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The Cauchy problem for a nonlinear Wheeler-DeWitt equation

by

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ABSTRACT. — In this paper we consider a nonlinear version of the simplified Wheeler-DeWitt equation which describes the minisuperspace model for the wave function ψ of a closed universe (*cf.* [3], [2], [1]). Following [1], where the linear case has been solved, we study this equation as an evolution equation in the scalar field $y \in \mathbf{R}$ with a scale factor $x \in]0, R[$. We solve the Cauchy problem for the initial data $\psi(x, 0)$ and $\frac{\partial \psi}{\partial y}(x, 0)$ and we study some decay properties and blow-up situations. A particular nonlinear version has been proposed in [6].

RÉSUMÉ. — Dans cet article nous considérons une version non linéaire de l'équation de Wheeler-DeWitt simplifiée qui décrit le modèle de mini-superespace pour la fonction d'onde ψ d'un univers fermé (*cf.* [3], [2], [1]). Dans l'esprit de [1], où nous avons résolu le cas linéaire, nous étudions cette équation comme une équation d'évolution dans le champ scalaire $y \in \mathbf{R}$ avec le facteur d'échelle $x \in]0, R[$. Nous résolvons le problème de Cauchy pour des données initiales $\psi(x, 0)$ et $\frac{\partial \psi}{\partial y}(x, 0)$ et nous étudions des propriétés de décroissance à l'infini et des situations d'explosion de la solution. Une version non linéaire particulière a été proposée dans [6].

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1. INTRODUCTION

We consider the following nonlinear model for the simplified Wheeler-DeWitt equation:

$$\frac{1}{x^2}\frac{\partial^2 \psi}{\partial y^2}-\frac{\partial^2 \psi}{\partial x^2}-\frac{p}{x}\frac{\partial \psi}{\partial x}+x^2\psi-k^2x^4\psi+gx^{q-2}|\psi|^r\psi=0 \qquad (1.1)$$

where $y\in\mathbf{R}$ is the scalar field, $x\in]0, \mathrm{R}[$ $(\mathrm{R}>0)$ is a scale factor, $p\in\mathbf{R}$, $k^2>0$, $g\in\mathbf{R}$, $r\geqq 1$ and $q\geqq \frac{1}{2}rp$ are given constants (p reflects the factor-ordering ambiguity and k^2 is a cosmological constant) and $\psi:]0, \mathrm{R}[\times\mathbf{R}\to\mathbf{C}$ is the wave function of the universe for the minisuperspace model. The equation (1.1) is equivalent, in the sense of distributions, to the following one:

$$\frac{\partial^2 \psi}{\partial y^2}-x^2\frac{\partial^2 \psi}{\partial x^2}-px\frac{\partial \psi}{\partial x}+x^4\psi-k^2x^6\psi+gx^{q}|\psi|^r\psi=0 \qquad (1.2)$$

which can be considered as an evolution equation in $y\in\mathbf{R}$.

Assuming that $\psi(x, 0)$ and $\frac{\partial\psi}{\partial y}(x, 0)$, belong to some suitable weighted Sobolev spaces (*cf.* [1]) we first prove the existence of a unique local solution for the correspondent Cauchy problem. Under the hypothesis

$$g>0 \quad\text{and}\quad k^2\mathrm{R}^2\leqq 1 \quad (\text{that is } x^4-k^2x^6\geqq 0), \qquad (1.3)$$

we will prove that this solution is a global solution.

