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On the existence of homoclinic solutions for almost periodic second order systems

by

Enrico SERRA

Dipartimento di Matematica del Politecnico,
Corso Duca degli Abruzzi, 24, I-10129 Torino.

Massimo TARALLO

Dipartimento di Matematica dell'Università,
Via Saldini, 50, I-20133 Milano.

and

Susanna TERRACINI

Dipartimento di Matematica del Politecnico,
Piazza Leonardo da Vinci, 32, I-20133 Milano.

ABSTRACT. – In this paper we prove the existence of at least one homoclinic solution for a second order Lagrangian system, where the potential is an almost periodic function of time. This result generalizes existence theorems known to hold when the dependence on time of the potential is periodic. The method is of a variational nature, solutions being found as critical points of a suitable functional. The absence of a group of symmetries for which the functional is invariant (as in the case of periodic potentials) is replaced by the study of problems "at infinity" and a suitable use of a property introduced by E. Séré.

RÉSUMÉ. – Dans cet article on démontre l'existence d'une solution homocline pour un système Lagrangien du deuxième ordre où le potentiel dépend du temps d'une façon quasi périodique. Ce résultat généralise le cas où le potentiel est une fonction périodique du temps. La méthode

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utilisée est variationnelle, les solutions étant trouvées comme points critiques d'une fonctionnelle. L'absence d'un groupe de symétries pour lequel la fonctionnelle est invariante (comme dans le cas des potentiels périodiques) est remplacée par l'étude des points critiques « à l'infini » et par une propriété introduite par E. Séré.

0. INTRODUCTION AND STATEMENT OF THE MAIN RESULT

In this paper we prove the existence of at least one homoclinic solution for a class of second order Lagrangian systems. In particular, we study the problem

$$(P) \qquad \begin{cases} -\ddot{u}(t) + u(t) = \alpha(t)\nabla G(u(t)) \\ \lim_{t\to\pm\infty} u(t) = \lim_{t\to\pm\infty} \dot{u}(t) = 0 \end{cases}$$

where α is a continuous positive almost periodic function (*see* Definition 0.3 below) and $G \in \mathcal{C}^2(\mathbf{R}^N;\mathbf{R})$, $N \geq 1$, satisfies suitable superquadraticity assumptions.

The main reason of interest for this kind of problem is to try to extend some results obtained in the study of homoclinic orbits, a subject which has received much attention in the last few years, especially when the potential is a periodic function of time. Indeed, starting from [7], [8] and [14] the problem of homoclinic and heteroclinic solutions has been widely investigated by people working with variational methods. Existence and powerful multiplicity results were given in [1], [2], [5], [8], [12], [13] for second order systems and in [7], [14]-[16] for the case of first order Hamiltonian systems. *See* also [6] for the asymptotically periodic case.

A second feature of interest is that this problem may serve as a model in the study of the existence of orbits of a conservative system, homoclinic to a given almost periodic solution. In this context *see* also the papers [3]-[4], [10],[11] (where the problem of homoclinics is seen from a different point of view) and the references therein.

We will prove the following result.

THEOREM 0.1. – *Assume that*

(G1) $G \in \mathcal{C}^2(\mathbf{R}^N;\mathbf{R})$ *and* $\alpha \in \mathcal{C}(\mathbf{R};\mathbf{R})$

(G2) *There exists* $\theta > 2$ *such that for all* $x \in \mathbf{R}^N \setminus \{0\}$,

$$0 < \theta G(x) \leq \nabla G(x)\cdot x\,;$$

(G3) *α is almost periodic, in the sense of Definition* 0.3, *and*

$$\underline{\alpha} = \inf_{t\in\mathbf{R}} \alpha(t) > 0.$$

Then Problem (P) admits at least one nonzero solution.

Remark 0.2. – We point out that the same result holds for systems of the form

$$-\ddot{u}(t) + A(t)u(t) = \nabla G(t, u(t)),$$

when $A(t)$ is a symmetric positive definite almost periodic matrix and $G(t, x)$ is almost periodic in t uniformly in x and satisfies (G2) uniformly in t. We shall work with the simpler problem (P) in order to avoid heavy technicalities.

Also note that superquadraticity of G (assumption (G2)) is standard when one deals with second order systems, as well as positivity of α. In particular therefore, Theorem 0.1 generalizes existence (though not multiplicity) results established in the periodic case.

We will study the existence of solutions to Problem (P) by means of a minimax procedure. Indeed let $H = H^1(\mathbf{R}; \mathbf{R}^N)$ and let $f : H \to \mathbf{R}$ be the functional defined by

$$\begin{aligned} f(u) &= \frac{1}{2}\int_{\mathbf{R}} [|\dot{u}(t)|^2 + |u(t)|^2]dt - \int_{\mathbf{R}} \alpha(t)G(u(t))dt \\ &= \frac{1}{2}||u||^2 - \int_{\mathbf{R}} \alpha(t)G(u(t))dt. \end{aligned}$$

It is readily seen, following for example [8] that if (G1)-(G3) hold (actually much less is enough) then $f \in \mathcal{C}^2(H; \mathbf{R})$ and

$$\nabla f(u) \cdot \varphi = \int_{\mathbf{R}} [\dot{u} \cdot \dot{\varphi} + u \cdot \varphi]dt - \int_{\mathbf{R}} \alpha \nabla G(u) \cdot \varphi dt,$$

so that critical points of f are weak (and, by regularity, strong) solutions to problem (P).

In the search for critical points of a functional like f one generally needs two different arguments. First, a change in the topology of some of the sublevel sets of f; secondly, some compactness property such as the Palais-Smale condition. In this context, it is easy to see that under assumption (G2) (and independently of any other assumption on α, as long as $\alpha \not\equiv 0$ is nonnegative) the function $z(t) \equiv 0$ is a strict local minimum for

f, and f is not bounded from below. This amounts to say that f satisfies the geometrical assumptions of the Mountain Pass Lemma.

In contrast, f does not satisfy in general the Palais-Smale condition, owing to the fact that the embedding of H into $L^\infty(\mathbf{R};\mathbf{R}^N)$ is not compact. However, if α is positive and bounded, it can be proved that for every Palais-Smale sequence u_n at a level $c \neq 0$ there exists a sequence $(t_n)_n \subset \mathbf{R}$ such that the sequence $u_n(\cdot + t_n)$ converges weakly to some limit $u_0 \not\equiv 0$.

These are the general features appearing in every problem concerning homoclinic solutions. Now, in order to highlight the difficulties that one has to face in the case when α is not a periodic function we first describe the easier problem with α *periodic* and then we examine the main differences between the two cases, from the point of view of existence results.

When α is periodic one can assume without loss of generality that the sequence t_n introduced above is made up of multiples of a period of α. Therefore, by the invariance of f, the sequence $u_n(\cdot + t_n)$ is still a Palais-Smale sequence for f and, as noted above, it contains some subsequence converging weakly to some $u_0 \not\equiv 0$. To conclude the proof it is enough to note that $\nabla f : H \to H$ is continuous for the weak topology: this shows that $\nabla f(u_0) = 0$.

If α is almost periodic but *not* periodic f is no longer invariant for the action of a group of translations. In particular this means that the sequence $u_n(\cdot + t_n)$ is *not* a Palais-Smale sequence for f, unless t_n is a sequence of ε_n-periods of α, with $\varepsilon_n \to 0$. To overcome this difficulty we will show that for those Palais-Smale sequences which satisfy the additional property $||u_n - u_{n-1}|| \to 0$, there exists a sequence t_n of ε_n-periods of α such that (a subsequence of) $u_n(\cdot + t_n)$ converges weakly to some $u_0 \not\equiv 0$. The weak continuity of ∇f then shows that u_0 is a critical point for f.

The accomplishment of this program involves a careful analysis of some qualitative properties of the solutions to the problems "at infinity". As an example, in the simplest case where $\alpha(t) = \alpha_1(t) + \alpha_2(t)$, with α_1 and α_2 periodic functions of periods T_1 and T_2 respectively ($\frac{T_1}{T_2} \notin \mathbf{Q}$), the family of the problems at infinity is given by the equations

$$(P_{\theta,\varphi}) \qquad -\ddot{u}(t) + u(t) = [\alpha_1(t+\theta) + \alpha_2(t+\varphi)]\nabla G(u(t)), \qquad \theta, \varphi \in \mathbf{R}.$$

We note that unless $\theta = \varphi$ this problem is *not* equivalent to (P).

Finally, we point out that the existence of Palais-Smale sequences with the further property $||u_n - u_{n-1}|| \to 0$ is due to E. Séré (*see* [14], [7]) where it has been used to find multiple homoclinic solutions in the periodic coefficient case. It seems to us that the full force of this property is well demonstrated in the study of problem (P), where it plays a central role; the fact that it is used here in a different way provides a new application of this general principle.

The proof of Theorem 0.1 is divided in a series of steps. In Section 1 we prove the existence of a Palais-Smale sequence u_n satisfying Séré's property $||u_n - u_{n-1}|| \to 0$. Section 2 is devoted to the study of the functional f and of its Palais-Smale sequences. Lastly, in Section 3 we examine some qualitative properties of the problems at infinity which will allow us to conclude the proof.

Before entering the proof of Theorem 0.1 we recall for the sake of completeness some definitions and properties concerning almost periodic functions. These are taken from [9], to where we refer the reader for further details and proofs.

DEFINITION 0.3. – (i) A set $P \subset \mathbf{R}$ is called relatively dense in $\mathbf{R}$ if there exists a number $\lambda > 0$ such that every interval of length λ contains at least one element of P. (ii) Let $\alpha : \mathbf{R} \to \mathbf{R}$ be a continuous function. Given $\varepsilon > 0$, a number $\tau \in \mathbf{R}$ is called an ε-period of α if

$$\sup_{t\in\mathbf{R}} |\alpha(t+\tau) - \alpha(t)| \leq \varepsilon.$$

(iii) A continuous function $\alpha : \mathbf{R} \to \mathbf{R}$ is called almost periodic if for every $\varepsilon > 0$, there exists a relatively dense set $P_\varepsilon \subset \mathbf{R}$ of ε-periods of α.

The next is a classical result on almost periodic functions and will be used at the core of our argument.

