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Remarks on $W^{1,p}$-stability of the conformal set in higher dimensions

by

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ABSTRACT. – In this paper, we study the stability of maps in $W^{1,p}$ that are close to the conformal set $K_1 = \mathbf{R}^+ \cdot SO(n)$ in an averaged sense as described in Definition 1.1. We prove that K_1 is $W^{1,p}$-compact for all $p \geq n$ but is not $W^{1,p}$-stable for any $1 \leq p < n/2$ when $n \geq 3$. We also prove a coercivity estimate for the integral functional $\int_{\mathbf{R}^n} d^p_{K_1}(\nabla\phi(x))\,dx$ on $W^{1,p}(\mathbf{R}^n;\mathbf{R}^n)$ for certain values of p lower than n using some new estimates for weak solutions of p-harmonic equations.

Key words: Weak stability, conformal set.

RÉSUMÉ. – Dans cet article, nous étudions la stabilité des applications dans $W^{1,p}$ qui sont proches de l'ensemble conforme $K_1 = \mathbf{R}^+ \cdot SO(n)$ dans un sens moyenné décrit dans la Définition 1.1. Nous prouvons que K_1 est $W^{1,p}$-compact pour $p \geq n$ mais n'est pas $W^{1,p}$-stable pour tout $1 \leq p < n/2$ si $n \geq 3$. Nous prouvons aussi une estimée de coercivité pour la fonctionnelle $\int_{\mathbf{R}^n} d^p_{K_1}(\nabla\phi(x))\,dx$ on $W^{1,p}(\mathbf{R}^n;\mathbf{R}^n)$ pour certaines valeurs de p inférieures à n en utilisant des estimées nouvelles pour des solutions faibles d'équations p-harmoniques.

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1. INTRODUCTION

Let $n \geq 2$ and $\mathcal{M}^{n\times n}$ denote the set of all real $n \times n$ matrices. For each $l \geq 1$, we consider the following subset K_l of $\mathcal{M}^{n\times n}$ in connection with the theory of l-quasiregular mappings in $\mathbf{R}^n$ (*see* Reshetnyak [21] and Rickman [22]),

$$K_l = \{A \in \mathcal{M}^{n\times n} \mid \|\mathcal{A}\|^n \leq l \det \mathcal{A}\}, \tag{1.1}$$

where $\|A\|$ is the norm of $A \in \mathcal{M}^{n\times n}$ viewed as a linear operator on $\mathbf{R}^n$, *i.e.*,

$$\|A\| = \max_{|h|=1} |A\,h| = \max_{|h|=1} \sqrt{h^T A^T Ah}. \tag{1.2}$$

When $l = 1$, set K_1 is the set of all conformal matrices, which will be called the *conformal* set in this paper. Note that $K_1 = \mathbf{R}^+ \cdot SO(n)$. We also consider the set $R(n)$ of all general orthogonal matrices in $\mathbf{R}^n$, *i.e.*,

$$R(n) = \{A \in \mathcal{M}^{n\times n} \mid A^T A = \lambda I \text{ for some } \lambda \geq 0\}. \tag{1.3}$$

Let Ω be a domain in $\mathbf{R}^n$, which is assumed throughout this paper to be bounded and smooth. We recall that a map $u \in W^{1,p}(\Omega; \mathbf{R}^n)$ is said to be (*weakly,* if $p < n$) l-quasiregular if $\nabla u(x) \in K_l$ for a.e.$x \in \Omega$, *see* [13], [14], [21] and [22]. The Liouville theorem asserts that every 1-quasiregular in $W^{1,n}(\Omega; \mathbf{R}^n)$ is conformal and thus is the restriction of a Möbius map if $n \geq 3$.

An important result proved in Iwaniec [13, Theorem 3] is that for each $n \geq 3$ and $l \geq 1$ there exists a $p_* = p(n,l) < n$ such that every weakly l-quasiregular map belonging to $W^{1,p_*}(\Omega; \mathbf{R}^n)$ belongs actually to $W^{1,n}(\Omega; \mathbf{R}^n)$ and is thus an l-quasiregular map as usually defined in [21] or [22]. Such higher integrability results depend on some new estimates for weak solutions of p-harmonic equations in Iwaniec [13], and Iwaniec and Sbordone [15].

In this paper, we shall study some properties pertaining to the stability of *weakly* quasiregular maps. We shall consider the stability of maps in $W^{1,p}(\Omega; \mathbf{R}^n)$ when their gradients are converging to the conformal set $K_1 = \mathbf{R}^+ \cdot SO(n)$ in the averaged sense described by (1.4) in Definition 1.1 below. The study is originated from a study of the structures of *Young measures* whose supports are *unbounded.* For references in this direction, we refer to [2], [3], [16], [17], [19], [23], [25], [26], [29], [30] and references therein.

