0}v_{n+1}(\omega;k),\qquad\frac{\delta}{\delta\mathrm{E}_0}h_{n+1,\mathrm{N}}(\omega;k)\qquad(2.7.34)$$

are finite sums of delta functions and we have

(i) $$\operatorname{supp}\frac{\delta}{\delta\mathrm{E}_0}\mathrm{E}_{n+1}(k)\subset\{k'\in\mathrm{B}\,|\,d_{n+1}(k',k)\leqq 1\}\qquad(2.7.35)$$

(ii) $$\operatorname{supp}\frac{\delta}{\delta\mathrm{E}_0}v_{n+1}(\omega;k)\subset\{k'\in\mathrm{B}\,|\,d_{n+1}(k',k)\leqq 1\}\qquad(2.7.36)$$

(iii) $$\operatorname{supp}\frac{\delta}{\delta\mathrm{E}_0}h_{n+1}(\omega;k)\subset\{k'\in\mathrm{B}\,|\,d_{n+1}(k',k)\leqq \mathrm{N}\}\qquad(2.7.37)$$

For all $k\in\mathrm{B}\diagdown\mathrm{J}_{n+1}$, *we have*

(iv) $$\left\|\frac{\delta}{\delta\mathrm{E}_0}(\mathrm{E}_{n+1}(k)-\mathrm{E}_n(k))\right\|_{\infty}\leqq 2^{\alpha(n+1)^2+1}\min(1,\,\varepsilon_n d(k,\mathrm{J}_{n+1})^{-1})\qquad(2.7.38)$$

(v) $$\sum_{\omega\in\mathscr{U}}\left\|\frac{\delta}{\delta\mathrm{E}_0}v_{n+1}(\omega;k)\right\|_{\infty}\leqq 2^{\alpha(n+1)^2}\sqrt{\varepsilon_n}\min(\varepsilon_n^{-7/4},\,d(k,\mathrm{J}_{n+1})^{-1})\qquad(2.7.39)$$

(vi) $$\left\|\frac{\delta}{\delta\mathrm{E}_0}\,\frac{d}{dk}(\mathrm{E}_{n+1}(k)-\mathrm{E}_n(k))\right\|\leqq 2^{2\alpha(n+1)^2}\min(\varepsilon_n^{-7/4},\,d(k,\mathrm{J}_{n+1})^{-1})\qquad(2.7.40)$$

(vii) $$\sum_{\omega\in\mathscr{U}}\left\|\frac{\delta}{\delta\mathrm{E}_0}\,\frac{d}{dk}v_{n+1}(\omega;k)\right\|\leqq 2^{2\alpha(n+1)^2}\sqrt{\varepsilon_n}\min(\varepsilon_n^{-7/2},\,d(k,\mathrm{J}_{n+1})^{-2})\qquad(2.7.41)$$

For all $N \geqq 2$ *and all* $k \in B \setminus J^N_{n+1}$, *we have*

$$\text{(viii)} \quad \sum_{\omega \in \mathcal{U}} \left\| \frac{\delta}{\delta E_0} h_{n+1,N}(\omega; k) \right\| \leqq \sqrt[4]{\varepsilon_n^N} \min(\varepsilon_n^{-7/4}, d(k, J^N_{n+1})^{-1}) \quad (2.7.42)$$

$$\text{(ix)} \quad \sum_{\omega \in \mathcal{U}} \left\| \frac{\delta}{\delta E_0} \frac{d}{dk} h_{n+1,N}(\omega; k) \right\| \leqq \sqrt[4]{\varepsilon_n^N} \min(\varepsilon_n^{-7/2}, d(k, J^N_{n+1})^{-2}). \quad (2.7.43)$$

2.8. Conclusions

In this subsection, we give the value of some of the constants introduced above and we show that Theorem 2 holds if the induction hypothesis $\mathcal{I}_i(n)$ holds for all $n \geqq 1$ and all $i = 1, \ldots, 10$.

The constants θ_i, $i = 1, \ldots, 34$, are given as follows:

$$\left.\begin{array}{c}
\theta_1 = 3, \quad \theta_2 = 15\theta_0 + 1, \quad \theta_3 = 6s + 2(r+1)\theta_2 + 2 \\
\theta_5 = \theta_3 + 1, \quad \theta_6 = (r+1)\theta_2 + 1, \quad \theta_7 = 2\theta_5 + 2 \\
\theta_8 = (r+3)\theta_3 + 3s + 1 + \theta_6 \quad \theta_9 = \theta_5 + 1 \\
\theta_{10} = \theta_3^{-1}(\theta_8 + 1), \quad \theta_{11} = 15 \\
\theta_{12} = 3 + \theta_8, \quad \theta_{13} = 4 + \theta_6 + \theta_8, \quad \theta_{14} = \theta_6 + 1 \\
\theta_{15} = \theta_{12} + \theta_{14}, \quad \theta_{16} = 2\theta_{15} + \theta_{12}, \quad \theta_{17} = 2\theta_{12} \\
\theta_{18} = \theta_{12} + \theta_{15}, \quad \theta_{19} = 2\theta_{14}, \quad \theta_{20} = 3\theta_0 + \theta_8 + \theta_{14} \\
\theta_{21} = 3\theta_0 + \theta_8 + 2\theta_{14}, \quad \theta_{22} = \theta_{20}, \quad \theta_{23} = \theta_{21} \\
\theta_{24} = \theta_{13}, \quad \theta_{25} = 2\theta_{13} + \theta_{22}, \quad \theta_{26} = \theta_{22} \\
\theta_{27} = 2\theta_8 + 3\theta_6 + 3\theta_0 + 6, \quad \theta_{28} = 5\theta_8 + 3\theta_6 + 6\theta_0 + 12 \\
\theta_{30} = 2\theta_8 + \theta_6 + 7, \quad \theta_{31} = 3\theta_0 + \theta_8, \quad \theta_{32} = \theta_{14} + \theta_{33} \\
\theta_{33} = \theta_6, \quad \theta_{34} = 2\theta_{14} + 2\theta_{32}.
\end{array}\right\} \quad (2.8.1)$$

We omit giving the explicit definition of the constant D_i, $i = 1, \ldots, 34$ and θ_4, α, ε_0 because it would be too cumbersome. We shall just state that they are the least positive constants fulfilling the inequalities in Lemma 2.1 and in the following sections. Since such constants have to satisfy a rather complex system of inequalities, I tried to state all the conditions in a very clear – though redundant – way.

We have

LEMMA 2.2. – *Let us suppose that there is a sequence* $\mathbb{U}_n$, $n = 0, 1, \ldots,$ *of unitary operators satisfying the induction hypothesis of the families* $\mathcal{I}_i(n+1)$, $i = 1, \ldots, 10$, *for all* $n \geqq 0$. *Then there is a unitary operator* $\mathbb{U}$ *on* $L^2(B)$

such that

$$\mathbb{U} = s - \lim_{n \to \infty} \prod_{m=0}^{n} \mathbb{U}_m \tag{2.8.2}$$

and Theorem 2 holds.

Proof. – It suffices to show that the limit in the strong sense in (2.8.2) exists. In fact, thanks to $\mathscr{I}_7(n)$, $\mathscr{I}_8(n)$ and $\mathscr{I}_9(n)$ we have

$$\lim_{n \to \infty} \mathbf{H}_n = \mathrm{E}_\infty \tag{2.8.3}$$

where $\mathbf{E}_\infty$ is the operator of multiplication by the function

$$\mathrm{E}_\infty(k) = \lim_{n \to \infty} \mathrm{E}_n(k) \tag{2.8.4}$$

and the convergence in (2.8.3) is in operator norm. Hence

$$\begin{aligned} \mathbb{U}^{-1} \mathbf{H}_0 \mathbb{U} &= s - \lim_{n \to \infty} \left(\prod_{m=0}^{n} \mathbb{U}_m \right)^{-1} \mathbf{H}_0 \left(\prod_{m'=0}^{n} \mathbb{U}_{m'} \right) \\ &= s - \lim_{\to \infty} \mathbf{H}_{n+1} = \mathbf{E}_\infty \end{aligned} \tag{2.8.5}$$

and Theorem 2 is satisfied.

To prove the existence of the strong limit (2.8.2) it suffices to prove that for all $f \in \mathrm{L}^2(\mathrm{B})$ we have

$$\lim_{n \to \infty} \lim_{\mathrm{N} \to \infty} \left\| \left(\prod_{m=n}^{\mathrm{N}} \mathbb{S}_m e^{\varepsilon_m \mathbf{R}_m} - 1 \right) f \right\|_2 = 0 \tag{2.8.6}$$

We have

$$\prod_{m=n}^{\mathrm{N}} \mathbb{S}_m e^{\varepsilon_m \mathbf{R}_m} = \left(\prod_{m=n}^{\mathrm{N}} \tilde{\mathbb{S}}_m^{(n)} \right) \left(\prod_{m'=n}^{\mathrm{N}} e^{\varepsilon_{m'} \mathbf{R}_{m'}} \right) \tag{2.8.7}$$

where $\tilde{\mathbb{S}}_n^{(m)} = \mathbb{S}_n$ and, if $m \geqq n+1$, we set

$$\tilde{\mathbb{S}}_m^{(n)} = \bar{e}^{\varepsilon_n \mathbf{R}_n} \dots \bar{e}^{\varepsilon_{n-1} \mathbf{R}_{m-1}} \mathbb{S}_m e^{\varepsilon_{m-1} \mathbf{R}_{m-1}} \dots e^{\varepsilon_n \mathbf{R}_n}. \tag{2.8.8}$$

We have

$$\begin{aligned} \left\| \prod_{m=n}^{\mathrm{N}} e^{\varepsilon_m \mathbf{R}_m} - 1 \right\| &\leqq \exp\left(3 \mathrm{D}_{13} \sum_{m=n}^{\mathrm{N}} 2^{\theta_{13} m} \varepsilon_m \right) - 1 \\ &= o(\sqrt{\varepsilon_n}) \end{aligned} \tag{2.8.9}$$

as $n \to \infty$. Moreover, if we split $\tilde{\mathbb{S}}_m^{(n)}$ as follows

$$\tilde{\mathbb{S}}_m^{(n)} = \mathbf{A}_m^{(n)} + \mathbf{B}_m^{(n)} \tag{2.8.10}$$

where

$$\mathbf{A}_m^{(n)} = \sum_{(3/2)^{i_1}+\ldots(3/2)^{i_k} \leqq (3/2)^m} \frac{1}{(i)!} \varepsilon_{i_1} \ldots \varepsilon_{i_k} [\ldots[\mathbb{S}_m, \mathbf{R}_{i_1}], \ldots \mathbf{R}_{i_k}] \quad (2.8.11)$$

and

$$(i)! = \prod_{m=1}^{\infty} (\#\{i_k = m\})!, \quad (2.8.12)$$

we have

$$\| \mathbf{B}_m^{(n)} \| = o(\sqrt{\varepsilon_n}) \quad (2.8.13)$$

as $n \to \infty$. Hence, it suffices to show that for all $f \in L^2(B)$ we have

$$\lim_{n \to \infty} \lim_{N \to \infty} \left\| \left(\prod_{m=n}^{N} \mathbf{A}_m^{(n)} - \mathbf{1} \right) f \right\| = 0. \quad (2.8.14)$$

The operator $\mathbf{A}_m^{(n)}$ can be written as follows.

$$\mathbf{A}_m^{(n)} = \sum_{\omega \in \mathscr{U}} \int_B dk\, a_m^{(n)}(\omega; k) \,|\, t_\omega k \rangle \langle k \,|. \quad (2.8.15)$$

If $\|\omega\| \geqq 2^{\theta_1 m+1}$ we have $a_m^{(n)}(\omega; k) = 0$. Moreover, we have

$$\operatorname{supp} a_m^{(n)}(\omega; .) \subset \{ t_\omega \mathscr{S}_m \,|\, \|\omega\| \leqq 2^{\theta_1 m+1} \} \quad (2.8.16)$$

and

$$\sum_{\omega} | a_m^{(n)}(\omega; k) | = 1 + o(\varepsilon_n) \quad (2.8.17)$$

as $n \to \infty$. If $\mathscr{A}_n$ is the set

$$\mathscr{A}_n = \bigcup_{m=n}^{\infty} \bigcup_{\omega \in \mathscr{U}} \operatorname{supp} a_m^{(n)}(\omega; .), \quad (2.8.18)$$

we have

$$l(\mathscr{A}_n) \leqq \sum_{m=n}^{\infty} D_{12}^{-1} D_3\, 2^{(\theta_1+\theta_3-\theta_{12})m+2}$$

$$= 4 D_{12}^{-1} D_3 \sum_{m=n}^{\infty} 2^{-[(r+2)\theta_3 - 3s - \theta_6 - 1]m}. \quad (2.8.19)$$

In particular, we see that $l(\mathscr{A}_n) \to 0$ as $n \to \infty$. Hence, for all $N > n$ we have

$$\left\| \left(\prod_{m=n}^{N} \mathbf{A}_m^{(n)} - \mathbf{1} \right) f \right\|_2 \leqq (1 + o(\sqrt{\varepsilon_n})) \left(\int_{\mathscr{A}_n} f^2 \, dk \right)^{1/2} \to 0$$

as $n \to \infty$.

Q.E.D.

3. NONRESONANCE CONDITIONS

3.1. Definition of the set Δ_{np}

Let $n \geqq 0$ and let us suppose that the first n renormalizations have been performed and that $\mathscr{I}(m)$ holds for $m \leqq n$. In this section, we define the sets Δ_{np} in (2.4.8) for $p \in \mathscr{P}_n$ in terms of the renormalized disersion law $E_n(z; k)$ and we prove that, with this choice, the family of induction hypothesis $\mathscr{I}_1(n+1)$ is true.

Notations. – Let $p \in \mathscr{P}_n$ and let X_{np} be the set

$$X_{np} = \underset{p' > \bar{p}}{\times} Q_{p'} \tag{3.1.1}$$

where

$$\bar{p} = \begin{cases} p & \text{if } \; p \geqq \delta_0^{-1}(n-1) \\ \theta_{10}\,\delta_0^{-1}(n-1) & \text{otherwise.} \end{cases} \tag{3.1.2}$$

The set Δ_{np} is constructed as the intersection of five sets defined below, *i. e.*

$$\Delta_{np} = \bigcap_{i=1,\ldots,5} \Delta_{np}^{(i)}. \tag{3.1.3}$$

The set $\Delta_{np}^{(i)}$ have the form

$$\Delta_{np}^{(i)} = \bigcup_{\bar{z} \in \Pi_{<\bar{p}} Z_n} \; \bigcup_{x \in X_{np}} \bar{z} \times \Delta_{np}^{(i)}(\bar{z}, x) \times x \tag{3.1.4}$$

with

$$\Delta_{np}^{(i)}(\bar{z}, x) \subset Q_{\bar{p}}. \tag{3.1.5}$$

If $\bar{z} \in \Pi_{<\bar{p}} Z_n$ and $x \in X_{np}$, the set $\Delta_{np}^{(i)}(\bar{z}, x)$, $i = 1, \ldots, 5$, is defined as the maximal subset of $Q_{\bar{p}}$ such that if

$$z \in (\bar{z} \times \Delta_{np}^{(i)}(\bar{z}, x) \times x) \cap Z_{n-1,\,p} \tag{3.1.6}$$

then the condition $\mathscr{C}_i(n, p)$ below is satisfied. Before stating such conditions, let us introduce the set $B_{np}(z)$ such that if $n \geqq 1$, then

$$B_{np}(z) = \{ k \in B \mid d(k, J_n(z)) \geqq D_9^{-1}\, 2^{-\theta_9 p - \alpha n^2} \} \tag{3.1.7}$$

and if $n=0$

$$B_{0\,p}=\varnothing. \tag{3.1.8}$$

We have

<u>*Condition $\mathscr{C}_1(n, p)$.*</u> *If $\omega\in\mathscr{U}_p\setminus\mathscr{U}_{p-1}$, we have*

$$|\mathscr{E}_n(z, \omega; j(z))|\geqq\frac{3}{2}D_5^{-1}2^{-\theta_5 p-n} \tag{3.1.9}$$

for all $j(z)\in J_n(z)$ and we have

$$|\mathscr{E}_n(z, \omega; k)|\geqq\frac{3}{4}D_5^{-1}2^{-\theta_5 p-n} \tag{3.1.10}$$

for all $k\in B\setminus B_{np}(z)$.

<u>*Condition $\mathscr{C}_2(n, p)$.*</u> *If $\omega\in\mathscr{U}_p\setminus\mathscr{U}_{p-1}$, we have*

$$\left|\frac{d}{dk}\bigg|_{k=k_0} E_n(z; k)\right|\geqq\frac{3}{2}D_6^{-1}2^{-\theta_6 p-n} \tag{3.1.11}$$

and

$$\left|\frac{d}{dk}\bigg|_{k=k_0} \mathscr{E}_n(z, \omega; k)\right|\geqq\frac{3}{2}D_6^{-1}2^{-\theta_6 p-n} \tag{3.1.11'}$$

for all points $k_0\in B_{np}(z)$ such that

$$|\mathscr{E}_n(z, \omega; k_0)|\leqq\frac{3}{2}D_7^{-1}2^{-\theta_7 p-n} \tag{3.1.12}$$

<u>*Condition $\mathscr{C}_3(n, p)$.*</u> *If $\omega\in\mathscr{U}_p\setminus\mathscr{U}_{p-1}$, we have*

$$|\mathscr{E}_n(z, \omega; \lambda(z))|\geqq\frac{3}{2}D_5^{-1}2^{-\theta_5 p-n} \tag{3.1.13}$$

for all

$$\lambda(z)\in\Lambda_n(z, \omega_0)\cap B_{np}(z)\cap t_{-\omega}B_{np}(z) \tag{3.1.14}$$

with $\omega_0\in\mathscr{U}_{p-1}\setminus\mathscr{U}_n$, such that

$$\left|\frac{d}{dk}\bigg|_{k=\lambda(z)} \mathscr{E}_n(z, \omega_0; k)\right|\geqq\frac{3}{2}D_6^{-1}2^{-\theta_6 p-n} \tag{3.1.15}$$

<u>*Condition $\mathscr{C}_4(n, p)$.*</u> *If $n\geqq1$ and $p\geqq\delta_0^{-1}(n-1)$, then for all $\omega_1\in\mathscr{U}_p\setminus\mathscr{U}_{p-1}$, $\omega_2\in\mathscr{U}_p\setminus\mathscr{U}_n$ and all pairs of distinct and not mutually conjugated zeros such that*

$$\lambda_1(z)\in\Lambda_n(z, \omega_1)\cap B_{np}(z)\cap t_{-\omega_1}B_{np}(z) \tag{3.1.16}$$

$$\lambda_2(z)\in\Lambda_n(z, \omega_2)\cap B_{np}(z)\cap t_{-\omega_2}B_{np}(z) \tag{3.1.17}$$

and such that

$$\left|\frac{d}{dk}\bigg|_{k=\lambda_i(z)} \mathscr{E}_n(z, \omega_i; k)\right|\geqq\frac{3}{2}D_6^{-1}2^{-\theta_6 p-n} \tag{3.1.18}$$

$i=1, 2$, *we have*

$$d(\lambda_1(z), t_\omega \lambda_2(z)) \geqq \frac{5}{2} D_3^{-1} 2^{-\theta_3 p} \qquad (3.1.19)$$

for all $\omega \in \mathcal{U}_n \cup \{0\}$.

Condition $\mathscr{C}_5(n, p)$. *For all* $\omega_1 \in \mathcal{U}_p \setminus \mathcal{U}_{p-1}$, $\omega_2 \in \mathcal{U}_p \setminus \mathcal{U}_n$ *and all pairs of distinct and not mutually conjugated zeros such that* (3.1.14), (3.1.15), (3.1.16) *hold and* (3.1.17) *is satisfied for all* $\omega \in \mathcal{U}_n \cup \{0\}$, *we have*

$$d(\lambda_1(z), t_{\omega_0} J_n(z)) \geqq \frac{3}{2} D_8^{-1} 2^{-\theta_8 p - n} \|\omega_0\|^{-\theta_0} \qquad (3.1.20)$$

and

$$d(\lambda_1(z), t_{\omega_0} \lambda_2(z)) \geqq \frac{3}{2} D_8^{-1} 2^{-\theta_0 p - n} \|\omega_0\|^{-\theta_0} \qquad (3.1.21)$$

for all $\omega_0 \in \mathcal{U}$.

3.2. Notations and Preliminary Lemmas

Let us start by giving the definition of the points $h_p(z, \beta)$ and of the integers $\beta_p(z)$ in (2.2.7) and (2.2.19). If $n \geqq 1$ and $p \in P_n$ is such that

$$p \geqq \delta_0^{-1}(n-1) \qquad (3.2.1)$$

let $H_p(z, \beta)$, $\beta \in \{1, \ldots, \beta_p(z)\}$, be a family of arcs partitioning the set $B_{np}(z)$ such that

$$\frac{1}{2} D_3^{-1} 2^{-\theta_3 p} \leqq |H_p(z, \beta)| \leqq D_3^{-1} 2^{-\theta_3 p} \qquad (3.2.2)$$

and such that

$$H_p(z, \beta) = H_p(z', \beta) \qquad (3.2.3)$$

for all $z, z' \in Z_n$ such that $J_n(z) = J_n(z')$. The point $h_p(z, \beta)$ is defined to be the center of $H_p(z, \beta)$.

LEMMA 3.1. – *If* $n \geqq 0$, $p \in P_n$ *is such that*

$$p \geqq \delta_0^{-1}(n-1) \qquad (3.2.4)$$

and $z \in F_p Z_{n-1, p}$, *then there exists a partition* $H_p(z, \beta)$, $\beta \in \{1, \ldots, \beta_p(z)\}$, *of* $B_{np}(z)$ *with the properties above.*

Proof. – If $n=0$, the existence of $H_p(z, \beta)$ is obvious. In case $n \geqq 1$, thanks to $\mathscr{I}_1(n)$ (vii) the shortest connected component of $B_{np}(z)$ is of length

$$\geqq \frac{1}{2} D_8^{-1} 2^{-\theta_8 (n+1) - n} - D_9^{-1} 2^{-\alpha n^2 - \theta_9 p}$$

$$\geqq D_3^{-1} 2^{-\theta_3 p} \qquad (3.2.6)$$

because ε is so small that

$$\delta_0^{-1}(n-1) \geqq \max(\theta_9^{-1}(\alpha n^2+\theta_8(n+1)+n+\log_2 2D_8),\\ \theta_3^{-1}(\theta_8(n+1)+n+\log_2 2D_8)), \tag{3.2.7}$$

for all $n \geqq 1$.

Q.E.D.

Notation. — If $p \geqq \delta_0^{-1}(n-1)$, let $\bar{J}_{np}(z)$ be the set of the points $k \in B$ such that

$$d(k, J_n(z)) \leqq 2 D_2^{-1} 2^{-\theta_2 p}. \tag{3.2.8}$$

LEMMA 3.2. — *If $p \geqq \delta_0^{-1}(n-1)$ and $\nu_0 \in \{1, \ldots, \nu_p\}$ is such that*

$$K_p(\nu_0) \not\subset \bar{J}_{np}(z) \tag{3.2.9}$$

for some $z \in Z_n$, then we have

$$J_n(z) = J_n(\tilde{z}) \tag{3.2.10}$$

for all $\tilde{z} \in Z_n$ such that $\tilde{z}^H = z^H$ and

$$\tilde{z}^K_{p'}(\nu) = z_{p'}(\nu) \tag{3.2.11}$$

for all $(p', \nu) \neq (p, \nu_0)$.

Proof. — This is a straightforward consequence of $\mathscr{I}_{10}(n)$ (i) and of the fact that if (3.2.9) holds then we have

$$J_n^p(z) \cap t_\omega K_p(\nu_0) = \varnothing \tag{3.2.12}$$

for all $\omega \in \mathscr{U}_n \cup \{0\}$.

Q.E.D.

LEMMA 3.3. — *If $p \geqq \delta_0^{-1}(n-1)$, $z \in F_n Z_{n-1}$, and $k_0 \in \bar{J}_{np}(z)$, then for all $\omega \in \mathscr{U}_p \setminus \mathscr{U}_{p-1}$, we have*

$$t_\omega k_0 \notin \bar{J}_{np}(z). \tag{3.2.13}$$

Proof. — The proof is by contradiction. Let $j_0(z)$ (resp. $j_1(z)$) be the closest point to k_0 (resp. to $t_\omega k_0$) belonging to the set $J_n^p(z)$. If (3.2.13) is false, then there are frequencies $\omega_0, \omega_1 \in \mathscr{U}_n$ such that

$$d(k_0, t_{\omega_0} j_0(z)) \leqq 2 D_2^{-1} 2^{-\theta_2 p} \tag{3.2.14}$$

and

$$d(t_\omega k_0, t_{\omega_1} j_1(z)) \leqq 2 D_2^{-1} 2^{-\theta_2 p}. \tag{3.2.15}$$

Hence, we have

$$d(j_0(z), t_{\omega_1-\omega-\omega_0} j_1(z))\\ \leqq d(j_0(z), t_{-\omega_0} k_0) + d(t_{-\omega_0} k_0, t_{\omega_1-\omega-\omega_0} j_1(z)) \leqq 4 D_2^{-1} 2^{-\theta_2 p}. \tag{3.2.16}$$

On the other hand, we have

$$\| \omega_1 - \omega - \omega_0 \| \leqq 2^{\theta_1 p} + 2^{\theta_1 n+1} \leqq 2^{\theta_{11} p} \tag{3.2.17}$$

because

$$\theta_{11} \geqq \theta_1 + 1. \qquad (3.2.18)$$

Hence, thanks to $\mathscr{I}_1(n)$ (vii) we have

$$d(j_1(z), t_{\omega_1-\omega-\omega_0} j_2(z)) \geqq \frac{1}{2} D_8^{-1} 2^{-\theta_8 n} \|\omega_1 - \omega - \omega_0\|^{-\theta_0}$$

$$\geqq \frac{1}{2} D_8^{-1} 2^{-\theta_0 \theta_{11} p - \theta_8 n} \geq 5 D_2^{-1} 2^{-\theta_2 p} \qquad (3.2.19)$$

where the last inequality holds because

$$\theta_2 \geqq \theta_0 \theta_{11} + 1 \qquad (3.2.20)$$

and ε is so small that

$$\delta_0^{-1}(n-1) \geqq \theta_8 n + \log_2\left(\frac{2}{5} D_8 D_2^{-1}\right) \qquad (3.2.21)$$

for all $n \geqq 1$.

Q.E.D.

3.3. The First Condition

We have

LEMMA 3.4. – *If $n \geqq 1$, $p \in \mathscr{P}_n$ is such that*

$$n+1 \leqq p < \delta_0^{-1}(n-1) \qquad (3.3.1)$$

and $z \in F_p Z_{n-1,p}$, then condition $\mathscr{C}_1(n, p)$ is satisfied.

Proof. – Due to $\mathscr{I}_7(n)$ (iii), we have

$$\sup_{k \in B} \left| \frac{d}{dk} \mathscr{E}_n(z, \omega; k) \right| \leqq 2^{\alpha n^2 + 1} \qquad (3.3.2)$$

Since

$$D_9 \geqq \frac{8}{3} D_5, \qquad \theta_9 \geqq \theta_5 + 1, \qquad (3.3.3)$$

we have

$$D_9^{-1} 2^{-\theta_9 p - \alpha n^2} \leqq \frac{3}{8} D_5^{-1} 2^{-\alpha n^2 - \theta_5 p - n}. \qquad (3.3.4)$$

Hence, if (3.1.7) holds for all $j(z) \in J_n(z)$, then (3.1.8) holds for all $k \in B \setminus B_{np}(z)$. Thus, it suffices to prove (3.1.7) for all $j(z) \in J_n(z)$. Thanks to $\mathscr{I}_1(n)$ (ii) and (iv), we have

$$|\mathscr{E}_{n-1}(z, \omega; j(z))| \geqq D_5^{-1} 2^{-\theta_5 p - n + 1} \qquad (3.3.5)$$

for all $j(z) \in \mathrm{J}_n(z)$. In virtue of $\mathscr{I}_7(n)$ (ii), we have

$$\begin{aligned}|\mathscr{E}_n(z,\omega;j(z))| &\geqq |\mathscr{E}_{n-1}(z,\omega;j(z))| - |\mathscr{E}_n(z,\omega;j(z)) - \mathscr{E}_{n-1}(z,\omega;j(z))| \\ &\geqq 2\,\mathrm{D}_5^{-1}\, 2^{-\theta_5 p - n} - 4\,\varepsilon_{n-1} \\ &\geqq 2\,\mathrm{D}_5^{-1}\, 2^{-\theta_5 p - n} - 4\,\mathrm{D}_4^{-1}\, 2^{-\theta_4 \delta_0^{-1}(n-1) - n} \\ &\geqq \frac{3}{2}\mathrm{D}_5^{-1}\, 2^{-\theta_5 p - n} \end{aligned} \tag{3.3.6}$$

because

$$\mathrm{D}_4 \geqq 8\,\mathrm{D}_5, \qquad \theta_4 \geqq \theta_5. \tag{3.3.7}$$

Q.E.D.