Furthermore, if $k^2\mathrm{R}^2\leqq\frac{2}{3}\left(\text{that is }\frac{d}{dx}(x^4-k^2x^6)\geqq 0\right)$, we obtain the decay property

$$\int_y^{y+1}\int_0^{\mathrm{R}} x^{q+p-2}|\psi(x, s)|^{r+2}\,dx\,ds\to 0 \qquad (y\to+\infty).$$

Finally we point out that, if $g<0$ and $k^2\mathrm{R}^2\leqq 1$, and for special initial data, there exists $y_0\in\mathbf{R}_+$ such that

$$\int_0^{\mathrm{R}} x^{p-2}|\psi(x, y)|^2\,dx\to+\infty \qquad (y\to y_0^-).$$

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2. SOME PROPERTIES OF THE ASSOCIATED LINEAR OPERATOR

Following [1] let us denote by $H^1_{p,0}$ the closure of $\mathscr{D}(]0, R[)$ in the space $\left\{u \in L^2_p \,|\, \dfrac{du}{dx} \in L^2_p\right\}$ where $L^2_p(]0, R[)$ is the L^2 space for the measure $d\mu = x^p\,dx$. We recall that we have $H^1_{p,0} \subsetneq L^2_{p-2}$ if $p \neq 1$ and $H^1_{p,0} \cap L^2_{p-2} = \{u \,|\, x^{p/2} u \in H^1_0\}$ where $H^1_0 = H^1_{0,0}$ is the usual Sobolev space.

Let us put $H = H^1_{p,0} \times L^2_{p-2}$,

$$B\psi = x^2 \frac{\partial^2 \psi}{\partial x^2} + p x \frac{\partial \psi}{\partial x}, \qquad V\psi = -x^4 \psi + k^2 x^6 \psi$$

$$D(A) = D(B) \times (H^1_{p,0} \cap L^2_{p-2}) \qquad \text{with} \quad D(B) = \{u \in H^1_{p,0} \,|\, Bu \in L^2_{p-2}\}.$$

Now, with $\psi_1 = \dfrac{\partial \psi}{\partial y}$, the linear equation associated to (1.2) can be written as follows

$$\frac{\partial}{\partial y}\begin{pmatrix}\psi \\ \psi_1\end{pmatrix} = \begin{pmatrix}0 & 1 \\ B & 0\end{pmatrix}\begin{pmatrix}\psi \\ \psi_1\end{pmatrix} + \begin{pmatrix}0 & 1 \\ V & 0\end{pmatrix}\begin{pmatrix}\psi \\ \psi_1\end{pmatrix} \tag{2.1}$$

In [1] we proved that $A : D(A) \to H$ defined by

$$A\begin{pmatrix}\psi \\ \psi_1\end{pmatrix} = \begin{pmatrix}0 & 1 \\ B & 0\end{pmatrix}\begin{pmatrix}\psi \\ \psi_1\end{pmatrix}$$

is skew-self-adjoint in H and that $D : H \to H$ defined by

$$D\begin{pmatrix}\psi \\ \psi_1\end{pmatrix} = \begin{pmatrix}0 & 0 \\ V & 0\end{pmatrix}\begin{pmatrix}\psi \\ \psi_1\end{pmatrix}$$

is continuous. Furthermore

$$\begin{aligned} D(B) \cap L^2_{p-2} &= \{u \in H^1_{p,0} \cap L^2_{p-2} \,|\, x^{p/2+1} u \in H^2\} \\ &= \{u \,|\, x^{p/2} u \in H^1_0 \text{ and } x^{p/2+1} u \in H^2\} \qquad (\textit{cf.}\ [1]). \end{aligned}$$

We denote by $S(y)$, $y \in \mathbf{R}$, the strongly continuous group of operators in H with infinitesimal generator $T = A + D : D(A) \to H$.

We need the following

PROPOSITION 2.1. – *Assume* $p = 1$ (*that is* $H = H^1_{1,0} \times L^2_{-1}$) *and put*

$$H_1 = (H^1_{1,0} \cap L^2_{-1}) \times L^2_{-1} \subsetneq H,\ S_1, \qquad S(y)_{|H_1}, \quad T_1 = T_{|D(T) \cap H_1}.$$

Then $S_1(y)$, $y \in \mathbf{R}$, *is a strongly continuous group of operators in* H_1 *with infinitesimal generator* T_1.