We need some notation to proceed. For a function f defined on $\mathcal{M}^{n\times n}$ we use $\mathcal{Z}(f)$ and $f^{\#}$ to denote the zero set and the quasiconvexification of f, respectively. For a given set $\mathcal{K} \subset \mathcal{M}^{n\times n}$, denote by $d_{\mathcal{K}}(A)$ the distance from A to $\mathcal{K}$ for all $A \in \mathcal{M}^{n\times n}$ (in any equivalent Euclidean norm), and let $\mathcal{K}^{\#}$ be the quasiconvex hull of $\mathcal{K}$. *See* Dacorogna [8], Yan [26] and Šverák [23] for the relevant definitions.

In this paper, we use the following definition, *see* also Zhang [30]. We refer to Ball [2], Kinderlehrer and Pedregal [17] and Tartar [25] for more connections of this definition with the Young measures theory.

DEFINITION 1.1. – *We say $\mathcal{K}$ is $W^{1,p}$-stable if for every sequence $u_j \rightharpoonup u_0$ in $W^{1,p}(\Omega;\mathbf{R}^n)$ satisfying*

$$\lim_{j\to\infty}\int_\Omega d^p_{\mathcal{K}}(\nabla u_j(x))\,dx = 0, \tag{1.4}$$

it follows that $\nabla u_0(x) \in \mathcal{K}$ for a.e. $x \in \Omega$. We say $\mathcal{K}$ is $W^{1,p}$-compact if every weakly convergent sequence $u_j \rightharpoonup u_0$ in $W^{1,p}(\Omega,\mathbf{R}^n)$ satisfying (1.4) *converges strongly to u_0 in $W^{1,1}(\Omega;\mathbf{R}^n)$. In terms of Young Measures, $\mathcal{K}$ is $W^{1,p}$-compact if and only if every $W^{1,p}$-gradient Young Measure supported on $\mathcal{K}$ is a* Dirac *Young Measure on $\mathcal{M}^{n\times n}$.*

It should be noted that in many cases $d^p_{\mathcal{K}}$ in (1.4) can be replaced by other functions f that vanish exactly on $\mathcal{K}$ and satisfy $0 \le f(A) \le C\,(|A|^p + 1)$. For example, when $\mathcal{K}$ is homogeneous, then $d^p_{\mathcal{K}}$ in (1.4) can be replaced by any non-negative homogeneous functions of degree p that vanish exactly on $\mathcal{K}$.

We also note that it follows from the result in Zhang [29]-[30] that if a *compact* set $\mathcal{K}$ is $W^{1,p}$-compact for *some* $p > 1$ then it is $W^{1,p}$-compact for *all* $p > 1$. One of the main purposes of this paper is to show that this result fails to hold for *unbounded* sets $\mathcal{K}$. Our counter-example is provided by the conformal set $K_1 = \mathbf{R}^+ \cdot SO(n)$ defined above. More precisely, we shall prove the following result.

THEOREM 1.2. – *Suppose $n \ge 3$. Then set $K_1 = \mathbf{R}^+ \cdot SO(n)$ is $W^{1,p}$-compact for all $p \ge n$, but not $W^{1,p}$-stable for any $1 \le p < n/2$.*

The $W^{1,n}$-compactness of K_1 follows from a stronger theorem (Theorem 3.1) proved by using the result of Evans and Gariepy [10] (*see* also Evans [9]) and the theory of polyconvex functions. Note that the $W^{1,p}$-compactness of K_1 for $p > n$ has been proved in Ball [3] using the Young measures and polyconvex functions; *see* also Kinderlehrer [16]. Using the similar techniques of *biting* Young measures, one can also prove the $W^{1,n}$-compactness of K_1 without using the result of [10]; but we do not pursue such a method in the present paper. For more on biting Young measures, we only refer to [6], [17] and [28].

It is also noted that $K_1 = \mathbf{R}^+ \cdot SO(n)$ is unbounded and contains no rank-one connections, but our theorem says that it may or may not support nontrivial gradient Young measures. This phenomenon also makes the conjecture in Tartar [25] more interesting for Young measures with unbounded supports; of course, this conjecture (in the case of compact supports) has been very well understood and resolved in Bhattacharya *et al.* [7], *see* also Šverák [24].

It has been proved in Yan [26] (also Zhang [29]) that if $\mathcal{K}$ is compact then $\mathcal{K}^\# = \mathcal{Z}(d_{\mathcal{K}}^\#)$. For $\mathcal{K} = \mathcal{K}_1$, the conformal set, if $n \geq 3$ it is easily seen from the proof of Theorem 3.3 that the set $R(n)$ is contained in $\mathcal{Z}(d_{\mathcal{K}}^\#)$. More recently, using this observation and the rank-one convex hulls, we have proved in Yan [27] that $d_{K_1}^\#$ actually must be identically zero. For more on the growth condition for *conformal energy* functions, we refer to the forthcoming paper Yan [27]. Therefore in general the previous result $\mathcal{K}^\# = \mathcal{Z}(d_{\mathcal{K}}^\#)$ does not hold for unbounded sets $\mathcal{K}$ (an example when $n = 2$ was given in Yan [26]).