THEOREM 0.4. (Bochner's criterion). – *Let $\mathcal{C}(\mathbf{R};\mathbf{R})$ be the space of continuous, bounded functions on the real line, endowed with the sup norm. A function $\alpha \in \mathcal{C}(\mathbf{R};\mathbf{R})$ is almost periodic if and only if the set of its translates $\{\alpha(\cdot + \tau) \ / \ \tau \in \mathbf{R}\}$ is precompact in $\mathcal{C}(\mathbf{R};\mathbf{R})$.*

Notations. – $H := H^1(\mathbf{R};\mathbf{R}^N)$ denotes the Sobolev space of L^2 vector-valued functions whose distributional derivative is (represented by) an L^2 function. This is a Hilbert space endowed with the norm $||u||^2 = \int_{\mathbf{R}} |\dot{u}|^2 dt + \int_{\mathbf{R}} |u|^2 dt$. We recall that H is continuously (though not compactly) embedded into $\mathcal{C}^0(\mathbf{R};\mathbf{R}^N)$ and $L^p(\mathbf{R};\mathbf{R}^N)$, for all $p \in [2, +\infty]$, and that in particular $||u||_\infty \leq ||u||$, for all $u \in H$.

By $u \cdot v$ we will denote both the scalar product in $\mathbf{R}^N$ and in H. The context will always rule out possible ambiguities.

Likewise, we use ∇ to denote both the gradient of an $\mathbf{R}^N$-valued function and the gradient of a functional defined over H, that is the unique element of H which represents the differential of the functional via the Riesz isomorphism.

1. AN ABSTRACT RESULT

The aim of this section is to restate a celebrated result due to E. Séré (*see* [14], [7]) in a form which will turn out to be useful for our purposes. Although we will give a proof of Theorem 1.2 below, we wish to make clear that we do it only for the convenience of the reader; nearly all the arguments used can be traced in the works [7], [14].

We start by recalling some definitions.

DEFINITION 1.1. – Let $(H, ||\cdot||)$ be a Hilbert space. With the term deformation of H we mean a continuous map $\eta : H \times [0,1] \to H$ such that $\eta(\cdot, 0) = Id_H$.

Given a functional $f : H \to \mathbf{R}$ and numbers $a, b \in \mathbf{R}$ we denote $f^a = \{u \in H \;/\; f(u) \leq a\}$, $f_b = \{u \in H \;/\; f(u) \geq b\}$, and $f_b^a = f^a \cap f_b$.

By minimax class for f at level $c \in \mathbf{R}$ we mean a class Γ of subsets of H such that

$$c = \inf_{A \in \Gamma} \sup_{x \in A} f(x).$$

We say that a minimax class Γ is invariant for a deformation η if $A \in \Gamma$ implies $\eta(A, t) \in \Gamma$, for all $t \in [0,1]$.

The following is the main result of this section.

THEOREM 1.2. – *Let $f \in C^2(H; \mathbf{R})$ and let Γ be a minimax class for f at level $c \in \mathbf{R}$. Assume that there exists $\varepsilon_0 > 0$ with the property that Γ is invariant for all deformations η such that $\eta(\cdot, t)$ is the identity outside $f_{c-2\varepsilon_0}^{c+2\varepsilon_0}$.*

Then for all $\varepsilon \in]0, \varepsilon_0[$, there exists a sequence $(u_n)_n \subset H$ such that

(i) $$\lim_{n\to\infty} f(u_n) \in [c - \varepsilon, c + \varepsilon],$$

(ii) $$\lim_{n\to\infty} \nabla f(u_n) = 0,$$

(iii) $$\lim_{n\to\infty} ||u_n - u_{n-1}|| = 0.$$

Proof. – The proof is a slight variant of well-known deformation techniques, so that we will be rather sketchy at some points.

Let $\chi : \mathbf{R} \to [0,1]$ be a C^2 cut-off function such that

$$\chi(s) = \begin{cases} 1 & \text{if } s \in [c - \varepsilon_0, c + \varepsilon_0] \\ 0 & \text{if } s \notin]c - 2\varepsilon_0, c + 2\varepsilon_0[\end{cases}$$

and consider the Cauchy problem

$$\begin{cases} \frac{d}{dt}\eta(u,t) = -\chi(f(\eta(u,t)))\frac{\nabla f(\eta(u,t))}{1+||\nabla f(\eta(u,t))||} \\ \eta(u,0) = u. \end{cases}$$

Notice that since the right-hand-side of the above differential equation is locally Lipschitz continuous and is uniformly bounded in norm by 1, the flow η is uniquely defined for all $u \in H$ and all $t \geq 0$, and of course is a continuous function. Moreover, since $\chi(f(u)) = 0$ if $f(u) \notin]c - 2\varepsilon_0, c + 2\varepsilon_0[$, we see that $\eta(\cdot, t)$ is the identity outside $f_{c-2\varepsilon_0}^{c+2\varepsilon_0}$, so that by assumption the class Γ is invariant for η.

Now let $\varepsilon \in]0, \varepsilon_0[$ and define a function $T : H \to \mathbf{R}$ by setting

$$T(u) = \begin{cases} \min\{t \geq 0 \ / \ f(\eta(u,t)) = c - \varepsilon\} & \text{if this set is not empty} \\ +\infty & \text{otherwise.} \end{cases}$$

It is not difficult to show that the function T is continuous at all points u where $T(u)$ is finite; we leave the details to the reader.

Let $\Omega_\varepsilon = \{u \in f^{c+\varepsilon} \ / \ T(u) < +\infty\}$, that is, the set of points in the sublevel $f^{c+\varepsilon}$ which are pushed by the flow η below the level $c - \varepsilon$ in a finite time. Note that T is continuous in Ω_ε.

We claim that $\Omega_\varepsilon \neq f^{c+\varepsilon}$.

Indeed if this is not the case, namely if T is finite at all points of $f^{c+\varepsilon}$, then consider the map $\hat{\eta} : f^{c+\varepsilon} \times [0,1] \to H$ defined by $\hat{\eta}(u,t) = \eta(u, T(u)t)$. Plainly, by Dugundji's theorem this map can be extended to another continuous map, still denoted by $\hat{\eta}$, such that $\hat{\eta}(u,t) = u$ for all $t \geq 0$ and all $u \notin f^{c+2\varepsilon_0}$. This and the fact that $T(u) = 0$ if $u \in f^{c-2\varepsilon_0}$ show that the class Γ is invariant for the deformation $\hat{\eta}$. Let $A \in \Gamma$ be a set such that $\sup_A f < c + \varepsilon$; in particular, therefore, $A \subset \Omega_\varepsilon$. Then by construction we have $\hat{\eta}(A, 1) \subset f^{c-\varepsilon}$, and this contradicts the fact that $\hat{\eta}(A, 1) \in \Gamma$.

This argument shows that there must be at least one point $u \in f^{c+\varepsilon}$ such that $T(u) = +\infty$, or, in other words, $f(\eta(u,t)) > c - \varepsilon$ for all $t > 0$. We now use this fact to conclude the proof. Let $g : [0, +\infty[\to \mathbf{R}$ be the function

$$g(t) = -\int_0^t \chi(f(\eta(u,t)))\frac{||\nabla f(\eta(u,t))||^2}{1 + ||\nabla f(\eta(u,t))||}dt.$$

Note that g is C^1 and that $f(\eta(u,t)) - f(u) = g(t)$, as one immediately sees from the definition of η. Moreover g is nonincreasing and is bounded from below, since

$$\inf_{t\geq 0} g(t) = \lim_{t\to+\infty} g(t) = \lim_{t\to+\infty} f(\eta(u,t)) - f(u) \geq -2\varepsilon.$$

Let $(s_n)_n \subset \mathbf{R}$ be a minimizing sequence for g such that $s_n \to +\infty$ and $|s_n - s_{n-1}| \to 0$. Applying Ekeland's variational principle to g yields a sequence t_n such that

$$(1.1) \qquad |t_n - s_n| \to 0, \qquad g(t_n) \to \inf_{[0,+\infty[} g, \qquad |g'(t_n)| \to 0.$$

Set $u_n = \eta(u, t_n)$. Then (1.1) says (also using the fact that $\eta(u,\cdot)$ is 1-Lipschitz continuous),

$$||u_n - u_{n-1}|| = ||\eta(u, t_n) - \eta(u, t_{n-1})|| \le |t_n - t_{n-1}| \to 0,$$

which proves (iii). Next, since $f(\eta(u,t)) \in [c-\varepsilon, c+\varepsilon]$ for all t, it is clear that also (i) holds. Finally because of (i) we see that $\chi(f(\eta(u,t))) = 1$ for all t, so that by (1.1),

$$o(1) = g'(t_n) = -\frac{||\nabla f(\eta(u,t_n))||^2}{1 + ||\nabla f(\eta(u,t_n))||},$$

that is, $||\nabla f(u_n)|| \to 0$, and the proof is complete. ■

Remark 1.3. – With the terminology of [7] we can say that under the assumptions of Theorem 1.2, there exists a $\overline{PS}$ sequence for f at some level between $c-\varepsilon$ and $c+\varepsilon$. Actually, with some slight changes one could obtain a better estimate on the level, namely $\lim f(u_n) \in [c, c+\varepsilon]$. Since we do not need this, we give no details. However it is also quickly seen that in general the estimate from above can be no better than this. Indeed if one takes $f(x) = \cos x + \cos \pi x$, and for Γ the class of one-point sets, one sees that $\inf_{\{x\}\in\Gamma} \sup_{x\in\{x\}} f(x) = \inf_{\mathbf{R}} f = -2$ but this value is never attained. One therefore sees that for each $\varepsilon > 0$ the flow lines converge to one of the local minima of f at some level strictly between -2 and $-2+\varepsilon$.

Theorem 1.2 above will be used only at the end of the proof. From now on we just keep in mind that whenever we speak of a Palais-Smale sequence u_n, it can be assumed without loss of generality that u_n is in reality a $\overline{PS}$ sequence, that is, it satisfies $||u_n - u_{n-1}|| \to 0$ as $n \to \infty$. We wish to point out that this property is *not* inherited by subsequences. A considerable amount of work in this paper is devoted to establish results which hold for an entire PS sequence, so that one can make use of the full force of the property $||u_n - u_{n-1}|| \to 0$.