Proof. – Let $(u_0, v_0) \in H_1 \cap D(T)$, $(u(y), v(y)) = S(y)(u_0, v_0)$.

Since $\frac{\partial}{\partial y}|u|^2 = 2\,\Re e\, u\bar{v}$ we obtain, for $\varepsilon > 0$,

$$\int_0^R x^{-1+\varepsilon} |u(y)|^2 dx \leqq \int_0^R x^{-1+\varepsilon} |u_0|^2 dx + 2\int_0^y \int_0^R \frac{|u(s)||v(s)|}{x^{(1/2)-(\varepsilon/2)}\, x^{(1/2)-(\varepsilon/2)}} dx\, ds$$
$$\leqq R^\varepsilon \int_0^R x^{-1} |u_0|^2 dx + 2 R^{\varepsilon/2} \int_0^y \| u(s) \|_{L^2_{-1+\varepsilon}} \| v(s) \|_{L^2_{-1}} ds.$$

Hence, by applying Gronwall's inequality and Fatou's lemma ($\varepsilon \to 0$), we get $u(y) \in L^2_{-1}$ and

$$\| u(y) \|_{L^2_{-1}} \leqq \| u_0 \|_{L^2_{-1}} + \int_0^y \| v(s) \|_{L^2_{-1}} ds \qquad (y \geqq 0)$$

and this inequality can be extended, by density, to $(u_0, v_0) \in H_1$. Since for $(u_0, v_0) \in H_1$ we have

$$\| v(s) \|_{L^2_{-1}} \leqq \| S(s)(u_0, v_0) \|_H \leqq M e^{\omega s} \| (u_0, v_0) \|_H$$

with $M > 0$, $\omega \geqq 0$. Hence for $(u_0, v_0) \in H_1$ we get

$$\| u(y) \|_{L^2_{-1}} \leqq c(y) (\| u_0 \|_{L^2_{-1}} + \| u_0 \|_{H^1_{1,0}} + \| v_0 \|_{L^2_{-1}}).$$

We also have $\| u(y) - u_0 \|_{L^2_{-1}} \leqq \int_0^y \| v(s) \|_{L^2_{-1}} ds.$

Now, let $\tilde{T}$ be the infinitesimal generator of S_1. It is easy to see that $D(\tilde{T}) \subset D(T) \cap H_1$ and that, in $D(\tilde{T})$, $\tilde{T} = T_1$. Let $(u_0, v_0) \in D(T) \cap H_1$.

We have $\frac{1}{y}(u(y) - u_0) \to v_0\ (y \to 0^+)$ in $H^1_{1,0}$.

We want to prove that $\frac{1}{y}(u(y) - u_0) \to v_0\ (y \to 0^+)$ in L^2_{-1}.

If $y_n \to 0^+$, we obtain

$$\limsup_{n\to\infty} \left\| \frac{1}{y_n}(u(y_n) - u_0) \right\|_{L^2_{-1}} \leqq \lim_{n\to\infty} \frac{1}{y_n} \int_0^{y_n} \| v(s) \|_{L^2_{-1}} ds = \| v_0 \|_{L^2_{-1}}.$$

Therefore there exists a subsequence y_{n_k} such that

$$\frac{1}{y_{n_k}}(u(y_{n_k}) - u_0) \rightharpoonup \eta \in L^2_{-1}\ (k \to \infty), \quad \text{weakly in } L^2_{-1}.$$

But, for $\varepsilon > 0$, $L^2_{-1} \subsetneq L^2_{-1+\varepsilon}$, $H^1_{1,0} \subsetneq H^1_{1+\varepsilon,0} \subsetneq L^2_{-1+\varepsilon}$, and so $\eta = v_0$. Hence $\frac{1}{y_{n_k}}(u(y_{n_k}) - u_0) \to v_0\ (k \to \infty)$ in L^2_{-1}.