The proof of Theorem 3.1 uses the polyconvex function $G(A)$ defined by (2.1) which vanishes exactly on the conformal set K_1 and is *uniformly strictly* quasiconvex in the term used by Evans [9] and Evans and Gariepy [10]. The proof using biting Young measures as in Ball [3] also uses such polyconvex functions. However for $p < n$ both proofs break down since there is no counterpart of polyconvex function $G(A)$ that vanishes exactly on K_1 and grows like $|A|^p$ when $p < n$, *see* Yan [27].

To study the case for $p < n$, we make use of some new estimates for p-harmonic equations obtained recently by Iwaniec [13, Theorem 1] (*see* also [15]). We shall prove the following coercivity result for the functional

$$\int_\Omega d_{K_1}^p(\nabla\phi(x))\,dx$$

on $W_0^{1,p}(\Omega;\mathbf{R}^n)$ for certain $p < n$, which follows obviously from Theorem 4.1.

THEOREM 1.3. – *Let $n \geq 3$ and $K_1 = \mathbf{R}^+ \cdot SO(n)$ be the conformal set. Then there exist constants $\alpha(n) < n < \beta(n)$ and $c_0(n) > 0$ such that for all $p \in [\alpha(n), \beta(n)]$*

$$c_0(n)\int_\Omega |\nabla\phi(x)|^p\,dx \leq \int_\Omega d_{K_1}^p(\nabla\phi(x))\,dx \leq \int_\Omega |\nabla\phi(x)|^p\,dx \qquad (1.5)$$

for all $\phi \in W_0^{1,p}(\Omega;\mathbf{R}^n)$.

This theorem implies that for certain values of p lower than n, any weakly convergent sequence $\{u_j\}$ in $W_0^{1,p}(\Omega; \mathbf{R}^n)$ that satisfies (1.4) must converge to 0 in $W^{1,p}(\Omega; \mathbf{R}^n)$. For functions ϕ with the conformal linear boundary conditions, we do not know whether a similar estimate like (1.5) can be obtained; see the remarks in the end of the paper.

Finally, we point out that the estimate like the second one of (1.5) can not be expected to hold for a constant $\alpha(n) < n/2$.

THEOREM 1.4. – *Let $\alpha(n) < n$ be any constant determined in the previous theorem. Then it follows that $\alpha(n) \geq n/2$.*

We now give the plan of the paper. In section 2, we review some notation and preliminaries that are needed to prove our main theorems. In section 3, we prove the $W^{1,p}$-compactness of the conformal set $K_1 = \mathbf{R}^+ \cdot SO(n)$ for $p \geq n$ and study the $W^{1,p}$-stability of K_1 for $p < n/2$. In section 4, we prove the coercivity property (1.5) of the integral functional $\int_{\mathbf{R}^n} d_K^p(\nabla\phi(x))\,dx$ on $W^{1,p}(\mathbf{R}^n; \mathbf{R}^n)$ for certain values of p lower than n. We also prove that such a coercivity estimate is not true for $p < n/2$. Finally, in section 5, we make some remarks regarding the $W^{1,p}$-compactness of set K_1 for certain lower values of $p < n$.

2. NOTATION AND PRELIMINARIES

For $n \geq 2$, let us define

$$G(A) = n^{-n/2}|A|^n - \det A \tag{2.1}$$

where $|A|^2 = \operatorname{tr}(A^T A)$. It is easily seen that $G(A) \geq 0$ is polyconvex and vanishes exactly on $K_1 = \mathbf{R}^+ \cdot SO(n)$ and is *uniformly strictly* quasiconvex in the sense defined by Evans and Gariepy in [10], also [9] and [11].

LEMMA 2.1. – *Let $G(A)$ be defined by* (2.1). *Then for any $\epsilon > 0$, there exists a constant $C_\epsilon > 0$ such that for all $A \in \mathcal{M}^{n\times n}$*

$$G(A) \leq C_\epsilon\, d_{K_1}^n(A) + \epsilon\,|A|^n. \tag{2.2}$$

Proof. – This follows easily from the homogeneity of $G(A)$ and $d_{K_1}^n(A)$. □

In order to use the estimates for p-harmonic tensors, we need some notation on exterior algebras and differential forms on $\mathbf{R}^n$. We follow the notation in Iwaniec and Martin [14].

Let $e_1, e_2, \cdots, e_n$ denote the standard basis of $\mathbf{R}^n$. For $l = 0, 1, \cdots, n$ we denote by $\Lambda^l = \Lambda^l(\mathbf{R}^n)$ the linear space of all l-tensors spanned by $\{e_I = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_l}\}$ for all ordered l-tuples $I = (i_1, i_2, \cdots, i_l)$ with $1 \leq i_1 < i_2 < \cdots < i_l \leq n$. Define $\Lambda^l = \{0\}$ if $l < 0$ or $l > n$. The Grassmann algebra $\Lambda = \oplus \Lambda^l$ is a graded algebra with respect to the exterior multiplication.