LEMMA 3.5. – *If* $n \geqq 0$, $p \in \mathscr{P}_n$ *is such that*

$$\delta_0^{-1}(n-1) \leqq p < \delta_0^{-1}(n) \tag{3.3.8}$$

and $\bar{z} \in \Pi_{<p} \mathrm{Z}_n$, $x \in \mathrm{X}_{np}$, *we have*

$$l(\mathrm{Q}_p \setminus \Delta_{np}^{(1)}(z, x)) \leqq \frac{1}{5} 2^{-p}. \tag{3.3.9}$$

Proof. – As in Lemma 3.4, also here (3.1.7) implies (3.1.8). If $\nu_0 \in \{1, \ldots, \nu_p\}$ and $\omega \in \mathscr{U}_p \setminus \mathscr{U}_{p-1}$, let

$$\Delta_{np}^{(1)}(\bar{z}, x, \omega, \nu_0) \subset \mathrm{Q}_p \tag{3.3.10}$$

be the set such that if

$$z \in \bar{z} \times \Delta_{np}^{(1)}(\bar{z}, x, \omega, \nu_0) \times x \tag{3.3.11}$$

then we have

$$|\mathscr{E}_n(z, \omega; j(z))| \geqq \frac{3}{2}\mathrm{D}_5^{-1}\, 2^{-\theta_5 p - n} \tag{3.3.12}$$

for all

$$j(z) \in \mathrm{K}_p(\nu_0) \cap \mathrm{J}_n(z). \tag{3.3.13}$$

Let us remark that due to (3.2.22) we have

$$\#(\mathrm{K}_p(\nu_0) \cap \mathrm{J}_n(z)) \leqq 1 \tag{3.3.14}$$

for all $\nu \in \{1, \ldots, \nu_p\}$. We propose to estimate the Lebesgue measure of the set (3.3.11).

In virtue of Lemma 3.3, if the set in (3.3.14) is nonvoid for some $z_p \in \mathrm{Q}_p$, then we have

$$t_\omega \mathrm{K}_p(\nu_0) \not\subset \bar{\mathrm{J}}_{np}(z) \tag{3.3.15}$$

for all $z_p \in Q_p$. Let $\nu_1 \in \{1, \ldots, \nu_p\}$ be such that

$$t_\omega j(z) \in K_p(\nu_1) \tag{3.3.16}$$

Thanks to Lemma 3.2, $J_n(z)$ and, consequently, the arcs $H_p(\beta, z)$ do not vary as $z_p(\nu_1)$ sweeps the interval $[-1, 1]$.

Let us fix a sequence $\bar{z}_p^K(\beta)$ defined for all $\beta \in \{1, \ldots, \beta_p(z)\}$ and a sequence $\bar{z}_p^k(\nu)$ defined for $\nu \in \{1, \ldots, \nu_p\} \setminus \{\nu_1\}$. Let

$$\Delta_{np}^{(1)}(\bar{z}, x, \bar{z}_p, \omega, \nu_0) \subset [-1, 1] \tag{3.3.17}$$

be the set of the values of $z_p^K(\nu_1)$ such that if $z_p^H = \bar{z}_p^H$ and $z_p^K(\nu) = \bar{z}_p^K(\nu)$ for $\nu \neq \nu_1$, then $z_p \in \Delta_{np}^{(1)}(\bar{z}, x, \omega, \nu_0)$. We have

$$l([-1, 1] \setminus \Delta_{np}^{(1)}(\bar{z}, x, \bar{z}_p, \omega, \nu_0)) \leqq 3\, D_2^{r+1} D_5^{-1} 2^{((r+1)\theta_2 - \theta_5)p - n}. \tag{3.3.18}$$

In virtue of Fubini's theorem, we also have

$$l(Q_p \setminus \Delta_{np}^{(1)}(\bar{z}, x, \omega, \nu_0)) \leqq 3\, D_2^{r+1} D_5^{-1} 2^{((r+1)\theta_2 - \theta_5)p - n}. \tag{3.3.19}$$

Hence

$$\begin{aligned} l(Q_p \setminus \Delta_{np}^{(1)}(\bar{z}, x)) &\leqq 3\,(\# \mathcal{U}_p \setminus \mathcal{U}_{p-1})\, \nu_p D_2^{r+1} D_5^{-1} 2^{((r+1)\theta_2 - \theta_5)p - n} \\ &\leqq 6\pi\, 2^s D_2^{r+2} D_5^{-1} 2^{(s\theta_1 + (r+2)\theta_2 - \theta_5)p} \leqq \frac{1}{5} 2^{-p} \end{aligned} \tag{3.3.20}$$

because

$$D_5 \geqq 30\pi\, 2^s D_2^{r+2}, \qquad \theta_5 \geqq 1 + s\theta_1 + (r+2)\theta_2. \tag{3.3.21}$$

Q.E.D.

3.4. The Second Condition

We have

LEMMA 3.6. – *If* $n \geqq 1$ *and* $p \in P_n$ *is such that*

$$n + 1 \leqq p < \delta_0^{-1}(n-1), \tag{3.4.1}$$

then condition $\mathscr{C}_2(n, p)$ *is satisfied for all* $z \in F_p Z_{n-1, p}$.

Proof. – Let $k_0 \in B$ be a point such that

$$|\mathscr{E}_n(z, \omega; k_0)| \leqq \frac{3}{2} D_7^{-1} 2^{-\theta_7 p - n} \tag{3.4.2}$$

Thanks to $\mathscr{I}_7(n)$ (ii), we have

$$\begin{aligned}|\mathscr{E}_{n-1}(z, \omega, k_0)| &\leqq |\mathscr{E}_n(z, \omega; k_0)| + |\mathscr{E}_n(z, \omega; k_0) - \mathscr{E}_{n-1}(z, \omega; k_0)| \\ &\leqq \frac{3}{2} D_7^{-1} 2^{-\theta_7 p - n} + 4 D_4^{-1} 2^{-\theta_4 \delta_0^{-1}(n-1) - n} \\ &\leqq D_7^{-1} 2^{-\theta_7 p - n + 1}. \end{aligned} \tag{3.4.3}$$

because

$$D_4 \geqq 8 D_7, \qquad \theta_4 \geqq \theta_7.$$

Thanks to $\mathscr{I}_7(n)$ (iv), we have

$$\begin{aligned}\left|\frac{d}{dk}\right|_{k=k_0} &(\mathscr{E}_n(z, \omega; k) - \mathscr{E}_{n-1}(z, \omega; k))| \\ &\leqq 2^{\alpha n^2} \min(1, \varepsilon_{n-1} d(k, J_n(z) \cup t_{-\omega} J_n(z))^{-1}) \\ &\leqq \varepsilon_{n-1} 2^{2\alpha n^2 + \theta_9 p}. \end{aligned} \tag{3.4.5}$$

From (3.4.3), (3.4.5) and $\mathscr{I}_1(n)$ (iii), we find

$$\begin{aligned}\left|\frac{d}{dk}\right|_{k=k_0} \mathscr{E}_n(z, \omega; k)| &\geqq 2 D_6^{-1} 2^{-\theta_6 p - n} - D_4^{-1} 2^{2\alpha n^2 + (\theta_9 - \theta_4)\delta_0^{-1}(n-1)} \\ &\geqq \frac{3}{2} D_6^{-1} 2^{-\theta_6 p - n} \end{aligned} \tag{3.4.6}$$

because

$$D_4 \geqq 2 D_6, \qquad \theta_4 \geqq \theta_9 + \theta_6 + 1 \tag{3.4.7}$$

and ε is so small that

$$\delta_0^{-1}(n-1) \geqq 2\alpha n^2 + n \tag{3.4.8}$$

for all $n \geqq 1$. Similarly, one can prove that (3.1.11) holds.

Q.E.D.

LEMMA 3.7. – *If $n \geqq 0$ and $p \in P_n$ is such that*

$$p \geqq \delta_0^{-1}(n-1), \tag{3.4.9}$$

then for all $\bar{z} \in \Pi_{<p} Z_n$ and all $x \in X_p$, we have

$$l(Q_p \setminus \Delta_{np}^{(2)}(\bar{z}, x)) \leqq \frac{1}{5} 2^{-p} \tag{3.4.10}$$

Proof. – For all $\omega \in \mathscr{U}_p \setminus \mathscr{U}_{p-1}$ and all $\nu \in \{1, \ldots, \nu_0\}$, let us introduce the set

$$\Delta_{np}^{(2)}(\bar{z}, x, \omega, \nu_0) \subset Q_p \tag{3.4.11}$$

defined as the maximal set such that if

$$z \in \bar{z} \times \Delta_{np}^{(2)}(\bar{z}, x, \omega, \nu_0) \times x \tag{3.4.12}$$

then we have

$$\left|\frac{d}{dk} \mathrm{E}_n(z; k)\right|, \left|\frac{d}{dk} \mathscr{E}_n)z, \omega; k)\right| \geqq \frac{3}{2} \mathrm{D}_6^{-1} 2^{-\theta_6 p-n} \tag{3.4.13}$$

for all $k \in \mathrm{B}_{np}(z) \cap t_{-\omega} \mathrm{B}_{np}(z) \cap \mathrm{K}_p(\nu)$ and we have

$$|\mathscr{E}_n(z, \omega; k)| \leqq \frac{3}{2} \mathrm{D}_7^{-1} 2^{-\theta_7 p-n} \tag{3.4.14}$$

Due to Lemma 3.3, we have either $\mathrm{K}_p(\nu) \not\subset \bar{\mathrm{J}}_{np}(z)$ or $t_{-\omega(\nu)} \not\subset \bar{\mathrm{J}}_{np}(z)$. The two cases being similar, we discuss only the situation in which $\mathrm{K}_p(\nu_0) \not\subset \bar{\mathrm{J}}_{np}(z)$. Let us fix a sequence $\bar{z}_p^{\mathrm{H}}(\beta)$ and a sequence $\bar{z}_p^{\mathrm{K}}(\nu)$ defined for $\nu \neq \nu_0$ and let

$$\Delta_{np}^{(2)}(\bar{z}, x, \bar{z}_p, \omega, \nu_0) \subset [-1, 1] \tag{3.4.15}$$

be the set of the values of $z_p^{\mathrm{K}}(\nu_0)$ such that if $z_p^{\mathrm{H}} = \bar{z}_p^{\mathrm{H}}$ and $z_p^{\mathrm{K}}(\nu) = \bar{z}_p^{\mathrm{K}}(\nu)$ for $\nu \neq \nu_0$, then we have

$$z_p \in \Delta_{np}^{(2)}(\bar{z}, x, \omega, \nu_0). \tag{3.4.16}$$

Thanks to $\mathscr{I}_{10}(m)$ (i), $m \leqq n$, if z_0 denotes the sequence with $z_p^{\mathrm{K}}(\nu_0) = 0$, we have

$$\mathscr{E}_n(z, \omega; k) = \mathscr{E}_n(z_0, \omega; k) + \mathrm{D}_2^{-(r+1)} 2^{-\theta_2 (r+1) p} z_p^{\mathrm{K}}(\nu_0). \tag{3.6.17}$$

Let us consider the set

$$\mathrm{G} = \{ k \in \mathrm{K}_p(\nu) \cap \mathrm{B}_{pn}(z) \cap t_{-\omega} \mathrm{B}_{pn}(z) \,|\, (3.4.13) \text{ fails to hold} \}. \tag{3.4.18}$$

If $k_0 \in \mathrm{G}$, let I_{k_0} be the maximal subinterval of $\mathrm{K}_p(\nu)$ containing k_0 and such that

$$\left|\frac{d}{dk} \mathscr{E}_n(z_0, \omega; k)\right| \leqq 2 \mathrm{D}_6^{-1} 2^{-\theta_6 p-n} \tag{3.4.19}$$

and let us consider the set

$$\bar{\mathrm{G}} = \bigcup_{k_0 \in \mathrm{G}} \mathrm{I}_{k_0}. \tag{3.4.20}$$

Let

$$\bar{\mathrm{G}} = \bigcup_{i \in \mathscr{J}} \bar{\mathrm{I}}_i \tag{3.4.21}$$

be the decomposition of $\bar{\mathrm{G}}$ into connected components. For all $i \in \mathscr{J}$, let $\mathrm{F}_i \subset \mathbf{R}$ be the interval of length

$$2 \mathrm{D}_6^{-1} 2^{-\theta_6 p-n} |\bar{\mathrm{I}}_i| \tag{3.4.22}$$

and having the same center of the interval

$$\mathscr{E}_n(z_0, \omega; \bar{I}_i). \tag{3.4.23}$$

We have

$$l(\bigcup_{i \in \mathscr{I}} F_i) \leqq 2 D_2^{-1} D_6^{-1} 2^{-(\theta_2+\theta_6)p-n}. \tag{3.4.24}$$

Let us consider the set of the $z_p^K(\nu_0)$ such that

$$-D_2^{(r+1)} 2^{-\theta_2(r+1)p} z_p^K(\nu_0) \notin \bigcup_{i \in \mathscr{I}} F_i. \tag{3.4.25}$$

If $z_p^K(\nu_0)$ satisfies this condition, then we have

$$\left|\frac{d}{dk}\mathscr{E}_n(z, \omega; k)\right| \geqq \frac{3}{2} D_6^{-1} 2^{-\theta_6 p-n} \tag{3.4.26}$$

for all k satisfying (3.4.14). In fact, in this case, $\Lambda_n(z, \omega) \cap \bar{G} = \varnothing$. In virtue of $\mathscr{I}_7(n)$ (iv), we have

$$\left|\frac{d^2}{dk^2}\mathscr{E}_n(z, \omega; k)\right| \geqq D_9 2^{2\alpha n^2+\theta_9 p} \tag{3.4.27}$$

for all $k \in B_{np}(z) \cap t_{-\omega} B_{np}(z)$. Hence, for all $k_0 \in G$ we have

$$\begin{aligned} |\mathscr{E}_n(z, \omega; k_0)| &\geqq \frac{1}{8} D_6^{-1} D_9^{-1} 2^{-\alpha n^2-(\theta_6+\theta_9)p-n} \\ &\geqq D_7^{-1} 2^{-\theta_7 p-n} \end{aligned} \tag{3.4.28}$$

because

$$D_7 = 8 D_6 D_9, \qquad \theta_7 = \theta_6 + \theta_9 + 1 \tag{3.4.29}$$

and ε is so small that

$$\delta_0^{-1}(n-1) \geqq 2\alpha n^2 \tag{3.4.30}$$

for all $n \geqq 1$. Similarly, one can treat the first of the conditions (3.4.13).

Thanks to (3.4.17) and (3.4.24), we have

$$l([-1, 1] \setminus \Delta_{np}^{(2)}(\bar{z}, x, \bar{z}_p, \omega, \nu_0)) \leqq 4 D_2^r D_6^{-1} 2^{(r\theta_2-\theta_6)p-n}. \tag{3.4.31}$$

Hence

$$\begin{aligned} l(Q_p \setminus \Delta_p^{(2)}(\bar{z}, x)) &\leqq (\# \mathscr{U}_p \setminus \mathscr{U}_{p-1}) \nu_p . 4 D_2^r D_6^{-1} 2^{(r\theta_2-\theta_6)p-n} \\ &\leqq 2^{s+3} \pi D_2^{r+1} D_6^{-1} 2^{((r+1)\theta_2-\theta_6)p-n} \\ &\leqq \frac{1}{5} 2^{-p}. \end{aligned} \tag{3.4.32}$$

because

$$D_6 = 5.2^{s+3}\pi D_2^{r+1}, \qquad \theta_6 = (r+1)\theta_2 + 1. \tag{3.4.33}$$

Q.E.D.

3.5. The Third Condition

We have

LEMMA 3.8. – *If $n \geqq 1$ and p is an integer such that*

$$n+1 \leqq p < \delta_0^{-1}(n-1), \tag{3.5.1}$$

then near all zeros $\lambda(z) \in \Lambda_n(z, \omega) \cap B_{np}(z) \cap t_{-\omega} B_{np}(z)$ with $\omega \in \mathcal{U}_p \setminus \mathcal{U}_{p-1}$ and $z \in F_p Z_{n-1, p}$, there exists a zero $\lambda'(z) \in \Lambda_{n-1}(z, \omega)$ such that

$$d(\lambda(z), \lambda'(z)) \leqq \varepsilon_{n-1} \tag{3.5.2}$$

Proof. – Thanks to $\mathcal{I}_7(n)$ (ii), we have

$$\begin{aligned} |\mathcal{E}_{n-1}(z, \omega; \lambda(z))| &\leqq \varepsilon_{n-1}^2 D_9 D_{30} 2^{\alpha n^2 + \theta_9 p + \theta_{30}(n+1)} \\ &\leqq D_7^{-1} 2^{-\theta_7 p - n} \end{aligned} \tag{3.5.3}$$

because ε is so small that

$$\varepsilon_{n-1}^2 \leqq D_7^{-1} D_9^{-1} D_{30}^{-1} 2^{-\alpha n^2 - (\theta_9 + \theta_7) p - n - \theta_{30}(n+1)} \tag{3.5.4}$$

for all $n \geqq 1$. Hence, thanks to $\mathcal{I}_1(n)$ (iii) we have

$$\left| \left. \frac{d}{dk} \right|_{k=\lambda(z)} \mathcal{E}_{n-1}(z, \omega; k) \right| \geqq D_6^{-1} 2^{-\theta_6 p - n}. \tag{3.5.5}$$

To fix the ideas, let us suppose that the derivative in (3.5.5) is positive. As k moves away from $\lambda(z)$ going to the left, the function $\mathcal{E}_{n-1}(z, \omega; k)$ decreases and the inequality (3.5.5) remains valid. Hence, there is a zero $\lambda'(z)$ of $\mathcal{E}_{n-1}(z, \omega; k)$ to the left of $\lambda(z)$ such that

$$\begin{aligned} d(\lambda(z), \lambda'(z)) &\leqq \varepsilon_{n-1}^2 D_6 D_9 D_{30} 2^{\alpha n^2 + (\theta_6 + \theta_9) p + \theta_{30}(n+1) + n} \\ &\leqq \varepsilon_{n-1} \end{aligned} \tag{3.5.6}$$

because ε is so small that

$$\varepsilon_n \leqq D_6^{-1} D_9^{-1} D_{30}^{-1} 2^{-\alpha n^2 - (\theta_6 + \theta_9) p - \theta_{30}(n+1) - n} \tag{3.5.7}$$

for all $n \geqq 1$.

Q.E.D.

LEMMA 3.9. – *If $n \geqq 1$ and $p \in P_n$ is such that*

$$n+1 \leqq p < \delta_0^{-1}(n-1), \tag{3.5.8}$$

then for all $z \in F_p Z_{n-1,p}$, *condition* $\mathscr{C}_3(n, p)$ *is satisfied.*

Proof. – Let $\omega \in \mathscr{U}_p \setminus \mathscr{U}_{p-1}$, $\omega_0 \in \mathscr{U}_{p-1} \setminus \mathscr{U}_n$ and let $\lambda(z)$ be a zero satisfying (3.1.12) and (3.1.13), Thanks to Lemma 3.9, there is a $\lambda'(z) \in \Lambda_{n-1}(z, \omega_0)$ at distance less than ε_{n-1} from $\lambda(z)$. Thanks to $\mathscr{I}_1(n)$ (iv), $\mathscr{I}_7(n)$ (ii) and $\mathscr{I}_7(n)$ (iii), we have

$$\begin{aligned} |\mathscr{E}_n(z, \omega; \lambda(z))| &\geqq |\mathscr{E}_{n-1}(z, \omega; \lambda'(z))| \\ &\quad - |\mathscr{E}_{n-1}(z, \omega; \lambda(z)) - \mathscr{E}_{n-1}(z, \omega; \lambda'(z))| \\ &\quad - |\mathscr{E}_n(z, \omega; \lambda(z)) - \mathscr{E}_{n-1}(z, \omega; \lambda(z))| \\ &\geqq D_5^{-1} 2^{-\theta_5 p - n + 1} - 2^{\alpha n^2 + 1} \varepsilon_{n-1} - 2\varepsilon_{n-1} \\ &\geqq 2 D_5^{-1} 2^{-\theta_5 p - n} - 4 D_4^{-1} 2^{\alpha n^2 - \theta_4 \delta_0^{-1}(n-1) - n} \\ &\geqq \frac{3}{2} D_5^{-1} 2^{-\theta_5 p - n} \end{aligned} \tag{3.5.9}$$

because

$$D_4 \geqq \frac{1}{8} D_5, \qquad \theta_4 \geqq \theta_5 + 1 \tag{3.5.10}$$

and ε is so small that

$$\delta_0^{-1}(n-1) \geqq \alpha n^2 \tag{3.5.11}$$

for all $n \geqq 1$.

Q.E.D.

LEMMA 3.10. – *Let* $n \geqq 0$ *and let* $p \in \mathscr{P}_n$ *be such that*

$$p \geqq \delta_0^{-1}(n-1). \tag{3.5.12}$$

If $\beta \in \{1, \ldots, \beta_p(z)\}$ *and* $H_p(z, \beta)$ *contains a zero*

$$\lambda(z) \in \Lambda_n(z, \omega_0) \cap B_{np}(z) \cap t_{-\omega_0} B_{np}(z) \tag{3.5.13}$$

such that

$$\left| \left. \frac{d}{dk} \right|_{k=\lambda(z)} \mathscr{E}_n(z, \omega; k) \right| \geqq \frac{3}{2} D_6^{-1} 2^{-\theta_6 p - n}, \tag{3.5.14}$$

then we have

$$\left| \frac{d}{dk} \mathscr{E}_n(z, \omega; k) \right| \geqq D_6^{-1} 2^{-\theta_6 p - n} \tag{3.5.15}$$

for all $k \in H_p(z, \beta)$.

Proof. — If $H_p(z, \beta)$ contains a zero satisfying (3.5.13), then we have

$$d(H_p(z, \beta), J_n(z) \cup t_{-\omega_0} J_n(z)) \geqq D_9^{-1} 2^{-\alpha n^2 + \theta_9 p} - D_3^{-1} 2^{-\theta_3 p}$$

$$\geqq \frac{1}{2} D_9^{-1} 2^{-\alpha n^2 - \theta_9 p} \qquad (3.5.16)$$

because

$$D_3 \geqq 2 D_6 D_9, \qquad \theta_3 \geqq \theta_6 + \theta_9 + 1 \qquad (3.5.17)$$

and ε is so small that

$$\delta_0^{-1}(n-1) \geqq 2^{3\alpha n^2} + n. \qquad (3.5.18)$$

Hence, thanks to $\mathcal{I}_7(n)$ (iv), we have

$$\left| \frac{d^2}{dk^2} \mathscr{E}_n(z, \omega; k) \right| \leqq 2 D_9 2^{3\alpha n^2 + \theta_9 p} \qquad (3.5.19)$$

for all $k \in H_p(z, \beta)$. Since

$$2 D_3^{-1} D_9 2^{3\alpha n^2 + (\theta_9 - \theta_3) p} \leqq \frac{1}{2} D_6^{-1} 2^{-\theta_6 p - n}, \qquad (3.5.20)$$

we find (3.5.15).

Q.E.D.

LEMMA 3.11. — *If $n \geqq 0$ and $p \in \mathscr{P}_n$ is such that*

$$p \geqq \delta_0^{-1}(n-1), \qquad (3.5.21)$$

we have

$$l(Q \setminus \Delta_{np}^{(3)}(\bar{z}, x)) \leqq \frac{1}{5} 2^{-p} \qquad (3.5.22)$$

for all $\bar{z} \in \Pi_{<p} Z_n$ and all $x \in X_p$.

Proof. — Let us fix $\bar{z} \in \Pi_{<p} Z_n$, $x \in X_p$, $\omega \in \mathscr{U}_p \setminus \mathscr{U}_{p-1}$, $\omega_0 \in \mathscr{U}_{p-1} \setminus \mathscr{U}_n$ and $\beta \in \{1, \ldots, \beta_p(z)\}$. Let

$$\Delta_{np}^{(3)}(\bar{z}, x, \beta, \omega, \omega_0) \setminus Q_p \qquad (3.5.23)$$

be the set of the $z_p \in Q_p$ such that if $H_p(z, \beta)$ contains a zero $\lambda(z)$ satisfying (3.1.14) and (3.1.15), then (3.1.13) holds. Thanks to Lemma 3.3, either $H_p(z, \beta)$ or $t_\omega H_p(z, \beta)$ is not contained in $\bar{J}_{np}(z)$. Let $\nu_0 \in \{1, \ldots \nu_p\}$ be such that $K_p(\nu_0)$ contains $\lambda(z)$ in the first case and $t_\omega \lambda(z)$ in the second. Let us fix a sequence $\bar{z}_p^H$ and a sequence $\bar{z}_p^K(\nu)$ defined for $\nu \in \{1, \ldots \nu_p\} \setminus \{\nu_0\}$. Let $\Delta_{np}^{(3)}(\bar{z}, \bar{z}_p, x, \beta, \omega, \iota_0) \setminus [-1, 1]$ be the set of the values of $z_p^K(\nu_0)$ such that if $z_p^H(\beta) = \bar{z}_p^H(\beta)$ for all $\beta \in \{1, \ldots \beta_p(z)\}$ and $z_p^K(\nu) = \bar{z}_p^K(\nu)$ for $\nu \in \{1, \ldots, \nu_p\} \setminus \{\nu_0\}$, then we have

$z_p \in \Delta_{np}^{(3)}(\bar{z}, x, \beta, \omega, \omega_0)$. We have

$$l(Q_p \setminus \Delta_{np}^{(3)}(\bar{z}, x, \beta, \omega, \omega_0)) \leqq l([-1, 1] \setminus \Delta_{np}^{(3)}(\bar{z}, \bar{z}_p, x, \beta, \omega, \omega_0))$$
$$\leqq 6\,D_2^{r+1}\,D_5^{-1}\,2^{((r+1)\theta_2-\theta_5)p-n} \qquad (3.5.25)$$

At this point, we need to estimate the number of the $\beta \in \{1, \ldots \beta_p(z)\}$ such that $H_p(z, \beta)$ contains a $\lambda(z) \in \Lambda_n(z, \omega_0)$ satisfying (3.1.13) and (3.1.14). Thanks to $\mathscr{I}_7(m)$ (iv), $m \leqq n$, if $k \in B$ we have

$$\left|\frac{d^2}{dk^2}\mathscr{E}_n(z, \omega_0; k)\right| \leqq 2^{2\alpha n^2+2}\,d(k, J_n \cup t_{-\omega_0} J_n)^{-1}. \qquad (3.5.26)$$

Let I be a connected component of the set $B \setminus (B_{np}(z) \cup t_{-\omega_0} B_{np}(z))$ and let $\tilde{I}$ be one half of I, *i.e.* an interval having an endpoint in common with I and the other one at the center of I. Let $\{\lambda_i\}_{i=0, 1, \ldots}$ be the eigenvalues $\in \Lambda_n(z, \omega_0) \cap \tilde{I}$ and satisfying (3.1.13), ordered so that $d(\lambda_i, J_n \cup t_{-\omega_0} J_n)$ increases with i. We have

$$d(\lambda_i, \lambda_{i+1}) \geqq \frac{1}{8} D_6^{-1}\,2^{-2\alpha n^2-\theta_6 p-n}\,d(\lambda_i, J_n \cup t_{-\omega_0} J_n) \qquad (3.5.27)$$

for all $i \geqq 0$. Hence

$$d(\lambda_0, \lambda_i) \geqq \left(1 + \frac{1}{8} D_6^{-1}\,2^{-2\alpha n^2-\theta_6 p-n}\right)^i d(\lambda_0, J_n \cup t_{-\omega_0} J_n)$$
$$\geqq D_9^{-1}\,2^{-\alpha n^2-\theta_9 p}\left(1 + \frac{1}{8} D_6^{-1}\,2^{-2\alpha n^2-\theta_6 p-n}\right)^i \qquad (3.5.28)$$

and we find

$$\#\{\lambda_i\} \leqq \left[\ln\left(1 + \frac{1}{8} D_6^{-1}\,2^{-2\alpha n^2-\theta_6 p-n}\right)\right]^{-1} \ln(2\pi D_9\,2^{\alpha n^2+\theta_9 p})$$
$$\leqq 8\,D_6\,2^{2\alpha n^2+\theta_6 p+n} \ln(2\pi D_9\,2^{\alpha n^2+\theta_9 p}) \qquad (3.5.29)$$

We can thus conclude that the number of β such that $H_p(z, \beta)$ contains a zero $\lambda(z) \in \Lambda_n(z, \omega_0)$ with the properties above is

$$\leqq 2^6 \pi D_3 D_6\,2^{2\alpha n^2+\theta_3 n+\theta_6 p+n} \ln(2\pi D_9\,2^{\alpha n^2+\theta_9 p})$$
$$\leqq (\alpha n^2 + \theta_9 p)\,2^{(\theta_6+1)p} \qquad (3.5.30)$$

because ε is so small that

$$\delta_0^{-1}(n-1) \geqq 2\alpha n^2 + \theta_3 n + n$$
$$+ \log_2(2^6 \pi D_3 D_6) + \log_2(\ln 2\pi D_9) \qquad (3.5.31)$$

for all $n \geqq 1$.