We conclude that $D(T) \cap H_1 \subset D(\tilde{T})$. □

3. THE CAUCHY PROBLEM FOR THE NONLINEAR EQUATION

In the remaining of the paper we put

$$X=\begin{cases} H=H^1_{p,0}\times L^2_{p-2}, & \text{if } p\neq 1 \\ H_1=(H^1_{1,0}\cap L^2_{-1})\times L^2_{-1}, & \text{if } p=1 \end{cases} \tag{3.1}$$

$$U=\begin{cases} T=A+D, \text{ with domain } D(A), & \text{if } p\neq 1 \\ T_1=T_{|H_1}, & \text{if } p=1 \end{cases} \tag{3.2}$$

and we will denote by $G(y)$, $y\in\mathbf{R}$, the strongly continuous group of operators generated by U in X.

Let

$$F\begin{pmatrix}\psi\\ \psi_1\end{pmatrix}=\begin{pmatrix}0\\ -gx^q|\psi|^r\psi\end{pmatrix},\qquad \psi_1=\frac{\partial\psi}{\partial y} \tag{3.3}$$

The nonlinear equation (1.2) can be written as follows

$$\frac{\partial}{\partial y}\begin{pmatrix}\psi\\ \psi_1\end{pmatrix}=U\begin{pmatrix}\psi\\ \psi_1\end{pmatrix}+F\begin{pmatrix}\psi\\ \psi_1\end{pmatrix} \tag{3.4}$$

and, since $H^1_{p,0}\cap L^2_{p-2}=\{u\,|\,x^{p/2}u\in H^1_0\}$ and $H^1_0\subsetneq L^\infty$, it is easy to prove that if $r\geqq 1$ and $q\geqq r\frac{p}{2}$, then F is a locally Lipschitz continuous function from X to X.

Furthermore X is an Hilbert space. Hence, by theorem 1.6 in [4] ch. 6, if we take

$$\begin{pmatrix}\psi_0\\ \psi_{1,0}\end{pmatrix}\in D(U),\qquad \text{that is}\quad \psi_0\in D(B)\cap L^2_{p-2},\quad \psi_{1,0}\in H^1_{1,0}\cap L^2_{p-2},$$

then there exists a unique local mild solution $\begin{pmatrix}\psi\\ \psi_1\end{pmatrix}$ of the corresponding Cauchy problem for the equation (3.4) which is a local strong solution, that is

$$\left.\begin{aligned} &\begin{pmatrix}\psi\\ \psi_1\end{pmatrix}\in\mathscr{C}(]-\varepsilon,\varepsilon[;D(U))\cap\mathscr{C}^1(]-\varepsilon,\varepsilon[;X),\\ &\begin{pmatrix}\psi\\ \psi_1\end{pmatrix}\text{ satisfies (3.4) for } y\in]-\varepsilon,\varepsilon[\quad\text{and}\quad \begin{pmatrix}\psi\\ \psi_1\end{pmatrix}(0)=\begin{pmatrix}\psi_0\\ \psi_{1,0}\end{pmatrix}.\end{aligned}\right\} \tag{3.5}$$

In order to obtain a result on the global existence of solution we need the following

LEMMA 3.1. — *Assume* $\begin{pmatrix} \psi_0 \\ \psi_{1,0} \end{pmatrix} \in \mathrm{D}(\mathrm{U})$ *and let* $\begin{pmatrix} \psi \\ \psi_1 \end{pmatrix}$ *be the corresponding local solution for the Cauchy problem. We have*

$$\begin{aligned} \mathrm{E}(y) = \frac{1}{2}\int_0^{\mathrm{R}} x^{p-2}\left|\frac{\partial\psi}{\partial y}\right|^2 dx + \frac{1}{2}\int_0^{\mathrm{R}} x^p\left|\frac{\partial\psi}{\partial x}\right|^2 dx + \frac{1}{2}\int_0^{\mathrm{R}} x^{p+2}|\psi|^2 dx \\ - \frac{k^2}{2}\int_0^{\mathrm{R}} x^{p+4}|\psi|^2 dx + \frac{g}{r+2}\int_0^{\mathrm{R}} x^{p+q-2}|\psi|^{r+2} dx \\ \equiv \mathrm{E}(0) = \mathrm{E}, \qquad y \in]-\varepsilon, \varepsilon[. \quad (3.6) \end{aligned}$$