For $\alpha = \sum \alpha^I e_I$ and $\beta = \sum \beta^I e_I$ in Λ the inner product is defined by

$$\langle \alpha, \beta \rangle = \sum_I \alpha^I \beta^I,$$

where the summation is taken over all l-tuples $I = (i_1, i_2, \cdots, i_l)$ and all integers $l = 0, 1, \cdots, n$.

The Hodge star operator $* : \Lambda \to \Lambda$ is then defined by the rule that

$$*1 = e_1 \wedge e_2 \wedge \cdots \wedge e_n$$

and

$$\alpha \wedge (*\beta) = \beta \wedge (*\alpha) = \langle \alpha, \beta \rangle\, (*1)$$

for all $\alpha, \beta \in \Lambda$. It is straightforward to see that $* : \Lambda^l \to \Lambda^{n-l}$ and the norm of $\alpha \in \Lambda$ is then given by the formula

$$|\alpha|^2 = \langle \alpha, \alpha \rangle = *(\alpha \wedge *\alpha) \in \Lambda^0.$$

For each $l = 0, 1, \cdots, n$, a differential form α of degree l defined on Ω

$$\alpha = \sum \alpha^I(x)\, dx_I = \sum \alpha^{i_1 i_2 \ldots i_l}(x)\, dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_l}$$

can be identified with a function $\alpha : \Omega \to \Lambda^l(\mathbf{R}^n)$ with the same coeffients $\{\alpha^I\}$. It is appropriate to introduce the space

$$\mathcal{D}'(\Omega; \Lambda) = \oplus \mathcal{D}'(\Omega; \Lambda^l)$$

of all differential forms whose coeffients are Schwartz distributions on Ω. We can also define $L^p(\Omega; \Lambda), W^{1,p}(\Omega; \Lambda)$ or other spaces by requiring all the coeffients belong to the suitable function spaces.

We shall make use of the exterior derivative

$$d : \mathcal{D}'(\Omega; \Lambda^l) \to \mathcal{D}'(\Omega; \Lambda^{l+1}), \quad l = 0, 1, ..., n,$$

and its formal adjoint operator, commonly called the Hodge codifferential,

$$d^* : \mathcal{D}'(\Omega; \Lambda^{l+1}) \to \mathcal{D}'(\Omega; \Lambda^l),$$

defined by $d^* = (-1)^{nl+1} * d *$ on $(l+1)$-forms.

The following observation will be useful in proof of Theorem 1.3.

LEMMA 2.2. – *Suppose $F \in K_1 = \mathbf{R}^+ \cdot SO(n)$. Let f_j be the j-th column (or row) vector of F for $j = 1, ..., n$, each being considered in Λ^1. Then*

$$|f_{i_1} \wedge \cdots \wedge f_{i_l}| = |f_1|^l, \quad 1 \leq i_1 < \cdots < i_l \leq n, \quad l = 1, 2, ..., n;$$

and

$$(-1)^{n-1} |f_1 \wedge \cdots \wedge f_{n-1}|^{\frac{2-n}{n-1}} (f_1 \wedge \cdots \wedge f_{n-1}) = *f_n.$$

Finally we need the following estimate on the weak solutions of nonhomogeneous p-harmonic equation in $\mathbf{R}^n$ proved in Iwaniec [13] and [12]. We refer to the recent paper of Iwaniec and Sbordone [15] for more discussions.

THEOREM 2.3. – *For each $p > 1$, there exists $\nu = \nu(n, p) \in (1, p)$ such that for every $s \geq \nu$ every weak solution u with $du \in L^s(\mathbf{R}^n; \Lambda)$ to the p-harmonic equation*

$$d^*[|g + du|^{p-2}(g + du)] = d^* h \quad \text{in } \mathbf{R}^n \tag{2.3}$$

satisfies for a constant $C(n, p, s) > 0$

$$\int_{\mathbf{R}^n} |du|^s \leq C(n, p, s) \int_{\mathbf{R}^n} \left(|g|^s + |h|^{\frac{s}{p-1}} \right). \tag{2.4}$$

Moreover, the constant $C(n, p, s)$ can be chosen independent of s for $\nu \leq s \leq n$.

Proof. – This is Theorem 1 in Iwaniec [13]. □

3. $W^{1,p}$-COMPACTNESS OF THE CONFORMAL SET K_1

Let $K_1 = \mathbf{R}^+ \cdot SO(n)$ be the conformal set defined before. In what follows, we assume $n \geq 3$. We first prove the $W^{1,n}$-compactness of the set K_1.

THEOREM 3.1. – *Suppose $u_j \rightharpoonup u_0$ in $W^{1,n}(\Omega; \mathbf{R}^n)$ and satisfies*

$$\lim_{j \to \infty} \int_\Omega d_{K_1}^n (\nabla u_j(x))\, dx = 0. \tag{3.1}$$

Then u_0 is a conformal map and moreover $u_j \to u_0$ in $W^{1,n}(\Omega; \mathbf{R}^n)$. Consequently, $K_1 = \mathbf{R}^+ \cdot SO(n)$ is $W^{1,n}$-compact.