On the basis of (3.5.25) and (3.5.30), we find

$$l(Q_p \setminus \Delta_{np}^{(3)}(\bar{z}, x)) \leqq 3 \,.\, 2^{2s+1} D_2^{r+1} D_5^{-1} (\alpha n^2 + \theta_9 p) 2^{(2\theta_1 s + (r+1)\theta_2 + \theta_6 + 1 - \theta_5) p - n} \leqq \frac{1}{5} 2^{-p} \quad (3.5.32)$$

because

$$D_5 \geqq 15 \,.\, 2^{2s+1} D_2^{r+1} (\alpha + \theta_9), \quad (3.5.33)$$

$$\theta_5 \geqq 2\theta_1 s + (r+1)\theta_2 + \theta_6 + 2 \quad (3.5.34)$$

Q.E.D.

3.6. The Fourth Condition

We have

LEMMA 3.12. – *If $n \geqq 0$ and $p \in \mathcal{P}_n$ is such that*

$$p \geqq \delta_0^{-1}(n-1), \quad (3.6.1)$$

then for all $\bar{z} \in \Pi_{<p} Z_n$ and all $x \in X_p$, we have

$$l(Q \setminus \Delta_{np}^{(4)}(\bar{z}, x)) \leqq \frac{1}{5} 2^{-p} \quad (3.6.2)$$

Proof. – Let us fix $\beta \in \{1, \ldots, \beta_p(z)\}$, $\omega_0 \in \mathcal{U}_n \cup \{0\}$, $\omega_1 \in \mathcal{U}_p \setminus \mathcal{U}_{p-1}$ and $\omega_2 \in \mathcal{U}_p \setminus \mathcal{U}_n$. If $\bar{z} \in \Pi_{<\delta_0^{-1}(n-1)} Z_n$ and $x \in X_p$, let

$$\Delta_{np}^{(4)}(\bar{z}, x, \beta, \omega_0, \omega_1, \omega_2) \subset Q_p \quad (3.6.3)$$

be the set of the $z_p \in Q_p$ such that if

$$\lambda_1(z) \in H_p(z, \beta) \cap \Lambda_n(z, \omega_1), \quad (3.6.4)$$

then we have

$$t_{\omega_0} \bar{H}_p(z, \beta) \cap \Lambda_n(z, \omega_2) = \varnothing. \quad (3.6.5)$$

Here $\bar{H}_p(z, \beta)$ is the closed arc having the same center $h_p(z, \beta)$ of $H_p(z, \beta)$ and length

$$|\bar{H}_p(z, \beta)| = 6 D_3^{-1} 2^{-\theta_3 p}. \quad (3.6.6)$$

SUBLEMMA 3.13. – *There are two indices $\nu_1, \nu_2 \in \{1, \ldots \nu_p\}$ such that*

$$K_p(\nu_1) \not\subset \bar{J}_{np}(z), \qquad K_p(\nu_2) \not\subset \bar{J}_{np}(z), \quad (3.6.7)$$

$$\bar{K}_p(\nu_1) \cap \bar{K}_p(\nu_2) = \varnothing \quad (3.6.8)$$

and are such that $K_p(\nu_1)$ contains either $H_p(z, \beta)$ or $t_{\omega_1} H_p(z, \beta)$ and $K_p(\nu_2)$ contains either $t_{\omega_0} H_p(z, \beta)$ or $t_{\omega_0+\omega_2} H_p(z, \beta)$.

Proof of Sublemma 3.13. — In case $\omega_0=0$, we have to consider only the situation in which $\omega_1\neq\omega_2$. If $h_p(z,\beta)\notin J_{np}(z)$, then either $t_{\omega_1}h_p(z,\beta)$ or $t_{\omega_2}h_p(z,\beta)\notin J_{np}(z)$, as one can see from the proof of Lemma 3.3. In case $\omega_0\neq 0$, $\omega_2=\omega_1+\omega_0$ and $t_{\omega_2}h_p(z,\beta)=t_{\omega_1+\omega_0}h_p(z,\beta)\notin \bar{J}_{np}(z)$, then either $h_p(z,\beta)$ or $t_{\omega_0}h_p(z,\beta)\notin \bar{J}_{np}$. In all other cases, thanks to Lemma 3.3, either $h_p(z,\beta)$ or $t_{\omega_1}h_p(z,\beta)\notin \bar{J}_{np}(z)$ and either $t_{\omega_0}h_p(z,\beta)$ or $t_{\omega_0+\omega_1}h_p(z,\beta)\notin \bar{J}_{np}(z)$. Moreover, these four points are separated from each other by a distance $>2D_2^{-1}2^{-\theta_2 p}$.

Q.E.D.

Let us return to the proof of Lemma 3.12. Let us choose the indices ν_1, ν_2 as indicated in the Sublemma and let us fix the sequence $\bar{z}_p^H\in Q_p^H$ and the sequence $\bar{z}_p^H(\nu)$ defined for $\nu\in\{1,\ldots\nu_p\}\setminus\{\nu_1,\nu_2\}$. Let

$$\Delta_{np}^{(4)}(\bar{z},\bar{z}_p,x,\beta,\omega,\omega_1,\omega_2)\subset[-1,1]^2 \tag{3.6.9}$$

be the set of the values of $(z_p^K(\nu_1),z_p^K(\nu_2))$ such that if $z_p^H=\bar{z}_p^H$ and $z_p^K(\nu)=\bar{z}_p^K(\nu)$ for $\nu\neq\nu_1,\nu_2$, we have $z_p\in\Delta_{np}^{(4)}(\bar{z},x,\beta,\omega,\omega_1,\omega_2)$. We propose to estimate the measure of the set in (3.6.9).

Let $(\hat{z}_p^K(\nu_1),\hat{z}_p^K(\nu_2))$ be a point not belonging to the set in (3.6.9) and let us consider the region

$$R=\{(\xi,\eta)\in[-1,1]^2 \text{ such that } |\eta-\hat{z}_p(\nu_2)|\leqq|\xi-\hat{z}_p(\nu_1)|\}. \tag{3.6.10}$$

As $(z_p^K(\nu_1),z_p^K(\nu_2))$ moves away from $(\hat{z}_p^K(\nu_1),\hat{z}_p^K(\nu_2))$ staying inside the region R, thanks to $\mathscr{I}_{10}(n)$ (iv), the renormalized dispersion law changes so that

$$\begin{aligned}&|\mathscr{E}_n(z,\omega_1;k)-\mathscr{E}_n(\hat{z},\omega_1;k)-(z_p^K(\nu_1)-\hat{z}_p(\nu_1))D_2^{-(r+1)}2^{-(r+1)p}|\\ &\qquad\leqq\frac{1}{2}\left(\sum_{m=0}^{n}\varepsilon_m 2^{\alpha m^2}\right)|z_p^K(\nu_2)-\hat{z}_p^K(\nu_2)|D_2^{-(r+1)}2^{-(r+1)\theta_2 p},\end{aligned} \tag{3.6.11}$$

for all $k\in K_p(\nu_1)$, where

$$\hat{z}=\bar{z}\times\bar{z}_p\times(\bar{z}_p(\nu_1),\bar{z}_p(\nu_2))\times x, \tag{3.6.12}$$
$$z=\bar{z}\times\bar{z}_p\times(z_p(\nu_1),z_p(\nu_2))\times x. \tag{3.6.13}$$

Since

$$\sup_{k\in H_p(\beta,\hat{z})}|\mathscr{E}_n(\hat{z},\omega_1;k)|\leqq 2D_3^{-1}2^{\alpha n^2-\theta_3 p} \tag{3.6.14}$$

and since

$$H_p(\beta,z)=H_p(\beta,\hat{z}) \tag{3.6.15}$$

for all $(z_p(\nu_1),z_p(\nu_2))\in[-1,1]^2$, we see that if $(z_p^K(\nu_1),z_p^K(\nu_2))\in R$ and

$$|z_p^K(\nu_1)-\hat{z}_p^K(\nu_2)|\geqq 4D_2^{r+1}D_3^{-1}2^{\alpha n^2+((r+1)\theta_2-\theta_3)p}, \tag{3.6.16}$$

then we have

$$H_p(\beta,z)\cap\Lambda_n(z,\omega_1)=\varnothing. \tag{3.6.17}$$

Analogously, one can study the region

$$R' = \{(\xi, \eta) \in [-1, 1]^2 \mid |\eta - \tilde{z}_p^K(\nu_2)| \geqq |\xi - \tilde{z}_p^K(\nu_1)|\}. \quad (3.6.18)$$

The conclusion is that the complementary set in $[-1, 1]^2$ of the set in (3.6.9), is contained in a square of area

$$64 D_2^{2(r+1)} D_3^{-2} 2^{2\alpha n^2 + 2((r+1)\theta_2 - \theta_3)p}. \quad (3.6.19)$$

Hence (3.6.19) gives an upper bound to the measure of the set $Q_p \setminus \Delta_{np}^{(4)}(\bar{z}, x, \beta, \omega, \omega_1, \omega_2)$ and we have

$$\begin{aligned} l(Q \setminus \Delta_{np}^{(4)}(\bar{z}, x)) &\leqq \beta_p(z)(\# \mathcal{U}_n)(\# \mathcal{U}_p \setminus \mathcal{U}_{p-1})(\# \mathcal{U}_p \setminus \mathcal{U}_n) \\ &\quad \times 64 D_2^{2(r+1)} D_3^{-2} 2^{2\alpha n^2 + 2((r+1)\theta_2 - \theta_3)p} \\ &\leqq 2^{7+3s} \pi D_2^{2(r+1)} D_3^{-1} 2^{2\alpha n^2 + (2\theta_1 s + 2(r+1)\theta_2 - \theta_3)p + \theta_1 sn} \\ &\leqq \frac{1}{5} 2^{-p} \end{aligned} \quad (3.6.20)$$

because

$$D_3 \geqq 5.2^{7+3s} \pi D_2^{2(r+1)} \quad (3.6.21)$$

$$\theta_3 \geqq 2\theta_1 s + 2(r+1)\theta_2 + 2 \quad (3.6.22)$$

and ε is so small that

$$\delta_0^{-1}(n-1) \geqq 2\alpha n^2 + \theta_1 sn, \quad (3.6.23)$$

for all $n \geqq 1$.

Q.E.D.

3.7. The Fifth Condition

We have

LEMMA 3.14. – *Let $n \geqq 0$, $p \in \mathcal{P}_n$, $\bar{z} \in \Pi_{<p} Z_n$, $x \in X_p$ and let us suppose that*

$$p \geqq \delta_0^{-1}(n-1). \quad (3.7.1)$$

Then we have

$$l(Q_p \setminus \Delta_{np}^{(5)}(\bar{z}, x)) \leqq \frac{1}{5} 2^{-p}. \quad (3.7.2)$$

Proof. – Let us fix a $\beta_0 \in \{1, \ldots \beta_p(z)\}$ and let

$$\Delta_{np}^{(5)}(\bar{z}, x, \beta_0) \subset Q_p \quad (3.7.3)$$

be the set of the $z_p \in Q_p$ such that in case $H_p(z, \beta_0)$ contains a zero $\lambda_1(z)$ satisfying (3.1.16) for some $\omega_1 \in \mathcal{U}_p \setminus \mathcal{U}_{p-1}$ and (3.1.18), then (3.1.20) and (3.1.21) hold for all distinct and not conjugated zeros satisfying

(3.1.17) for some $\omega_2 \in \mathcal{U}_p \setminus \mathcal{U}_n$, (3.1.19) for all $\omega \in \mathcal{U}_n \cup \{0\}$ and (3.1.18).

Let us fix the sequences $\bar{z}_p^{\mathrm{K}}$ and $\bar{z}_p^{\mathrm{H}}(\beta)$ for $\beta \in \{1, \ldots, \beta_p(z)\} \setminus \{\beta_0\}$. As $\bar{z}_p^{\mathrm{H}}(\beta_0)$ sweeps the interval $[-1, 1]$, the unrenormalized dispersion law $\mathrm{E}_0(z; k)$ is modified on the arc of center $h_p(z, \beta_0)$ and of size $(3/2)\mathrm{D}_3^{-1} 2^{-\theta_3 p}$. If $\mathrm{H}_p(z, \beta_0)$ contains a zero $\lambda_1(z)$ with the properties above, then thanks to Lemma 3.10 this is the only zero of $\mathscr{E}_n(z, \omega_1; k)$ in $\mathrm{H}_p(z, \beta_0)$ and thanks to Condition $\mathscr{C}_4(n, p)$, as $\bar{z}_p^{\mathrm{H}}(\beta_0)$ varies, neither $\mathrm{J}_n(z)$ nor the zeros $\lambda_2(z)$ with the properties above, move. Let

$$\Delta_{np}^{(5)}(\bar{z}, \bar{z}_p, x, \beta_0) \subset [-1, 1] \tag{3.7.4}$$

be the set of the values of $z_p^{\mathrm{H}}(\beta_0)$ such that if $z_p^{\mathrm{K}} = \bar{z}_p^{\mathrm{K}}$ and $z_p^{\mathrm{H}}(\beta) = \bar{z}_p^{\mathrm{H}}(\beta)$ for $\beta \neq \beta_0$, then $z_p \in \Delta_{np}^{(5)}(\bar{z}, x, \beta_0)$. Thanks to Lemma 3.10, we have

$$\begin{aligned} l(\mathrm{Q} \setminus \Delta_{np}^{(5)}(\bar{z}, x)) &\leqq \sum_{\beta_0 = 1}^{\beta_p} l([-1, 1] \setminus \Delta_{np}^{(5)}(\bar{z}, \bar{z}_p, x, \beta_0)) \\ &\leqq \mathrm{D}_3^{r+1} 2^{(r+1).\theta_3 p} . \beta_p [\# \mathrm{J}_n(z) \cup \bigcup_{\omega_2 \in \mathcal{U}_p \setminus \mathcal{U}_n} \Lambda_n(z, \omega_2)] \\ &\quad \times \mathrm{D}_6 2^{\theta_6 p} . 3 \mathrm{D}_8^{-1} 2^{-\theta_8 p - n} \sum_{\omega_0 \in \mathcal{U}} \| \omega_0 \|^{-\theta_0} \\ &\leqq 3 . 2^{s+2} \pi^2 \mathrm{D}_3^{r+3} \mathrm{D}_6 \mathrm{D}_8^{-1} 2^{(s\theta_1 + (r+3)\theta_3 + \theta_6 - \theta_8) p - n} \sum_{\omega_0 \in \mathcal{U}} \| \omega_0 \|^{-\theta_0} \\ &\leqq \frac{1}{5} 2^{-p} \end{aligned} \tag{3.7.5}$$

because

$$\mathrm{D}_8 \geqq 15 . 2^{s+2} \pi^2 \mathrm{D}_3^{r+3} \mathrm{D}_6 \sum_{\omega_0 \in \mathcal{U}} \| \omega_0 \|^{-\theta_0} \tag{3.7.6}$$

$$\theta_8 \geqq (r+3)\theta_3 + s\theta_1 + \theta_6 + 1. \tag{3.7.7}$$

Q.E.D.

LEMMA 3.15. – *If* $n \geqq 1$, $p \in \mathscr{P}_n$ *is such that*

$$n + 1 \leqq p < \delta_0(n-1), \tag{3.7.8}$$

$z \in \mathrm{Z}_{n-1, p}$, $\omega_1 \in \mathcal{U}_p \setminus \mathcal{U}_{p-1}$, $\omega_2 \in \mathcal{U}_p \setminus \mathcal{U}_n$ *and* $\lambda_1(z)$, $\lambda_2(z)$ *are two distinct, not conjugated zeros satisfying* (3.1.16), (3.1.17) *and* (3.1.18), *then we have*

$$d(\lambda_1(z), t_{\omega_0} \mathrm{J}_n(z)) \geqq \frac{3}{2} \mathrm{D}_8^{-1} 2^{-\theta_8 p - n} \| \omega_0 \|^{-\theta_0} \tag{3.7.9}$$

and

$$d(\lambda_1(z), t_{\omega_0}(z)) \geqq \frac{3}{2} D_8^{-1} 2^{-\theta_8 p-n} \| \omega_0 \|^{-\theta_0} \tag{3.7.10}$$

for all $\omega_0 \in \mathcal{U}$ *such that*

$$\| \omega_0 \| \leqq (4 D_8 2^{\theta_8 p+n} \varepsilon_{n-1})^{-\theta_0}. \tag{3.7.11}$$

Proof. – Let $\lambda'_1(z) \in \Lambda_{n-1}(z, \omega_1)$ and $\lambda'_2(z) \in \Lambda_{n-1}(z, \omega_2)$ be the zeros at distance $\leqq \varepsilon_{n-1}$ from $\lambda_1(z)$ and $\lambda_2(z)$, respectively, whose existence is established in Lemma 3.8. Thanks to $\mathcal{I}_1(n)$ (vi), we have

$$\begin{aligned} d(\lambda_1(z), t_{\omega_0}\lambda_2(z)) &\geqq d(\lambda'_1(z), t_{\omega_0}\lambda'_2(z)) - d(\lambda_1(z), \lambda'_1(z)) - d(\lambda_2(z), \lambda'_2(z)) \\ &\geqq 2 D_8^{-1} 2^{-\theta_8 p-n} \| \omega_0 \|^{-\theta_0} - 2\varepsilon_{n-1} \\ &\geqq \frac{3}{2} D_8^{-1} 2^{-\theta_8 p-n} \| \omega_0 \|^{-\theta_0} \end{aligned} \tag{3.7.12}$$

because

$$\varepsilon_{n-1} \leqq \frac{1}{4} D_8^{-1} 2^{-\theta_8 p-n} \| \omega_0 \|^{-\theta_0}. \tag{3.7.13}$$

for all ω_0 satisfying (3.7.11). (3.7.10) can be proven in a similar way.

Q.E.D.

Lemma 3.16. – *If* $n \geqq 1$, $p \in P_n$ *is such that*

$$n+1 \leqq p < \delta_0^{-1}(n-1), \tag{3.7.14}$$

$\bar{z} \in \Pi_{<\bar{p}} Z_n$ *and* $x \in X_{\bar{p}}$, *then we have*

$$l(Q_{\bar{p}} \setminus \Delta_{np}^{(5)}(\bar{z}, x)) \leqq \frac{1}{5} 2^{-\bar{p}}, \tag{3.7.15}$$

where $\bar{p}$ *is defined in* (3.1.2).

Proof. – Thanks to Lemma 3.15, one can define $\Delta_{np}^{(5)}(\bar{z}, x)$ as the maximal subset of $Q_{\bar{p}}$ such that if

$$z \in (\bar{z} \times \Delta_{np}^{(5)}(\bar{z}, x) \times x) \cap Z_{n-1, p}, \tag{3.7.16}$$

then condition $\mathcal{C}_5(n, p)$ holds for all $\omega_0 \in \mathcal{U}$ such that

$$\| \omega_0 \| > (4 D_8 2^{\theta_8 p+n} \varepsilon_{n-1})^{-\theta_0} \tag{3.7.17}$$

Let us denote with $\bar{\mathcal{U}} \subset \mathcal{U}$ the set of the $\omega_0 \in \mathcal{U}$ satisfying (3.7.17).

Let us fix a $\beta_0 \in \{1, \ldots, \beta_{\bar{p}}(z)\}$ and let

$$\Delta_{np}^{(5)}(\bar{z}, x, \beta_0) \subset Q_p \tag{3.7.18}$$

be the set of the $z_{\bar{p}} \in Q_{\bar{p}}$ such that if $H_{\bar{p}}(z, \beta_0)$ contains a zero $\lambda_1(z)$ satisfying (3.1.16) for some $\omega_1 \in \mathcal{U}_p \setminus \mathcal{U}_{p-1}$ and (3.1.18) and such that

(3.1.20) and (3.1.21) hold for all distinct and not conjugated zeros $\lambda_2(z)$ satisfying (3.1.17) for some $\omega_2 \in \mathcal{U}_p \setminus \mathcal{U}_n$ and (3.1.18). Thanks to Lemma 3.9, there are two roots $\lambda'_1(z) \in \Lambda_{n-1}(z, \omega_1)$ and $\lambda'_2(z) \in \Lambda_{n-1}(z, \omega_2)$ such that

$$d(\lambda_1(z), \lambda'_1(z)), \qquad d(\lambda_2(z), \lambda'_2(z)) \leqq \varepsilon_{n-1}. \tag{3.7.19}$$

Moreover, thanks to $\mathcal{I}_1(n)$ (vi), we have

$$d(\lambda'_1(z), t_\omega \lambda'_2(z)) \geqq D_8^{-1} 2^{-\theta_8 p-n+1} . 2^{-\theta_0 \theta_1 n} \tag{3.7.20}$$

for all $\omega \in \mathcal{U}_n$. We have

$$\begin{aligned} d(\lambda_1(z), t_\omega \lambda_2(z)) &\geqq 2 . D_8^{-1} 2^{-\theta_8 p-\theta_0 \theta_1 n-n} - 2\varepsilon_{n-1} \\ &\geqq 2 D_8^{-1} 2^{-\theta_8 p-\theta_0 \theta_1 n-n} - 2 D_4^{-1} 2^{-\theta_4 \delta_0^{-1}(n-1)-n} \\ &\geqq D_8^{-1} 2^{-\theta_8 p-(\theta_0 \theta_1+1) n} \end{aligned} \tag{3.7.21}$$

for all $\omega \in \mathcal{U}_n$, because

$$D_4 \geqq 2 D_8, \qquad \theta_4 \geqq \theta_8 + 1 \tag{3.7.25}$$

and ε is so small that

$$\delta_0^{-1}(n-1) \geqq \theta_0 \theta_1 n \tag{3.7.26}$$

for all $n \geqq 1$. We have

$$D_8^{-1} 2^{-\theta_8 p-(\theta_0 \theta_1+1) n} \geqq 3 D_3^{-1} 2^{-\theta_3 \bar{p}} \tag{3.7.27}$$

because

$$\theta_{10} = \theta_3^{-1}(\theta_8 + 1) \tag{3.7.28}$$

and ε is so small that

$$\delta_0^{-1}(n-1) \geqq (1+\theta_0 \theta_1) n + \log_2(3 D_3^{-1} D_8) \tag{3.7.29}$$

for all $n \geqq 1$. Hence, the arc $H_{\bar{p}}^-(z, \beta_0)$ contains at most one zero $\lambda_1(z)$ with the properties above. Moreover, in case this happens, the set

$$\bigcup_{\omega \in \mathcal{U}_n \cup \{0\}} t_\omega H_{\bar{p}}^-(z, \beta_0) \tag{3.7.30}$$

contains no zero $\lambda_2(z)$ with the properties above and it has void intersection with the principal jump set $J_n^p(z)$. As in Lemma 3.14, we can conclude that as $\bar{z}_p^H(\beta_0)$ varies, neither $J_n(z)$ nor the zeros $\lambda_2(z)$ move. By repeating the arguments in the proof of Lemma 3.14, we find

$$\begin{aligned} &l(Q_p \setminus \Delta_{np}^{(5)}(\bar{z}, x)) \\ &\leqq 3 . 2^{s+2} \pi^2 D_3^{r+3} D_6 D_8^{-1} 2^{(s \theta_1 + (r+1) \theta_3 + \theta_6 - \theta_8) p-n} \sum_{\omega_0 \in \mathcal{U}} \|\omega_0\|^{-\theta_0}. \end{aligned} \tag{3.7.31}$$

We have

$$\sum_{\omega_0 \in \bar{\mathcal{U}}} \|\omega_0\|^{-\theta_0} \leqq \sum_{m=m_0}^{\infty} m^{s-1-\theta_0} \leqq \sum_{m=m_0}^{\infty} m^{-2}$$
$$\leqq m_0 (m_0-1)^{-2} \sum_{m=m_0}^{\infty} \left(\frac{m_0-1}{m}\right)^2 m_0^{-1}$$
$$\leqq m_0 (m_0-1)^{-2} \int_1^{\infty} dt\, t^{-2} \leqq m_0^{-1} \tag{3.7.32}$$

where

$$m_0 = (4\,D_8\, 2^{\theta_8 p+n}\, \varepsilon_{n-1})^{-\theta_0} \tag{3.7.33}$$

and we use the inequality

$$\theta_0 > s+1. \tag{3.7.34}$$

Hence, we find

$$l(Q_p \setminus \Delta_{np}^{(5)}(\bar{z}, x)) \leqq 3\,.\,2^{s+2+2\theta_0} \pi^2 D_3^{r+3} D_4^{-\theta_0} D_6 D_8^{\theta_0-1}$$
$$\times 2^{(s\theta_1+(r+3)\theta_3+\theta_6+(\theta_0-1)\theta_8)p+(\theta_0-1)n-\theta_4\theta_0\delta_0^{-1}(n-1)} \leqq \frac{1}{5} 2^{-\bar{p}} \tag{3.7.29}$$

because

$$D_4 \geqq (3\,.\,2^{s+2+2\theta_0} \pi^2 D_3^{r+3} D_6 D_8^{\theta_0-1})^{\theta_0^{-1}} \tag{3.7.30}$$
$$\theta_4 \geqq \theta_0^{-1}(\theta_{10}+s\theta_1+(r+3)\theta_3+\theta_6+(\theta_0-1)(\theta_8+1)) \tag{3.7.31}$$

Q.E.D.