Remark 1.4. – We wish to make clear the difference with $\mathbf{Z}$-invariant functionals, such as those associated to the periodic in time potentials. Indeed we cannot say that the $\overline{PS}$ condition is satisfied, and therefore we cannot prove a real deformation Lemma as in the papers [8], [14]. We only prove that the existence of a $\overline{PS}$ sequence implies the existence of a solution to our problem. The underlying reason is that when the potential is periodic in time one uses assumptions like finiteness or discreteness (modulo translations) of solutions to problems at infinity to prove the deformation Lemma. In that case the problems at infinity coincide with the original problem and these assumptions are reasonable. In our case, on the contrary, the structure of critical points at infinity is much more complicated, and this type of assumptions would not only be an arbitrary imposition, but they could even be not satisfied *a priori* in some situations.

2. SOME BASIC PROPERTIES

In this section we will state some of the properties that we will use in proving the main result. From now on, in the statement of propositions, we tacitly assume that (G1)-(G3) hold. It is clear that many results hold without the totality of these assumptions, but it seems to us that there is no need to specify each time the minimal conditions that could be used.

For future reference note that (G2) implies that $G(x) = o(|x|^2)$ and $\nabla G(x) = o(|x|)$ as $x \to 0$; these facts will be used repeatedly.

DEFINITION 2.1. – If α satisfies (G3), then it is bounded above by some constant $\overline{\alpha}$, see [9]. In the sequel we will denote by A_α the set

$$A_\alpha = \{\beta \in \mathcal{C}(\mathbf{R}, \mathbf{R}) \;/\; \underline{\alpha} \leq \beta(t) \leq \overline{\alpha},\ \forall t \in \mathbf{R}\}.$$

Let $f : H \to \mathbf{R}$ be the functional defined in the introduction. As a first result concerning f we have

PROPOSITION 2.2. – *∇f is weakly continuous, in the sense that*

$$if\ u_n \rightharpoonup u\ weakly\ in\ H \quad then \quad \nabla f(u_n) \rightharpoonup \nabla f(u)\ weakly\ in\ H.$$

Proof. – Let $J(u) = \int_{\mathbf{R}} \alpha(t) G(u(t)) dt$. Since $f(u) = \frac{1}{2}||u||^2 - J(u)$, and the quadratic part has the desired property, we only have to check that the same holds for J. To this aim, pick $\varphi \in H$, fix $\varepsilon > 0$ and let $R_\varepsilon > 0$ be so large that $\int_{|t|>R_\varepsilon} |\varphi|^2 dt < \varepsilon^2$. Then we have

$$\begin{aligned}|(\nabla J(u_n) - \nabla J(u)) \cdot \varphi| &= \left|\int_{\mathbf{R}} \alpha[\nabla G(u_n) - \nabla G(u)] \cdot \varphi dt\right| \\ &\leq \overline{\alpha} \int_{-R_\varepsilon}^{R_\varepsilon} |\nabla G(u_n) - \nabla G(u)||\varphi| dt \\ &\quad + \varepsilon\overline{\alpha}\left(\int_{|t|>R_\varepsilon} |\nabla G(u_n) - \nabla G(u)|^2 dt\right)^{\frac{1}{2}}\end{aligned}$$

Now the first integral tends to zero as $n \to +\infty$ because $u_n \to u$ strongly in L^∞_{loc}; the second integral is bounded independently of n (since u_n is bounded in H). The fact that ε is arbitrary concludes the proof. ■

We now list some of the geometric features of the functional f.

PROPOSITION 2.3. – (i) $u \equiv 0$ *is a strict local minimum for* f; (ii) *there exist* $\rho > 0$ *and* $\sigma > 0$ *such that* $||u|| = \rho$ *implies* $f(u) \geq \sigma$; *there exists* $v \in H$ *such that* $f(v) < 0$.

The proof is straightforward and we omit it. We note that Proposition 2.3 states that the functional f verifies the geometric assumptions of the Mountain Pass Lemma.

The next property shows that some estimates hold uniformly in A_α. In order to make this point clear, and also because we will soon need it, we introduce some notation.

DEFINITION 2.4. – For every $\beta \in A_\alpha$ we define a functional $f(\beta, \cdot) \in C^2(H, \mathbf{R})$ by setting

$$\begin{aligned}f(\beta, u) &= \frac{1}{2}\int_{\mathbf{R}} [|\dot{u}(t)|^2 + |u(t)|^2] dt - \int_{\mathbf{R}} \beta(t) G(u(t)) dt \\ &= \frac{1}{2}||u||^2 - \int_{\mathbf{R}} \beta(t) G(u(t)) dt.\end{aligned}$$

The expression $\nabla f(\beta, u)$ stands for $\nabla_u f(\beta, u)$ and it is understood that $f(\alpha, u)$ will be denoted simply by $f(u)$.

PROPOSITION 2.5. – *For* $\beta \in A_\alpha$, *let* $\mathcal{K}_\beta = \{u \in H / u \not\equiv 0,\ \nabla f(\beta, u) = 0\}$ *be the set of nonzero critical points of* $f(\beta, \cdot)$. *Then we have:*

(i) $$\inf_{\beta \in A_\alpha} \inf_{u \in \mathcal{K}_\beta} ||u|| > 0$$

(ii) $$\inf_{\beta \in A_\alpha} \inf_{u \in \mathcal{K}_\beta} f(\beta, u) > 0$$

Proof. – Let $\delta > 0$ be such that $1 - \overline{\alpha}\delta > 0$ and let $\sigma > 0$ be so small that $|x| \leq \sigma$ implies $|\nabla G(x)| \leq \delta |x|$ (this is possible by (G2)). Now let $u \in H$ verify $||u|| \leq \sigma$. Since $||u||_\infty \leq ||u||$, we have that $|\nabla G(u(t)) \cdot u(t)| \leq \delta |u(t)|^2$ for all t. Now for any $\beta \in A_\alpha$ there results

$$\begin{aligned}\nabla f(\beta, u) \cdot u = ||u||^2 - \int_{\mathbf{R}} \beta \nabla G(u) \cdot u dt \geq ||u||^2 - \delta \int_{\mathbf{R}} \beta |u|^2 dt \\ \geq ||u||^2 - \overline{\alpha}\delta ||u||_2^2 \geq (1 - \overline{\alpha}\delta)||u||^2 > 0.\end{aligned}$$

This shows that if $||u||$ is small, then u cannot be a critical point of any $f(\beta, \cdot)$.

To prove (ii), just note that if u is critical for some $f(\beta, \cdot)$, then

$$f(\beta, u) = f(\beta, u) - \frac{1}{\theta} \nabla f(\beta, u) \cdot u \geq \left(\frac{1}{2} - \frac{1}{\theta}\right) ||u||^2,$$

so that invoking (i) the proof is complete. ■

We now begin the study of the Palais-Smale sequences for the functionals of the form $f(\beta, \cdot)$. To this aim we will first prove some technical lemmas which show that the behavior of these functionals with respect to some limit operations is somewhat uniform on A_α. The type of uniformity is the same as that of the previous proposition.

LEMMA 2.6. – *Let $u_0 \in H$ and let $(v_n)_n \subset H$ be a sequence such that $v_n \rightharpoonup 0$ weakly in H. Then*

$$\text{(i)} \sup_{\beta \in A_\alpha} \left| \int_{\mathbf{R}} \beta [G(v_n + u_0) - G(v_n) - G(u_0)] dt \right| \to 0 \quad \text{as} \quad n \to \infty;$$

$$\text{(ii)} \sup_{\beta \in A_\alpha} \sup_{||\varphi||=1} \left| \int_{\mathbf{R}} \beta [\nabla G(v_n + u_0) - \nabla G(v_n) - \nabla G(u_0)] \cdot \varphi dt \right| \to 0 \quad \text{as} \quad n \to \infty;$$

Proof. – Fix $\varepsilon > 0$ and let $R > 0$ be free for the moment. Split the integral in (i) as $\int_{-R}^{R} + \int_{|t|>R}$: note that for any fixed R the first integral tends to zero as $n \to \infty$, because of the strong L^∞_{loc} convergence of v_n to zero. Therefore it is enough to show that, given $\varepsilon > 0$, we can find $R = R_\varepsilon$ such that the second integral is, say, less than ε for all n.

By the Mean Value Theorem we have, for some convenient numbers $\xi_n^t \in]0,1[$,

$$\left|\int_{|t|>R} \beta[G(v_n+u_0) - G(v_n) - G(u_0)]dt\right|$$
$$\leq \overline{\alpha}\int_{|t|>R} |\nabla G(v_n+\xi_n^t u_0)\cdot u_0|dt + \overline{\alpha}\int_{|t|>R}|G(u_0)|dt$$

for all $\beta \in A_\alpha$.

Now, by (G2), for all $M > 0$, there exists $K_M > 0$ depending only on M, such that $|x| \leq M$ implies both $|G(x)| \leq K_M|x|^2$ and $|\nabla G(x)| \leq K_M|x|$. Choose M so large that (by boundedness) $|u_0(t)| \leq M$ and $|v_n(t) + \xi_n^t u_0(t)| \leq M$ for all $t \in R$ and all n. Then

$$\overline{\alpha}\int_{|t|>R} |\nabla G(v_n+\xi_n^t u_0)\cdot u_0|dt + \overline{\alpha}\int_{|t|>R}|G(u_0)|dt$$
$$\leq \overline{\alpha}K_M\int_{|t|>R}|v_n(t)+\xi_n^t u_0(t)||u_0|dt + \overline{\alpha}K_M\int_{|t|>R}|u_0|^2dt$$
$$\leq \overline{\alpha}K_M\left(\sup_n ||v_n||_2\left(\int_{|t|>R}|u_0|^2dt\right)^{\frac{1}{2}} + \int_{|t|>R}|u_0|^2dt\right)$$

This shows that given $\varepsilon > 0$ we only have to take $R = R_\varepsilon$ so large that the last quantity is less than ε, which is possible since the function $R \mapsto \int_{|t|>R}|u_0|^2dt$ tends to zero as $R \to +\infty$; part (i) is proved.