Proof. – Let $G(A)$ is defined by (2.1). By Lemma 2.1 and (3.1), since $\{\|\nabla u_j\|_{L^n(\Omega)}\}$ is bounded, we easily obtain

$$\lim_{j\to\infty} I(u_j) \equiv \lim_{j\to\infty} \int_\Omega G(\nabla u_j(x))\, dx = 0. \tag{3.2}$$

Since $G(A)$ is polyconvex and satisfies $0 \le G(A) \le C\,|A|^n$, therefore by the theorem of Acerbi-Fusco [1], the functional

$$I(u) = \int_\Omega G(\nabla u(x))\, dx$$

is weakly lower semicontinuous on $W^{1,n}(\Omega; \mathbf{R}^n)$ (*see* also Ball and Murat [4] and Morrey [18]) thus it follows that

$$0 = I(u_0) \le \liminf_{j\to\infty} I(u_j) = 0$$

which implies u_0 is a conformal map and $u_j \to u_0$ in $W^{1,n}(\Omega; \mathbf{R}^n)$ by the result of Evans and Gariepy [10] since $G(A)$ is uniformly strictly quasiconvex. Finally by definition it follows that K_1 is $W^{1,n}$-compact. □

The $W^{1,n}$-compactness of K_1 can also be proved by using *biting* Young measures as in Ball [2] using Young measures for $p > n$. However, both methods do not work anymore for $p < n$ mainly because in this case there is no counterpart of the polyconvex function $G(A)$ vanishing exactly on K_1 and with growth like $|A|^p$; *see* Yan [27].

Before considering the $W^{1,p}$-compactness of set K_l for $1 < p < n$, we make some remark about the non-$W^{1,p}$-compactness for a general set $\mathcal{K} \subset \mathcal{M}^{n\times n}$ and $1 < p < \infty$.

Let $A \in \mathcal{M}^{n\times n}$, we consider the following system of equations or differential relations,

$$\left.\begin{array}{ll} u \in W^{1,p}(\Omega; \mathbf{R}^n), & \\ \nabla u(x) \in \mathcal{K}, & \text{for a.e.} x \in \Omega, \\ u(x) = A\,x, & \text{for } x \in \partial\Omega. \end{array}\right\} \tag{3.3}$$

Generally, the solvability of (3.3) relies heavily on the structure of $\mathcal{K}$. It is expected that nontrivial solutions of (3.3) (if exist) should be highly oscillatory if set $\mathcal{K}$ does not have certain *nice* structures.

When $\mathcal{K}$ is the compatible two-well in two dimensions, Müller and Šverák [19] recently proved that for certain matrices $A \notin \mathcal{K}$, Problem (3.3) has Lipschitz solutions.

We now prove the following result. The argument is closely related to that in Ball and Murat [4].

THEOREM 3.2. – *Suppose* u, $\mathcal{K}$ *and* A *solve system* (3.3). *Then* $A \in \mathcal{Z}(d_{\mathcal{K}}^{\#})$, *and* $\mathcal{K}$ *is not* $W^{1,p}$*-stable if* $A \notin \mathcal{K}$.

Proof. – First we remark that without loss of generality we can assume Ω to be the unit cell Q_0 centered at origin, since otherwise, using the Vitali covering and the affine boundary condition of $u(x)$, one can construct a solution $v \in W^{1,p}(Q_0; \mathbf{R}^n)$ to a system similar to (3.3) only with Ω being replaced by Q_0.

For each $k = 1, 2, \cdots$, we divide Q_0 into 2^{nk} sub-cells with side 2^{-k}, and denote these sub-cells by $\{Q_j^k\}$ with $1 \leq j \leq 2^{nk}$. Suppose

$$Q_j^k \equiv a_j^k + 2^{-k} Q_0, \quad j = 1, 2, \cdots, 2^{nk}. \tag{3.4}$$

We now define a map $u^k : Q_0 \to \mathbf{R}^n$ as follows,

$$u^k(x) = \begin{cases} A\, a_j^k + 2^{-k} u(2^k(x - a_j^k)), & \text{if } x \in Q_j^k \text{ for some } j, \\ A\, x, & \text{for other } x \in Q_0. \end{cases} \tag{3.5}$$

It is easily seen that $\nabla u^k(x) \in \mathcal{K}$ for a.e. $x \in Q_0$ and $u^k \in W^{1,p}(Q_0; \mathbf{R}^n)$. It is also easy to see for all functions $W(A)$ defined on $\mathcal{M}^{n\times n}$ that

$$\int_{Q_0} W(\nabla u^k(x))\, dx = \int_{Q_0} W(\nabla u(x))\, dx. \tag{3.6}$$

A calculation also shows that (*see* e.g., Ball and Murat [4, Corollary A. 2])

$$u^k \rightharpoonup u_0 \text{ in } W^{1,p}(Q_0; \mathbf{R}^n) \quad \text{as } k \to \infty, \tag{3.7}$$

where $u_0(x) \equiv A\, x$. Since $d_{\mathcal{K}}(\nabla u^k(x)) = 0$, therefore, it follows from theorem on weak lower semicontinuity (*see* [1], [4], [8] and [18]) that $\nabla u_0(x) \equiv A \in Z(d_{\mathcal{K}}^{\#})$. If $A \notin \mathcal{K}$, then $\nabla u_0(x) \notin \mathcal{K}$, thus by definition and (3.7), this shows that $\mathcal{K}$ is not $W^{1,p}$-stable. We thus complete the proof. □