3.8. Proof of the Induction Hypothesis of the Family $\mathscr{I}_1(n+1)$

We have

LEMMA 3.17. – *The induction hypothesis of the family* $\mathscr{I}_1(n+1)$, *hold.*

Proof. – (i) Follows from Lemmas 3.5, 3.7, 3.11, 3.12, 3.14 and 3.16. To prove the next induction hypothesis let us remark that for all $p \geqq n+1$ we have

$$\sum_{p'=\delta_0^{-1}(p-1)}^{\infty} D_2^{-(r+1)} 2^{-(r+1)\theta_2 p'} + D_3^{-(r+1)} 2^{-(r+1)\theta_3 p'}$$
$$\leqq 2\, D_2^{-(r+1)} 2^{-(r+1)\theta_2 \delta_0^{-1}(p-1)} (1-2^{-(r+1)\theta_2})^{-1}$$
$$\leqq 2^{-(r+1)\theta_2\delta_0^{-1}(p-1)} \tag{3.8.1}$$

(ii) If $z\in Z_{np}$, we have

$$|\mathscr{E}_n(z,\omega;j(z))|\geqq\frac{3}{2}D_5^{-1}2^{-\theta_5 p-n} \tag{3.8.2}$$

for all $\omega\in\mathscr{U}_p\setminus\mathscr{U}_{p-1}$ and $j(z)\in J_n(z)$. Hence, thanks to $\mathscr{I}_{10}(n)$ (iv), if $z\in F_p Z_{np}$ we have

$$\begin{aligned}|\mathscr{E}_n(z,\omega;j(z))|&\geqq\frac{3}{2}D_5^{-1}2^{-\theta_5 p-n}-2^{\alpha n^2-(r+1)\theta_2\delta_0^{-1}(p-1)}\\&\geqq D_5^{-1}2^{-\theta_5 p-n}\end{aligned} \tag{3.8.3}$$

because ε is so small that

$$\delta_0^{-1}(p-1)\geqq\frac{1}{(r+1)\theta_2}(\alpha p^2+\theta_5 p+p+1). \tag{3.8.4}$$

for all $p\geqq 1$.

(iii) Let $\omega\in\mathscr{U}_p\setminus\mathscr{U}_{p-1}$ and $k_0\in B$ be such that

$$|\mathscr{E}_n(z,\omega;k_0)|\leqq D_7^{-1}2^{-\theta_7 p-n} \tag{3.8.5}$$

for some $z\in F_p Z_{np}$ and let $z'\in Z_{np}$ be such that

$$\Pi_{<\delta_0^{-1}(p-1)}z'=\Pi_{<\delta_0^{-1}(p-1)}z.$$

Thanks to $\mathscr{I}_{10}(n)$ (iv), we have

$$\begin{aligned}|\mathscr{E}_n(z',\omega;k_0)|&\leqq D_7^{-1}2^{-\theta_7 p-n}+2^{\alpha n^2-(r+1)\theta_2\delta_0^{-1}(p-1)}\\&\leqq\frac{3}{2}D_7^{-1}2^{-\theta_7 p-n}\end{aligned} \tag{3.8.6}$$

because ε is so small that

$$\delta_0^{-1}(p-1)\geqq(r+1)^{-1}\theta_2^{-1}[\alpha p^2+(\theta_7+1)p+\log_2 D_7] \tag{3.8.7}$$

for all $p\geqq 1$. Hence, we have

$$\left|\frac{d}{dk}\mathscr{E}_n(z',\omega;k_0)\right|\geqq\frac{3}{2}D_6^{-1}2^{-\theta_6 p-n}. \tag{3.8.8}$$

Thanks to $\mathscr{I}_{10}(m)$ (vi), $m\leqq n$, we have

$$\begin{aligned}\left|\frac{d}{dk}\mathscr{E}_n(z,\omega;k_0)\right|&\geqq\frac{3}{2}D_6^{-1}2^{-\theta_6 p-n}-D_9 2^{3\alpha n^2+\theta_3 p-(r+1)\theta_2\delta_0^{-1}(p-1)+1}\\&\geqq D_6^{-1}2^{-\theta_6 p-n}\end{aligned} \tag{3.8.9}$$

because ε is so small that

$$\delta_0^{-1}(p-1)\geqq(r+1)^{-1}\theta_2^{-1}[3\alpha p^2+(\theta_6+\theta_9)p+\log_2 D_6 D_9^{-1}+2]. \tag{3.8.10}$$

(iv) If $z \in Z_{np}$, then $\Lambda_n(z, \omega_0) \subset B_{np}$ for all $\omega_0 \in \mathscr{U}_{p-1} \setminus \mathscr{U}_n$ and we have

$$|\mathscr{E}_n(z, \omega; \lambda(z))| \geqq \frac{3}{2} D_5^{-1} 2^{-\theta_5 p-n} \tag{3.8.11}$$

for all $\omega \in \mathscr{U}_p \setminus \mathscr{U}_{p-1}$ and $\lambda(z) \in \Lambda_n(z, \omega_0)$. If $z \in F_p Z_{np}$, we have

$$|\mathscr{E}_n(z, \omega; \lambda(z))| \geqq D_5^{-1} 2^{-\theta_5 p-n} \tag{3.8.12}$$

because ε is so small that (3.8.4) holds $\forall p \geqq 1$.

(v) Let $p \geqq \delta_0^{-1}(n-1)$, $z \in F_p Z_{np}$, $\omega_1 \in \mathscr{U}_p \setminus \mathscr{U}_{p-1}$, $\omega_2 \in \mathscr{U}_p \setminus \mathscr{U}_n$ and $\lambda_1(z) \in \Lambda_n(z, \omega_1)$, $\lambda_2(z) \in \Lambda_n(z, \omega_2)$ be two roots satisfying the conditions in $\mathscr{I}_1(n+1)$ (v). If $z' \in Z_{np}$ is such that $\Pi_{<\delta_0^{-1}(p-1)} z' = \Pi_{<\delta_0^{-1}(p-1)} z$, we have

$$d(\lambda_1(z'), t_{\omega_0}(z')) \geqq \frac{5}{2} D_3^{-1} 2^{-\theta_3 p} \tag{3.8.13}$$

for all $\omega_0 \in \mathscr{U}_n \cup \{0\}$. Hence

$$\begin{aligned} d(\lambda_1(z), t_{\omega_0}\lambda_2(z)) &\geqq \frac{5}{2} D_3^{-1} 2^{-\theta_3 p} - D_6 2^{\theta_6 p+n-(r+1)\theta_2 \delta_0^{-1}(p-1)} \\ &\geqq 2 D_3^{-1} 2^{-\theta_3 p} \end{aligned} \tag{3.8.14}$$

because ε is so small that

$$\delta_0^{-1}(p-1) \geqq (r+1)^{-1} \theta_2^{-1}((\theta_6+\theta_3+1)p+1+\log_2 D_3 D_6) \tag{3.8.15}$$

for all $p \geqq 1$.

(vi) is true by definition of Z_{np}. Finally, also (vii) holds. In fact, if $z \in F_n Z_{n+1}$ and $z' \in Z_{n+1}$ is such that $\Pi_{<\delta_0^{-1}(n)} z' = \Pi_{<\delta_0^{-1}(n)} z$, we have

$$\begin{aligned} d(j(z), J_{n+1}(z')) &\leqq D_6 2^{\theta_6(n+1)+n-(r+1)\theta_2 \delta_0^{-1}(n)} \\ &\leqq \frac{1}{4} D_8^{-1} 2^{-(\theta_8+\theta_{11})(n+1)-n} \end{aligned} \tag{3.8.16}$$

for all $j(z) \in J_{n+1}(z)$, because ε is so small that

$$\delta_0^{-1}(n) \geqq (r+1)^{-1} \theta_2^{-1}((\theta_6+\theta_8+\theta_{11})(n+1) + 2n + \log_2(4 D_6 D_8)). \tag{3.8.17}$$

Q.E.D.

4. SINGULAR SETS

In this section, we consider the singular sets $\mathscr{S}_n$ and $\bar{\mathscr{S}}_n$ introduced in Section 2.5 and we prove that if $\mathscr{I}(m)$ holds for all $m \leqq n$, then also $\mathscr{I}_2(n+1)$ is true.

LEMMA 4.1. – *For all $j \in J^p_{n+1}$, we have*

$$\bar{C}_n(j) \cap J'_{n+1} = \{j\} \tag{4.1}$$

Proof. – Thanks to $\mathscr{I}_1(n+1)$ (vii), we know that

$$d(j, J^p_{n+1} \setminus \{j, t_{\omega_n(j)} j\}) \geqq \frac{1}{2} D_8^{-1} 2^{-\theta_8(n+1)-n} \geqq 2 D_{12}^{-1} 2^{-\theta_{12} n} \tag{4.2}$$

because

$$D_{12} \geqq 4 D_8 2^{\theta_8}, \qquad \theta_{12} \geqq \theta_8 + 1. \tag{4.3}$$

Moreover, since $\omega_n(j) \in \mathscr{U}_{n+1}$, we have

$$\begin{aligned} d(j, t_{\omega_n(j)} j) &\geqq D_0^{-1} \| \omega_n(j) \|^{-\theta_0} \\ &\geqq D_0^{-1} 2^{-\theta_0 \theta_1 (n+1)} \geqq 2 D_{12}^{-1} 2^{-\theta_{12} n} \end{aligned} \tag{4.4}$$

because

$$D_{12} \geqq D_0 . 2^{\theta_0 \theta_1 + 1}, \qquad \theta_{12} \geqq \theta_0 \theta_1. \tag{4.5}$$

Q.E.D.

LEMMA 4.2. – *For all pairs of distinct, nonconjugated jump points $j, j' \in J^p_{n+1}$ and all $\omega_0 \in \mathscr{U}$ such that*

$$\| \omega_0 \| \leqq 2^{\theta_{11}(n+1)} \tag{4.6}$$

we have

$$\bar{C}_n(j) \cap t_{\omega_0} \bar{C}_n(j') = \varnothing. \tag{4.7}$$

thanks to $\mathscr{I}_1(n+1)$ (iii), we have

$$\left| \frac{d}{dk} \mathscr{E}_n(\omega; k) \right| \geqq D_6^{-1} 2^{-\theta_6(n+1)-n} \tag{4.12}$$

when $k = k_0$. To fix the ideas, let us suppose that the derivative in (4.12) has the same sign of $\mathscr{E}_n(\omega; k_0)$. If k moves away from k_0 going to $\mathscr{I}_1(n+1)$ (iii), (4.12) still holds. Hence, there exists a zero $\lambda \in \Lambda_n(\omega)$ such that

$$\begin{aligned} d(k_0, \lambda) &\leqq D_6 D_{13}^{-1} 2^{\theta_6(n+1)+n-\theta_{13} n} \\ &\leqq \frac{1}{2} D_{12}^{-1} 2^{-\theta_{12} n} \end{aligned} \tag{4.13}$$

because

$$D_{13} \geqq D_6 D_{12} 2^{\theta_6 + 1}, \qquad \theta_{13} \geqq \theta_6 + \theta_{12} + 1. \tag{4.14}$$

Hence $k_0 \in \bar{C}_n(\lambda)$ and $\omega = \omega_n(\lambda)$.

In case $\omega \in \mathcal{U}_n$, if (4.10) does not hold then thanks to $\mathcal{I}_7(n)$ (ii) we have

$$\begin{aligned} |\mathcal{E}_{n-1}(\omega; k_0)| &\leqq D_{13}^{-1} 2^{-\theta_{13} n} + 4\varepsilon_{n-1} \\ &\leqq 2 D_{13}^{-1} 2^{-\theta_{13} n} \leqq D_{13}^{-1} 2^{-\theta_{13}(n-1)} \end{aligned} \tag{4.15}$$

because ε is so small that

$$\varepsilon_{n-1} \leqq \frac{1}{4} D_{13}^{-1} 2^{-\theta_3 n} \tag{4.16}$$

for all $n \geqq 1$. Hence, thanks to $\mathcal{I}_2(n)$ (iii), there is a jump point $j \in J_n^p$ such that $k_0 \in \bar{C}_n(j)$ and $\omega = \omega_n(j)$. Moreover, thanks to $\mathcal{I}_2(n)$ (iv) we have

$$d(k_0, j) \leqq 2 D_{13}^{-1} D_{14} 2^{(\theta_{14} - \theta_{13}) n} \leqq \frac{1}{2} D_{12}^{-1} 2^{-\theta_{12} n} \tag{4.17}$$

because

$$D_{13} \geqq 4 D_{12} D_{14}, \qquad \theta_{13} \geqq \theta_{12} + \theta_{14} \tag{4.18}$$

Q.E.D.

LEMMA 4.4. – *If $j \in J_{n+1}^p$ and $k \in \bar{C}_n(j)$, then we have*

$$|\mathcal{E}_n(\omega_n(j); k)| \geqq D_{14}^{-1} 2^{-\theta_{14} n} d(k, j) \tag{4.19}$$

Proof. – If $j \in J_{n+1}^p \setminus J_n^p$, then thanks to the proof of Lemma 4.3, the inequality (4.12) holds for all $k \in \bar{C}_n(j)$. Hence, we have

$$\begin{aligned} |\mathcal{E}_n(\omega_n(j); k)| &\geqq D_6^{-1} 2^{-\theta_6 (n+1) - n} d(k, j) \\ &\geqq D_{14}^{-1} 2^{-\theta_{14} n} d(k, j) \end{aligned} \tag{4.20}$$

because

$$D_{14} \geqq D_6 2^{\theta_6}, \qquad \theta_{14} \geqq \theta_6 + 1. \tag{4.21}$$

In case $j \in J_n^p$ then thanks to $\mathcal{I}_2(n)$ (iv) we have

$$|\mathcal{E}_{n-1}(\omega_n(j); k)| \geqq D_{14}^{-1} 2^{-\theta_{14}(n-1)} d(k, j). \tag{4.22}$$

If j is n-degenerate we have

$$\begin{aligned} |\mathcal{E}_n(\omega_n(j); k)| &\geqq \frac{1}{2} |\mathcal{E}_{n-1}(\omega_n(j); k)| \\ &\geqq \frac{1}{2} D_{14}^{-1} 2^{-\theta_{14}(n-1)} d(k, j) \geqq D_{14} 2^{-\theta_{14} n} d(k, j) \end{aligned} \tag{4.23}$$

where we apply $\mathscr{I}_7(n)$ (vii). On the other hand, if j is n-regular, (4.19) follows directly from $\mathscr{I}_7(n)$ (vi).

Q.E.D.

LEMMA 4.5. – *If* $d_{n+1}(k, k')$ *is the semidistance in Section* 2.5, *we have*

$$\sup\{ d_0(k, k') \mid k, k' \in B \quad \text{and} \quad d_{n+1}(k, k') \leqq 1 \} \leqq 2^{\theta_1 (n+2)} \tag{4.24}$$

Proof. – Let us fix an integer $N \geqq 1$, a point $k \in B$ and let $\omega \in \mathscr{U}$ be such that

$$\|\omega\| > 2^{N \theta_1 (n+2)} \tag{4.25}$$

We have to prove that

$$d_{n+1}(k, t_\omega k) > N. \tag{4.26}$$

Let $\omega_1, \ldots, \omega_m$ be a set of frequencies in $\mathscr{U}$ which minimizes the sum

$$\frac{1}{3} \sum_{l=0}^{m-1} d_n(t_{\omega_l} k, t_{\omega_{l+1}} k) g_n(t_{\omega_l} k, t_{\omega_{l+1}} k) \tag{4.27}$$

under the constraints

$$\omega_0 = 0, \tag{4.28}$$

$$\omega_1 + \ldots + \omega_m = \omega, \tag{4.29}$$

and

$$\|\omega_{l+1} - \omega_l\| \leqq 2^{\theta_1 (n+1)}, \tag{4.30}$$

for all $l = 0, \ldots, m-1$. Thanks to $\mathscr{I}_2(n)$ (v), we have

$$d_n(t_{\omega_l} k, t_{\omega_{l+1}} k) \geqq 1 \tag{4.31}$$

for all $l = 0, \ldots, m-1$. We also have

$$2^{\theta_1 (n+1)} m \geqq 2^{N \theta_1 (n+2)} \tag{4.32}$$

Furthermore, if l_0 and l_1 are two integers $\in \{0, \ldots, m-1\}$ such that the function g_n in (4.27) vanishes, we have

$$|l_1 - l_0| \, 2^{\theta_1 (n+1)} \geqq 2^{\theta_{11} (n+1)}, \tag{4.33}$$

where we use $\mathscr{I}_2(n+1)$ (ii). Hence, we have

$$\begin{aligned} d_{n+1}(k, t_\omega k) &= \frac{1}{3} \sum_{l=0}^{m-1} d_n(t_{\omega_l} k, t_{\omega_{l+1}} k) g_n(t_{\omega_l} k, t_{\omega_{l+1}} k) \\ &\geqq \frac{1}{3} 2^{N \theta_1 (n+2) - \theta_1 (n+1)} (1 - 2^{\theta_1 n - \theta_{11} (n+1)}) \\ &\geqq \frac{1}{6} 2^{N \theta_1} > N \end{aligned} \tag{4.34}$$

because

$$\theta_{11} \geqq \theta_1, \qquad \theta_1 > \log_2 6. \tag{4.35}$$

Q.E.D.

As a consequence of the five lemmas above, we have

COROLLARY 4.6. – *The induction hypothesis in the family $\mathscr{I}_2(n+1)$, hold.*

5. THE SINGULAR PART OF $\mathbb{U}_n$

In this section, we construct the singular part $\mathbb{S}_n$ of $\mathbb{U}_n$ – *see* (2.6.1) – and we prove the induction hypothesis in the families $\mathscr{I}_3(n+1)$, $\mathscr{I}_4(n+1)$, $\mathscr{I}_5(n+1)$ and $\mathscr{I}_6(n+1)$.

5.1. Gap Estimates

Let us recall from Section 2.6 that $\mathbb{S}_n$ is the operator of the form

$$\mathbb{S}_n = \prod_{j \in J_{n+1}^{p+}} \mathbb{S}_n(j) \tag{5.1.1}$$

where $\mathbb{S}_n(j)$ is given by

$$\mathbb{S}_n(j) = \int_{B \setminus (\bar{C}_n(j) \cup \bar{C}_n(t_n j))} dk \, | k \rangle \langle k | + \int_{\bar{C}_n(j)} dk \, S_n(j; k) \tag{5.1.2}$$

and $S_n(j; k)$ is an O(2) operator of the form

$$\begin{aligned} S_n(j; k) = {} & \cos\theta_n(j; k) \, | k \rangle \langle k | - \sin\theta_n(j; k) \, | k \rangle \langle t_n k | \\ & + \sin\theta_n(j; k) \, | t_n k \rangle \langle k | + \cos\theta_n(j; k) \, | t_n k \rangle \langle t_n k |. \end{aligned} \tag{5.1.3}$$

If $k \in \mathring{C}_n(j)$, the funtion $\theta_n(j; k)$ has two bounded derivatives with respect to k and it is such that $S_n(j; k)$ diagonalizes the operator

$$\begin{aligned} F_n(j; k) = {} & E_n(k) \, | k \rangle \langle k | + w_n(j; k) \, | k \rangle \langle t_n k | \\ & + w_n(j; k) \, | t_n k \rangle \langle k | + E_n(t_n k) \, | t_n k \rangle \langle t_n k |. \end{aligned} \tag{5.1.4}$$

Notations. – Let us introduce the following functions defined on $\bar{\mathscr{S}}_n$:

$$\mathscr{E}_n(k) = \mathscr{E}_n(\omega_n(k); k) \tag{5.1.5}$$

$$w_n(k) = \sum_{j \in J_{n+1}^p} w_n(j; k) \tag{5.1.6}$$

$$\eta_n(k)=\left(\frac{1}{4}\mathscr{E}_n(k)^2+\varepsilon_n^2 w_n(k)^2\right)^{1/2}-\frac{1}{2}|\mathscr{E}_n(k)| \qquad (5.1.7)$$

$$\tilde{E}_n^s(k)=E_n(k)+\eta_n(k)\,\mathrm{sgn}\,\mathscr{E}_n(k) \qquad (5.1.8)$$

In (5.1.8) and in the following, we adopt the convention according to which sgn $0=+1$.

The following result derives from a simple calculation.

LEMMA 5.1. – *For all $j\in J_{n+1}^{p+}$ and all $k\in\bar{C}_n(j)$, the eigenvalues of the operator $F_n(j;k)$ are $\tilde{E}_n^s(k)$ and $\tilde{E}_n^s(t_n k)$.*

We have

LEMMA 5.2. – *For all $j\in J_{n+1}^p$ and all $k\in\mathring{C}_n(j)$, we have*

$$\left(\frac{1}{4}\mathscr{E}_n(k)^2+\varepsilon_n^2\,\omega_n(k)^2\right)^{1/2}\geqq\frac{1}{2}D_{14}^{-1}\,2^{-\theta_{14}n}\max(\varepsilon_n^{7/4},\,d(k,j)) \qquad (5.1.9)$$

Proof. – If $j\in J_{n+1}^p$ is $(n+1)$-regular, then either $j\in J_n^p$ and this result follows from $\mathscr{I}_7(n)$ (vi), or we have

$$\varepsilon_n|\omega_n(j;j)|\geqq 2\,\varepsilon_n^{7/4}. \qquad (5.1.10)$$

In this case, thanks to $\mathscr{I}_8(n)$ (v), we have

$$\varepsilon_n|\omega_n(j;j)|\geqq\varepsilon_n^{7/4} \qquad (5.1.11)$$

for all $k\in[j-\varepsilon_m^{-1}\varepsilon_n^{7/4},\,j+\varepsilon_m^{-1}\varepsilon_n^{7/4}]$. On the other hand, for all $j\in J_{n+1}^p$ and all $k\in\bar{C}_n(j)$ such that $d(k,j)>\varepsilon_m^{-1}\varepsilon_n^{7/4}$, thanks to $\mathscr{I}_2(n+1)$ (iv) we have

$$|\mathscr{E}(k)|\geqq D_{14}^{-1}\,2^{-\theta_{14}n}\,d(k,j) \qquad (5.1.12)$$

Q.E.D.

LEMMA 5.3. – *For all $j\in J_{n+1}^p$ and all $k\in\mathring{C}_n(j)$, we have*

(i) $$|\eta_n(k)|\leqq\varepsilon_n|\omega_n(k)|\min\left(1,\,\frac{1}{2}\varepsilon_n D_{14}\,2^{\theta_{14}n}\,d(k,j)^{-1}\right) \qquad (5.1.13)$$

(ii) $$\left|\frac{d}{dk}\eta_k(k)\right|\leqq 2^{\alpha n^2+1}\min\left(1,\,\frac{1}{2}\varepsilon_n D_{14}\,2^{\theta_{14}n}\,d(k,j)^{-1}\right) \qquad (5.1.14)$$

(iii) $$\left|\frac{d^2}{dk^2}\eta_n(k)\right|\leqq 5\,D_{14}\,2^{2\alpha n^2+\theta_{14}n}\min(\varepsilon_n^{-7/4},\,d(k,j)^{-1}) \qquad (5.1.15)$$

Proof. – (i) In case it results

$$\varepsilon_n|\omega_n(k)|\leqq\frac{1}{2}|\mathscr{E}_n(k)|, \qquad (5.1.16)$$

by applying the inequality $\sqrt{1+x}-1 \leqq \frac{1}{2}x$ valid for all $x \geqq 0$, we find

$$|\eta_n(k)| \leqq \frac{1}{2}\varepsilon_n^2 |\mathscr{E}_n(k)|^{-1} |w_n(k)|^2 \leqq \frac{1}{4}\varepsilon_n |w_n(k)|. \qquad (5.1.17)$$

If k is such that

$$\frac{1}{2} D_{14}^{-1} 2^{-\theta_{14} n} d(k, j) \geqq \varepsilon_n, \qquad (5.1.18)$$

then thanks to $\mathscr{I}_2(n+1)$ (iv) and $\mathscr{I}_8(n)$ (ii), (5.1.16) holds. In fact, we have

$$\frac{1}{2} |\mathscr{E}_n(k)| \geqq \frac{1}{2} D_{14}^{-1} 2^{-\theta_{14} n} d(k, j) \geqq \varepsilon_n \geqq \varepsilon_n |w_n(k)|. \qquad (5.1.19)$$

Hence, from the first inequality in (5.1.17) we find

$$|\eta_n(k)| \leqq \frac{1}{2}\varepsilon_n^2 |w_n(k)| D_{14} 2^{\theta_{14} n} d(k, j)^{-1}. \qquad (5.1.20)$$

On the other hand, if (5.1.17) fails to hold, then we can use the inequality $\sqrt{1+x^2}-x \leqq 1$ valid for all $x \geqq 0$ and we find

$$\begin{aligned} |\eta_n(k)| \leqq \varepsilon_n |w_n(k)| \Bigg[\left(1 + \frac{1}{4}\varepsilon_n^{-1} w_n(k)^{-2} \mathscr{E}_n(k)^2\right)^{1/2} & \\ - \frac{1}{2}\varepsilon_n^{-1} |w_n(k)^{-1} \mathscr{E}_n(k)| \Bigg] & \\ \leqq \varepsilon_n |w_n(k)|. & \qquad (5.1.21) \end{aligned}$$

(ii) We have

$$\begin{aligned} \frac{d}{dk}\eta_n(k) &= \left(\frac{1}{4}\mathscr{E}_n(k)^2 + \varepsilon_n^2 w_n(k)^2\right)^{-1/2} \left(\frac{1}{4}\mathscr{E}_n(k)\mathscr{E}_n'(k) + \varepsilon_n^2 w_n(k) w_n'(k)\right) \\ &\qquad - \frac{1}{2}\mathscr{E}'(k) \operatorname{sgn} \mathscr{E}_n(k) \\ &= \left(\frac{1}{4}\mathscr{E}_n(k)^2 + \varepsilon_n^2 w_n(k)^2\right)^{-1/2} \cdot \left(\frac{-1}{2}\eta_n(k)\mathscr{E}_n'(k) + \varepsilon_n^2 w_n(k) w_n'(k)\right). \qquad (5.1.22) \end{aligned}$$

Thanks to $\mathscr{I}_8(n)$ (iii), we have

$$|w_n'(k)| \leqq 2^{\alpha n^2} \varepsilon_{n-1}^{1/2} \min(\varepsilon_{n-1}^{-7/4}, d(k, j)^{-1}) + 2^4 D_{12} 2^{\theta_{12} n} \qquad (5.1.23)$$

Hence, we have

$$\left|\frac{d}{dk}\eta_n(k)\right| \leqq \left(\frac{1}{4}\mathscr{E}_n(k)^2+\varepsilon_n^2 w_n(k)^2\right)^{-1/2} \varepsilon_n |w_n(k)|$$
$$\times\left[2^{\alpha n^2}\min\left(1, \frac{1}{2}\varepsilon_n D_{14} 2^{\theta_{14} n} d(k, j)^{-1}\right)\right.$$
$$\left.+2^{\alpha n^2}\min(1, \varepsilon_n d(k, j)^{-1})+2^4\varepsilon_n D_{12} 2^{\theta_{12} n}\right]$$
$$\leqq 2^{\alpha n^2+1}\min\left(1, \frac{1}{2}\varepsilon_n D_{14} 2^{\theta_{14} n} d(k, j)^{-1}\right) \qquad (5.1.24)$$

because

$$d(k, j)^{-1} \geqq D_{12} 2^{\theta_{12} n}, \qquad D_{14} \geqq 2^5+2. \qquad (5.1.25)$$

(iii) We have

$$\left|\frac{d^2}{dk^2}\eta_k(k)\right|$$
$$=\left|-\frac{1}{2}\left(\frac{1}{4}\mathscr{E}_n(k)^2+\varepsilon_n^2 w_n(k)^2\right)^{-3/2}\left(\frac{1}{4}\mathscr{E}_n(k)\mathscr{E}'_n(k)+\varepsilon_n^2 w_n(k) w'_n(k)\right)^2\right.$$
$$+\left(\frac{1}{4}\mathscr{E}_n(k)^2+\varepsilon_n^2 w_n(k)^2\right)^{-1/2}\left(\frac{1}{4}\mathscr{E}'_n(k)^2+\varepsilon_n^2 w'_n(k)^2\right)$$
$$\left.+\left(\frac{1}{4}\mathscr{E}_n(k)^2+\varepsilon_n^2 w_n(k)^2\right)^{-1/2}\left(-\frac{1}{2}\eta_n(k)\mathscr{E}''_n(k)+\varepsilon_n^2 w_n(k) w''_n(k)\right)\right|$$
$$\leqq 3\left(\frac{1}{4}\mathscr{E}_n(k)^2+\varepsilon_n^2 w_n(k)^2\right)^{-1/2}\left(\frac{1}{4}\mathscr{E}'_n(k)^2+\varepsilon_n^2 w'_n(k)^2\right)$$
$$+\left(\frac{1}{2}|\mathscr{E}''_n(k)|+\varepsilon_n|w''_n(k)|\right) \qquad (5.1.26)$$

Thanks to $\mathscr{I}_7(n)$ (v), $\mathscr{I}_8(n)$ (iv) and Lemma 5.2, we have

$$\left|\frac{d^2}{dk^2}\eta_k(k)\right| \leqq D_{14} 2^{2\alpha n^2+\theta_{14} n+2}\min(\varepsilon_n^{-7/4}, d(k, j)^{-1})$$
$$+\frac{1}{2}2^{2\alpha n^2}\min(\varepsilon_{n-1}^{-7/4}, d(k, j)^{-1})$$
$$+2^{2\alpha n^2}\varepsilon_{n-1}^2\min(\varepsilon_{n-1}^{-15/4}, d(k, j)^{-2})$$
$$+2^5 D_{12} 2^{\alpha n^2+\theta_{12} n}+2^8 D_{12}^2 2^{2\theta_{12} n}\varepsilon_n$$
$$\leqq 5 D_{14} 2^{2\alpha n^2+\theta_{14} n}\min(\varepsilon_n^{-7/4}, d(k, j)^{-1}) \qquad (5.1.27)$$

because

$$D_{14} \geqq \frac{5}{2} + 2^5 \tag{5.1.28}$$

and ε is so small that

$$\varepsilon_n \leqq 2^{-8} D_{12}^{-1} 2^{-\theta_{12} n} \tag{5.1.29}$$

for all $n \geqq 0$.

