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Estimates for solutions of nonlinear variational inequalities

by

M. R. POSTERARO

Dip. di Matematica ed Appl. "Renato Caccioppoli",
Università di Napoli "Federico II",
Complesso Monte S.Angelo - 80126 Napoli, Italia.

ABSTRACT. – We prove a pointwise estimate for the solution u of a nonlinear variational inequality in terms of a function which is solution of a suitable variational inequality with spherically symmetric data. Using this result a lower bound for the measure of the set $\{x : u(x) = 0\}$ and *a priori* estimates for the L^p-norm and the $W_0^{1,p}$-norm of u are obtained. An existence result is also given.

RÉSUMÉ. – On démontre une estimation ponctuelle pour la solution u d'une inéquation variationnelle non linéaire en fonction de la solution d'une inéquation variationnelle opportune dont les données sont à symétrie sphérique. En utilisant ce résultat on obtient une borne inférieure de la mesure de l'ensemble $\{x : u(x) = 0\}$ et une estimation *a priori* des normes L^p et $W_0^{1,p}$ de u. De plus est donné un résultat d'existence.

1. INTRODUCTION

Let Ω be an open bounded set of $\mathbf{R}^n$. We consider the operator

$$Lu = Au + H(x, \nabla u) + g(x, u) \tag{1.1}$$

Work partially supported by MURST (40%).

where $Au = -\text{div}\, a(x, u, \nabla u)$ and

$$a : \Omega \times \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}^n, \qquad H : \Omega \times \mathbf{R}^n \to \mathbf{R}, \qquad g : \Omega \times \mathbf{R} \to \mathbf{R}$$

are Carathéodory functions which satisfy the following conditions:

(i) $a(x, \eta, \xi)\, \xi \geq |\xi|^p$ for a.e. $x \in \Omega$, $\forall \xi \in \mathbf{R}^n$, $p > 1$

(ii) $|H(x, \xi)| \leq B|\xi|^{p-1}$ for a.e. $x \in \Omega$, $\forall \xi \in \mathbf{R}^n$

(iii) $g(x, \eta)\eta \geq 0$ for a.e. $x \in \Omega$, $\forall \eta \in \mathbf{R}$.

Let us assume that there exists a function $u \in W_0^{1,p}(\Omega)$ solution of the variational inequality

$$(1.2) \qquad < Lu, v - u >$$

$$= \int_\Omega \left[a(x, u, \nabla u)(\nabla U - \nabla u) + H(x, \nabla u)(U - u) + g(x, u)(U - u)\right] dx$$

$$\geq \int_\Omega f(x)(U - u)\, dx \qquad \forall U \in W_0^{1,p}(\Omega),\ U, u \geq 0$$

In this paper we will prove a pointwise comparison between the function u solution of (1.2) and the solution of a suitable variational inequality defined in a ball with spherically symmetric data. More precisely we consider the problem

$$\int_{\Omega^\#} |\nabla v|^{p-2} v_{x_i} (V - v)_{x_i} + B|\nabla v|^{p-2} v_{x_i} \frac{x_i}{|x|}(V - v)$$

$$\geq \int_{\Omega^\#} f^\#(x)(V - v)\, dx \qquad \forall V \in W_0^{1,p}(\Omega^\#),\ V, v \geq 0$$

where $\Omega^\#$ is the ball of $\mathbf{R}^n$ centered at the origin with measure $|\Omega|$ and $f^\#(x)$ is the spherically symmetric decreasing rearrangement of f. In § 2 we will prove that this problem has a unique spherically symmetric solution $v(x) = v^\#(x)$ and $u^\#(x) \leq v(x)$ holds a.e. in $\Omega^\#$. This result allows us to obtain sharp estimates for u in terms of the function v, moreover we can find an optimal lower bound for the measure of the coincidence set of u.

The proof of this result is based on properties of the level sets of u and uses as main tools the isoperimetric inequality (*see* [11]) and the coarea formula (*see* [14]). The method was introduced by Talenti ([22]) who get a comparison result for the solution of a linear elliptic equation, and then was developed by many authors in different directions (*see* for example [23], [24], [1], [13] for linear and nonlinear elliptic equations). In this contest the first results for variational inequalities are due to Bandle-Mossino ([5]) and Maderna-Salsa ([17]); other results can be found in [2], [19] and [20].

Using this comparison result we give a priori estimates for the L^p-norm of u and of ∇u in terms of the norm of f in suitable Lorentz spaces (*see* also [23], [24], [6], [12] in the case of equations). As a conseguence we obtain also an existence result for problem (1.2). Other existence results for problems involving operator of the type (1.1) can be found in [9], [12] for elliptic equations and in [10], [8], for variational inequalities.