It is proved in [13] that there exists a $p_* = p(n, l) < n$ for each $n \geq 3$ and $l \geq 1$ such that every weakly l-quasiregular map belonging to $W^{1,p_*}(\Omega; \mathbf{R}^n)$ belongs actually to $W^{1,n}(\Omega; \mathbf{R}^n)$; thus is an l-quasiregular map as usually defined in [21] and [22]. The general conjecture is that $p_* = \frac{nl}{l+1}$; *see* also [14]. From this it follows that problem (3.8) can not have a solution when $p \geq p_* = p(n)$ and $l = 1$ unless $A \in K_1$.

The following results are based on the existence of *weakly* l-quasiregular maps that are not l-quasiregular when $n \geq 3$. Recall that $R(n)$ is the set of all general orthogonal matrices in $\mathbf{R}^n$ defined by (1.3). See also Iwaniec and Martin [14, section 12].

THEOREM 3.3. – *Let* $1 \le p < \frac{nl}{l+1}$ *and* $A \in R(n)$ *with* $\det A = -1$. *Then the following problem has a solution:*

$$\left.\begin{array}{l} u \in W^{1,p}(B;\mathbf{R}^n), \\ \nabla u(x) \in K_l, \text{ for a.e. } x \in B, \\ u(x) = Ax, \text{ for } x \in \partial B, \end{array}\right\} \tag{3.8}$$

where B *is the unit open ball in* $\mathbf{R}^n$.

Proof. – For a given $l \ge 1$, define a radial map $\Phi_l : B \to \mathbf{R}^n$ as follows:

$$\Phi_l(x) = \left(\frac{1}{|x|}\right)^{1+\frac{1}{l}} x \quad \text{for } x \in B. \tag{3.9}$$

When $l = 1$, Φ_1 is the inversion with respect to the unit sphere.

It is easily seen that $\Phi_l(x) = x$ for $x \in \partial B$ and that

$$\nabla\Phi_l(x) = \left(\frac{1}{|x|}\right)^{1+\frac{1}{l}} \left(I - \frac{l+1}{l}\frac{x}{|x|} \otimes \frac{x}{|x|}\right).$$

Thus

$$\|\nabla\Phi_l(x)\|^n = |x|^{-n(1+\frac{1}{l})} = -l \det \nabla\Phi_l(x) \quad \text{for } x \in B\backslash\{0\}. \tag{3.10}$$

For $A \in R(n)$ with $\det A = -1$, define $u(x) = \Phi_l(Ax)$ for $x \in B$. We claim that u solves (3.8) for any $1 \le p < \frac{nl}{l+1}$.

Note that $\nabla u(x) = \nabla\Phi_l(Ax)A$ for $x \in B\backslash\{0\}$ and $|Ax| = |x|$ for any $x \in \mathbf{R}^n$. Therefore by (3.10), it follows that $\|\nabla u(x)\|^n = l \det \nabla u(x)$ for $x \in B\backslash\{0\}$ and $u(x) = Ax$ if $x \in \partial B$. What is left to check is $u \in W^{1,p}(B;\mathbf{R}^n)$ for any $1 \le p < \frac{nl}{l+1}$. Our calculation shows

$$\|\nabla u(x)\|^p = |x|^{-p(1+\frac{1}{l})} \quad \text{for } x \in B\backslash\{0\}.$$

Thus

$$\int_B \|\nabla u(x)\|^p\, dx = \frac{l\omega_n}{ln - p(l+1)} < \infty,$$

where ω_n is the area of ∂B. Thus $u \in W^{1,p}(B;\mathbf{R}^n)$ for any $1 \le p < \frac{nl}{l+1}$. We thus complete the proof. □

Combining Theorems 3.2 and 3.3, we have proved the following corollary.

COROLLARY 3.4. – *For any* $l \ge 1$ *and* $1 \le p < \frac{nl}{l+1}$, *the set* K_l *is not* $W^{1,p}$*-stable. Moreover,* $R(n) \subset \mathcal{Z}(d^{\#}_{K_l})$.

As mentioned in the introduction, using rank-one convex hulls, we can prove $R(n)^{\#} \equiv \mathcal{M}^{n\times n}$ for $n \ge 3$. Therefore the previous corollary actually implies that $d^{\#}_{K_l}$ must be identically zero. *See* Yan [27] for more.

4. THE COERCIVITY OF $\int_{\mathbf{R}^n} d^p_{K_1}(\nabla u(x))\,dx$ ON $W^{1,p}(\mathbf{R}^n;\mathbf{R}^n)$

Let $K = K_1 = \mathbf{R}^+ \cdot SO(n)$ be the conformal set. This section is devoted to proving the following result.