The proof of part (ii) is analogous: after splitting the corresponding integral as above, and keeping in mind that $v_n \to 0$ in L^∞_{loc}, the only relevant part is to show that for every $\varepsilon > 0$, there exists R_ε such that

$$\int_{|t|>R_\varepsilon} |\beta(\nabla G(v_n+u_0) - \nabla G(v_n) - \nabla G(u_0))\cdot\varphi|dt \leq \varepsilon$$

for all $||\varphi|| = 1$ and all $\beta \in A_\alpha$. But this is readily accomplished, using as above the fact that $|\nabla G(x)| \leq K_M|x|$ and, this time, also the fact that ∇G is lipschitz continuous (with lipschitz constant L_M) on the compact set $\{|x| \leq M\}$. Then plainly, by Hölder inequality,

$$\int_{|t|>R_\varepsilon} |\beta(\nabla G(v_n+u_0) - \nabla G(v_n) - \nabla G(u_0))\cdot\varphi|dt$$
$$\leq \overline{\alpha}\int_{|t|>R}|\nabla G(v_n+u_0) - \nabla G(v_n)||\varphi|dt + \int_{|t|>R}|\nabla G(u_0)||\varphi|dt$$
$$\leq \overline{\alpha}(L_M+K_M)\left(\int_{|t|>R}|u_0|^2dt\right)^{\frac{1}{2}}$$

and the conclusion follows as above. ■

Lemma 2.6 will be immediately used to prove the following basic result.

PROPOSITION 2.7. – *Let $(v_n)_n \subset H$ be a sequence such that $v_n \rightharpoonup v_0$ weakly in H. Then*

(i) $\sup_{\beta \in A_\alpha} |f(\beta, v_n - v_0) - f(\beta, v_n) + f(\beta, v_0)| \to 0 \quad$ as $\quad n \to \infty$;

(ii) $\sup_{\beta \in A_\alpha} ||\nabla f(\beta, v_n - v_0) - \nabla f(\beta, v_n) + \nabla f(\beta, v_0)||_H \to 0$ as $\quad n \to \infty$;

Proof. – Let us begin by (i). We have, as $n \to \infty$,

$$f(\beta, v_n - v_0) - f(\beta, v_n) + f(\beta, v_0) = o(1) + \int_{\mathbf{R}} \beta[G(v_n - v_0) - G(v_n) + G(v_0)]dt.$$

Setting $v_n - v_0 = z_n$, we see that $z_n \rightharpoonup 0$ weakly in H and therefore

$$\sup_{\beta \in A_\alpha} |f(\beta, v_n - v_0) - f(\beta, v_n) + f(\beta, v_0)| \leq o(1) + \sup_{\beta \in A_\alpha} \int_{\mathbf{R}} |\beta[G(z_n + v_0) - G(z_n) - G(v_0)]|dt,$$

and this quantity tends to zero as $n \to \infty$, by Lemma 2.6, part (i).

We now verify that (ii) holds. To this aim, note that for every $\varphi \in H$,

$$|\nabla f(\beta, v_n - v_0) \cdot \varphi - \nabla f(\beta, v_n) \cdot \varphi + \nabla f(\beta, v_0) \cdot \varphi| = \left| \int_{\mathbf{R}} \beta[\nabla G(v_n - v_0) - \nabla G(v_n) + \nabla G(v_0)] \cdot \varphi dt \right|$$

If, as above, we set $v_n - v_0 = z_n$, we see that

$$\sup_{\beta \in A_\alpha} ||\nabla f(\beta, v_n - v_0) - \nabla f(\beta, v_n) + \nabla f(\beta, v_0)||_H \leq \sup_{\beta \in A_\alpha} \sup_{||\varphi||=1} \left| \int_{\mathbf{R}} \beta[\nabla G(z_n + v_0) - \nabla G(z_n) - \nabla G(v_0)] \cdot \varphi dt \right|,$$

which tends to zero by Lemma 2.6, part (ii). ■

We are now almost ready to describe the Palais-Smale sequences of the functional f. The next result is the first step towards a complete characterization. Analogous results in this direction can be found in almost every paper on homoclinic solutions, *see* e.g. [5]-[8], etc.

PROPOSITION 2.8. – *Let* $(u_n)_n \subset H$ *be a Palais-Smale sequence for* f *at level* $c \in \mathbf{R}$*, that is,*

$$f(u_n) \to c \quad \text{and} \quad \nabla f(u_n) \to 0 \quad \text{in} \quad H, \quad \text{as} \quad n \to \infty.$$

Then there exist a subsequence (still denoted u_n*) and* $u_0 \in H$ *such that*

(i) $u_n \rightharpoonup u_0$ *weakly in* H;

(ii) $\nabla f(u_0) = 0$;

(iii) $u_n - u_0$ *is a Palais – Smale sequence for* f *at level* $c - f(u_0)$.

Proof. – The proof is a standard technique in the variational approach to homoclinic solutions. Being very short, we report it for completeness.

Since

$$c + o(1) + ||u_n|| o(1) = f(u_n) - \frac{1}{\theta}\nabla f(u_n) \cdot u_n \geq (\frac{1}{2} - \frac{1}{\theta})||u_n||^2,$$

we see that u_n is bounded in H. Then there exist a subsequence, still denoted u_n, such that $u_n \rightharpoonup u_0$ weakly in H (and therefore strongly in L^∞_{loc}), for some $u_0 \in H$.

Now since $u_n \rightharpoonup u_0$ weakly in H and $\nabla f(u_n) \to 0$ in H, by Proposition 2.2, for all $\varphi \in H$,

$$0 = \lim_{n \to \infty} \nabla f(u_n) \cdot \varphi = \nabla f(u_0) \cdot \varphi,$$

which proves (i). To prove (ii) it is enough to invoke Proposition 2.7, with $\beta = \alpha$:

$$f(u_n - u_0) = f(u_n) - f(u_0) + o(1) = c - f(u_0) + o(1).$$

Finally, since $\nabla f(u_n) = o(1)$ in H and $\nabla f(u_0) = 0$, we have

$$\nabla f(u_n - u_0) = \nabla f(u_n - u_0) - \nabla f(u_n) + \nabla f(u_0) + o(1) = o(1)$$

by Proposition 2.7. The proof is complete. ■

Remark 2.9. – The full generality of Proposition 2.7 (uniformity over A_α) has not yet been taken into account, but it soon will.

Remark 2.10. – The way Proposition 2.8 will be used in the sequel is the following. The functional f satisfies the geometric assumptions of the Mountain Pass Lemma (Proposition 2.3); by Theorem 1.2 one can find a Palais-Smale (actually $\overline{PS}$) sequence u_n for f at some level $c > 0$.

By Proposition 2.8 there is a subsequence u_n converging weakly to some $u_0 \in H$, which is a critical point for f. If $u_0 \not\equiv 0$, then we have found a solution to our problem, and there is nothing left to say. Therefore in what follows we shall always assume, without loss of generality, that if u_n is a PS sequence at some level $c > 0$, then $u_n \rightharpoonup 0$ weakly in H.

We now turn to a nonvanishing property of PS sequences.

PROPOSITION 2.11. – *Let u_n be a PS sequence for f at some level $c > 0$. Then there exists $a > 0$ such that* $\liminf_{n\to\infty} ||u_n||_\infty \geq a$.

Proof. – Assume for contradiction that $\liminf_{n\to\infty} ||u_n||_\infty = 0$. Then for some subsequence u_{n_k} we have $\lim_{k\to\infty} ||u_{n_k}||_\infty = 0$. Choose a number $\varepsilon > 0$ such that $1 - \overline{\alpha}\varepsilon > 0$, and note that by (G2), there exists $\sigma > 0$ such that $|x| \leq \sigma$ implies $|\nabla G(x)| \leq \varepsilon|x|$. Now for k large enough we have $||u_{n_k}||_\infty \leq \sigma$, so that

$$o(1) = \nabla f(u_{n_k})\cdot u_{n_k} \geq \int_{\mathbf{R}} [|\dot{u}_{n_k}|^2 + |u_{n_k}|^2 - \overline{\alpha}\varepsilon|u_{n_k}|^2]dt \geq (1-\overline{\alpha}\varepsilon)||u_{n_k}||^2.$$

But then $u_{n_k} \to 0$ strongly in H, and therefore $c = \lim_{k\to\infty} f(u_{n_k}) = 0$, which is false. ■

The previous result will be used through the following proposition.

COROLLARY 2.12. – *Let u_n ($u_n \rightharpoonup 0$) be a PS sequence for f at some level $c > 0$. Then there exists a sequence $(\tau_n)_n \subset \mathbf{R}$ such that:*

(i) *no subsequence of $u_n(\cdot + \tau_n)$ tends to zero weakly in H;*

(ii) *no subsequence of τ_n is bounded, that is, $|\tau_n| \to +\infty$.*

Proof. – For each n let τ_n be a point such that $|u_n(\tau_n)| = ||u_n||_\infty$ and set $v_n(t) = u_n(t + \tau_n)$. Then we must show that no subsequence of v_n tends to zero weakly in H. Indeed if some $v_{n_k} \rightharpoonup 0$ in H, then $v_{n_k} \to 0$ in L^∞_{loc}, so that in particular $v_{n_k}(0) \to 0$. But then $||u_{n_k}||_\infty = |v_{n_k}(0)| \to 0$, against Proposition 2.11.

Next, to prove that $|\tau_n| \to +\infty$, recall first that we are dealing with the case $u_n \rightharpoonup 0$ weakly in H, as it is pointed out in Remark 2.10. Then assume for contradiction that there is some subsequence τ_{n_k} which is bounded. In this case, by uniform convergence of u_{n_k} to zero on compact sets we have $||u_{n_k}||_\infty = |u_{n_k}(\tau_{n_k})| \to 0$, which again is false. ■

The next result is fundamental for the study of the Palais-Smale sequences for functionals of the form $f(\beta, \cdot)$. It shows that functionals $f(\beta, \cdot)$ with different β's are uniformly close in a C^1 sense whenever the functions β's are close in L^∞.