Q.E.D.

The following result has a proof very similar to that of the preceeding lemma.

LEMMA 5.4. – *For all $j \in J_{n+1}^p$ and all $k \in \mathring{C}_n(j)$, we have*

$$\text{(i)} \quad \left\| \frac{\delta}{\delta E_0} \eta_n(k) \right\|_\infty \leqq 2^{\alpha n^2 + 1} \min\left(1, \frac{1}{2} \varepsilon_n D_{14} 2^{\theta_{14} n} d(k, j)^{-1}\right) \tag{5.1.30}$$

$$\text{(ii)} \quad \left\| \frac{\delta}{\delta E_0} \frac{d}{dk} \eta_n(k) \right\|_\infty \leqq 5 . D_{14} 2^{2 \alpha n^2 + \theta_{14} n} \min(\varepsilon_n^{-7/4}, d(k, j)^{-1}). \tag{5.1.31}$$

5.2. The Operator $S_n(j, k)$

We have

LEMMA 5.5. – *For all $j \in J_{n+1}^{p+}$ and all $k \in \mathring{C}_n(j)$, the function $\theta_n(j; k)$ satisfies the following differential equation*

$$\frac{d}{dk} \theta_n(j, k) = (\tilde{E}_n^s(k) - \tilde{E}_n^s(t_n k))^{-1} \times \left(\varepsilon_n w_n'(k) \cos 2\theta_n(j; k) + \frac{1}{2} \mathscr{E}_n'(k) \sin 2\theta_n(j; k)\right). \tag{5.2.1}$$

Proof. – Let $u_1(j; k)$ and $u_2(j; k)$ be the eigenvectors of the operator $F_n(j; k)$ in (5.1.4) that correspond to the eigenvalues $\tilde{E}_n^s(k)$ and $\tilde{E}_n^s(t_n k)$, respectively, and have euclidean norm one. If $k \in \mathring{C}_n(j)$, then for all $k' \in B$ close enough to k, we have

$$u_1(j; k') = \frac{1}{2\pi i c(k')} \int_{\mathscr{C}} dz (z - F_n(j; k'))^{-1} u_1(j; k), \tag{5.2.2}$$

where $\mathscr{C} \subset \mathbb{C}$ is a small circle enclosing $\tilde{E}_n^s(k)$ but not $\tilde{E}_n^s(t_n k)$ and $c(k)$ is a normalization factor. By differentiating with respect to k' at $k = k'$, we find

$$\frac{d}{dk} u_1(j;k)$$

$$= \frac{1}{2\pi i} \int_{\mathscr{C}} dz\, (z - F_n(j;k))^{-1} F'_n(j;k) (z - F_n(j;k))^{-1} u_1(j;k)$$

$$= (\tilde{E}^s_n(k) - \tilde{E}^s_n(t_n k))^{-1} \langle u_2(j;k) | F'_n(j;k) | u_j(j;k) \rangle u_2(j;k). \tag{5.2.3}$$

By introducing the expressions (5.1.4) and

$$u_1(j;k) = \begin{pmatrix} \cos\theta_n(j;k) \\ \sin\theta_n(j;k) \end{pmatrix}, \qquad u_2(j;k) = \begin{pmatrix} -\sin\theta_n(j;k) \\ \cos\theta_n(j;k) \end{pmatrix} \tag{5.2.4}$$

into (5.2.3), we find (5.2.1).

Q.E.D.

LEMMA 5.6. – *For all* $j \in J^{p+}_{n+1}$ *and all* $k \in \mathring{C}_n(j)$, *we have*

(i)
$$\left| \frac{d}{dk} \theta_n(j;k) \right| \leqq \frac{3}{2} D_{14} 2^{\alpha n^2 + \theta_{14} n} \min(\varepsilon_n^{-7/4}, d(k, j)^{-1}) \tag{5.2.5}$$

(ii)
$$\left| \frac{d^2}{dk^2} \theta_n(j;k) \right| \leqq 6 D_{14}^2 2^{2\alpha n^2 + 2\theta_{14} n} \min(\varepsilon_n^{-7/2}, d(k,j)^{-2}). \tag{5.2.6}$$

Proof. – (i) Thanks to Lemma 5.2, we have

$$\begin{aligned} |\tilde{E}^s_n(t_n k) - \tilde{E}^s_n(k)| &= |\mathscr{E}_n(k)| + 2\eta_n(k) \\ &= 2\left(\frac{1}{4}\mathscr{E}_n(k)^2 + \varepsilon_n^2 w_n(k)^2\right)^{1/2} \\ &\geqq D_{14}^{-1} 2^{-\theta_{14} n} \max(\varepsilon_n^{7/4}, d(k,j)). \end{aligned} \tag{5.2.7}$$

Hence, thanks to $\mathscr{I}_7(n)$ (iii), $\mathscr{I}_8(n)$ (iii) and Lemma 5.5, we have

$$\left| \frac{d}{dk} \theta_n(j;k \right| \leqq \frac{3}{2} D_{14} 2^{\alpha n^2 + \theta_{14} n} \min(\varepsilon_n^{-7/4}, d(k,j)^{-1}) \tag{5.2.8}$$

(ii) By differentiating in (5.2.1), we find

$$\begin{aligned} \left| \frac{d^2}{dk^2} \theta_n(j;k) \right| &\leqq 3 D_{14}^2 2^{2\alpha n^2 + 2\theta_{14} n} \min(\varepsilon_n^{-7/2}, d(k,j)^{-2}) \\ &\quad + \frac{9}{4} D_{14}^2 2^{2\alpha n^2 + 2\theta_{14} n} \min(\varepsilon_n^{-7/2}, d(k,j)^{-2}) \\ &\quad + D_{14} 2^{2\alpha n^2 + \theta_{14} n} \cdot \left(\frac{1}{2}\min(\varepsilon_{n-1}^{-7/4}, d(k,j)^{-1}) + \min(\varepsilon_{n-1}^{-7/4}, \varepsilon_{n-1}^{8/4} d(k,j)^{-2})\right) \\ &\leqq 6 D_{14}^2 2^{\alpha n^2 + 2\theta_{14} n} \min(\varepsilon_n^{-7/2}, d(k,j)^{-2}) \end{aligned} \tag{5.2.9}$$

because

$$D_{14} \geqq 2. \tag{5.2.10}$$

Q.E.D.

LEMMA 5.7. – *For all* $j \in J_{n+1}^{p+}$ *and all* $k \in \bar{C}_n(j) \setminus C_n(j)$, *we have*

(i) $$|\theta_n(j;k)| \leqq D_{15} 2^{\theta_{15} n} \varepsilon_n \tag{5.2.11}$$

(ii) $$\left|\frac{d}{dk}\theta_n(j;k)\right| \leqq D_{16} 2^{\theta_{16} n} \varepsilon_n \tag{5.2.12}$$

(iii) $$\left|\frac{d^2}{dk^2}\theta_n(j;k)\right| \leqq D_{17} 2^{\alpha n^2 + \theta_{17} n} \sqrt{\varepsilon_{n-1}} \tag{5.2.13}$$

Proof. – Let us write the operator $F_n(j;k)$ as follows.

$$F_n(j;k) = A_n(j;k) + B_n(j;k) \tag{5.2.14}$$

where

$$A_n(j;k) = E_n(k) | k \rangle \langle k | + E_n(t_n k) | t_n k \rangle \langle t_n k | \tag{5.2.15}$$

and

$$B_n(j;k) = w_n(j;k) | k \rangle \langle t_n k | + w_n(j;k) | t_n k \rangle \langle k |. \tag{5.2.16}$$

Let $u_1(j;k)$ and $u_2(j;k)$ be the eigenvectors of $F_n(j;k)$ given in (5.2.4). The orthogonal projection $P_1(j;k)$ onto $u_1(j;k)$ is given by the contour integral

$$P_1(j;k) = \frac{1}{2\pi i} \int_{\mathscr{C}_1(j;k)} dz (z - A_n(j;k) - B_n(j;k))^{-1} \tag{5.2.17}$$

where

$$\mathscr{C}_1(j;k) = \left\{ z \in \mathbb{C} \; \| z - E_n(k) | = \frac{1}{2} | E_n(k) - E_n(t_n k) | \right\}. \tag{5.2.18}$$

By expanding (5.2.17) in geometric series and using the bound

$$\| B_n(j;k) \| = | \varepsilon_n w_n(j;k) | \leqq \varepsilon_n \tag{5.2.19}$$

deriving from $\mathscr{I}_8(n)$ (ii), we find

$$\begin{aligned}
&\left\| P_1(j;k) - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\| \\
&\leqq \sum_{m=1}^{\infty} \left\| \frac{1}{2\pi i} \int_{\mathscr{C}_1} dz (z - A_n(j;k))^{-1} [B_n(j;k)(z - A_n(j;k)]^{-1} \right\| \\
&\leqq \sum_{m=1}^{\infty} 2 | E_n(k) - E_n(t_n k) |^{-m} \varepsilon_n^m \\
&\leqq 4 D_{12} D_{14} 2^{(\theta_{12} + \theta_{14}) n} \varepsilon_n (1 - 4 D_{12} D_{14} 2^{(\theta_{12} + \theta_{14}) n} \varepsilon_n)^{-1} \\
&\leqq \frac{1}{2} D_{15} 2^{\theta_{15} n} \varepsilon_n
\end{aligned} \tag{5.2.20}$$

because

$$D_{15}=16\,D_{12}\,D_{14},\qquad \theta_{15}=\theta_{12}+\theta_{14},\tag{5.2.21}$$

and ε is so small that

$$\varepsilon\leqq\frac{1}{8\sqrt{2}}D_{12}^{-1}\,D_{14}^{-1}\,2^{-(\theta_{12}+\theta_{14})\,n}\tag{5.2.22}$$

for all $n\geqq 0$. Hence, we have

$$|\sin\theta_n(j;\,k)|\leqq\frac{1}{2}D_{15}\,2^{\theta_{15}\,n}\,\varepsilon_n.\tag{5.2.23}$$

Thanks to (5.2.22), we also have

$$|\theta_n(j;\,k)|\leqq 2\,|\sin\theta_n(j;\,k)|\leqq D_{15}\,2^{\theta_{15}\,n}\,\varepsilon_n.\tag{5.2.24}$$

(ii) Thanks to Lemma 5.5, (i) and $\mathscr{I}_8(n)$ (iii), we have

$$\begin{aligned}\left|\frac{d}{dk}\theta_n(j;\,k)\right|&\leqq\frac{1}{8}\varepsilon_n\,D_{15}\,2^{\theta_{15}\,n}(2^4\,D_{12}\,2^{\theta_{12}\,n}+\sqrt{\varepsilon_{n-1}}\,2\,D_{12}\,2^{\alpha\,n^2+\theta_{12}\,n}\\&\quad+2\,D_{12}\,D_{15}\,\varepsilon_{n-1}\,2^{\alpha\,n^2+(\theta_{12}+\theta_{15})\,n}\\&\leqq D_{16}\,2^{\theta_{16}\,n}\,\varepsilon_n\end{aligned}\tag{5.2.25}$$

because

$$D_{16}=3\,D_{12}\,D_{15}+\frac{1}{4}D_{12}\,D_{15}^2,\qquad \theta_{16}=2\,\theta_{15}+\theta_{12}\tag{5.2.26}$$

and ε is so small that

$$\varepsilon_{n-1}\leqq 2^{-2\,\alpha\,n^2}\tag{5.2.27}$$

for all $n\geqq 1$.

(iii) Finally, by differentiating in (5.2.1) and using $\mathscr{I}_7(n)$ (v) and $\mathscr{I}_8(n)$ (iv), we find

$$\begin{aligned}\left|\frac{d^2}{dk^2}\theta_n(j;\,k)\right|&\leqq\frac{1}{8}D_{15}\,D_{16}\,2^{\alpha\,n^2+(\theta_{15}+\theta_{16})\,n}\,\varepsilon_n+D_{16}^2\,2^{2\,\alpha\,n^2+2\,\theta_{16}\,n}\,\varepsilon_n\\&\quad+\frac{1}{8}D_{15}\,2^{\theta_{15}\,n}(64\,D_{12}^2\,2^{\theta_{12}\,n}\,\varepsilon_n+32\sqrt{\varepsilon_{n-1}}\,D_{12}^2\,2^{\alpha\,n^2+2\,\theta_{12}\,n}\\&\quad+2\sqrt{\varepsilon_{n-1}}\,D_{12}\,2^{\alpha\,n^2+\theta_{12}\,n}+D_{15}\,2^{2\,\alpha\,n^2+\theta_{15}\,n}\,\varepsilon_n)\\&\leqq D_{17}\,2^{\alpha\,n^2+\theta_{17}\,n}\sqrt{\varepsilon_{n-1}}\end{aligned}\tag{5.2.28}$$

because

$$D_{17}=4\,D_{15}\,D_{12}^2+\frac{1}{4}D_{15}\,D_{12}+1,\qquad \theta_{17}=2\,\theta_{12}\tag{5.2.29}$$

and ε is so small that

$$\varepsilon_n^{5/6} \leqq 2^{\alpha n^2+\theta_{17} n}\left(\frac{1}{8} D_{15} D_{16} 2^{\alpha b^2+(\theta_{15}+\theta_{16}) n}+D_{16}^2 2^{2 \alpha n^2+\theta_{16} n} + \frac{1}{8} D_{15}^2 2^{2 \alpha n^2+2 \theta_{15} n}+8 D_{12}^2 D_{15} 2^{(\theta_{12}+\theta_{15}) n}\right) \quad (5.2.30)$$

for all $n \geqq 1$.

Q.E.D.

LEMMA 5.8. — *If* $j \in J_{n+1}^{p+}$ *is an* $(n+1)$*-degenerate jump of order* m, *the function* $\theta_n(j, k)$ *defined above for* $k \in \mathring{C}_n(j)$ *can be extended to* $\bar{C}_n(j)$ *in such a way that it has the following properties:*

(i) *If* $d(k, j) \leqq \varepsilon_m^{-1} \varepsilon_n^{7/4}$, *we have*

$$|E_n^s(t_n k)-E_n^s(k)| \geqq |E_n(t_n k)-E_n(k)| \quad (5.2.31)$$

where $E_n^s(k)$ *is the function defined in* (2.6.18);

(ii) *If* $\frac{1}{2} \varepsilon_m^{-1} \varepsilon_n^{7/4} \leqq d(k, j) \leqq \frac{1}{2} \varepsilon_m^{-1} \varepsilon_{n-1}^{7/4}$, *we have*

$$|\theta_n(j; k)| \leqq D_{14} 2^{\theta_{14} m+4} \varepsilon_m \leqq \varepsilon_m^{3/4} \quad (5.2.32)$$

and

$$\left|\frac{d}{dk} \theta_n(j; k)\right| \leqq 2 D_{14}^2 D_{31} 2^{(2 \theta_{14}+\theta_{31}) m} d(k, j)^{-1} \varepsilon_m \leqq \varepsilon_m^{3/4} d(k, j)^{-1} \quad (5.2.33)$$

(iii) *If* $\frac{1}{2} \varepsilon_m^{-1} \varepsilon_n^{7/4} \leqq d(k, j) \leqq \varepsilon_m^{-1} \varepsilon_n^{7/4}$, *we have*

$$\left|\frac{d^2}{dk^2} \theta_n(j; k)\right| \leqq \varepsilon_m^{5/2} \varepsilon_n^{-7/2} \quad (5.2.34)$$

(iv) *If* $d(k, j) \geqq \frac{1}{2} \varepsilon_m^{-1} \varepsilon_{n-1}^{7/4}$, *we have*

$$|\theta_n(j; k)| \leqq \varepsilon_m^{-1/4} \varepsilon_{n-1}^{7/4} d(k, j)^{-1} \quad (5.2.35)$$

and

$$\left|\frac{d}{dk} \theta_n(j; k)\right| \leqq \varepsilon_m^{-1/4} \varepsilon_{n-1}^{7/4} d(k, j)^{-2} \quad (5.2.36)$$

(v) *If* $d(k, j) \leqq \frac{1}{2} \varepsilon_m^{-1} \varepsilon_n^{7/4}$, *we have*

$$\theta_n(j; k)=0. \quad (5.2.37)$$

Proof. — Let $\bar{\theta}_n(j; k)$, $k \in \bar{C}_n(j)$, be defined as the function such that if we replace $\theta_n(j; k)$ with $\bar{\theta}_n(j; k)$ in the definition (5.1.3) of $\mathbb{S}_n(j)$, then this operator diagonalizes $F_n(j; k)$ exactly. Let us suppose that $\bar{\theta}_n(j; k)$ is chosen so that it results continuous in k and it vanishes at the endpoints of $\bar{C}_n(j)$. A rather straightforward calculation gives the following result:

$$\bar{\theta}_n(j;k)=\frac{1}{2}\cos^{-1}\left[\left(\frac{1}{4}\mathscr{E}_n(k)^2+\varepsilon_n^2 w_n(k)^2\right)^{-1/2}\cdot\frac{1}{2}\mathscr{E}_n(k)\right]. \quad (5.2.38)$$

Since $\mathscr{E}_n(k)\neq 0$ for all $k\in\bar{C}_n(j)\setminus\{j\}$, we have

$$-\frac{\pi}{4}<\bar{\theta}_n(j;k)<\frac{\pi}{4}. \quad (5.2.39)$$

Moreover, we have

$$\begin{aligned}E_n^s(t_n k)-E_n^s(k)&=\mathscr{E}_n(k)\cos 2\theta_n(j;k)-2\varepsilon_n w_n(k)\sin 2\theta_n(j;k)\\&=2\left(\frac{1}{4}\mathscr{E}_n(k)^2+\varepsilon_n^2 w_n(k)^2\right)^{1/2}\cos(2\theta_n(j;k)-\theta_n(j;k)) \quad (5.2.40)\end{aligned}$$

for all $k\in\bar{C}_n(j)\setminus\{j\}$. If $k\in\bar{C}_n(j)\setminus\mathring{C}_n(j)$, we set

$$\theta_n(j;k)=\bar{\theta}_n(j;k)\,f(2\varepsilon_n^{-7/4}(k-j+\varepsilon_n^{7/4}))\,f(2\varepsilon_n^{-7/4}(k-j-\varepsilon_n^{7/4})) \quad (5.2.41)$$

where f is the function defined in (2.2.12). (v) is clearly true. Moreover, we have

$$|\theta_n(j;k)|<|\bar{\theta}_n(j;k)|<\frac{\pi}{4} \quad (5.2.42)$$

for all $k\in\bar{C}_n(j)\setminus\mathring{C}_n(j)$. Hence

$$0\leqq 2|\bar{\theta}_n(j;k)-\theta_n(j;k)|\leqq 2|\bar{\theta}_n(j;k)|<\frac{\pi}{2} \quad (5.2.43)$$

and from (5.2.40) we find

$$\begin{aligned}|E_n^s(t_n k)-E_n^s(k)|&\geqq 2\left(\frac{1}{4}\mathscr{E}_n(k)^2+\varepsilon_n^2 w_n(k)^2\right)^{1/2}\cos(2\bar{\theta}_n(j;k))\\&\geqq|\mathscr{E}_n(k)| \quad (5.2.44)\end{aligned}$$

To prove (ii), let us notice that if $\frac{1}{2}\varepsilon_m^{-1}\varepsilon_n^{7/4}\leqq d(k,j)\leqq\frac{1}{2}\varepsilon_m^{-1}\varepsilon_{n-1}^{7/4}$ and $B_n(j;k)$ is the matrix in (5.2.16), then we have

$$\|B_n(j;k)\|=|\varepsilon_n w_n(j;k)|\leqq\varepsilon_n^{7/4}+\varepsilon_m d(k,j). \quad (5.2.45)$$

This derives from $\mathscr{I}_8(n)$ (v). Thanks to $\mathscr{I}_2(n+1)$ (iv), if we expand in geometric series as in (5.2.20), we find

$$\begin{aligned}|\bar{\theta}_n(j;k)|&\leqq 2\left\|P_1(j;k)-\begin{pmatrix}1&0\\0&0\end{pmatrix}\right\|\\&\leqq\sum_{r=1}^{\infty}2(D_{14}2^{\theta_{14}m}d(k,j))^{-1})(2\varepsilon_n^{7/4}+\varepsilon_m d(k,j))^r\\&\leqq 16D_{14}2^{\theta_{14}m}\varepsilon_m\leqq\varepsilon_m^{3/4} \quad (5.2.46)\end{aligned}$$

because ε is so small that

$$\varepsilon_m^{1/4} \leqq \frac{1}{4} D_{14}^{-1} 2^{-\theta_{14} m} \tag{5.2.47}$$

for all $m \geqq 0$. Thanks to (5.2.1), $\mathscr{I}_7(n)$ (ix) and $\mathscr{I}_8(n)$ (vi), in this case we also have

$$\begin{aligned} \left| \frac{d}{dk} \bar{\theta}_n(j; k) \right| &\leqq D_{14} 2^{\theta_{14} m} d(k, j)^{-1} (\varepsilon_m + D_{14} D_{31} 2^{(\theta_{14}+\theta_{31}) m} \varepsilon_m) \\ &\leqq \varepsilon_m^{3/4} d(k, j)^{-1} \end{aligned} \tag{5.2.48}$$

because ε is so small that

$$\varepsilon_m^{1/4} \leqq \frac{1}{2} D_{14}^{-2} D_{31}^{-1} 2^{-(2\theta_{14}+\theta_{31}) m}. \tag{5.2.49}$$

If $\frac{1}{2} \varepsilon_m^{-1} \varepsilon_n^{7/4} \leqq d(k, j) \leqq \varepsilon_m^{-1} \varepsilon_n^{7/4}$, then thanks to $\mathscr{I}_7(n)$ (ix) and $\mathscr{I}_8(n)$ (vi), we have

$$\begin{aligned} \left| \frac{d}{dk^2} \bar{\theta}_n(j; k) \right| &\leqq D_{14}^2 2^{2\theta_{14} m + \alpha m^2} \varepsilon_m^2 \varepsilon_n^{-7/2} (\varepsilon_m + D_{14} D_{31} 2^{(\theta_{14}+\theta_{31}) m} \varepsilon_m) \\ &\quad + D_{14} 2^{\theta_{14} m} \varepsilon_m \varepsilon_n^{-7/4} (\varepsilon_m + 4 \varepsilon_m^{5/2} \varepsilon_m^{-7/4} + 2^{2\alpha n^2} \\ &\quad + 66 D_{14}^2 D_{31}^2 2^{(2\theta_{14}+2\theta_{31}) m} \varepsilon_m^2 \varepsilon_n^{-7/4}) \\ &\leqq \varepsilon_m^{5/2} \varepsilon_n^{-7} \end{aligned} \tag{5.2.50}$$

because ε is so small that

$$\varepsilon_m^{1/2} \leqq \frac{1}{75} D_{14}^{-3} D_{31}^{-2} 2^{-(3\theta_{14}+2\theta_{31}) m + \alpha m^2} \tag{5.2.51}$$

for all $m \geqq 0$.

Finally, if $d(k, j) \geqq \frac{1}{2} \varepsilon_m^{-1} \varepsilon_{n-1}^{7/4}$, then thanks to $\mathscr{I}_8(n)$ (viii), we have

$$\| B_n(j; k) \| \leqq | D_{14}^2 D_{31} 2^{(2\theta_{14}+\theta_{31}) m + 3} \varepsilon_{n-1}^{7/4} \tag{5.2.52}$$

and hence

$$|\theta_n(j;k)| \leqq \sum_{r=1}^{\infty} 4(D_{14}2^{\theta_{14}m}d(k,j)^{-1})(D_{14}^2 D_{31} 2^{(2\theta_{14}+\theta_{31})m+3}\varepsilon_{n-1}^{7/4})^r$$
$$\leqq \varepsilon_m^{-1/4}\varepsilon_{n-1}^{7/4} d(k,j)^{-1} \qquad (5.2.53)$$

and (5.2.36) follows.

Q.E.D.