THEOREM 4.1. – *For each $n \geq 3$, there exists $\alpha(n) < n$ such that for all $p \geq \alpha(n)$*

$$\int_{\mathbf{R}^n} |\nabla\phi(x)|^p\,dx \leq C(n,p)\int_{\mathbf{R}^n} d^p_K(\nabla\phi(x))\,dx \tag{4.1}$$

for all $\phi \in W^{1,p}(\mathbf{R}^n;\mathbf{R}^n)$. Moreover, $1 \leq C(n,p) \leq C(n) < \infty$ for $\alpha(n) \leq p \leq n$.

Proof. – We have only to prove (4.1) for $\phi \in C_0^\infty(\mathbf{R}^n;\mathbf{R}^n)$. Let us assume

$$\nabla\phi(x) = A(x) - B(x),\ \ A(x) \in K_1, |B(x)| = d_K(\nabla\phi(x)), a.e. \tag{4.2}$$

We can also assume B has compact support and is bounded. Let ϕ_i be the i-th coordinate function of ϕ, then $d\phi_i$ is a 1-form. Let $\beta_i(x)$ be the i-th row vector of $B(x)$ considered as a 1-form. Since $\nabla\phi(x) + B(x) \in K_1$ thus by Lemma 2.2, we have

$$\begin{aligned}&|(d\phi_{i_1} + \beta_{i_1}) \wedge \cdots \wedge (d\phi_{i_l} + \beta_{i_l})|\\ &\quad = |d\phi_1 + \beta_1|^l,\ 1 \leq i_1 < \cdots < i_l \leq n, \quad l = 1, 2, ..., n;\end{aligned} \tag{4.3}$$

and

$$\begin{aligned}&|(d\phi_1 + \beta_1) \wedge \cdots \wedge (d\phi_{n-1} + \beta_{n-1})|^{\frac{2-n}{n-1}}(d\phi_1 + \beta_1) \wedge \cdots \wedge (d\phi_{n-1} + \beta_{n-1})\\ &= (-1)^{n-1} * (d\phi_n + \beta_n).\end{aligned} \tag{4.4}$$

Let

$$u = \phi_{n-1}d\phi_1 \wedge \cdots \wedge d\phi_{n-2}, \quad du = (-1)^n d\phi_1 \wedge \cdots \wedge d\phi_{n-1},$$

$$g = (-1)^n[((d\phi_1 + \beta_1) \wedge \cdots \wedge (d\phi_{n-1} + \beta_{n-1})) - (d\phi_1 \wedge \cdots \wedge d\phi_{n-1})],$$

$$h = - * \beta_n.$$

Then it follows from (4.4) and $d^* * = *d$ on 1-forms that

$$d^*[|g + du|^{p-2}(g + du)] = d^*h \quad \text{in } \mathbf{R}^n, \text{ where } p = \frac{n}{n-1}. \tag{4.5}$$

Therefore by Theorem 2.3, there exists $1 < \nu < \frac{n}{n-1}$ such that for all $s \geq \nu$,

$$\int_{\mathbf{R}^n} |du|^s \leq C(n,s) \int_{\mathbf{R}^n} \Big(|g|^s + |h|^{s(n-1)} \Big). \tag{4.6}$$

By (4.3) and definition of du and g, it follow that for each $j = 1, 2, ..., n$,

$$|d\phi_j|^{(n-1)s} \leq |g + du|^s + |\beta_j|^{(n-1)s} \leq |g|^s + |du|^s + |\beta_j|^{(n-1)s}$$

and

$$|g|^s \leq \sum |d\phi_{i_1} \wedge \cdots \wedge d\phi_{i_l}|^s |\beta_{j_1} \wedge \cdots \wedge \beta_{j_m}|^s,$$

where the summation is over all $l + m = (n-1)$ and $l > 0$, $m > 0$ and $i_l \leq (n-1)$. From this we have

$$|g|^s \leq \epsilon \sum_{j=1}^{n-1} (|d\phi_j|^{s(n-1)}) + C(\epsilon)|B|^{s(n-1)},$$

where $\epsilon > 0$ is arbitrary. Combining these pointwise estimates, integrating over $\mathbf{R}^n$ and using (4.6), we obtain for each $j = 1, 2, ..., n$,

$$\int_{\mathbf{R}^n} |d\phi_j|^{s(n-1)} \leq \epsilon \sum_{j=1}^{n-1} \int_{\mathbf{R}^n} |d\phi_j|^{s(n-1)} + C(n,s,\epsilon) \int_{\mathbf{R}^n} |B|^{s(n-1)},$$

for a different arbitrary $\epsilon > 0$, to be chosen later. Summing this inequality for j from 1 to n and choosing $\epsilon = 1/(2n)$, it follows that

$$\sum_{j=1}^{n} \int_{\mathbf{R}^n} |d\phi_j|^{s(n-1)} \leq C(n,s) \int_{\mathbf{R}^n} |B|^{s(n-1)}, \tag{4.7}$$

for all $s \geq \nu$. Now let $\alpha(n) = \nu(n-1)$ then $\alpha(n) < n$. For this $\alpha(n)$ it is easy to *see* (4.1) follows from (4.7). The proof is thus completed. □