LEMMA 2.13. – *Let $\beta_1, \beta_2 \in L^\infty(\mathbf{R}; \mathbf{R})$ and let B be a bounded subset of H. Then there exists a constant S, depending only on B, such that for all $u \in B$,*

(i) $$|f(\beta_1, u) - f(\beta_2, u)| \leq S||\beta_1 - \beta_2||_\infty,$$

(ii) $$||\nabla f(\beta_1, u) - \nabla f(\beta_2, u)||_H \leq S||\beta_1 - \beta_2||_\infty.$$

Proof. – Let us start with (i):

$$|f(\beta_1, u) - f(\beta_2, u)| = \left| \int_{\mathbf{R}} (\beta_2 - \beta_1) G(u) dt \right| \leq ||\beta_1 - \beta_2||_\infty \sup_{u \in B} \int_{\mathbf{R}} G(u) dt.$$

Now since B is bounded there exists a constant $K = K(B)$ such that for all $u \in B, ||u||_\infty \leq ||u|| \leq K$. By the assumption on G there exists $M = M(K)$ such that $|x| \leq K$ implies $|G(x)| \leq M|x|^2$. Therefore $\sup_{u \in B} \int_{\mathbf{R}} G(u) dt \leq \sup_{u \in B} M \int_{\mathbf{R}} |u|^2 dt \leq KM$; setting $S = KM$ we obtain

$$\forall u \in B, \qquad |f(\beta_1, u) - f(\beta_2, u)| \leq S||\beta_1 - \beta_2||_\infty.$$

For the second part, with a similar calculation we have

$$||\nabla f(\beta_1, u) - \nabla f(\beta_2, u)||_H \leq ||\beta_1 - \beta_2||_\infty \sup_{u \in B} \left(\int_{\mathbf{R}} |\nabla G(u)|^2 dt \right)^{\frac{1}{2}},$$

and the conclusion follows as above, by the assumption on G. ■

Remark 2.14. – The situation that we will soon meet is the following. Suppose we have a bounded sequence $(u_n)_n \subset H$ and a sequence $(\beta_n)_n \subset L^\infty$ such that $\beta_n \to \beta$ in L^∞. Then Lemma 2.13 allows us to say that as $n \to \infty$,

$$|f(\beta_n, u_n) - f(\beta, u_n)| \to 0, \quad \text{and} \quad ||\nabla f(\beta_n, u_n) - \nabla f(\beta, u_n)||_H \to 0.$$

Before we proceed any further we need to introduce some notation. For every $\tau \in \mathbf{R}$ we define an isometry $T_\tau : L^\infty \to L^\infty$ (and also $T_\tau : H \to H$) by setting

$$(T_\tau u)(t) = u(t + \tau).$$

With some trivial changes of variable, it is immediate to see that

$$f(\beta, T_\tau u) = f(T_{-\tau}\beta, u),$$

so that in particular $f(T_\tau\beta, T_\tau u) = f(\beta, u)$, and that

$$\nabla f(\beta, T_\tau u) \cdot \varphi = \nabla f(T_{-\tau}\beta, u) \cdot T_{-\tau}\varphi,$$

which also yields $\nabla f(T_\tau\beta, T_\tau u) \cdot T_\tau\varphi = \nabla f(\beta, u) \cdot \varphi$. In the sequel, with abuse of notation, we will denote by $\nabla f(\beta, u) \circ T_\tau$ the unique element in H such that $\nabla f(\beta, u) \circ T_\tau \cdot \varphi = \nabla f(\beta, u) \cdot T_\tau\varphi$, for all $\varphi \in H$.

The next lemma is the final step towards the description of the Palais-Smale sequences of f. We recall that we have proved so far that if u_n is a PS sequence for f at a level $c > 0$, then we can assume that $u_n \rightharpoonup 0$ weakly in H, and that there exists a sequence $(\tau_n)_n \subset \mathbf{R}$ with the properties that $|\tau_n| \to \infty$ and that no subsequence of $T_{\tau_n} u_n$ tends to zero weakly in H.

LEMMA 2.15. – *Let u_n ($u_n \rightharpoonup 0$) be a PS sequence for f at a level $c > 0$. Then there exist a function $\beta_1 \in A_\alpha$, a function $v_1 \in H$, $v_1 \not\equiv 0$, a sequence τ_n of real numbers such that for a subsequence of $T_{\tau_n} u_n$, still denoted $T_{\tau_n} u_n$, the following properties are satisfied:*

(i) $T_{\tau_n} u_n \rightharpoonup v_1$;

(ii) $\nabla f(\beta_1, v_1) = 0$;

(iii) $|\tau_n| \to \infty$;

(iv) $(u_n - T_{-\tau_n} v_1)_n$ *is a PS sequence for f at level $c - f(\beta_1, v_1)$.*

Proof. – Let $(\tau_n)_n \subset \mathbf{R}$ be a sequence given by Corollary 2.12. Then (iii) is trivially satisfied and we must show that the remaining properties also hold true.

The sequence $T_{\tau_n} u_n$ is bounded in H, and therefore it contains some subsequence, still denoted $T_{\tau_n} u_n$ such that $T_{\tau_n} u_n \rightharpoonup v_1$ weakly, for some $v_1 \in H$; note that by Corollary 2.12, $v_1 \not\equiv 0$. Therefore (i) is satisfied. Consider the (sub) sequence $T_{\tau_n}\alpha$: by Bochner's criterion, there exist still another subsequence, again denoted $T_{\tau_n}\alpha$ and a function $\beta_1 \in A_\alpha$ such that

$$T_{\tau_n}\alpha \to \beta_1 \quad \text{uniformly in} \quad \mathbf{R}.$$

Summing up, we see that by passing to convenient subsequences, we can make sure that we have both

$$T_{\tau_n} u_n \rightharpoonup v_1 \quad \text{weakly in} \quad H \quad \text{and} \quad T_{\tau_n}\alpha \to \beta_1 \quad \text{uniformly in} \quad \mathbf{R}.$$

Setting $T_{\tau_n} u_n = v_n$ we have that $v_n \rightharpoonup v_1$ in H; let us prove that (ii) and (iv) hold.

For all $\varphi \in H$, by Proposition 2.2 we can compute

$$\nabla f(\beta_1, v_1) \cdot \varphi = \lim_n \nabla f(\beta_1, v_n) \cdot \varphi =$$
$$\lim_n [\nabla f(\beta_1, v_n) \cdot \varphi - \nabla f(T_{\tau_n} \beta_1, v_n) \cdot \varphi] + \lim_n \nabla f(T_{\tau_n} \beta_1, v_n) \cdot \varphi = 0,$$

because the first term vanishes by Lemma 2.13, while the second is zero because

$$\nabla f(T_{\tau_n} \beta_1, v_n) \cdot \varphi = \nabla f(u_n) \cdot T_{\tau_n} \varphi = o(1),$$

since u_n is a PS sequence for f and $T_{\tau_n} \varphi$ is bounded. (ii) is proved.

Let us turn to (iv): we have

$$\begin{aligned} & f(u_n - T_{-\tau_n} v_1) - f(u_n) + f(\beta_1, v_1) \\ & \quad = [f(T_{\tau_n} \alpha, v_n - v_1) - f(T_{\tau_n} \alpha, v_n) + f(T_{\tau_n} \alpha, v_1)] \\ & \quad\quad - [f(T_{\tau_n} \alpha, v_1) - f(\beta_1, v_1)] = o(1) \end{aligned}$$

as one immediately sees by using Proposition 2.7 and Lemma 2.13. This shows that $f(u_n - T_{-\tau_n} v_1)$ tends to $c - f(\beta_1, v_1)$.

Finally, note that

$$\begin{aligned} & \nabla f(u_n - T_{-\tau_n} v_1) + o(1) \\ & \quad = \nabla f(u_n - T_{-\tau_n} v_1) - \nabla f(u_n) + \nabla f(T_{\tau_n} \alpha, v_1) \circ T_{\tau_n} \\ & \quad = \nabla f(T_{\tau_n} \alpha, v_n - v_1) \circ T_{\tau_n} - \nabla f(T_{\tau_n} \alpha, v_n) \circ T_{\tau_n} \\ & \quad\quad + \nabla f(T_{\tau_n} \alpha, v_1) \circ T_{\tau_n} = o(1) \end{aligned}$$

by Proposition 2.7. The proof is complete. ∎

We can now prove the main result of this section. This is the result that appears in almost every paper on homoclinic solutions, *see* e.g. [8]. In our case, though, it takes a slightly different form.

PROPOSITION 2.16 (Representation lemma). – *Let u_n ($u_n \rightharpoonup 0$) be a PS sequence for f at a level $c > 0$. Then there exist a number $q \in \mathbf{N}$, depending only on c, q functions $\beta_i \in A_\alpha$, q functions $v_i \in H$, $v_i \not\equiv 0$, a subsequence still denoted u_n and q sequences θ_n^i of real numbers such that*

(i) $$||u_n - \sum_{i=1}^{q} T_{\theta_n^i} v_i|| \to 0 \quad \text{as} \quad n \to \infty;$$

(ii) $$\nabla f(\beta_i, v_i) = 0, \qquad \forall i = 1, \ldots q;$$

(iii) $$c = \sum_{i=1}^{q} f(\beta_i, v_i);$$

(iv) $$|\theta_n^j - \theta_n^k| \to \infty, \qquad \forall j \neq k, \qquad \text{as} \qquad n \to \infty.$$

Proof. – Applying Lemma 2.15 we find a subsequence of u_n, a function $\beta_1 \in A_\alpha$, a function $v_1 \in H$, $v_1 \not\equiv 0$, a sequence τ_n such that (setting $\theta_n^1 = -\tau_n$), we have

$$\begin{aligned} &\nabla f(\beta_1, v_1) = 0 \\ &|\theta_n^1| \to \infty \quad \text{as} \quad n \to \infty, \\ &f(u_n - T_{\theta_n^1} v_1) \to c - f(\beta_1, v_1) \\ &\nabla f(u_n - T_{\theta_n^1} v_1) \to 0 \quad \text{in} \quad H. \end{aligned}$$

Therefore $u_n - T_{\theta_n^1} v_1$ is a PS sequence for f at level $c - f(\beta_1, v_1)$; this implies that $f(\beta_1, v_1) \leq c$. Two cases may present.

Case I: $f(\beta_1, v_1) = c$. But then $f(u_n - T_{\theta_n^1} v_1) \to 0$, which implies that $||u_n - T_{\theta_n^1} v_1|| \to 0$, so that the proposition is proved with $q = 1$.