It is straightforward to extend the arguments above to prove the following result:

LEMMA 5.9. – *For all* $k \in \bar{C}_n(j) \setminus \{j\}$, *we have*

(i) $$\left\| \frac{\delta}{\delta E_0} \theta_n(j;k) \right\|_\infty \leqq D_{14} 2^{\alpha n^2 + \theta_{14}n+4} \min(\varepsilon_n^{-7/4}, d(k,j)^{-1}) \qquad (5.2.54)$$

(ii) $$\left\| \frac{\delta}{\delta E_0} \frac{d}{dk} \theta_n(j;k) \right\|_\infty \leqq D_{14}^2 2^{2\alpha n^2 + 2\theta_{14}n+8} \min(\varepsilon_n^{-7/2}, d(k,j)^{-2}) \qquad (5.2.55)$$

Moreover, for all $k \in \bar{C}_n(j) \setminus C_n(j)$, *we have*

(iii) $$\left\| \frac{\delta}{\delta E_0} \theta_n(j;k) \right\|_\infty \leqq D_{16} 2^{\theta_{16}n} \varepsilon_n \qquad (5.2.56)$$

(iv) $$\left\| \frac{\delta}{\delta E_0} \frac{d}{dk} \theta_n(j;k) \right\|_\infty \leqq D_{17} 2^{\alpha n^2 + \theta_{17}n} \sqrt{\varepsilon_{n-1}} \qquad (5.2.57)$$

5.3. Proof of the Induction Hypothesis $\mathcal{I}_3(n+1)$, $\mathcal{I}_4(n+1)$, $\mathcal{I}_5(n+1)$ and $\mathcal{I}_6(n+1)$

We have

LEMMA 5.10. – *For all* $j \in J_{n+1}^{p+}$ *and all* $k \in \check{C}_n(j)$, *we have*

(i) $$|E_n^s(k) - E_n(k)| \leqq \min(\varepsilon_n, \varepsilon_n^2 D_{18} 2^{\theta_{18}n} d(k,j)^{-1}) \qquad (5.3.1)$$

(ii) $$\left| \frac{d}{dk}(E_n^s(k) - E_n(k)) \right| \leqq 2^{\alpha n^2+2} \min\left(1, \frac{1}{2}\varepsilon_n D_{14} 2^{\theta_{14}n} d(k,j)^{-1}\right) \qquad (5.3.2)$$

(iii) $$\left| \frac{d^2}{dk^2}(E_n^s(k) - E_n(k)) \right| \leqq 6 D_{14} 2^{2\alpha n^2 + \theta_{14}n} \min(\varepsilon_n^{-7/4}, d(k,j)^{-1}) \qquad (5.3.3)$$

(iv) $$|E_n^s(t_n k) - E_n^s(k)| \geqq \frac{1}{2} D_{14}^{-1} 2^{-\theta_{14}n} \max(\varepsilon_n^{7/4}, d(k,j)) \qquad (5.3.4)$$

Proof. – If $k \in \mathring{C}_n(j)$, we have

$$E_n^s(k) = E_n(k) + (\operatorname{sgn} \mathscr{E}_n(k))\,\eta_n(k) + \varepsilon_n (\operatorname{sgn} w_n(k))(1 - \psi_n(j; k))\, v_n(w_n(k); k) \sin 2\theta_n(j, k) \qquad (5.3.5)$$

The third term on the right hand side has support in the set

$$\tilde{A}_n(j) = \bar{C}_n(j) \setminus C_n(j) \qquad (5.3.6)$$

and it can be estimated with the help of Lemma 5.7. Thanks also to Lemma 5.3, we have

$$\begin{aligned} |E_n^s(k) - E_n(k)| &\leqq \varepsilon_n \min\left(1, \frac{1}{2}\varepsilon_n D_{14} 2^{\theta_{14} n} d(k, j)^{-1}\right) \\ &\quad + 2\varepsilon_n^2 D_{15} 2^{\theta_{15} n} 1_{\tilde{A}_n(j)}(k) \\ &\leqq \min(\varepsilon_n, \varepsilon_n^2 D_{18} 2^{\theta_{14} n} d(k, j)^{-1}) \end{aligned} \qquad (5.3.7)$$

because

$$D_{18} = \frac{1}{2} D_{14} + 4 D_{12} D_{15}, \qquad \theta_{18} = \max(\theta_{14}, \theta_{12} + \theta_{15}). \qquad (5.3.8)$$

This proves (i). We also have

$$\begin{aligned} &\left|\frac{d}{dk}(E_n^s(k) - E_n(k)\right| \\ &\qquad \leqq 2^{\alpha n^2 + 1} \min\left(1, \frac{1}{2}\varepsilon_n D_{14} 2^{\theta_{14} n} d(k, j)^{-1}\right) \\ &\qquad\quad + 1_{\tilde{A}_n(j)}(k)\,[\varepsilon_n^2 D_{12} D_{15} 2^{(\theta_{12} + \theta_{15}) n + 5} + \varepsilon_n^2 D_{16} 2^{\theta_{16} n} \\ &\qquad\quad + 2\varepsilon_n D_{15} 2^{\alpha n^2 + \theta_{15} n} \min(1, \sqrt{\varepsilon_{n-1}}\, d(k, j)^{-1})] \\ &\qquad \leqq 2^{\alpha n^2 + 2} \min\left(1, \frac{1}{2}\varepsilon_n D_{14} 2^{\theta_{14} n} d(k, j)^{-1}\right) \end{aligned} \qquad (5.3.9)$$

because ε is so small that

$$\varepsilon_n \leqq \frac{1}{4} D_{14} 2^{\theta_{14} n} (D_{12} D_{15} 2^{(\theta_{12} + \theta_{15}) n + 5} + D_{16} 2^{\theta_{16} n})^{-1} \qquad (5.3.10)$$

for all $n \geqq 0$ and

$$\varepsilon_{n-1} \leqq 2^{-8} D_{12}^{-2} D_{14}^2 2^{2(\theta_{12} + \theta_{14}) n} \qquad (5.3.11)$$

for all $n \geqq 1$. This proves (ii).

We have

$$\begin{aligned} \left|\frac{d^2}{dk^2}(E_n^s(k) - E_n(k))\right| &\leqq 5 D_{14} 2^{2\alpha n^2 + \theta_{14} n} \min(\varepsilon_n^{-7/4}, d(k, j)^{-1}) \\ &\quad + \varepsilon_n D_{19} 2^{2\alpha n^2 + \theta_{19} n} 1_{\tilde{A}_n(j)}(k) \\ &\leqq 6 D_{14} 2^{2\alpha n^2 + \theta_{14} n} \min(\varepsilon_n^{-7/4}, d(k, j)^{-1}) \end{aligned} \qquad (5.3.12)$$

because

$$D_{19}=D_{14}^2, \qquad \theta_{19}=2\theta_{14} \tag{5.3.13}$$

and ε is so small that

$$\varepsilon_n \leqq 2 D_{12} D_{14}^{-1} 2^{(\theta_{12}-\theta_{14})n} \tag{5.3.14}$$

for all $n \geqq 0$.

Finally, from (5.3.5) we find

$$\begin{aligned} &|E_n^s(t_n k)-E_n^s(k)| \\ &\quad \geqq 2\left(\frac{1}{4}\mathscr{E}_n(k)^2+\varepsilon_n^2 w_n(k)^2\right)^{1/2} - 4\varepsilon_n^2 D_{15} 2^{\theta_{15} n} 1_{\tilde{A}_n(j)}(k) \\ &\quad \geqq D_{14}^{-1} 2^{-\theta_{14} n} \max(\varepsilon_n^{7/4}, d(k,j)) - 4\varepsilon_n^2 D_{15} 2^{\theta_{15} n} 1_{\tilde{A}_n(j)}(k) \\ &\qquad\qquad \geqq \frac{1}{2} D_{14}^{-1} 2^{-\theta_{14} n} \max(\varepsilon_n^{7/4}, d(k,j)) \end{aligned} \tag{5.3.15}$$

because ε is so small that

$$\varepsilon_n \leqq \frac{1}{8} D_{12}^{-1} D_{14}^{-1} D_{15}^{-1} 2^{-(\theta_{12}+\theta_{14}+\theta_{15})n} \tag{5.3.16}$$

for all $n \geqq 0$.

Q.E.D.

LEMMA 5.11. – *If $j \in J_{n+1}^{p+}$ is $(n+1)$ degenerate of order m, the following are true:*

(i) *If $d(k,j) \leqq \frac{1}{2}\varepsilon_m^{-1}\varepsilon_n^{7/4}$, we have*

$$E_n^s(k)=E_n(k) \tag{5.3.17}$$

(ii) *If $k \in \bar{C}_n(j)$, we have*

$$|E_n^s(k)-E_n(k)| \leqq \varepsilon_n \tag{5.3.18}$$

and

$$\left|\frac{d}{dk}(E_n^s(k)-E_n(k))\right| \leqq \varepsilon_m \tag{5.3.19}$$

(iii) *If $\frac{1}{2}\varepsilon_m^{-1}\varepsilon_n^{7/4} \leqq d(k,j) \leqq \varepsilon_m^{-1}\varepsilon_n^{7/4}$, we have*

$$\left|\frac{d^2}{dk^2}(E_n^s(k)-E_n(k))\right| \leqq \varepsilon_m \varepsilon_n^{-7/4}. \tag{5.3.20}$$

Proof. – (5.3.17) derives from (5.2.37). Moreover, for all $k \in \bar{C}_n(j)$ we have

$$\begin{aligned} E_n^s(k) = {} & E_n(k)+2\varepsilon_n(\operatorname{sgn} w_n(k))\, v_n(w_n(k); k)\sin 2\theta_n(j;k) \\ & + (E_n(t_n k)-E_n(k))\sin^2\theta_n(j;k). \end{aligned} \tag{5.3.21}$$

Thanks to $\mathscr{I}_7(n)$ (viii), we have

$$|E_n^s(k)-E_n(k)| \leqq 6\varepsilon_n^{7/4} + D_{31} 2^{\theta_{31} m}\varepsilon_n^{7/4} \leqq \varepsilon_n \tag{5.3.22}$$

because ε is so small that

$$\varepsilon_n^{1/2} \leqq \frac{1}{2}(6 + D_{31}\, 2^{\theta_{31}\, n})^{-1} \tag{5.3.23}$$

for all $n \geqq 0$. If $\frac{1}{2}\varepsilon_m^{-1}\,\varepsilon_n^{7/4} \leqq d(k, j) \leqq \frac{1}{2}\varepsilon_m^{-1}\,\varepsilon_{n-1}^{7/4}$, we have

$$\begin{aligned} &\left|\frac{d}{dk}(E_n^s(k) - E_n(k))\right| \\ &\leqq 4\,\varepsilon_m^{7/4} + 4\,(\varepsilon_n^{7/4} + \varepsilon_m\, d(k, j))\,.\,\varepsilon_m^{3/4}\, d(k, j)^{-1} \\ &\qquad + D_{31}\, 2^{\theta_{31}\, m+1}\,\varepsilon_m^{3/2} + 2\, D_{31}\, 2^{\theta_{31}\, m}\, d(k, j)\,.\,\varepsilon_m^{3/2}\,.\,\varepsilon_m^{3/4}\, d(k, j)^{-1} \\ &\leqq \varepsilon_m. \end{aligned} \tag{5.3.24}$$

On the other hand, if $d(k, j) \geqq \frac{1}{2}\varepsilon_m^{-1}\,\varepsilon_{n-1}^{7/4}$, we have

$$\begin{aligned} \left|\frac{d}{dk}(E_n^s(k) - E_n(k))\right| &\leqq D_{14}^3\, D_{31}\, 2^{(3\,\theta_{14} + \theta_{31})\, m+8}\,\varepsilon_m^2 \\ &\quad + 3\,\varepsilon_{n-1}^{7/4}\,\varepsilon_m^{3/4} + D_{14}\, D_{31}\, 2^{(\theta_{14} + \theta_{31})\, m+5}\,\varepsilon_n^{7/4} \\ &\quad + D_{14}^2\, D_{31}\, 2^{(\theta_{31} + 2\,\theta_{14})\, m+9}\,\varepsilon_m^2 \leqq \varepsilon_m \end{aligned} \tag{5.3.26}$$

because ε is so small that

$$\varepsilon_m \leqq D_{14}^{-3}\, D_{31}^{-1}\, 2^{-(3\,\theta_{14} + \theta_{31})\, m-10} \tag{5.3.27}$$

for all $m \geqq 0$. Similarly, one can prove (iii).

Q.E.D.

LEMMA 5.12. — *The induction hypotheses of the family $\mathscr{I}_3(n+1)$, hold.*

Proof. — (i) is obvious. (ii) follows from Lemma 5.10 (i) and Lemma 5.11 (i). Also (iii) and (iv) follow from the two preceding lemmas because

$$D_{22} = 4 + D_{20}, \qquad \theta_{22} = \theta_{20} \tag{5.3.28}$$

and

$$D_{23} = 6\, D_{14} + D_{21}, \qquad \theta_{23} = \max(\theta_{14}, \theta_{21}). \tag{5.3.29}$$

(v) follows from Lemma 5.10 (iv) and (vi) from Lemma 5.8 (ii) and (iii). Since (viii) follows from Lemma 5.11 (ii), the only hypothesis left to prove is (vii).

Let $j \in J_{n+1}^p$ be an $(n+1)$-regular jump of order m and let us set $m' = m$ if $n' = n$, $m' = n$ otherwise. If $k \in \bar{C}_n(j)$ is such that

$$d(k, j) \geqq D_{32}\, 2^{\alpha\, m'^2 + \theta_{32}\, m'}\,\varepsilon_n\,|\,w_n(j)\,|, \tag{5.3.30}$$

then we have

$$\begin{aligned}&2\,|\mathscr{E}_n(k)|^{-1}\,\varepsilon_n\,|w_n(k)|\\&\quad 2\,D_{14}\,2^{\theta_{14}m}\,d(k,j)^{-1}\,(\varepsilon_n\,|w_n(j)|+\varepsilon_m\,d(k,j))\\&\quad \leqq 3\,D_{14}\,D_{32}^{-1}\,2^{(\theta_{14}-\theta_{32})\,m'-\alpha\,m'^2}\end{aligned} \tag{5.3.31}$$

because ε is so small that

$$\varepsilon_{m'}\leqq D_{32}^{-1}\,2^{-\theta_{32}\,m'-\alpha\,m'^2} \tag{5.3.32}$$

for all $m'\geqq 0$. Hence

$$\begin{aligned}&\left|\frac{d}{dk}\eta_n(k)\right|\\&\leqq\left(\frac{1}{4}\mathscr{E}_n(k)^2+\varepsilon_n^2\,w_n(k)^2\right)^{-1/2}\left(\frac{1}{2}|\eta_n(k)\,\mathscr{E}'_n(k)|+\varepsilon_n^2\,|w_n(k)\,w'_n(k)|\right)\\&\leqq\left|\frac{2\,\varepsilon_n\,w_n(k)}{\varepsilon_n(k)}\right|\,|\mathscr{E}'_n(j)|+\varepsilon_n\,|w'_n(k)|\\&\leqq 6\,D_{14}\,D_{32}^{-1}\,2^{(\theta_{14}-\theta_{32})\,m'}+\varepsilon_{m'}\leqq D_{33}^{-1}\,2^{-\theta_{33}\,m'}\end{aligned} \tag{5.3.33}$$

because

$$D_{32}=12\,D_{14}\,D_{33},\qquad \theta_{32}=\theta_{14}+\theta_{33} \tag{5.3.34}$$

and ε is so small that

$$\varepsilon_{m'}\leqq\frac{1}{2}D_{33}^{-1}\,2^{-\theta_{33}\,m'} \tag{5.3.35}$$

for all $m'\geqq 0$.

Let us now suppose that (5.3.30) fails to hold. In this case, we have

$$\begin{aligned}&\frac{d^2}{dk^2}\eta_n(k)\\&=\left(\frac{1}{4}\mathscr{E}_n(k)^2+\varepsilon_n^2\,w_n(k)^2\right)^{-3/2}\left[\frac{1}{2}\left(\frac{1}{4}\mathscr{E}_n(k)\,\mathscr{E}'_n(k)-\varepsilon_n^2\,w_n(k)\,w'_n(k)\right)^2\right.\\&\qquad\left.+\frac{\varepsilon_n^2}{4}\mathscr{E}_n(k)^2\,w'_n(k)^2+\frac{\varepsilon_n^2}{4}\mathscr{E}'_n(k)^2\,w_n(k)^2\right]\\&\quad+\left(\frac{1}{4}\mathscr{E}_n(k)^2+\varepsilon_n^2\,w_n(k)^2\right)^{-1/2}\left(-\frac{1}{2}\eta_n(k)\,\mathscr{E}''_n(k)+\varepsilon_n^2\,w_n(k)\,w''_n(k)\right)\\&\geqq\left(\frac{1}{4}\mathscr{E}_n(k)^2+\varepsilon_n^2\,w_n(k)^2\right)^{P-3/2}\cdot\frac{1}{4}\mathscr{E}'_n(k)^2\,\varepsilon_n^2\,w_n(k)^2\\&\quad-\left(\frac{1}{4}\mathscr{E}_n(k)^2+\varepsilon_n^2\,w_n(k)^2\right)^{-1/2}\left(\frac{1}{2}|\eta_n(k)\,\mathscr{E}''_n(k)|+\varepsilon_n^2\,|w_n(k)\,w''_n(k)|\right)\end{aligned}$$

$$\geqq D_{14}^{-2} 2^{-2\theta_{14} m' - 3\alpha m'^2 + 1} (\varepsilon_n |w_n(j)| + \varepsilon_{m'} d(k, j))^2 d(k, j)^{-3}$$
$$-\left(\frac{1}{4}\mathscr{E}_n(k)^2 + \varepsilon_n^2 w_n(k)^2\right)^{-1/2} |\varepsilon_n w_n(k)| (|\mathscr{E}_n''(k)| + |\varepsilon_n w_n''(k)|) \quad (5.3.36)$$

If j has height $n'=n$, then we have

$$\frac{d^2}{dk^2}\eta_n(k) \geqq D_{14}^{-2} D_{32}^{-3} 2^{-(2\theta_{14}+3\theta_{32})m - 6\alpha m^2 - 2} \varepsilon_n^{-1} |w_n(j)|^{-1} - 2^{2\alpha n^2+1}$$
$$\geqq \left(n+\frac{1}{2}\right)^{-1} D_{34}^{-1} 2^{-\theta_{34} m - 6\alpha m^2} \varepsilon_n^{-1} (j)|^{-1} \quad (5.3.37)$$

because

$$D_{34} = 12 D_{14}^2 D_{32}^3, \qquad \theta_{34} = 2\theta_{14} + 3\theta_{32} \quad (5.3.38)$$

and ε is so small that

$$\varepsilon_n \leqq D_{14}^{-2} D_{32}^{-3} 2^{-(2\theta_{14}+3\theta_{32})n - 8\alpha n^2 - 3} \quad (5.3.39)$$

for all $n \geqq 0$. On the other hand, if $n'<n$, *i.e.* if $j \in J_n^p$ and it is n-regular, we have

$$\frac{d^2}{dk^2}\eta_n(k) \geqq -2|\mathscr{E}_n(k)|^{-1} |\varepsilon_n w_n(k)| (|\mathscr{E}_n''(k)| + |\varepsilon_n w_n''(k)|)$$
$$\geqq -D_{14} 2^{\theta_{14} n' + 2} \varepsilon_{n'}^{-1} |w_{n'}(j)|^{-1} . \varepsilon_n . 2^{2\alpha n^2 + 1}$$
$$\geqq \left[\left(n+\frac{1}{2}\right)^{-1} - n^{-1}\right] D_{34}^{-1} 2^{-\theta_{34} m - 6\alpha m^2} \varepsilon_{n'}^{-1} |w_{n'}(j)|^{-1} \quad (5.3.40)$$

because ε is so small that

$$\varepsilon_n \leqq n^{-2} D_{14}^{-1} D_{34}^{-1} 2^{-(\theta_{14}+\theta_{34})n - 8\alpha n^2 - 4} \quad (5.3.41)$$

for all $n \geqq 1$. This completes the proof of $\mathscr{I}_3(n+1)$ (vii).

Q.E.D.

LEMMA 5.13. – *The induction hypotheses of the family $\mathscr{I}_4(n+1)$ hold.*

Proof. – (i), (ii) and (iii) are obvious. To prove (iv), (v) and (vi), let us notice that if $j \in J_{n+1}^p$, $\omega \neq \omega_n(j)$ and $k \in \bar{C}_n(j)$, we have

$$v_n^s(\omega, k) = v_n(\omega; k) \cos\theta_n(j; k)$$
$$+ (\operatorname{sgn} \omega_n(k)) v_n(\omega - \omega_n(k); t_n k) \sin\theta_n(j; k). \quad (5.3.42)$$

On the other hand, if $\omega = \omega_n(j)$, the function $v_n^s(\omega_n(j); k)$ vanishes for $k \in C_n(j)$ and if $k \in \bar{C}_n(j) \diagdown C_n(j)$ we have

$$v_n^s(\omega_n(k); k) = v_n(\omega_n(k); k) \cos^2\theta_n(j; k)$$
$$-\frac{1}{2}(\operatorname{sgn} \omega_n(k)) \mathscr{E}_n(k) \sin 2\theta_n(j; k)). \quad (5.3.43)$$

Hence, thanks to $\mathcal{I}_8(n)$ (ii), we have

$$\sum_{\omega} |v_n^s(\omega; k)| \leqq 2 + D_{15} 2^{\theta_{15} n} \varepsilon_n \leqq 3 \tag{5.3.44}$$

because ε is so small that

$$\varepsilon_n \leqq D_{15}^{-1} 2^{-\theta_{15} n} \tag{5.3.45}$$

for all $n \geqq 0$. to Lemmas 5.7 and 5.8 and to $\mathcal{I}_8(n)$ (iii), we have

$$\begin{aligned}\sum_{\omega} \left| \frac{d}{dk} v_n^s(\omega; k) \right| &\leqq D_{14} 2^{\alpha n^2 + \theta_{14} n + 5} \min(\varepsilon_n^{-7/4}, d(k, j)^{-1}) \\ &\quad + 2^{\alpha n^2 + 1} \min(\varepsilon_{n-1}^{-7/4}, d(k, j)^{-1}) \\ &\quad + 1_{A_n(j)} (D_{12} 2^{\alpha n^2 + \theta_{12} n + 1} \sqrt{\varepsilon_{n-1}} + D_{15} 2^{\alpha n^2 + \theta_{15} n} \varepsilon_n + 3 D_{16} 2^{\theta_{16} n} \varepsilon_n) \\ &\leqq D_{14} 2^{\alpha n^2 + \theta_{14} n + 6} \min(\varepsilon_n^{-7/4}, d(k, j)^{-1})\end{aligned} \tag{5.3.46}$$

because

$$D_{14} \geqq 2^{-4} + 1 \tag{5.3.47}$$

and

$$\sqrt{\varepsilon_{n-1}} \leqq D_{12} 2^{\theta_{12} n} (D_{12}^{-1} 2^{-\alpha n^2 - \theta_{12} n - 1} + D_{15} 2^{\alpha n^2 + \theta_{15} n} + 3 D_{16} 2^{\theta_{16} n})^{-1} \tag{5.3.48}$$

for all $n \geqq 1$. Finally, thanks to $\mathcal{I}_8(n)$ (iv) we have

$$\begin{aligned}\sum_{\omega} \left| \frac{d^2}{dk^2} v_n^s(\omega; k) \right| &\leqq D_{14}^2 2^{2\alpha n^2 + 2\theta_{14} n + 11} \min(\varepsilon_n^{-7/2}, d(k, j)^{-2}) \\ &\quad + 1_{A_n(j)}(k) (D_{12}^2 2^{2\alpha n^2 + 2\theta_{12} n + 2} \sqrt{\varepsilon_{n-1}} + D_{12} D_5 2^{2\alpha n^2 + (\theta_{12} + \theta_{15}) n + 1} \varepsilon_n \\ &\quad + 3 D_{12} D_{16} 2^{\alpha n^2 + (\theta_{12} + \theta_{16}) n} \varepsilon_n \sqrt{\varepsilon_{n-1}} + 2 D_{16} 2^{\alpha n^2 + \theta_{16} n} \varepsilon_n \\ &\quad + 4 D_{16}^2 2^{2\theta_{16} n} \varepsilon_n^2 + 2 D_{17} 2^{\alpha n^2 + \theta_{17} n} \sqrt{\varepsilon_{n-1}}) \\ &\leqq D_{14}^2 2^{2\alpha n^2 + 2\theta_{14} n + 12} \min(\varepsilon_n^{-7/2}, d(k, j)^{-2})\end{aligned}$$

because

$$\begin{aligned}\sqrt{\varepsilon_{n-1}} \leqq D_{12}^2 2^{2\theta_{12} n} (&D_{12}^2 2^{2\alpha n^2 + 2\theta_{12} n + 2} + D_{12} D_{15} 2^{2\alpha n^2 + (\theta_{12} + \theta_{15}) n + 1} \\ &+ 3 D_{12} D_{16} 2^{\alpha n^2 + (\theta_{12} + \theta_{16}) n} + 2 D_{16} 2^{\alpha n^2 + \theta_{16} n} \\ &+ 4 D_{16}^2 2^{2\theta_{16} n} + 2 D_{17} 2^{\alpha n^2 + \theta_{17} n})^{-1}\end{aligned} \tag{5.3.49}$$

for all $n \geqq 1$.

Q.E.D.

LEMMA 5.14. – *The induction hypotheses of the family $\mathcal{I}_5(n+1)$, hold.*

Proof. – (i) and (ii) are obvious and (iii), (iv) and (v) can be proven as $\mathcal{I}_4(n+1)$ (iv), (v) and (vi), respectively. If $N=2$, the function $h_{n2}^s(\omega_n(k); k)$ defined by (2.6.24) requires a special consideration when $k \in \bar{C}_n(j) \setminus \mathring{C}_n(j)$, j being a $(n+1)$-degenerate jump point in J_{n+1}^p. In this case, we have

$$\varepsilon_n^2 h_{n2}^s(\omega_n(k); k) = (\varepsilon_n v_n(\omega_n(k); k) + \varepsilon_n^2 h_{n\,2}(\omega_n(k); k)) \cos^2 \theta_n(j; k) - \frac{1}{2}\operatorname{sgn}(\omega_n(k)) \mathscr{E}_n(k) \sin 2\theta_n(j; k). \quad (5.3.50)$$

If $d(k, j) \leqq \frac{1}{2} \varepsilon_m^{-1} \varepsilon_n^{7/4}$, we have $\theta_n(j; k) = 0$. Hence we have

$$|\varepsilon_n^2 h_{n2}^s(w_n(k); k)| \leqq \varepsilon_n^{7/4} + \varepsilon_m d(k, j) + \varepsilon_n^2 \leqq 2\varepsilon_n^{7/4} + \varepsilon_m d(k, j) \quad (5.3.51)$$

and

$$\left|\frac{d}{dk} \varepsilon_n^2 h_{n2}^s(\omega_n(k); k)\right| \leqq \varepsilon_m(2 - n^{-1}) + \varepsilon_n^2 \leqq \varepsilon_m\left(1 - \frac{1}{n + (1/2)}\right) \quad (5.3.52)$$

because ε is so small that

$$\varepsilon_n \leqq n^{-2} \quad (5.3.53)$$

for all $n \geqq 0$. Similarly, one can prove (viii).

On the other hand, if $\frac{1}{2} \varepsilon_m^{-1} \varepsilon_n^{7/4} \leqq d(k, j) \leqq \varepsilon_m^{-1} \varepsilon_n^{7/4}$, we have

$$\varepsilon_n^2 h_{n2}^s(\omega_n(k); k) \leqq \varepsilon_n^{7/4} + 2\varepsilon_n^{7/4} + \varepsilon_n^2 + D_{14} D_{31} 2^{(\theta_{14} + \theta_{31}) m + 4} \varepsilon_n^{7/4} \leqq D_{14} D_{31} 2^{(\theta_{14} + \theta_{31}) m + 5} \varepsilon_n^{7/4} \quad (5.3.54)$$

Moreover, we have

$$\left|\frac{d}{dk} \varepsilon_n^2 h_{n2}^2(\omega_n(k); k)\right| \leqq 2\varepsilon_m + \varepsilon_n^2 + 6\varepsilon_n^{7/4} \varepsilon_m^{3/4} + D_{14}^2 D_{31} 2^{(2\theta_{14} + \theta_{31}) m + 1} \varepsilon_m + 16 D_{14} D_{31} 2^{(\theta_{14} + \theta_{31}) m} \varepsilon_n^{7/4} \leqq D_{14}^2 D_{31} 2^{(2\theta_{14} + \theta_{31}) m + 2} \varepsilon_m \quad (5.3.55)$$

and

$$\left|\frac{d^2}{dk^2}\varepsilon_n^2 h_{n2}^s(\omega_n(k); k)\right|$$

$$\leqq (2\varepsilon_m+\varepsilon_n^2)(1+3D_{14}^2 D_{31} 2^{(2\theta_{14}+\theta_{31})m}\varepsilon_n^{-7/4})$$

$$+2\varepsilon_n^{7/4}.\varepsilon_m^{5/2}\varepsilon_n^{-7/2}+2^{2\alpha m^2}\varepsilon_m^{3/4}$$

$$+2D_{14}^2 D_{31} 2^{(2\theta_{14}+\theta_{31})m}\varepsilon_n^{-7/4}+D_{31}2^{\theta_{31}m}\varepsilon_m^{-1}\varepsilon_n^{7/4}.\varepsilon_m^{5/2}\varepsilon_n^{-7/2}$$

$$\leqq D_{14}^2 D_{31} 2^{(2\theta_{14}+\theta_{31})m+2}\varepsilon_n^{-7/4} \tag{5.3.56}$$

Q.E.D.

Finally, let us mention without proof that estimates very similar to the ones above, permit us to prove the following:

LEMMA 5.15. – *The induction hypotheses of the family $\mathscr{I}_6(n+1)$, hold.*

6. THE REGULAR PART OF $\mathbb{U}_n$

In this section we consider the regular part $\exp(\varepsilon_n \mathbf{R}_n)$ of the unitary transformation $\mathbf{U}_n$ and we prove the induction hypothesis $\mathscr{I}_7(n+1)$, $\mathscr{I}_8(n+1)$, $\mathscr{I}_9(n+1)$ and $\mathscr{I}_{10}(n+1)$ in Section 2.7.