Proof. — If we multiply the equation (1.2) by $x^{p-2}\dfrac{\partial\bar{\psi}}{\partial y}$, take the real part and integrate in x (over]0, R[) we obtain $\dfrac{d}{dy}\mathrm{E}(y)=0$, since $\left(x^p \dfrac{\partial\psi}{\partial x}\dfrac{\partial\bar{\psi}}{\partial y}\right)(y) \in \mathrm{W}_0^{1,1}$. □

Hence, we have

THEOREM 3.1. — *Assume* (1.3) *and* $\begin{pmatrix} \psi_0 \\ \psi_{1,0} \end{pmatrix} \in \mathrm{D}(\mathrm{U})$.

Then there exists a unique global solution $\begin{pmatrix} \psi \\ \psi_1 \end{pmatrix}$ *of the corresponding Cauchy problem for the equation* (1.2) *in the sense of* (3.5) (*for* $y \in]-\infty, +\infty[$).

Proof. — By (3.6) we have

$$\int_0^{\mathrm{R}} x^{p-2}|\psi_1|^2 dx + \int_0^{\mathrm{R}} x^p\left|\frac{\partial\psi}{\partial x}\right|^2 dx \leqq \mathrm{E}.$$

In the special case $p=1$, this implies, reasoning as in the first part of the proof of proposition 2.1, that

$$\|\psi(y)\|_{\mathrm{L}^2_{-1}} \leqq \|\psi_0\|_{\mathrm{L}^2_{-1}} + \int_0^y \|\psi_1(s)\|_{\mathrm{L}^2_{-1}} ds \leqq \|\psi_0\|_{\mathrm{L}^2_{-1}} + \mathrm{E}y \; (y \geqq 0). \quad □$$

Remark. — The theorem 3.1 can be extended to more general situations, for example, if gx^q is replaced by a function $g(x, y) \geqq 0$, g continuous, $\dfrac{\partial g}{\partial y}$ continuous and such that

$$g(x, y) \leqq c(y)x^q, \qquad \left|\frac{\partial g}{\partial y}(x, y)\right| \leqq \min(c_1(y)x^q, c_2(y)g(x, y)),$$

with $c, c_1 \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbf{R})$, $c_2 \in \mathrm{L}^1_{\mathrm{loc}}(\mathbf{R}\setminus\{0\})$, $c, c_1, c_2 \geqq 0$. Furthermore, the condition $k^2\mathrm{R}^2 \leqq 1$ can be replaced by $\mathrm{R}^4 - k^2\mathrm{R}^6 + \dfrac{1}{4}(p-1)^2 > 0$ if $p \neq 1$.

4. DECAY PROPERTIES AND A BLOW-UP CASE

THEOREM 4.1. – *Let us assume* $g>0$ *and* $k^2 R^2 \leqq \frac{2}{3}$ *and let* $\begin{pmatrix}\psi_0\\ \psi_{1,0}\end{pmatrix} \in D(U)$ *and*

$$\psi \in \mathscr{C}(\mathbf{R};\, D(B) \cap L^2_{p-2}) \cap \mathscr{C}^1(\mathbf{R};\, H^1_{p,0} \cap L^2_{p-2}) \cap \mathscr{C}^2(\mathbf{R};\, L^2_{p-2})$$

be the unique global solution of the corresponding Cauchy problem for the equation (1.2). *Then we have*

$$(2-3k^2R^2)\int_0^y\int_0^R x^{p+2}|\psi|^2\,dx\,ds + g\frac{2q-rp+r}{2(r+2)}\int_0^y\int_0^R x^{p+q-2}|\psi|^{r+2}\,dx\,ds \leqq 4E, \quad (4.1)$$

$\forall y \in \mathbf{R}_+$, *where* E *is defined by* (3.6).