Case II: $f(\beta_1, v_1) = c_1 < c$. In this case the sequence $u_n^1 := u_n - T_{\theta_n^1} v_1$ is a PS sequence for f at level $c - c_1 > 0$. Applying Lemma 2.16 we find a subsequence of u_n^1, a function $\beta_2 \in A_\alpha$, a function $v_2 \in H$, $v_2 \not\equiv 0$, a sequence τ_n^2 such that (setting $\theta_n^2 = -\tau_n^2$), we have

$$\begin{aligned} &\nabla f(\beta_2, v_2) = 0 \\ &|\theta_n^2| \to \infty \quad \text{as} \quad n \to \infty, \\ &f(u_n^1 - T_{\theta_n^2} v_2) \to c - c_1 - f(\beta_2, v_2) \\ &\nabla f(u_n^1 - T_{\theta_n^2} v_2) \to 0 \quad \text{in} \quad H. \end{aligned}$$

Proceeding as above, if $f(\beta_2, v_2) = c - c_1$, then the proposition is proved with $q = 2$.

Otherwise we iterate the application of Lemma 2.15, starting with the PS sequence $u_n^2 := u_n^1 - T_{\theta_n^2} v_2$. To prove that this procedure ends, it is enough to show that for some $q \in \mathbf{N}$, $f(\beta_q, v_q) = c - c_1 - \ldots - c_{q-1}$. But this follows plainly from Proposition 2.5: for all $i = 1, 2, \ldots$ we have

$$c_i = f(\beta_i, v_i) \geq \inf_{\beta \in A_\alpha} \inf_{u \in \mathcal{K}_\beta} f(\beta, u) =: b > 0$$

so that after at most $q := \left[\frac{c}{b}\right]$ steps we obtain $f(\beta_q, v_q) = 0$.

Finally, to see that for all $j \neq k$ we have $|\theta_n^j - \theta_n^k| \to \infty$, we can work exactly as in [8], and therefore we omit the details. ■

3. THE KEY ARGUMENT

We now come to description of the fundamental argument which will allow us to find a solution to problem (P). As in the previous section we will proceed by a series of simple steps.

We begin by fixing some notation.

DEFINITION 3.1. – Recalling that $A_\alpha = \{\beta \in \mathcal{C}(\mathbf{R},\mathbf{R}) \ / \ \underline{\alpha} \leq \beta(t) \leq \overline{\alpha}, \forall t \in \mathbf{R}\}$, we define

$$\mathcal{K}_\infty = \{v \in H \ / \ v \not\equiv 0, \ \exists \beta \in A_\alpha, \nabla f(\beta, v) = 0\},$$

and

$$Q_\infty = \{\varphi \in H^1(\mathbf{R},\mathbf{R}) \ / \ \varphi(t) = \sum_{\text{finite}} |T_{\theta_i} v_i(t)|^2, \ v_i \in \mathcal{K}_\infty, \ \theta_i \in \mathbf{R}, \ \forall i\}$$

A few comments are in order. The representation lemma says that the Palais-Smale sequences of f are sums of solutions to problems $\nabla f(\beta, \cdot) = 0$ (where β is a uniform limit of translates of α), up to negligible quantities in H. These β's clearly belong to A_α. Actually the set A_α is larger than the set of uniform limits of translates of α, but this fact will not bother the rest of the argument.

Also note that the functions in Q_∞ are not elements of H, but are sums of squares of elements of H. Since $H^1(\mathbf{R};\mathbf{R})$ is a Banach algebra, Q_∞ is contained in $H^1(\mathbf{R};\mathbf{R})$. The set Q_∞ is also larger than necessary, but its use simplifies some proofs.

Finally we alert the reader that in the sequel we will denote by $||\cdot||$ both the norm in H and the norm in $H^1(\mathbf{R};\mathbf{R})$. The context will rule out any possible ambiguity.

The first result concerns a qualitative property of elements of Q_∞.

PROPOSITION 3.2. – *There exists $\delta > 0$ such that for all $\varphi \in Q_\infty$,*

$$0 < \varphi(t) < 2\delta \qquad \textit{implies} \qquad \varphi''(t) > 0.$$

In particular, we see that below a uniform quantity, no function in Q_∞ can have a local maximum.

Proof. – First note that since the elements of Q_∞ are built using solutions of equations of the form $\nabla f(\beta, v) = 0$, they are $\mathcal{C}^2$ functions, so that their second derivative is well defined. An easy computation shows that if

$$\varphi(t) = \sum_{i=1}^{p} |T_{\theta_i} v_i(t)|^2,$$

then

$$\varphi''(t) = 2\sum_{i=1}^{p} |T_{\theta_i}\dot{v}_i(t)|^2 + 2\sum_{i=1}^{p} T_{\theta_i}v_i(t)\cdot T_{\theta_i}\ddot{v}_i(t) \geq 2\sum_{i=1}^{p} T_{\theta_i}v_i(t)\cdot T_{\theta_i}\ddot{v}_i(t).$$

Since $\nabla f(\beta_i, v_i) = 0$, $\forall i = 1, \ldots, p$, for some $\beta_i \in A_\alpha$, we have (replacing $\ddot{v}_i$ in the last expression)

$$\varphi''(t) \geq 2\sum_{i=1}^{p} |T_{\theta_i}v_i(t)|^2 - 2\overline{\alpha}\sum_{i=1}^{p} \nabla G(T_{\theta_i}v_i(t)) \cdot T_{\theta_i}v_i(t).$$

Now let $\varepsilon > 0$ be so small that $1 - \overline{\alpha}\varepsilon > 0$ and take $\delta > 0$ such that $|x| \leq \sqrt{2\delta}$ implies $|\nabla G(x)| \leq \varepsilon|x|$; this is possible by (G2).

Let $t \in \mathbf{R}$ be a point where $0 < \varphi(t) \leq 2\delta$; then we also have, for each $i = 1, \ldots, p$, that $|T_{\theta_i}v_i(t)| \leq \sqrt{2\delta}$. But then $|\nabla G(T_{\theta_i}v_i(t))| \leq \varepsilon|T_{\theta_i}v_i(t)|$, so that

$$\varphi''(t) \geq 2(1 - \varepsilon\overline{\alpha})\sum_{i=1}^{p} |T_{\theta_i}v_i(t)|^2 > 0,$$

and the proof is complete. ■

Remark 3.3. – Note that in Proposition 3.2 we actually proved that there exists $\delta > 0$ such that for all $\varphi \in Q_\infty$,

$$0 < \varphi(t) < 2\delta \qquad \text{implies} \qquad \varphi''(t) > 2(1 - \varepsilon\overline{\alpha})\varphi(t).$$

This slightly stronger statement will be used below.

The next proposition shows that if u_n is a Palais-Smale sequence for f, then not only its subsequences are close (in H) to sums of solutions to problems of the type $\nabla f(\beta, \cdot) = 0$, but also that their squares are close to Q_∞, always in a H^1 sense.

PROPOSITION 3.4. – *Let u_n be a sequence as in the representation lemma, that is, assume*

$$u_n - \sum_{i=1}^{q} T_{\theta_n^i} v_i \to 0 \quad \textit{strongly in} \;\; H,$$

for some $v_i \in A_\alpha$ and $\theta_n^i \in \mathbf{R}$. Then

$$|u_n|^2 - \sum_{i=1}^{q} |T_{\theta_n^i} v_i|^2 \to 0 \quad \textit{strongly in} \;\; H^1(\mathbf{R}, \mathbf{R}).$$

Proof. – The proof is divided in a series of steps.

Step 1. Let $u, v \in H$ and let θ_n^1, θ_n^2 be sequences of real numbers such that $|\theta_n^1 - \theta_n^2| \to \infty$ as $n \to \infty$. Then

$$T_{\theta_n^1} u \cdot T_{\theta_n^2} v \to 0 \quad \text{strongly in } H^1(\mathbf{R}; \mathbf{R}).$$

Indeed, let $\varepsilon > 0$ be fixed; then, setting $\Delta_n = \theta_n^2 - \theta_n^1$, we have

$$\int_{\mathbf{R}} |T_{\theta_n^1} u \cdot T_{\theta_n^2} v|^2 dt = \int_{|u(t)|<\varepsilon} |u \cdot T_{\Delta_n} v|^2 dt + \int_{|u(t)|\geq\varepsilon} |u \cdot T_{\Delta_n} v|^2 dt \leq \varepsilon^2 ||v||_{L^2(\mathbf{R},\mathbf{R})} + o(1),$$

since $v(\cdot + \Delta_n) \to 0$ in L^∞_{loc}. This shows that $T_{\theta_n^1} u \cdot T_{\theta_n^2} v \to 0$ in $L^2(\mathbf{R}, \mathbf{R})$.

Next, with the same change of variable, we have

$$\text{(3.0)} \quad \int_{\mathbf{R}} \left| \frac{d}{dt}(T_{\theta_n^1} u \cdot T_{\theta_n^2} v) \right|^2 dt \leq 2 \int_{\mathbf{R}} |\dot{u} \cdot T_{\Delta_n} v|^2 dt + 2 \int_{\mathbf{R}} |T_{-\Delta_n} u \cdot \dot{v}|^2 dt.$$

Taking $\varepsilon > 0$, we just have to choose a compact K_ε such that $\int_{\mathbf{R}\setminus K_\varepsilon} |\dot{u}|^2 dt \leq \varepsilon$ to see that

$$\int_{\mathbf{R}} |\dot{u} \cdot T_{\Delta_n} v|^2 dt \leq \int_{K_\varepsilon} |\dot{u}|^2 |T_{\Delta_n} v|^2 dt + \int_{\mathbf{R}\setminus K_\varepsilon} |\dot{u}|^2 |T_{\Delta_n} v|^2 dt \leq o(1) + \varepsilon ||v||_\infty.$$

For the second integral in (3.0) the estimate is the same, and this concludes the proof of Step 1.

Step 2. Let u_n, v_n be bounded sequences in H, and assume that $||u_n - v_n|| \to 0$ as $n \to \infty$. Then

$$|u_n|^2 - |v_n|^2 \to 0 \quad \text{strongly in } H^1(\mathbf{R}, \mathbf{R}).$$

Since u_n and v_n are bounded in H, and therefore in L^∞, we have

$$\left\| |u_n|^2 - |v_n|^2 \right\|_{L^2(\mathbf{R},\mathbf{R})} \leq \sup_n (||u_n||_\infty + ||v_n||_\infty) \int_{\mathbf{R}} \big| |u_n| - |v_n| \big|^2 dt \leq C ||u_n - v_n||,$$

which shows the first part.