As explained in Section 2.7, the operator $\mathbf{R}_n$ is skewsymmetric and has the form

$$\mathbf{R}_n=\sum_{\omega\in\mathscr{U}\setminus\{0\}}\int_B dk\, r_n(\omega; k)\,|\,t_\omega k\rangle\langle k\,| \tag{6.1}$$

where

$$r_n(\omega; k)=-\mathscr{E}_n^s(\omega; k)\, v_n^s(\omega; k). \tag{6.2}$$

We have

LEMMA 6.2. – *The following are true*

(i) *For all $k\in B$, we have*

$$\sum_\omega |r_n(\omega; k)|\leqq D_{24}2^{\theta_{24}n} \tag{6.3}$$

For all $k\in B$ such that $d_n(k, \bar{\mathscr{S}}>1$, we have

(ii)
$$\sum_\omega\left|\frac{d}{dk}r_n(\omega; k)\right|\leqq D_{25}2^{\theta_{25}n} \tag{6.4}$$

(iii)
$$\sum_\omega\left|\frac{d^2}{dk^2}r_n(\omega; k)\right|\leqq D_{28}2^{2\alpha n^2+\theta_{28}n} \tag{6.5}$$

For all $k \in B \setminus J_{n+1}$, *we have*

$$\text{(iv)} \quad \sum_\omega \left| \frac{d}{dk} r_n(\omega; k) \right| \leqq D_{26} 2^{\alpha n^2 + \theta_{26} n} \min(\varepsilon_n^{-7/4}, d(k, j)^{-1}) \tag{6.6}$$

$$\text{(v)} \quad \sum_\omega \left| \frac{d^2}{dk^2} r_n(\omega; k) \right| \leqq D_{27} 2^{2 \alpha n^2 + \theta_{27} n} \min(\varepsilon_n^{-7/2}, d(k, j)^{-2}) \tag{6.7}$$

Proof. – From $\mathscr{I}_2(n+1)$ (iii) and $\mathscr{I}_4(n+1)$ (iv), we find

$$\sum_\omega |r_n(\omega; k)| \leqq 3 . D_{13} 2^{\theta_{13} n} = D_{24} 2^{\theta_{24} n} \tag{6.8}$$

because

$$D_{24} = 3 D_{13}, \qquad \theta_{24} = \theta_{13}. \tag{6.8'}$$

If $k \in B$ is such that $d_n(k, \mathring{\mathscr{S}}) > 1$, then thanks to $\mathscr{I}_8(n)$ (iii), (iv), (v) and (vi) and from $\mathscr{I}_4(n+1)$ (ii), (iii), we have

$$\sum_\omega \left| \frac{d}{dk} v_n(\omega; k) \right| \leqq \max(1, D_{12} 2^{\alpha n^2 + \theta_{12} n + 1} \sqrt{\varepsilon_{n-1}}) \leqq 1 \tag{6.9}$$

and

$$\sum_\omega \left| \frac{d^2}{dk^2} v_n(\omega; k) \right| \leqq \max(1, D_{12}^2 2^{2 \alpha n^2 + 2 \theta_{12} n + 2} \sqrt{\varepsilon_{n-1}}) \leqq 1 \tag{6.10}$$

because ε is so small that

$$\varepsilon_{n-1} \leqq D_{12}^{-1} 2^{-\alpha n^2 - \theta_{12} n - 1} \tag{6.11}$$

for all $n \geqq 1$. Moreover, thanks to $\mathscr{I}_7(n)$ (viii), (ix) and to $\mathscr{I}_3(n+1)$ (iii) and (iv), we have

$$\begin{aligned} \sup_{\omega \in \mathscr{U}_{n+1}} \left| \frac{d^2}{dk^2} \mathscr{E}_n^s(\omega, k) \right| &\leqq 2 D_{31} 2^{\theta_{31} n} + 4 D_{12} D_{22} 2^{\alpha n^2 + (\theta_{12} + \theta_{23}) n} \varepsilon_n \\ &\leqq 3 D_{31} 2^{\theta_{31} n} \end{aligned} \tag{6.12}$$

and

$$\begin{aligned} \sup_{\omega \in \mathscr{U}_{n+1}} \left| \frac{d^2}{dk^2} \mathscr{E}_n^s(\omega, k) \right| &\leqq 2^{2 \alpha n^2 + 1} + 4 D_{12} D_{23} 2^{2 \alpha n^2 + (\theta_{12} + \theta_{23}) n} \\ &\leqq 5 D_{12} D_{23} 2^{2 \alpha n^2 + (\theta_{12} + \theta_{23}) n} \end{aligned} \tag{6.13}$$

because ε is so small that

$$\varepsilon_n \leqq \frac{1}{4} D_{12}^{-1} D_{22}^{-1} D_{31} 2^{-\alpha n^2 + (\theta_{31} - \theta_{12} - \theta_{22}) n} \tag{6.14}$$

for all $n \geqq 0$. Hence we have

$$\sum_{\omega} \left| \frac{d}{dk} r_n(\omega; k) \right| \leqq \sum_{\omega} \left(\mathscr{E}_n^s(\omega; k)^{-2} \left| v_n^s(\omega; k) \frac{d}{dk} \mathscr{E}_n^s(\omega; k) \right| + \left| \mathscr{E}_n^s(\omega; k)^{-1} \frac{d}{dk} v_n^s(\omega; k) \right| \right)$$

$$\leqq 9 D_{13}^2 D_{31} 2^{(2\theta_{13}+\theta_{31})n} + 3 D_{13} 2^{\theta_{13} n}$$

$$\leqq D_{25} 2^{\theta_{25} n} \tag{6.15}$$

because

$$D_{25} = 9 D_{13}^2 D_{31} + 3 D_{13}, \qquad \theta_{25} = 2\theta_{13} + \theta_{22}. \tag{6.16}$$

This proves (ii). We also have

$$\sum_{\omega} \left| \frac{d^2}{dk^2} r_n(\omega; k) \right| \leqq 15 D_{13}^2 D_{12} D_{23} 2^{2\alpha n^2 + (\theta_{12} + 2\theta_{13} + \theta_{23}) n} + 27 D_{13}^3 D_{31}^2 2^{(3\theta_{13} + 2\theta_{31}) n} + 6 D_{13}^2 D_{31} 2^{(\theta_{31} + 2\theta_{13}) n} + D_{13} 2^{\theta_{31} n}$$

$$\leqq D_{28} 2^{2\alpha n^2 + \theta_{28} n} \tag{6.17}$$

because

$$D_{28} = 15 D_{13}^2 D_{13} D_{23} + 27 D_{13}^3 D_{31}^2 + 6 D_{13}^2 D_{31} + D_{13} \tag{6.18}$$

$$\theta_{28} = \max(\theta_{12} + 2\theta_{13} + \theta_{23},\ 3\theta_{13} + 2\theta_{31}). \tag{6.19}$$

This proves (iii).

If $k \in B \setminus J_{n+1}$, we have

$$\sum_{\omega} \left| \frac{d}{dk} r_{n+1}(\omega; k) \right| \leqq 7 D_{13}^2 D_{22} 2^{\alpha n^2 + \theta_{22} n} + D_{13} D_{14} 2^{\alpha n^2 + \theta_{14} n + 6} \min(\varepsilon_n^{-7/4}, d(k, j)^{-1})$$

$$\leqq D_{26} 2^{\alpha n^2 + \theta_{26} n} \min(\varepsilon_n^{-7/4}, d(k, j)^{-1}) \tag{6.20}$$

because

$$D_{26} = 7 D_{13}^2 D_{22} + 2^6 D_{13} D_{14}, \qquad \theta_{26} = \max(\theta_{22}, \theta_{14}). \tag{6.21}$$

Here, we make use of $\mathscr{I}_7(n)$ (iii), $\mathscr{I}_3(n+1)$ (iii) and $\mathscr{I}_4(n+1)$ (v). Finally on the basis of $\mathscr{I}_7(n)$ (v), $\mathscr{I}_3(n+1)$ (iv) and $\mathscr{I}_4(n+1)$ (vi), we find

$$\sum_{\omega} \left| \frac{d^2}{dk^2} r_n(\omega; k) \right| \leqq 19 D_{13}^3 D_{22}^2 2^{2\alpha n^2 + 2\theta_{22} n}$$

$$+ 7 D_{13}^2 D_{23} 2^{2\alpha n^2 + \theta_{23} n} \min(\varepsilon_n^{-7/4}, d(k, j)^{-1})$$

$$+ 4 D_{13} D_{14} D_{22} 2^{2\alpha n^2 + (\theta_{13} + \theta_{14} + \theta_{22}) n + 6} \min(\varepsilon_n^{-7/4}, d(k, j)^{-1})$$

$$+ D_{13} D_{14}^2 2^{2\alpha n^2 + 2\theta_{14} n + 12} \min(\varepsilon_n^{-7/2}, d(k, j)^{-2})$$

$$\leqq D_{27} 2^{2\alpha n^2 + \theta_{27} n} \min(\varepsilon_n^{-7/2}, d(k, j)^{-2}) \tag{6.22}$$

because

$$D_{27} = 19\, D_{13}^3 D_{22}^2 + 7\, D_{13}^2 D_{23} + 2^8 D_{13} D_{14} D_{22} + 2^{12} D_{13} D_{14}^2 \quad (6.23)$$

$$\theta_{27} = \max(2\,\theta_{22},\ \theta_{23},\ \theta_{13} + \theta_{14} + \theta_{22},\ 2\,\theta_{14}). \quad (6.24)$$

Q.E.D.

LEMMA 6.2. – *For all* $k \in B$, *we have*

(i) $$|E_{n+1}(k) - E_n(k)| \leqq \min(2\,\varepsilon_n,\ \varepsilon_n^2 D_{30} 2^{\theta_{30} n} d(k, J_{n+1}^p)^{-1}) \quad (6.25)$$

(ii) $$\sum_{\omega} |v_{n+1}(\omega; k)| \leqq 1 \quad (6.26)$$

(iii) $$\sum_{\omega} |h_{n+1, N}(\omega; k)| \leqq 1 \quad (6.27)$$

for all $N \geqq 2$.

Proof. – (i) Due to $\mathscr{I}_3(n+1)$ (ii), Lemma 6.1 (i), $\mathscr{I}_4(n+1)$ (iv) and $\mathscr{I}_5(n+1)$ (iii), we have

$$\begin{aligned} |E_{n+1}(k) - E_n(k)| &\leqq \min(\varepsilon_n,\ \varepsilon_n^2 D_{18} 2^{\theta_{18} n} d(k, J_{n+1}^p)^{-1}) \\ &\quad + 2\, D_{24}^2 2^{2\,\theta_{24} n} \varepsilon_n^2 + \frac{1}{3!} 2^3 D_{24}^3 2^{3\,\theta_{24} n} \varepsilon_n^3 \\ &\quad + 3\, D_{24} 2^{\theta_{24} n} \varepsilon_n^2 + 6\, D_{24}^2 2^{2\,\theta_{24} n} \varepsilon_n^3 \\ &\quad + 5\,\varepsilon_n^2 + 10\, D_{24} 2^{\theta_{24} n} \varepsilon_n^3 + 5\,\varepsilon_n^3 \\ &\leqq \min(2\,\varepsilon_n,\ \varepsilon_n^2 D_{30} 2^{\theta_{30} n} d(k, J_{n+1}^p)^{-1}) \end{aligned} \quad (6.28)$$

because

$$D_{30} = D_{18} + 2\, D_{24}^2 + 3\, D_{24} + 6, \qquad \theta_{30} = \max(\theta_{18},\ 2\,\theta_{24}) \quad (6.29)$$

and ε is so small that

$$\varepsilon_n \leqq D_{30} 2^{\theta_{30} n} \left(\frac{1}{3!} 2^3 D_{24}^3 2^{3\,\theta_{24} n} + 6\, D_{24}^2 2^{2\,\theta_{24} n} + 10\, D_{24} 2^{\theta_{24} n} + 5 \right)^{-1} \quad (6.30)$$

for all $n \geqq 0$.

(ii) From (2.7.8), we find

$$\begin{aligned} \sum_{\omega} |v_{n+1}(\omega; k)| \leqq \sqrt{\varepsilon_n} \Big(& 2\, D_{24}^2 2^{2\,\theta_{24} n} + \frac{1}{3!} 2^3 D_{24}^3 2^{3\,\theta_{24} n} \varepsilon_n \\ & + 3\, D_{24} 2^{\theta_{24} n} + 6\, D_{24}^2 2^{2\,\theta_{24} n} \varepsilon_n + 5 \\ & + 10\, D_{24} 2^{\theta_{24} n} \varepsilon_n + 5\,\varepsilon_n \Big) \leqq 1 \end{aligned} \quad (6.31)$$

because ε_n satisfies (6.25) and we have

$$\varepsilon_n \leqq (2\, D_{24}^2 2^{2\,\theta_{24} n} + 3\, D_{24} 2^{\theta_{24} n} + 5 + D_{30} 2^{\theta_{30} n})^{-2} \quad (6.32)$$

for all $n \geqq 0$.

(iii) From (2.7.10) we find

$$\begin{aligned}
&\sum_{\omega} |h_{n+1,\mathrm{N}}(\omega; k)| \\
&\leqq \sqrt{\varepsilon_n^{\mathrm{N}}} \Bigg[\frac{1}{(2\mathrm{N})!}(2\mathrm{D}_{24}2^{\theta_{24}n})^{2\mathrm{N}} + \frac{1}{(2\mathrm{N}+1)!}(2\mathrm{D}_{24}2^{\theta_{24}})^{2\mathrm{N}+1}\varepsilon_n \\
&\quad + \frac{3}{(2\mathrm{N}-1)!}(2\mathrm{D}_{24}2^{\theta_{24}n})^{2\mathrm{N}-1} + \frac{3}{(2\mathrm{N})!}(2\mathrm{D}_{24}2^{\theta_{24}n})^{2\mathrm{N}}\varepsilon_n \\
&\quad + 5\sum_{\mathrm{N}'=2}^{2\mathrm{N}} \frac{1}{(2\mathrm{N}-\mathrm{N}')!}(2\mathrm{D}_{24}2^{\theta_{24}n})^{2\mathrm{N}-\mathrm{N}'} \\
&\quad + \varepsilon_n \sum_{\mathrm{N}'=2}^{2\mathrm{N}+1} \frac{1}{(2\mathrm{N}+1-\mathrm{N}')!}(2\mathrm{D}_{24}2^{\theta_{24}n})^{2\mathrm{N}+1-\mathrm{N}'}\Bigg] \\
&\leqq \sqrt[4]{\varepsilon_n^{\mathrm{N}}}(1+\varepsilon_n)\exp(2\sqrt[4]{\varepsilon_n}\,\mathrm{D}_{24}2^{\theta_{24}n}) \leqq 1
\end{aligned} \tag{6.33}$$

because ε is so small that

$$\varepsilon_n \leqq \min\left(\frac{1}{16}e^{-4}, \frac{1}{16}\mathrm{D}_{24}^{-4}2^{-4\theta_{24}n}\right) \tag{6.34}$$

for all $n \geqq 0$.

Q.E.D.

LEMMA 6.3. — *For all* $k \in \mathrm{B} \setminus \mathrm{J}_{n+1}$, *we have*

(i) $$\left|\frac{d}{dk}(\mathrm{E}_{n+1}(k) - \mathrm{E}_n(k))\right| \leqq 2^{\alpha(n+1)^2-1}\min(1, \varepsilon_n d(k, \mathrm{J}_{n+1})^{-1}) \tag{6.35}$$

(ii) $$\sum_{\omega}\left|\frac{d}{dk}v_{n+1}(\omega; k)\right| \leqq 2^{\alpha(n+1)^2}\sqrt{\varepsilon_n}\min(\varepsilon_n^{-7/4}, d(k, \mathrm{J}_{n+1})^{-1}) \tag{6.36}$$

(iii) *For all* $\mathrm{N} \geqq 2$ *and all* $k \in \mathrm{B} \setminus \mathrm{J}_{n+1}^{\mathrm{N}}$, *we have*

$$\sum_{\omega}\left|\frac{d}{dk}h_{n+1,\mathrm{N}}(\omega; k)\right| \leqq \sqrt[4]{\varepsilon_n^{\mathrm{N}}}\min(\varepsilon_n^{-7/4}, d(k, \mathrm{J}_{n+1}^{\mathrm{N}})^{-1}) \tag{6.37}$$

Proof. – (i) Thanks to $\mathscr{I}_7(n)$ (iii), $\mathscr{I}_3(n+1)$ (iii), $\mathscr{I}_4(n+1)$ (v) and Lemma 6.1, from the expression (2.7.6) for $E_{n+1}(k)$ we find

$$\begin{aligned}
&\left|\frac{d}{dk}(E_{n+1}(k)-E_n^s(k))\right| \\
&\quad\leqq 4\varepsilon_n^2(4D_{24}^2 2^{2\theta_{24}n}+30D_{24}2^{\theta_{24}n}+20) \\
&\qquad\times[2D_{22}2^{\alpha n^2+\theta_{22}n}\min(1,\varepsilon_n d(k,J_{n+1})^{-1}) \\
&\qquad+3D_{14}2^{\alpha n^2+\theta_{14}n+6}\min(\varepsilon_n^{-7/4},d(k,J_{n+1})^{-1})] \\
&\qquad+20\varepsilon_n^2\left(\frac{3}{2}D_{24}2^{\theta_{24}n}+3\right)D_{26}2^{\alpha n^2+\theta_{26}n}\min(\varepsilon_n^{-7/4},d(k,J_{n+1})^{-1}) \\
&\quad\leqq\frac{1}{4}2^{\alpha(n+1)^2}\varepsilon_n^2\min(\varepsilon_n^{-7/4},d(k,J_{n+1})^{-1})
\end{aligned}\tag{6.38}$$

because α is so large that

$$\begin{aligned}
2^{\alpha-2}\geqq 4(4D_{24}^2+30D_{24}+20)(2D_{22}+3.2^6D_{14}) \\
+20D_{26}\left(\frac{3}{2}D_{24}2^{\theta_{24}n}+3\right)
\end{aligned}\tag{6.39}$$

$$\alpha\geqq\frac{1}{2}\max(2\theta_{24}+\theta_{22},2\theta_{24}+\theta_{14},\theta_{24}+\theta_{26})\tag{6.40}$$

and ε is so small that

$$\varepsilon_n\leqq\frac{1}{2}D_{24}^{-1}2^{-\theta_{24}n}\tag{6.41}$$

for all $n\geqq 0$. Since we also have

$$2^\alpha\geqq D_{22},\qquad \alpha\geqq\frac{1}{2}\theta_{22}\tag{6.42}$$

(6.35) follows from (6.38) and $\mathscr{I}_3(n+1)$ (iii). (ii) can be proven in the same way.

(iii) If we start from (2.7.10), we find

$$\begin{aligned}
&\sum_\omega\left|\frac{d}{dk}h_{n+1,N}(\omega;k)\right| \\
&\quad\leqq 5\varepsilon_n^{3N/8}(1+\varepsilon_n)\exp(2D_{24}\varepsilon_n^{1/8}2^{\theta_{24}n}) \\
&\qquad\times[D_{26}2^{\alpha n^2+\theta_{26}n}\min(\varepsilon_n^{-7/4},d(k,J_{n+1}^N)^{-1}) \\
&\qquad+2^{\alpha n^2}+3D_{14}2^{\alpha n^2+\theta_{14}n+6}\min(\varepsilon_n^{-7/4},d(k,J_{n+1}^N)^{-1})] \\
&\quad\leqq\sqrt[4]{\varepsilon_n^N}\min(\varepsilon_n^{-7/4},d(k,J_{n+1}^N)^{-1})
\end{aligned}\tag{6.43}$$

because ε is so small that

$$\varepsilon_n^{1/8} \leqq \min\left(\frac{1}{2} D_{24}^{-1} 2^{-\theta_{24} n}, \frac{1}{20e} 2^{-\alpha n^2} (D_{26} 2^{\theta_{26} n} + 1 + 3 \cdot 2^6 D_{14} 2^{\theta_{14} n})^{-1}\right) \quad (6.44)$$

for all $n \geqq 0$.

Q.E.D.

LEMMA 6.4. – *For all* $k \in B \setminus J_{n+1}$, *we have*

(i)
$$\left|\frac{d^2}{dk^2}(E_{n+1}(k) - E_n(k))\right| \leqq 2^{2\alpha(n+1)^2 - 1} \min(\varepsilon_n^{-7/4}, d(k, J_{n+1})^{-1}) \quad (6.45)$$

(ii)
$$\sum_\omega \left|\frac{d^2}{dk^2} v_{n+1}(\omega; k)\right| \leqq 2^{2\alpha(n+1)^2} \sqrt{\varepsilon_n} \min(\varepsilon_n^{-7/2}, d(k, J_{n+1})^{-2}) \quad (6.46)$$

(iii) *For all* $N \geqq 2$ *and all* $k \in B \setminus J_{n+1}^N$, *we have*

$$\sum_\omega \left|\frac{d^2}{dk^2} h_{n+1, N}(\omega; k)\right| \leqq \sqrt[4]{\varepsilon_n^N} \min(\varepsilon_n^{-7/2}, d(k, J_{n+1}^N)^{-2}) \quad (6.47)$$

Proof. – Thanks to $\mathcal{I}_7(n)$ (v), $\mathcal{I}_3(n+1)$ (iv), $\mathcal{I}_4(n+1)$ (vi) and Lemma 6.1, we have

$$\begin{aligned}
&\left|\frac{d^2}{dk^2}(E_{n+1}(k) - E_n^s(k))\right| \\
&\leqq \varepsilon_n^2 (4 D_{24}^2 2^{2\theta_{24} n} + 30 D_{24} 2^{\theta_{24} n} + 20) \\
&\quad \times [2 D_{23} 2^{2\alpha n^2 + \theta_{23} n} \min(\varepsilon_n^{-7/4}, d(k, j)^{-1}) \\
&\quad + 3 D_{14}^2 2^{2\alpha n^2 + 2\theta_{14} n + 12} \min(\varepsilon_n^{-7/2}, d(k, j)^{-2})] \\
&\quad + 4\varepsilon_n^2 \left(\frac{3}{2} D_{24} 2^{\theta_{24} n} + 3\right) D_{26} 2^{\alpha n^2 + \theta_{26} n} \min(\varepsilon_n^{-7/4}, d(kj)^{-1}) \\
&\quad \times [2 D_{22} 2^{\alpha n^2 + \theta_{22} n} + 3 D_{14} 2^{\alpha n^2 + \theta_{14} n + 6} \min(\varepsilon_n^{-7/4}, d(k, j)^{-1})] \\
&\quad + 20\varepsilon_n^2 \left[\left(\frac{3}{2} D_{24} 2^{\theta_{24} n} + 3\right) D_{27} 2^{\alpha n^2 + \theta_{27} n} + D_{26}^2 2^{2\alpha n^2 + 2\theta_{26} n}\right] \\
&\quad \times \min(\varepsilon_n^{-7/2}, d(k, J_{n+1})^{-2}) \\
&\leqq \frac{1}{4} 2^{2\alpha(n+1)^2} \varepsilon_n^2 \min(\varepsilon_n^{-7/2}, d(k, J_{n+1})^{-2})
\end{aligned} \quad (6.48)$$

because α is so large that

$$\begin{aligned}2^{2\alpha-2} \geqq & (4D_{24}^2+30D_{24}+20)(2D_{23}+3D_{14}^2)\\ &+4\left(\frac{3}{2}D_{24}+3\right)D_{26}(2D_{22}+3.2^6D_{14})\\ &+20\left[\left(\frac{3}{2}D_{24}+3\right)D_{27}+D_{26}^2\right]\end{aligned} \tag{6.49}$$

and

$$\alpha \geqq \frac{1}{4}(2\theta_{24}+\theta_{23},\ 2\theta_{24}+2\theta_{14},\ \theta_{24}+\theta_{26} \\ +\theta_{22},\ \theta_{24}+\theta_{26}+\theta_{14},\ \theta_{14},\ \theta_{24}+\theta_{26}). \tag{6.50}$$

By using (6.43) and $\mathscr{I}_4(n+1)$ (iv), we find (6.45). Also (6.46) is a straightforward consequence of (6.48). Moreover, we have

$$\begin{aligned}\sum_{\omega}\left|\frac{d^2}{dk^2}h_{n+1,\mathrm{N}}(\omega;k)\right| & \\ \leqq 5\,\varepsilon^{(3\mathrm{N}/8)}&(1+\varepsilon_n)\exp(2D_{24}\varepsilon_n^{1/8}2^{\theta_{24}n})\\ &\times[(2D_{23}2^{2\alpha n^2+\theta_{23}n}+2D_{26}2^{2\alpha n^2+\theta_{26}n})\min(\varepsilon_n^{-7/4},\ d(k,\mathrm{J}_{n+1}^{\mathrm{N}})^{-1})\\ &+(3.D_{14}^2 2^{2\alpha n^2+2\theta_{14}n+12}+6D_{14}D_{26}2^{2\alpha n^2+(\theta_{14}+\theta_{26})n+6}\\ &+D_{26}^2 2^{2\alpha n^2+2\theta_{26}n}+D_{27}2^{2\alpha n^2+\theta_{27}n})\min(\varepsilon_n^{-7/2},\ d(k,\mathrm{J}_{n+1}^{\mathrm{N}})^{-2})]\\ \leqq \sqrt[4]{\varepsilon_n^{\mathrm{N}}}\min(\varepsilon_n^{-7/2},\ d&(k,\mathrm{J}_{n+1}^{\mathrm{N}})^{-2})\end{aligned} \tag{6.51}$$

because ε is so small that

$$\begin{aligned}\varepsilon_n^{1/8} \leqq \min\Bigg(\frac{1}{2}D_{24}^{-1}2^{-\theta_{24}n},\ & \frac{1}{10e}[2D_{23}2^{2\alpha n^2+\theta_{23}n}+2D_{26}2^{2\alpha n^2+\theta_{26}n}\\ &+3D_{14}^2 2^{2\alpha n^2+2\theta_{14}n+12}+6D_{14}D_{26}2^{2\alpha n^2+(\theta_{14}+\theta_{26})n+6}\\ &+D_{26}^2 2^{2\alpha n^2+2\theta_{26}n}+D_{27}2^{2\alpha n^2+\theta_{24}n}]^{-1}\Bigg)\end{aligned} \tag{6.52}$$

for all $n \geqq 0$.

Q.E.D.