Proof. – Let $v = x^{p/2}\psi$. We have $v \in \mathscr{C}^1(\mathbf{R};\, H^1_0) \cap \mathscr{C}^2(\mathbf{R};\, L^2_{-2})$, $xv \in \mathscr{C}(\mathbf{R};\, H^2)$ and the equation (1.2) takes the form

$$\frac{\partial^2 v}{\partial y^2} - x^2\frac{\partial^2 v}{\partial x^2} + \frac{p}{4}(p-2)v + x^4 v - k^2x^6 v + gx^{q-(1/2)pr}|v|^r v = 0. \quad (4.2)$$

Let us multiply the equation (4.2) by $\frac{1}{2}\frac{\bar v}{x^2} - \frac{1}{x}\frac{\partial \bar v}{\partial x}$ take the real part and integrate over]0, R[. Taking in account that

$$\Re e\int_0^R \frac{\partial^2 v}{\partial y^2}\frac{\bar v}{x^2}\,dx = \Re e\frac{d}{dy}\int_0^R\frac{\partial v}{\partial y}\frac{\bar v}{x^2}\,dx - \int_0^R\frac{1}{x^2}\left|\frac{\partial v}{\partial y}\right|^2 dx$$

and

$$\Re e\int_0^R \frac{\partial^2 v}{\partial y^2}\frac{\partial \bar v}{\partial x}\frac{1}{x}\,dx = \Re e\frac{d}{dy}\int_0^R\frac{\partial v}{\partial y}\frac{\partial \bar v}{\partial x}\frac{1}{x}\,dx - \frac{1}{2}\int_0^R\frac{1}{x}\frac{\partial}{\partial x}\left|\frac{\partial v}{\partial y}\right|^2 dx$$

and integrating by parts we obtain (recall that if $u \in H^1$ and $u(0)=0$ then $x^{-(1/2)}u \to 0\,(x\to 0^+)$):

$$0 = \frac{1}{2}\frac{d}{dy}\Re e\int_0^R\frac{1}{x^2}\frac{\partial v}{\partial y}\bar v\,dx - \frac{d}{dy}\Re e\int_0^R\frac{1}{x}\frac{\partial v}{\partial y}\frac{\partial \bar v}{\partial x}\,dx + \int_0^R(2x^2-3k^2x^4)|v|^2\,dx + g\frac{2q-rp+r}{2(r+q)}\int_0^R x^{q-(1/2)pr-2}|v|^{r+2}\,dx + \frac{1}{2}R\left|\frac{\partial v}{\partial x}\right|^2(R) - \frac{1}{2}\lim_{x\to 0^+}\left(x\left|\frac{\partial v}{\partial x}\right|^2\right) + \frac{1}{4}\lim_{x\to 0^+}\frac{\partial|v|^2}{\partial x}. \quad (4.3)$$

But

$$x\frac{\partial v}{\partial x}\in \mathrm{H}^1 \qquad \text{and} \qquad x\frac{\partial v}{\partial x}=\frac{\partial}{\partial x}(xv)-\frac{xv}{x}\to 0\,(x\to 0^+).$$

Hence

$$x\left|\frac{\partial v}{\partial x}\right|^2\to 0\,(x\to 0^+)$$

and

$$\frac{\partial |v|^2}{\partial x}=2\,\Re e\left[(x^{-1/2}\bar{v})\left(x^{1/2}\frac{\partial v}{\partial x}\right)\right]\to 0\,(x\to 0^+)$$