To see that the same holds for the derivatives it suffices to compute

$$\begin{aligned}\left|\left|\frac{d}{dt}(|u_n|^2 - |v_n|^2)\right|\right|_{L^2(\mathbf{R},\mathbf{R})} &\\ \leq 2||u_n \cdot (\dot{u}_n - \dot{v}_n)||_{L^2(\mathbf{R},\mathbf{R})} + 2||\dot{v}_n \cdot (u_n - v_n)||_{L^2(\mathbf{R},\mathbf{R})} &\\ \leq 2\sup_n ||u_n||_\infty^2 \int_\mathbf{R} |\dot{u}_n - \dot{v}_n|^2 dt + 2||u_n - v_n||_\infty \int_\mathbf{R} |\dot{v}_n|^2 dt \to 0.\end{aligned}$$

Step 3. Conclusion. Adding and subtracting the same quantity we have

$$\begin{aligned}\left|\left||u_n|^2 - \sum_{i=1}^p |T_{\theta_n^i} v_i|^2\right|\right| \leq \left|\left| |u_n|^2 - \left|\sum_{i=1}^p T_{\theta_n^i} v_i\right|^2 \right|\right| &\\ + \left|\left| \left|\sum_{i=1}^p T_{\theta_n^i} v_i\right|^2 - \sum_{i=1}^p |T_{\theta_n^i} v_i|^2 \right|\right|\end{aligned}$$

Now the first term in the right-hand-side tends to zero as $n \to \infty$, by step 2. For the second term note that squaring gives

$$\left|\left| \left|\sum_{i=1}^p T_{\theta_n^i} v_i\right|^2 - \sum_{i=1}^p |T_{\theta_n^i} v_i|^2 \right|\right| = \left|\left| \sum_{i\neq j} T_{\theta_n^i} v_i \cdot T_{\theta_n^j} v_j \right|\right| \to 0$$

by step 1 (recall that $|\theta_n^i - \theta_n^j| \to \infty$ as $n \to \infty$). ■

The next definition introduces the fundamental tool for the conclusion of the proof.

DEFINITION 3.5. – For all $\varphi \in Q_\infty$ we define a set of real numbers $Z(\varphi)$ by letting

$$Z(\varphi) = \{t \in \mathbf{R} \,/\, \varphi(t) = \delta\},$$

where δ is the number introduced in Proposition 3.2. Note that since δ can be taken as small as we please, by Proposition 2.5, we can assume without loss of generality that $Z(\varphi) \neq \emptyset$, for all $\varphi \in Q_\infty$.

Next we define a function $\mathcal{T} : Q_\infty \to \mathbf{R}$ by

$$\mathcal{T}(\varphi) = \max Z(\varphi).$$

Remark that $\mathcal{T}$ is well defined, since for all $\varphi \in Q_\infty$, $Z(\varphi)$ is compact and nonempty.

We now study some properties of $Z(\varphi)$ and $\mathcal{T}$.

PROPOSITION 3.6. – *For all $\varphi \in Q_\infty$, the set $Z(\varphi)$ is discrete.*

Proof. – Let $t^* \in Z(\varphi)$, and let $\mathcal{U}_{t^*}$ be a neighborhood of t^* such that $\forall t \in \mathcal{U}_{t^*}$, $\varphi(t) < 2\delta$. This neighborhood exists by continuity of φ. By Proposition 3.2 in $\mathcal{U}_{t^*}$ we have $\varphi'' > 0$, so that φ' is strictly increasing. If t^* is not isolated, then there exists a monotone (increasing for example) sequence $t_n \to t^*$, with $t_n \in Z(\varphi)$. But then, for all n there exists $\eta_n \in]t_n, t_{n+1}[$ such that $\varphi'(\eta_n) = 0$. Thus φ' cannot be strictly increasing, as it should. ■

The preceding proposition allows us to define a function $\mathcal{T}_1 : Q_\infty \to R$ by setting it equal to the predecessor of $\mathcal{T}(\varphi)$ in $Z(\varphi)$, namely, $\mathcal{T}_1(\varphi) = \max\{Z(\varphi) \setminus \{\mathcal{T}(\varphi)\}\ \}$.

PROPOSITION 3.7. – *For all $\varphi \in Q_\infty$, there exists $\xi \in]\mathcal{T}_1(\varphi), \mathcal{T}(\varphi)[$ such that $\varphi(\xi) \geq 2\delta$.*

Proof. – Let ξ be a point such that $\varphi(\xi) = \max_{t \geq \mathcal{T}_1(\varphi)} \varphi(t)$. Clearly, $\xi > \mathcal{T}_1(\varphi)$, since otherwise $\mathcal{T}(\varphi)$ would be a point of local maximum, and we know that there are no such points where $\varphi < 2\delta$. Therefore $\varphi(\xi) \geq 2\delta$, and since by definition of $\mathcal{T}$, $\varphi(t) < \delta$ for all $t > \mathcal{T}(\varphi)$, it must be $\xi \in]\mathcal{T}_1(\varphi), \mathcal{T}(\varphi)[$. ■

PROPOSITION 3.8. – *Let B be a bounded (in $H^1(\mathbf{R};\mathbf{R})$) subset of Q_∞. Then*

$$\inf_{\varphi \in B} (\mathcal{T}(\varphi) - \mathcal{T}_1(\varphi)) =: \mu > 0.$$

Proof. – By Proposition 3.7 we can find, for each φ, a point $\xi \in]\mathcal{T}_1(\varphi), \mathcal{T}(\varphi)[$ where $\varphi(\xi) \geq 2\delta$. But then

$$\delta \leq |\varphi(\xi) - \varphi(\mathcal{T}_1(\varphi))| \leq \int_{\mathcal{T}_1(\varphi)}^{\xi} |\varphi'| dt \leq \sqrt{\xi - \mathcal{T}_1(\varphi)} ||\dot{\varphi}||_{L^2}.$$

Therefore,

$$\mathcal{T}(\varphi) - \mathcal{T}_1(\varphi) \geq \xi - \mathcal{T}_1(\varphi) \geq \frac{\delta}{\sup_B ||\varphi'||_{L^2}} =: \mu > 0,$$

because B is bounded. ■

Remark 3.9. – The argument used in the last proposition can be applied, without any changes to prove the following stronger statement. Let B be a bounded subset of Q_∞; then

$$\inf_{\varphi \in B} \inf\{|t - s| \ / \ \varphi(t) = \delta,\ \varphi(s) = 2\delta\ \} =: \nu > 0.$$

This result will be referred to in the next propositions.

We wish to prove that the function $\mathcal{T}$ enjoys some continuity properties. The main estimate we need is given by the following result.

PROPOSITION 3.10. – *Let B be a bounded subset of Q_∞. Then there exist $\rho > 0$ and $\gamma > 0$ such that*

$$(3.1) \qquad \varphi'(t) \leq -\gamma, \qquad \forall t \in [\mathcal{T}(\varphi) - \rho, \mathcal{T}(\varphi) + \rho], \quad \forall \varphi \in B.$$

Proof. – We know that by Remark 3.3 there exists a constant $b := 2(1 - \overline{\alpha}\varepsilon) > 0$ such that

$$0 < \varphi(t) < 2\delta \qquad \text{implies} \qquad \varphi''(t) > b\varphi(t), \qquad \forall \varphi \in Q_\infty.$$

Let $\eta = \eta(\varphi) = \max\{t \in \mathbf{R} \ / \ \varphi(t) = 2\delta\}$. This number is well defined, as it was the case for $\mathcal{T}(\varphi)$ (by compactness).

We claim that for all $t > \eta$ we have $\varphi'(t) < 0$. Indeed, suppose for contradiction that there exists $t_1 > \eta$ where $\varphi'(t_1) \geq 0$; in this case it is plainly seen that there also exists $t^* > \eta$ where $\varphi'(t^*) = 0$ (it can't be $\varphi'(t) \geq 0 \ \forall t > \eta$ because φ tends to zero at infinity). By definition of η, we see that $0 < \varphi(t^*) < 2\delta$, so that t^* is a strict local minimum for φ. Therefore in the interval $[t^*, +\infty[$ there must be at least one local maximum. Since φ at local maxima must be larger than 2δ there is also a point $t_2 \in [t^*, +\infty[$ where $\varphi(t_2) = 2\delta$, and this contradicts the definition of η.

This and Remark 3.9 allow us to say that there exists $\nu > 0$ such that

$$\varphi'(t) < 0 \qquad \forall t \in [\mathcal{T}(\varphi) - \nu, +\infty[, \qquad \forall \varphi \in B.$$

Consider now the function $E_\varphi : [\mathcal{T}(\varphi) - \nu, +\infty[\to \mathbf{R}$ given by

$$E_\varphi(t) = \frac{1}{2}|\varphi'(t)|^2 - \frac{b}{2}|\varphi(t)|^2;$$

differentiating we see that $E'_\varphi(t) = \varphi'(t)(\varphi''(t) - b\varphi(t)) < 0$ for all $t \in [\mathcal{T}(\varphi) - \nu, +\infty[$. Therefore E_φ is decreasing, and since $\lim_{t \to +\infty} E_\varphi(t) \geq 0$, we see that it must be $E_\varphi(t) \geq 0$ for all $t \in [\mathcal{T}(\varphi) - \nu, +\infty[$.

Now let $\rho > 0$ be so small that

$$\frac{\delta}{2} < \varphi(t) < \frac{3\delta}{2}, \qquad \forall t \in [\mathcal{T}(\varphi) - \rho, \mathcal{T}(\varphi) + \rho], \qquad \forall \varphi \in B.$$

Such ρ exists by virtue of the same argument of Remark 3.9. In particular, in $[\mathcal{T}(\varphi) - \rho, \mathcal{T}(\varphi) + \rho]$ we have $E_\varphi(t) \geq 0$, for all $\varphi \in B$. Thus in this interval we have

$$|\varphi'(t)|^2 \geq b|\varphi(t)|^2 \geq b\frac{\delta^2}{4},$$

and since $\varphi' < 0$ we obtain that $\varphi'(t) \leq -\sqrt{b}\frac{\delta}{2} =: -\gamma$, for all $t \in [\mathcal{T}(\varphi) - \rho, \mathcal{T}(\varphi) + \rho]$ and all $\varphi \in B$. ■

We can now show that the function $\mathcal{T}$ enjoys some continuity property which we will use in the last step.