LEMMA 6.5. – *If* $j \in \mathrm{J}_{n+1}^p$ *is an* $(n+1)$*-degenerate jump point of order* m *and* $k \in \mathrm{B}$ *such that*

$$d_{n+1}\left(k,\left[j-\frac{1}{2}\varepsilon_m^{-1}\varepsilon_n^{7/4},\ j+\frac{1}{2}\varepsilon_m^{-1}\varepsilon_n^{7/4}\right]\right) \leqq 13 \tag{6.53}$$

then we have

(i) $$\left|\frac{d}{dk}\mathrm{E}_n^s(k)\right| \leqq D_{31}2^{\theta_{31}n} \tag{6.54}$$

(ii)
$$\left|\frac{d^2}{dk^2} E_n^s(k)\right| \leqq D_{28} 2^{2\alpha n^2 + \theta_{28} + 1} \tag{6.55}$$

(iii)
$$\sum_{\omega} \left|\frac{d}{dk} v_n^s(\omega; k)\right| \leqq \varepsilon_m \tag{6.56}$$

(iv)
$$\sum_{\omega} \left|\frac{d^2}{dk^2} v_n^s(\omega; k)\right| \leqq \varepsilon_m \tag{6.57}$$

If $\sum'$ *denotes the sum over* $\omega \in \mathcal{U}$ *such that* $\omega \neq \omega_n(k)$ *in case* $N=2$ *and* $k \in \mathcal{S}_n$, *we have*

(v)
$$\sum{}' \left|\frac{d}{dk} h_{nN}^s(\omega; k)\right| \leqq 1 \tag{6.58}$$

(vi)
$$\sum{}' \left|\frac{d^2}{dk^2} h_{nN}^s(\omega; k)\right| \leqq 1 \tag{6.59}$$

Proof. – If (6.53) holds, then thanks to $\mathcal{I}_1(n+1)$ (viii) and $\mathcal{I}_2(n+1)$ (v) we have either

$$d(k, J_{n+1}) \leqq \frac{1}{2} \varepsilon_n^2 \tag{6.60}$$

or

$$d(k, J_n^N) \geqq \frac{1}{2} D_8^{-1} 2^{-\theta_8(n+1)-n} 2^{-(\theta_1(n+1)+5)\theta_0} - \frac{1}{2} \varepsilon_n^2$$
$$\geqq D_{28}^{-1} 2^{-\theta_{28} n} \tag{6.61}$$

because

$$D_{28} \geqq 4 D_8 2^{\theta_1 \theta_0 + 5\theta_0 + \theta_8}, \qquad \theta_{28} \geqq \theta_0 \theta_1 + \theta_8, \tag{6.62}$$

and ε is so small that

$$\varepsilon_n \leqq \sqrt{2} D_{28}^{-1/2} 2^{-(1/2)\theta_{28} n} \tag{6.63}$$

for all $n \geqq 0$. In both cases we have

$$E_n^s(k) = E_n(k), \tag{6.64}$$
$$v_n^s(\omega; k) = v_n(\omega; k) \tag{6.65}$$
$$h_{nN}^2(\omega; k) = h_{nN}(\omega; k) \tag{6.66}$$

unless $k \in \mathcal{S}_n$, $N=2$ and $\omega = \omega_n(k)$. This follows from $\mathcal{I}_2(n+1)$ (ii) because

$$\theta_{11} \geqq 5\theta_1. \tag{6.67}$$

In case (6.60) holds, this lemma follows from $\mathcal{I}_4(n)$, $\mathcal{I}_8(n)$ and $\mathcal{I}_9(n)$. Otherwise, thanks to $\mathcal{I}_7(n)$ (iv), $n' \leqq n$, we have

$$\left|\frac{d}{dk} E_n^s(k)\right| \leqq \frac{1}{2} + \sum_{n'=1}^{n} 2^{\alpha n'^2 - 1} \varepsilon_{n'} d(k, J_{n'})^{-1}$$
$$\leqq d(k, J_n)^{-1} \leqq D_{28} 2^{\theta_{23} n}$$
$$\leqq D_{31} 2^{\theta_{31} n} \tag{6.68}$$

because

$$D_{31} \geqq D_{28}, \qquad \theta_{31} \geqq \theta_{28} \tag{6.69}$$

and ε is so small that

$$\sum_{m=1}^{\infty} 2^{\alpha m^2} \varepsilon_m \leqq 1. \tag{6.70}$$

We also have

$$\begin{aligned}\left|\frac{d^2}{dk^2} E_0(k)\right| &\leqq \frac{1}{2} + \sum_{m=1}^{n} 2^{2\alpha m^2} d(k, J_n)^{-1} \\ &\leqq 2^{2\alpha n^2+1} d(k, J_n)^{-1} \leqq D_{28}\, 2^{2\alpha n^2+\theta_{28} n+1}\end{aligned} \tag{6.71}$$

because α is so large that

$$\sum_{m=1}^{\infty} 2^{-\alpha m^2} \leqq 1. \tag{6.72}$$

Thanks to $\mathscr{I}_8(n)$ (iii), (iv), (v) and (vi), we have

$$\sum_{\omega} \left|\frac{d}{dk} v_n^s(\omega; k)\right| \leqq \max(1, 2^{\alpha(n+1)^2} \sqrt{\varepsilon_n}\, D_{28}\, 2^{\theta_{28} n}) \tag{6.73}$$

and

$$\sum_{\omega} \left|\frac{d^2}{dk^2} v_n^s(\omega; k)\right| \leqq \max(1, 2^{2\alpha(n+1)^2} \sqrt{\varepsilon_n}\, D_{28}\, 2^{2\theta_{28} n}) \tag{6.74}$$

because ε is so small that

$$\sqrt{\varepsilon_n} \leqq 2^{-2\alpha(n+1)^2} D_{28}^{-2}\, 2^{-2\theta_{28} n}. \tag{6.75}$$

The same argument proves (v) and (vi).

Q.E.D.

LEMMA 6.6. – *The induction hypotheses of the family $\mathscr{I}_7(n+1)$, hold.*

Proof. – (i), (ii), (iii), (iv) and (v) are contained in Lemmas 6.2, 6.3 and 6.4. (vi) follows from $\mathscr{I}_7(n)$ (vi) and (vii) and from $\mathscr{I}_4(m)$ (iv), $m \leqq n$. In fact, from (6.23) and (6.24) we see that

$$\begin{aligned}&|E_{n+1}(t_n k) - E_{n+1}(k)| \\ &\qquad \geqq \frac{1}{2} D_{14}^{-1}\, 2^{-\theta_{14} n} \max(\varepsilon_n^{7/4}, d(k, j)) - \varepsilon_n^2 D_{30}\, 2^{\theta_{30} n} \\ &\qquad\qquad \geqq D_{14}^{-1}\, 2^{-\theta_{14}(n+1)} \max(\varepsilon_n^{7/4}, d(k, j))\end{aligned} \tag{6.76}$$

because ε is so small that

$$\varepsilon_n^{1/4} \leqq \frac{1}{4} D_{14}^{-1} D_{30}^{-1}\, 2^{-(\theta_{14}+\theta_{30}) n} \tag{6.77}$$

for all $n \geqq 0$ and

$$\theta_{14} \geqq 2. \tag{6.78}$$

If $j \in J_{n+1}^{p}$ and $k \in B$ is such that

$$d_{n+1}\left(k, \left[j - \frac{1}{2}\varepsilon_n^{7/4}, j + \frac{1}{2}\varepsilon_n^{7/4}\right]\right) \leqq 1, \tag{6.79}$$

we have

$$\begin{aligned}\left|\frac{d}{dk}(E_{n+1}(k) - E_n(k))\right| &\leqq 4\varepsilon_n^2 (4 D_{24}^2 2^{2\theta_{24} n} + 30 D_{24} 2^{\theta_{24} n} + 20)(D_{31} 2^{\theta_{31} n} + 3) \\ &\quad + 20\varepsilon_n^2 \left(\frac{3}{2} D_{24} 2^{\theta_{24} n} + 3\right) D_{25} 2^{\theta_{25} n} \leqq \varepsilon_n\end{aligned} \tag{6.80}$$

because ε is so small that

$$\varepsilon_n \leqq \left[4(4 D_{24}^2 2^{2\theta_{24} n} + 30 D_{24} 2^{\theta_{24} n} + 20)(D_{31} 2^{\theta_{31} n} + 3) + 20\left(\frac{3}{2} D_{24} 2^{\theta_{24} n} + 3\right) D_{25} 2^{\theta_{25} n}\right]^{-1} \tag{6.81}$$

for all $n \geqq 0$. Hence, we have

$$\begin{aligned}|E_{n+1}(t_n k) - E_{n+1}(k)| &\geqq (D_{14}^{-1} 2^{-\theta_{14} n} - \varepsilon_n) d(k, j) \\ &\geqq D_{14}^{-1} 2^{-\theta_{14}(n+1)} d(k, j)\end{aligned} \tag{6.82}$$

because

$$\varepsilon_n \leqq D_{14}^{-1} 2^{-\theta_{14}(n+1)} \tag{6.83}$$

for all $n \geqq 0$. This proves (vii). Also (viii) follows from (6.75). Finally, we have

$$\left|\frac{d^2}{dk^2}(E_{n+1}(k) - E_n(k))\right| \leqq \frac{1}{4} 2^{2\alpha(n+1)^2} \varepsilon_n^2, \tag{6.84}$$

thanks to (6.43), (6.44) and (6.45). This proves (ix).

(x) Let us suppose that $k \in B$ is such that

$$\left|\frac{d}{dk} E_{n+1}(k)\right| \leqq \frac{1}{2} D_6^{-1} 2^{-\theta_6(n+1)}. \tag{6.85}$$

If $k \notin \bigcup_{m=0}^{\infty} \bar{\mathscr{S}}_m$, then we have

$$\begin{aligned}\left|\frac{d}{dk} E_n(k)\right| &\leqq \frac{1}{2} D_6^{-1} 2^{-\theta_6(n+1)} + \frac{1}{4} D_{12} 2^{\alpha n^2 + \theta_{12} n} \varepsilon_n^2 \\ &\leqq \frac{1}{2} D_6^{-1} 2^{-\theta_6 n}.\end{aligned} \tag{6.86}$$

Hence

$$\left|\frac{d^2}{dk^2}\mathrm{E}_{n+1}(k)\right| \geqq \frac{1}{16}[1+n^{-1}]\lambda_1 - \mathrm{D}_{12}^2\, 2^{2\alpha n^2+2\theta_{12} n}\varepsilon_n^2$$
$$\geqq \frac{1}{16}[1+(n+1)^{-1}]\lambda_1 \tag{6.87}$$

because ε is so small that

$$\varepsilon_n^2 \leqq \min(\mathrm{D}_6^{-1}\mathrm{D}_{12}^{-1}\, 2^{-\alpha n^2-(\theta_6+\theta_{12})n-1},\ \mathrm{D}_{12}^{-2} n^{-2}\, 2^{2\alpha n^2+2\theta_{12} n}) \tag{6.88}$$

for all $n \geqq 1$.

On the other hand, suppose $k \in \bigcup_{m=0}^{n} \bar{\mathscr{S}}_m$ and let m_0 be the least integer such that $k \in \bar{\mathscr{S}}_{m_0}$. Then, there must be a $(n+1)$-regular jump $j \in \mathrm{J}_{n+1}^p$ of height n' and order $m \geqq m_0$, such that

$$d(k, j) \leqq \mathrm{D}_{32}\, 2^{\alpha m^2+\theta_{32} m}\varepsilon_{n'} \,|w_{n'}(j)|. \tag{6.89}$$

In fact, if this were not true, then thanks to $\mathscr{I}_3(m')$ (vii), $m_0+1 \leqq m' \leqq n+1$, we would have

$$\left|\frac{d}{dk}\mathrm{E}_{n+1}(k)\right| \geqq \mathrm{D}_6^{-1}\, 2^{-\theta_6 m_0} - \frac{1}{4}\sum_{m'=m_0+1}^{n+1}\left(\mathrm{D}_{33}^{-1}\, 2^{-\theta_{33} m'} + \frac{1}{4} 2^{\alpha m'^2}\varepsilon_{m'}\right)$$
$$> \frac{1}{2}\mathrm{D}_6^{-1}\, 2^{-\theta_6 m_0} \tag{6.90}$$

because

$$\mathrm{D}_{33} = 2\mathrm{D}_6, \qquad \theta_{33} = \theta_6 \tag{6.91}$$

and ε is so small that

$$\varepsilon_{m'} \leqq 4\,\mathrm{D}_{33}^{-1}\, 2^{-\theta_{33}-\alpha m'^2} \tag{6.92}$$

for all $m' \geqq 0$. But (6.90) is absurd. Hence, thanks to $\mathscr{I}_4(n+1)$ (vii), we have

$$\left|\frac{d^2}{dk^2}\mathrm{E}_{n+1}(k)\right| \geqq \left[1+\left(n+\frac{1}{2}\right)^{-1}\right]\mathrm{D}_{34}^{-1}\, 2^{-\theta_{34} m-6\alpha m^2}\varepsilon_{n'}^{-1}\,|w_{n'}(j)|^{-1}$$
$$-\frac{1}{4}\cdot 2^{2\alpha(n+1)^2}\varepsilon_n^2(\varepsilon_{n'}^{-1}\,|w_{n'}(j)|^{-1})$$
$$\geqq [1+(n+1)^{-1}]\,\mathrm{D}_{34}^{-1}\, 2^{-\theta_{34} m-6\alpha m^2}\varepsilon_{n'}^{-1}\,|w_{n'}(j)|^{-1}. \tag{6.93}$$

(xi) If $\bar{\varepsilon} \geqq \varepsilon_n^{1/4}$, then thanks to (ii) we have

$$\left| \int_B dk \left(\frac{\bar{\varepsilon}}{(E_{n+1}(k)-x)^2+\bar{\varepsilon}^2} - \frac{\bar{\varepsilon}}{(E_n(k)-x)^2+\bar{\varepsilon}^2} \right) \right| \leqq \int_B dk \frac{2\bar{\varepsilon}(1+x)\,|E_{n+1}(k)-E_n(k)|}{[(E_{n+1}(k)-x)^2+\bar{\varepsilon}^2]\,[(E_n(k)-x)^2+\bar{\varepsilon}^2]} \leqq 12\pi\bar{\varepsilon}^{-3}\varepsilon_n \leqq 12\pi\varepsilon_n^{1/4}. \tag{6.94}$$

(xii) It is easy to convince ourselves that there exists a smooth function $\bar{E}_{n+1}(k)$ agreeing with $E_{n+1}(k)$ for $k \in B'$ and such that

$$\left| \frac{d}{dk}(\bar{E}_{n+1}(k) - E_0(k)) \right| \leqq \sum_{m=0}^{n} 2 D_{12} 2^{\alpha m^2 + \theta_{12} m + 2} \varepsilon_m^2 \leqq \varepsilon \tag{6.95}$$

and

$$\left| \frac{d^2}{dk^2}(\bar{E}_{n+1}(k) - E_0(k)) \right| \leqq \sum_{m=0}^{n} 2 D_{12}^2 2^{2\alpha m^2 + 2\theta_{12} m + 4} \varepsilon_m^2 \leqq \varepsilon \tag{6.96}$$

for all $k \in B$. Hence, thanks to Lemma 2.1, we have that either

$$\left| \frac{d^2}{dk^2} \bar{E}_{n+1}(k) \right| \geqq \frac{1}{2}\lambda_1 - \varepsilon \tag{6.97}$$

or

$$\left| \frac{d}{dk} \bar{E}_{n+1}(k) \right| \geqq \frac{1}{2}\lambda_2 - \varepsilon. \tag{6.98}$$

Since

$$\varepsilon \leqq \min\left(\frac{1}{4}\lambda_1, \frac{1}{4}\lambda_2 \right), \tag{6.99}$$

$\bar{E}_{n+1}(k)$ has the same number c_0 of critical points of $E_{n+1}(k)$ and we have

$$\sup_{0<\bar{\varepsilon}<1} \int dx \left[\int_{B'} dk \frac{\bar{\varepsilon}}{(E_{n+1}(k)-x)^2+\bar{\varepsilon}^2} \right]^{3/2} \leqq \sup_{0<\bar{\varepsilon}<1} \int dx \left[\int_B dk \frac{\bar{\varepsilon}}{(\bar{E}_{n+1}(k)-x)^2+\bar{\varepsilon}^2} \right]^{3/2} \leqq 2^7 \rho_7^{-4} + 2^6 \pi^{-1} c_0^{3/2} (\rho_0 \lambda_2^{-1} + \rho_0^{1/2} \lambda_1^{-1/2}) \tag{6.100}$$

(xiii) Let $j \in J_{n+1}^p$ be a jump of order m and height $n' \leqq n$ and let $\tilde{E}_{n+1}(k)$ be a smooth function defined for $k \in \bar{C}_m(j) \setminus \{j\}$ and agreeing with $E_{n+1}(k)$ for all

$$k\in\bar{C}_m(j)\setminus(\{j\}\cup\bigcup_{m'=m+1}^{n}\bigcup_{\substack{j'\in J^p_{m'+1}\setminus J^p_{m'}\\ j'\neq j}}\bar{C}_{m'}(j'))\tag{6.101}$$

and such that

$$\begin{aligned}\left|\frac{d}{dk}\left(\tilde{E}_{n+1}(k)-E_{m_0}(k)\right)\right| &\leqq \sum_{m'=m+1}^{n} D_{12}\,2^{\alpha m'^2+\theta_{12}m'+3}\,\varepsilon^2_{m'}\\ &\leqq \varepsilon_{m+1}\end{aligned}\tag{6.102}$$

and

$$\begin{aligned}\left|\frac{d^2}{dk^2}\left(\tilde{E}_{n+1}(k)-E_{m}(k)\right)\right| &\leqq \sum_{m'=m+1}^{n} D^2_{12}\,2^{2\alpha m'^2+2\theta_{12}m'+5}\,\varepsilon^2_{m'}\\ &\leqq \varepsilon_{m+1}.\end{aligned}\tag{6.103}$$

Hence, thanks to $\mathscr{I}_7(m)$ (x), we have that either

$$\left|\frac{d}{dk}\tilde{E}_{n+1}(k)\right|\geqq\frac{1}{2}D_6^{-1}\,2^{-\theta_6 m}-\varepsilon_{m+1}\geqq\frac{1}{4}D_6^{-1}\,2^{-\theta_6 m}\tag{6.104}$$

or

$$\left|\frac{d^2}{dk^2}\tilde{E}_{n+1}(k)\right|\geqq\frac{1}{2}D_{34}^{-1}\,2^{-\theta_{34}m-6\alpha m^2}\,\varepsilon_{n'}^{-1}\tag{6.105}$$

because $|w_{n'}(j)|\leqq 1$ and ε is so small that

$$\varepsilon_{m+1}\leqq\frac{1}{4}D_6^{-1}\,2^{-\theta_6 m}\tag{6.106}$$

for all $m\geqq 0$. If $x\in\mathbf{R}$, let us define the set

$$\mathscr{A}_n(x)=\{k\in\bar{C}_m(j):|\tilde{E}_{n+1}(k)-x|\leqq\varepsilon_n^{1/10}\}\tag{6.107}$$

for all $x\in\mathbf{R}$. Since $0<\bar{\varepsilon}<\varepsilon_n^{1/4}$, we have

$$\int_{\bar{C}_n(j)\setminus\mathscr{A}_n(x)} dk\frac{\bar{\varepsilon}}{(\tilde{E}_{n+1}(k)-x)^2+\bar{\varepsilon}^2}\leqq\varepsilon_n^{1/20}.\tag{6.108}$$

Each set $\mathscr{A}_n(x)$ has at most four connected components $\mathscr{A}_{n\alpha}(x)$, $\alpha=1,\ldots,4$. If

$$d(\mathscr{A}_{n\alpha}(x),j)\geqq D_{32}\,2^{\alpha m^2+\theta_{32}m}\,\varepsilon_{n'}\,|w_{n'}(j)|\tag{6.109}$$

then we have

$$\int_{\mathscr{A}_{n\alpha}(x)} dk\frac{\bar{\varepsilon}}{(\tilde{E}_{n+1}(k)-x)^2+\bar{\varepsilon}^2}\leqq 4\pi^{-1}D_6\,2^{\theta_6 m_0}.\tag{6.110}$$

Otherwise, we have

$$\int_{\mathscr{A}_{n\alpha}(x)} dk\frac{\bar{\varepsilon}}{(\tilde{E}_{n+1}(k)-x)^2-\bar{\varepsilon}^2}\leqq\frac{1}{2\pi}D_{34}^{1/2}\,2^{1/2\,\theta_{34}m+3\alpha m^2}\,\varepsilon_{n'}.\tag{6.111}$$

Moreover, the measure of the set X of the $x \in \mathbf{R}$ such that $\mathcal{A}_n(x)$ is nonvoid is $\leqq D_6 D_{12}^{-1} 2^{(\theta_6-\theta_{12})m}$. Hence, we find

$$\int dx \left[\int dk \frac{\bar{\varepsilon}}{(\tilde{E}_{n+1}(k) - x^2 + \bar{\varepsilon}^2} \right]^{3/2} \leqq 6\, \varepsilon_n^{3/40} + 16\,\pi^{-1} D_6^2 D_{12}^{-1} 2^{(2\,\theta_6 - \theta_{12})m} + \pi^{-1} D_{34}^{1/2} 2^{1/2\,\theta_{34} m + 3\,\alpha m^2} \varepsilon_{n'} \leqq 7\, D_6^2 D_{12}^{-1} 2^{(2\,\theta_6 - \theta_{12})m} \tag{6.112}$$

because ε is so small that

$$6\,\varepsilon_m^{3/40} + \pi^{-1} D_{34}^{1/2} 2^{1/2\,\theta_{34} m + 3\,\alpha m^2} \varepsilon_m \leqq D_6^2 D_{12}^{-1} 2^{(2\,\theta_6 - \theta_{12})m} \tag{6.113}$$

for all $m \geqq 0$ and $m \leqq n' \leqq n$.

(xiv) Let us introduce the set

$$\bar{\mathcal{S}}'_m = \bigcup \bar{C}_m(j)' \tag{6.114}$$

where the union is over all $(n+1)$-regular jumps $j \in J_{n+1}^p$ of order m. Thanks to (xiii), we have

$$\int dx \left[\int_{\bar{\mathcal{S}}'_m} dk \frac{\bar{\varepsilon}}{(E_{n+1}(k) - x)^2 + \bar{\varepsilon}^2} \right]^{3/2} \leqq 7\,\pi\, D_6^3 D_{12}^{-1} 2^{(3\,\theta_6 + 5\,\theta_1 - \theta_{12} - 1)m + \theta_6 + s} \leqq 2^{-m} \tag{6.115}$$

because

$$D_{12} \geqq 7\,\pi\, D_6^3 . 2^{\theta_6 + 5}, \qquad \theta_{12} \geqq 3\,\theta_6 + s\,\theta_1. \tag{6.116}$$

In virtue of Holder's inequality, we have

$$\int dx \left[\sum_{m=0}^{n+1} \int_{\bar{\mathcal{S}}'_m} dk \frac{\bar{\varepsilon}}{(E_{n+1}(k) - x)^2 + \bar{\varepsilon}^2} \right]^{3/2} \leqq \left(\sum_{m=0}^{n+1} 2^{-(3/2)m} \right)^{1/2} \sum_{m=0}^{n+1} 2^{1/2\,m} \int dx \left[\int_{\bar{\mathcal{S}}'_m} dk \frac{\bar{\varepsilon}}{(E_{n+1}(k) - x)^2 + \bar{\varepsilon}^2} \right]^{3/2} \leqq (1 - 2^{-3/2})^{-1/2} (1 - 2^{-1/2})^{-1} \leqq 8 \tag{6.117}$$

Q.E.D.

LEMMA 6.7. – *The induction hypothesis of the family $\mathcal{I}_8(n+1)$, hold.*

Proof. – (i) follows from $\mathcal{I}_2(n+1)$ (v). (ii), (iii) and (iv) follow from Lemma 6.2, 6.3 and 6.4.

Let us suppose that $k \in B$ is such that

$$d_{n+1}\left(k, \left[j - \frac{\varepsilon_m^{-1} \varepsilon_n^{7/4}}{2}, j + \frac{\varepsilon_m^{-1} \varepsilon_n^{7/4}}{2}\right]\right) \leqq 1. \tag{6.118}$$

Thanks to $\mathscr{I}_5(n+1)$ (vii) we have

$$\begin{aligned}\left|\frac{d}{dk}\varepsilon_{n+1} v_{n+1}(\omega_n(k); k)\right| &\leqq 2-\left(n+\frac{1}{2}\right)^{-1}+4\varepsilon_n^2(4D_{24}^2 2^{2\theta_{24}n}+30D_{24}2^{\theta_{24}n}+20)(D_{31}2^{\theta_{31}n}+3)\\ &\quad+20\varepsilon_n^2\left(\frac{3}{2}D_{24}2^{\theta_{24}n}+3\right)D_{25}2^{\theta_{25}n}\\ &\leqq 2-n^{-1}\end{aligned} \tag{6.119}$$

because ε is so small that

$$\sqrt{\varepsilon_n}\leqq\left(n^{-1}-\left(n+\frac{1}{2}\right)^{-1}\right)\Bigg[4(4D_{24}^2 2^{2\theta_{24}n}+30D_{24}2^{\theta_{24}n}+20)(D_{31}2^{\theta_{31}n}+3)+20\left(\frac{3}{2}D_{24}2^{\theta_{24}n}+3\right)D_{25}2^{\theta_{25}n}\Bigg] \tag{6.120}$$

for all $n\geqq 0$. This condition implies also the following bound:

$$\sum_{\omega\neq\omega_n(k)}\left|\frac{d}{dk}v_{n+1}(\omega; k)\right|\leqq 1. \tag{6.121}$$

If k satisfies (6.85), then thanks to $\mathscr{I}_5(n+1)$ (viii) we also have

$$\begin{aligned}\left|\frac{d^2}{dk^2}\varepsilon_{n+1} v_{n+1}(\omega; k)\right| &\leqq 1-\left(n+\frac{1}{2}\right)^{-1}+\frac{1}{4}2^{2\alpha(n+1)^2}\varepsilon_n^2\\ &\leqq 1-n^{-1}\end{aligned} \tag{6.122}$$

because

$$\sqrt{\varepsilon_n}\leqq 4.2^{-2\alpha(n+1)^2} \tag{6.123}$$

for all $n\geqq 0$. This bound also implies $\mathscr{I}_8(n+1)$ (vi), *i.e.*

$$\sum_{\omega\neq\omega_n(k)}\left|\frac{d^2}{dk^2}v_{n+1}(\omega; k)\right|\leqq 1 \tag{6.124}$$

Q.E.D.

LEMMA 6.8. – *The induction hypotheses of the family $\mathscr{I}_9(n+1)$, hold.*
Proof. – (i) Follows from $\mathscr{I}_2(n+1)$ (v) and (ii), (iii) and (iv) follow from Lemmas 6.2, 6.3 and 6.4 (v) and (vi) follow from (iii) and (iv) in case $N\geqq 14$.

Let us suppose that $N\leqq 13$ and let $k\in B$ be such that

$$d_{n+1}\left(k, \left[j-\frac{1}{2}\varepsilon_n^{7/4}, j+\frac{1}{2}\varepsilon_n^{7/4}\right]\right)\leqq 13 \tag{6.125}$$

Thanks to Lemmas 6.1 and 6.5, we have

$$\sum_{\omega \in \mathcal{U}} \left| \frac{d}{dk} h_{n+1, \mathrm{N}}(\omega; k) \right| \leqq 5 \varepsilon_n^{3\mathrm{N}/8} (1+\varepsilon_n) \exp(2 \mathrm{D}_{24} \varepsilon_n^{1/8} 2^{\theta_{24} n}) \times [\mathrm{D}_{31} 2^{\theta_{31} n} + 3 + \mathrm{D}_{25} 2^{\theta_{28} n}] \leqq 1 \qquad (6.126)$$

because (6.44) holds and

$$\varepsilon_n^{1/4} \leqq \frac{1}{10 e} (\mathrm{D}_{31} 2^{\theta_{31} n} + 3 + \mathrm{D}_{25} 2^{\theta_{25} n})^{-1} \qquad (6.127)$$

for all $n \geqq 0$. We also have

$$\sum_{\omega \in \mathcal{U}} \left| \frac{d^2}{dk^2} h_{n+1, \mathrm{N}}(\omega; k) \right| \leqq 5 \varepsilon_n^{3\mathrm{N}/8} (1+\varepsilon_n) \exp(2 \mathrm{D}_{24} \varepsilon_n^{1/8} 2^{\theta_{24} n}) \times [\mathrm{D}_{28} 2^{2\alpha n^2 + \theta_{28} n + 1} + \mathrm{D}_{25} \mathrm{D}_{31} 2^{(\theta_{25}+\theta_{31}) n} + 3 + 6 \mathrm{D}_{25} 2^{\theta_{25} n} + \mathrm{D}_{28} 2^{2\alpha n^2 + \theta_{28} n}] \leqq 1 \qquad (6.128)$$

because

$$\varepsilon_n^{1/4} \leqq \frac{1}{10 e} (\mathrm{D}_{28} 2^{2\alpha n^2 + \theta_{28} n + 1} + \mathrm{D}_{25} \mathrm{D}_{31} 2^{(\theta_{25}+\theta_{31}) n} + 3 + 6 \mathrm{D}_{25} 2^{\theta_{25} n} + \mathrm{D}_{28} 2^{2\alpha n^2 + \theta_{28} n})^{-1} \qquad (6.129)$$

Q.E.D.

Finally, let us mention without proof that estimates very similar to the ones above, permit us to prove the following result

LEMMA 6.9. – *The induction hypothesis of the family $\mathcal{I}_{10}(n+1)$, holds.*

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