From (4.3) we obtain for $\psi=x^{-(p/2)}v$ and with $k^2\,\mathrm{R}^2\leqq\frac{2}{3}$:

$$\begin{aligned}\theta=(2-3k^2\,\mathrm{R}^2)\int_0^{\mathrm{R}} x^{p+2}|\psi|^2\,dx+g\frac{2q-rp+r}{2(r+2)}\int_0^{\mathrm{R}} x^{p+q-2}|\psi|^{r+2}\,dx\\ \leqq -\frac{1}{2}\frac{d}{dy}\Re e\int_0^{\mathrm{R}} x^{p-2}\frac{\partial\psi}{\partial y}\bar{\psi}\,dx+\frac{d}{dy}\Re e\int_0^{\mathrm{R}} x^{p-1}\frac{\partial\psi}{\partial y}\frac{\partial\bar{\psi}}{\partial x}\,dx\\ +\frac{d}{dy}\Re e\frac{p}{2}\int_0^{\mathrm{R}} x^{p-2}\frac{\partial\psi}{\partial x}\bar{\psi}\,dx.\end{aligned}$$

Hence, for $y>0$,

$$\begin{aligned}&\int_0^y \theta(s)\,ds\\ &\leqq\left[\int_0^{\mathrm{R}} x^{p-2}\left|\frac{\partial\psi}{\partial y}\right|^2 dx+\frac{1}{2}\int_0^{\mathrm{R}} x^p\left|\frac{\partial\psi}{\partial x}\right|^2 dx+\frac{1}{2}\frac{(p-1)^2}{4}\int_0^{\mathrm{R}} x^{p-2}|\psi|^2\,dx\right](y)\\ &\quad+\left[\int_0^{\mathrm{R}} x^{p-2}|\psi_{1,0}|^2\,dx+\frac{1}{2}\int_0^{\mathrm{R}} x^p\left|\frac{\partial\psi_0}{\partial x}\right|^2 dx\right.\\ &\quad\left.+\frac{1}{2}\frac{(p-1)^2}{4}\int_0^{\mathrm{R}} x^{p-2}|\psi_0|^2\,dx\right]\leqq 4\,\mathrm{E},\end{aligned}$$

since for $p\neq 1$ we have, by Hardy's inequality (*cf.* (2.1) in [1]),

$$\frac{(p-1)^2}{4}\int_0^{\mathrm{R}} x^{p-2}|\psi|^2\,dx\leqq\int_0^{\mathrm{R}} x^p\left|\frac{\partial\psi}{\partial x}\right|^2 dx. \qquad \square$$

COROLLARY. – *Under the hypothesis of theorem* 4.1 *we have*

$$\int_y^{y+1}\int_0^{\mathrm{R}} x^{p+q-2}|\psi|^{2+r}\,dx\,ds\to 0\,(y\to+\infty).$$

Remark. – By the reversibility in y we have similar results for $y\in\mathbf{R}_-$.

Finally we can point out a blow-up situation. The proof of the following result is similar to the one in the example of **X**.13 in [5] for the nonlinear wave equation and so we omit it.

PROPOSITION 4.1. — *Assume* $g<0$, $k^2 R^2 \leqq 1$ *and let* $\begin{pmatrix}\psi_0\\ \psi_{1,0}\end{pmatrix} \in D(U)$ *be real and such that* $E \leqq 0$ *and* $\int_0^R x^{p-2} \psi_0 \psi_{1,0}\, dx > 0$. *Let* ψ *be the local solution of the corresponding Cauchy problem for the equation* (1.2). *Then there exists* $y_0 \in \mathbf{R}_+$ *such that*

$$\lim_{y \to y^0} \int_0^R x^{p-2} \psi^2(x, y)\, dx = +\infty.$$

Remark. — Like in the quoted example for the nonlinear wave equation, it is easy to check that $y_0 \leqq \frac{2}{r}\left(\int_0^R x^{p-2} \psi_0^2\, dx\right)\left(\int_0^R x^{p-2} \psi_0 \psi_{1,0}\, dx\right)^{-1}$.

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