THEOREM 4.2. – *Let $\alpha(n) < n$ be any constant determined in the previous theorem. Then it follows that $\alpha(n) \geq n/2$.*

Proof. – We suppose $\alpha(n) < n/2$. Let $\Phi_1 : B_1 \to \mathbf{R}^n$ be the inversion with respect to the unit sphere as defined by (3.9). Let $A \in R(n)$ with $\det A = -1$. Define $u(x) = \Phi_1(Ax)$ for $x \in B_1$. Then

$$\nabla u(x) = \nabla \Phi_1(Ax)\, A \in K_1 = \mathbf{R}^+ \cdot SO(n), \quad a.e.\ x \in B_1.$$

Now let $\rho \in C_0^\infty(\mathbf{R}^n)$ with $\rho(x) = 1$ for $x \in B_{1/2}$ and $\rho(x) = 0$ for $x \notin B_1$, and

$$0 \leq \rho(x) \leq 1, \ |\nabla\rho(x)| \leq 2.$$

Let $\phi(x) = \rho(x)(u(x) - c)$, where c is a constant to be chosen later. Then $\phi \in W^{1,p}(\mathbf{R}^n; \mathbf{R}^n)$ for all $1 \leq p < n/2$.

For $0 < \epsilon < \frac{n}{2} - \alpha(n)$, applying (4.1) to $\phi \in W^{1,\frac{n}{2}-\epsilon}(\mathbf{R}^n; \mathbf{R}^n)$, we obtain

$$\begin{aligned}\int_{B_{1/2}} & |\nabla u(x)|^{\frac{n}{2}-\epsilon}\, dx \\ & \leq C(n) \int_{B_1} d_K^{\frac{n}{2}-\epsilon}\big(\nabla\rho(x) \otimes (u(x) - c) + \rho(x)\nabla u(x)\big)\, dx,\end{aligned}$$

from which and using $d_K(A + B) \leq d_K(A) + |B|$ it follows that

$$\begin{aligned}\int_{B_{1/2}} |\nabla u|^{\frac{n}{2}-\epsilon} & \leq C(n) \int_{B_1} |u - c|^{\frac{n}{2}-\epsilon} \\ & \leq C(n)\Big(\int_{B_1} |\nabla u|^{\frac{n^2-2n\epsilon}{3n-2\epsilon}}\Big)^{\frac{3n-2\epsilon}{2n}}, \qquad (4.8)\end{aligned}$$

where we have chosen $c = \frac{1}{|B_1|}\int_{B_1} u$ and applied the Sobolev inequality. In (4.8), letting $\epsilon \to 0$ we would have

$$\int_{B_{1/2}} |\nabla u(x)|^{n/2}\, dx \leq C(n)\Big(\int_{B_1} |\nabla u(x)|^{n/3}\, dx\Big)^{3/2} < \infty,$$

which is a contradiction, since $u \notin W^{1,n/2}(B_{1/2}; \mathbf{R}^n)$ as we showed before. We have thus completed the proof. □

5. A CONCLUDING REMARK

As we mentioned before, it is proved in Iwaniec [13, Theorem 3] that there exists a minimal $p_* = p(n) \in [n/2, n)$ for each $n \geq 3$ such that if a map $u(x)$ belonging to $W^{1,p_*}(\Omega; \mathbf{R}^n)$ satisfies $\nabla u(x) \in K_1 = \mathbf{R}^+ \cdot SO(n)$ a.e. then it belongs actually to $W^{1,n}(\Omega; \mathbf{R}^n)$. Note that $p_* = n/2$ when n is even, by the results in [14].

For a weakly convergent unperturbed sequence $\{u_j\}$ in $W^{1,p_*}(\Omega; \mathbf{R}^n)$ with $\nabla u_j(x) \in K_1$ for a.e. $x \in \Omega$, the strong convergence follows easily from Theorem 3.1 and this higher integrability result.

Now, if we only assume the distance from $\nabla u_j(x)$ to the conformal set is small and approaches zero as $j \to \infty$, then we do not usually have the higher integrability for $u_j \in W^{1,p_*}(\Omega; \mathbf{R}^n)$. In the even dimensions, there are some linear structures (*see* [14] and [27]) among the subdeterminants of half dimension size, that may compensate some loss of the stability due to the weak convergence of $\{u_j\}$. But I have not come up with the definite results in this aspect even in even dimensions. Therefore, it would be interesting to consider the following problem.

PROBLEM 5.1. – *Determine whether* $K_1 = \mathbf{R}^+ \cdot SO(n)$ *is* $W^{1,p}$*-compact for some* $p < n$. *If it is, whether the minimal value of such* p *is equal to* p_* *given above.*

Remark. – Most recently, in Müller, Šverák and Yan [20], it is proved that for *even* dimensions $n \geq 4$ the minimal $\alpha(n)$ in Problem 5.1 is $n/2$.

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