Proposition 3.11. – *The function $\mathcal{T} : Q_\infty \to \mathbf{R}$ is locally Lipschitz continuous on bounded subsets of Q_∞.*

Proof. – Precisely we shall show that given a bounded subset B of Q_∞ there exists a constant $\sigma > 0$ such that

$$|\mathcal{T}(\varphi) - \mathcal{T}(\psi)| \leq \frac{1}{\gamma}||\varphi - \psi||, \qquad \forall \varphi, \psi \in B, \quad ||\varphi - \psi|| \leq \sigma,$$

where $\gamma = \gamma(B)$ is the constant provided by Proposition 3.10.

Let $\nu > 0$ be the number defined in Remark 3.9 and let ρ be given as in Proposition 3.10. Note that we can assume without loss of generality that $\rho < \nu$: the inequality (3.1) holds with the same γ.

Let $\varphi \in B$; first of all we see that by Proposition 3.10,

$$\varphi(\mathcal{T}(\varphi) + \rho) - \varphi(\mathcal{T}(\varphi)) = \int_{\mathcal{T}(\varphi)}^{\mathcal{T}(\varphi)+\rho} \varphi' dt \leq -\gamma\rho.$$

This and a similar computation show that

$$\varphi(\mathcal{T}(\varphi) + \rho) \leq \delta - \gamma\rho \qquad \text{and} \qquad \varphi(\mathcal{T}(\varphi) - \rho) \geq \delta + \gamma\rho.$$

Let $\sigma < \min(\gamma\rho, \delta)$, and let $\psi \in B$ verify $||\varphi - \psi|| \leq \sigma$. Then

$$\psi(\mathcal{T}(\varphi)-\rho) = \psi(\mathcal{T}(\varphi)-\rho)-\varphi(\mathcal{T}(\varphi)-\rho)+\varphi(\mathcal{T}(\varphi)-\rho) \geq -\sigma+\delta+\gamma\rho > \delta,$$

and similarly, $\psi(\mathcal{T}(\varphi) + \rho) < \delta$. Therefore there exists $t^* \in]\mathcal{T}(\varphi) - \rho, \mathcal{T}(\varphi) + \rho[$ such that $\psi(t^*) = \delta$.

We claim that $t^* = \mathcal{T}(\psi)$. Indeed if $t^* \neq \mathcal{T}(\psi)$, then it must be $t^* \leq \mathcal{T}_1(\psi)$. Now by Proposition 3.7 there is a point $\xi \in]\mathcal{T}_1(\psi), \mathcal{T}(\psi)[$ where $\psi(\xi) = 2\delta$. But since $\xi - t^* \geq \nu$, then $\xi \geq \nu + t^* \geq \rho + \mathcal{T}(\varphi) - \rho = \mathcal{T}(\varphi)$. The function φ is decreasing for $t > \mathcal{T}(\varphi) - \rho$, so that $\varphi(\xi) \leq \varphi(\mathcal{T}(\varphi)) = \delta$. Therefore

$$\delta \leq \psi(\xi) - \varphi(\xi) \leq ||\psi - \varphi|| \leq \sigma < \delta,$$

which is a contradiction. This means that it must be $t^* > \mathcal{T}_1(\psi)$, and so, necessarily, $t^* = \mathcal{T}(\psi)$; the claim is proved.

Finally, if $\mathcal{T}(\psi) \geq \mathcal{T}(\varphi)$, we find

$$0 = \psi(\mathcal{T}(\psi)) - \varphi(\mathcal{T}(\varphi)) = \psi(\mathcal{T}(\psi)) - \varphi(\mathcal{T}(\psi)) + \varphi(\mathcal{T}(\psi)) - \varphi(\mathcal{T}(\varphi))$$
$$\leq ||\varphi - \psi|| + \int_{\mathcal{T}(\varphi)}^{\mathcal{T}(\psi)} \varphi' dt \leq ||\varphi - \psi|| - \gamma(\mathcal{T}(\psi) - \mathcal{T}(\varphi))$$

and likewise, if $\mathcal{T}(\psi) < \mathcal{T}(\varphi)$, then $0 \leq ||\varphi - \psi|| - \gamma(\mathcal{T}(\varphi) - \mathcal{T}(\psi))$. These two inequalities show that

$$|\mathcal{T}(\varphi) - \mathcal{T}(\psi)| \leq \frac{1}{\gamma}||\varphi - \psi||, \qquad \forall \varphi, \psi \in B, \quad ||\varphi - \psi|| \leq \sigma,$$

and the proof is complete. ■

The following two propositions contain the last properties we need.

PROPOSITION 3.12. – *Let u_n be a Palais-Smale sequence for f at some level $c > 0$. Then*

$$dist\,(|u_n|^2, Q_\infty) \to 0 \quad \text{as} \quad n \to \infty, \tag{3.2}$$

where dist is the $H^1(\mathbf{R}, \mathbf{R})$ distance.

Proof. – If (3.2) is false then for some subsequence, still denoted u_n, we have

$$\lim_{n\to\infty} dist\,(|u_n|^2, Q_\infty) > 0. \tag{3.3}$$

Passing (if necessary) to another subsequence, u_n, by the representation lemma we know that

$$\left|\left|u_n - \sum_{i=1}^{q} T_{\theta_n^i} v_i\right|\right| \to 0 \qquad \text{as} \qquad n \to \infty,$$

for some suitable q, v_i, θ_n^i. By Proposition 3.4 we have

$$|u_n|^2 - \sum_{i=1}^{q} |T_{\theta_n^i} v_i|^2 \to 0 \quad \text{strongly in } H^1(\mathbf{R}, \mathbf{R}),$$

and this shows that $dist\,(|u_n|^2, Q_\infty) \to 0$, contradicting (3.3). ■

PROPOSITION 3.13. – *Let u_n be a Palais-Smale sequence for f at some level $c > 0$. Assume moreover that*

$$||u_n - u_{n-1}|| \to 0 \quad \text{as} \quad n \to \infty.$$

Then there exists a sequence $(\tau_n)_n \subset \mathbf{R}$ such that

(i) $$\liminf_{n\to\infty} |T_{\tau_n} u_n(0)| > 0,$$

(ii) $$\lim_{n\to\infty} |\tau_n - \tau_{n-1}| = 0$$

Proof. – Since $dist\,(|u_n|^2, Q_\infty) \to 0$, there exists $\varphi_n \in Q_\infty$ such that $||\, |u_n|^2 - \varphi_n|| \to 0$. Let $\tau_n = \mathcal{T}(\varphi_n)$. To begin with, we have

$$||\varphi_n - \varphi_{n-1}|| \leq ||\varphi_n - |u_n|^2|| + ||\, |u_n|^2 - |u_{n-1}|^2|| + ||\varphi_{n-1} - |u_{n-1}|^2|| \to 0,$$

by the hypothesis and Step 2 of Proposition 3.4. Now by uniform continuity of $\mathcal{T}$ on bounded sets we obtain

$$|\tau_n - \tau_{n-1}| = |\mathcal{T}(\varphi_n) - \mathcal{T}(\varphi_{n-1})| \to 0.$$

To complete the proof we just have to note that since $||\, |u_n|^2 - \varphi_n||_\infty \to 0$, then

$$|T_{\tau_n} u_n(0)|^2 = |u_n(\tau_n)|^2 - \varphi_n(\tau_n) + \varphi_n(\tau_n) = o(1) + \varphi_n(\mathcal{T}(\varphi_n)) = o(1) + \delta$$

as $n \to \infty$, which proves (i). ■

With the last proposition we are in a position to conclude the proof of Theorem 0.1.

End of the proof of Theorem. 0.1. – Since the functional f satisfies the geometric assumptions of the Mountain Pass lemma (Proposition 2.3), the application of Theorem 1.2 yields a $\overline{PS}$ sequence, namely a sequence $u_n \rightharpoonup 0$ such that

(i) $$\lim_{n\to\infty} f(u_n) > 0,$$

(ii) $$\lim_{n\to\infty} \nabla f(u_n) = 0,$$

(iii) $$\lim_{n\to\infty} ||u_n - u_{n-1}|| = 0.$$

By Proposition 3.13 we know that there exists a sequence τ_n (it can be assumed without loss of generality that $|\tau_n| \to \infty$) such that $|\tau_n - \tau_{n-1}| \to 0$ and $T_{\tau_n} u_n(0)$ has no subsequences converging to zero. Set $v_n = T_{\tau_n} u_n$.

The almost periodicity of the function α implies that there exists a sequence $(\sigma_k)_k \subset \mathbf{R}$ such that $|\sigma_k| \to \infty$ and $||\alpha(\cdot + \sigma_k) - \alpha||_\infty \to 0$. Since $|\tau_n - \tau_{n-1}| \to 0$ as $n \to \infty$, we can extract from τ_n a subsequence

τ_{n_k} such that $|\sigma_k - \tau_{n_k}| \to 0$ as $k \to \infty$. Moreover, since v_{n_k} is bounded, it contains some subsequence v_{n_k} such that

$$v_{n_k} \rightharpoonup v \not\equiv 0 \qquad \text{weakly in} \quad H.$$

We claim that v is the desired solution to problem (P) Indeed note that

$$||\alpha(\cdot + \tau_{n_k}) - \alpha||_\infty \leq ||\alpha(\cdot + \tau_{n_k}) - \alpha(\cdot + \sigma_k)||_\infty + ||\alpha(\cdot + \sigma_k) - \alpha||_\infty \to 0,$$

because the first term tends to zero by uniform continuity of α, and the second by definition of σ_k. Then for all $\varphi \in H$ we have, by Remark 2.14 and weak continuity of the gradient,

$$\nabla f(\alpha, v) \cdot \varphi = \lim_k \nabla f(\alpha, v_{n_k}) \cdot \varphi = \lim_k \nabla f(T_{\tau_{n_k}} \alpha, v_{n_k}) \cdot \varphi, \tag{3.4}$$

so that with the familiar changes of variable we obtain from (3.4)

$$\nabla f(\alpha, v) \cdot \varphi = \lim_k \nabla f(\alpha, u_{n_k}) \cdot T_{-\tau_{n_k}} \varphi = 0,$$

because u_{n_k} is a Palais-Smale sequence for $f = f(\alpha, \cdot)$. The fact that v does not vanish identically concludes the proof. ■

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