2. COMPARISON RESULTS

Firstly we recall some definitions about rearrangements. If Ω is an open bounded set of $\mathbf{R}^n$, we will denote by $|\Omega|$ its measure and by $\Omega^\#$ the ball of $\mathbf{R}^n$ centered at the origin whose measure is $|\Omega|$. Moreover if φ is a measurable function,

$$\mu(t) = |\{x \in \Omega : \varphi(x) > t\}|, \qquad t \in \mathbf{R},$$

is the distribution function of φ and

$$\varphi^*(s) = \sup\{t \geq 0 : \mu(t) > s\}, \qquad s \in [0, |\Omega|],$$

is its decreasing rearrangement. If C_n is the measure of the n-dimensional unit ball,

$$\varphi^\#(x) = \varphi^\#(|x|) = \varphi^*(C_n|x|^n), \qquad x \in \Omega^\#,$$

is the spherically symmetric decreasing rearrangement of $\varphi(x)$. For an exhaustive treatment of rearrangements we refer, for example, to [4], [18].

Let us consider the problem

$$\text{(2.1)} \qquad \int_{\Omega^\#} |\nabla v|^{p-2} v_{x_i} (V - v)_{x_i} + B|\nabla v|^{p-2} v_{x_i} \frac{x_i}{|x|} (V - v) \geq \int_{\Omega^\#} f^\#(V - v) \qquad \forall V \in W_0^{1,p}(\Omega^\#),\ V, v \geq 0$$

We will prove that such a problem has a unique spherically symmetric solution which is decreasing with respect to the radius.

If R is the radius of $\Omega^{\#}$ the function

$$w(x) = w(|x|) = \int_{|x|}^{R} \operatorname{sign}\left(\int_0^{\rho} \exp(-Br) f^{\#}(r) r^{n-1}\, dr\right) \times \left|\rho^{1-n} \exp(B\rho) \int_0^{\rho} \exp(-Br) f^{\#}(r) r^{n-1}\, dr\right|^{\frac{1}{p-1}} d\rho.$$

is the unique spherically symmetric solution of the Dirichlet problem

$$(2.2) \qquad \begin{cases} L^{\#}w = -(|\nabla w|^{p-2} w_{x_i})_{x_i} \\ +B|\nabla w|^{p-2} w_{x_i} \frac{x_i}{|x|} = f^{\#}(x) \quad \text{in } \Omega^{\#} \\ w = 0 \qquad \text{on } \partial\Omega^{\#} \end{cases}$$

The derivative of $w(\rho)$ is given by

$$-w_{\rho}(\rho) = \operatorname{sign}\left(\int_0^{\rho} \exp(-Br) f^{\#}(r) r^{n-1}\, dr\right) \times \left|\rho^{1-n} \exp(B\rho) \int_0^{\rho} \exp(-Br) f^{\#}(r) r^{n-1}\, dr\right|^{\frac{1}{p-1}}.$$

Firstly we observe that, if $f_{+}(x) = \{\max f(x), 0\} \equiv 0$, then $v(x) = 0$ is solution of (2.1); if $\int_0^{R} \exp(-Br) f^{\#}(r) r^{n-1}\, dr \geq 0$ then $w(x)$ is solution of (2.1). Let us consider the non trivial case $f_{+}(x) \not\equiv 0$ and $\int_0^{R} \exp(-Br) f^{\#}(r) r^{n-1}\, dr < 0$. In such a case there exists $0 < \bar{\rho} \leq R$ such that

$$(2.3) \qquad \begin{aligned} &\int_0^{\bar{\rho}} \exp(-Br) f^{\#}(r) r^{n-1}\, dr = 0 \\ &\int_0^{|x|} \exp(-Br) f^{\#}(r) r^{n-1}\, dr > 0 \qquad \text{if} \quad x < \bar{\rho} \\ &\int_0^{|x|} \exp(-Br) f^{\#}(r) r^{n-1}\, dr < 0 \qquad \text{if} \quad x > \bar{\rho}. \end{aligned}$$

This means that $w(x)$ takes the minimum value on the boundary of the sphere of radius $\bar{\rho}$. Let us put $-K = \min_{x \in \Omega^\#} w(x) = w(\bar{\rho})$

If $v(x)$ is solution of the variational inequality (2.1), then in the set $E = \{x : v(x) > 0\}$ $v(x)$ verifies the equation

$$\begin{cases} L^\# v = f^\# \\ v(x) = 0 \qquad \text{on } \partial E. \end{cases}$$

Therefore $v(x)$ is such that

$$v(|x|) = \begin{cases} (w(|x|) + h)_+ & \text{if } |x| < \bar{\rho} \\ 0 & \text{if } |x| \geq \bar{\rho} \end{cases}$$

where $h \leq K$. We want to show that the only possibility is $h = K$. To this aim we choose as test function in (2.1)

$$V(|x|) = \begin{cases} w(|x|) + K & \text{if } |x| < \bar{\rho} \\ 0 & \text{if } |x| \geq \bar{\rho} \end{cases}$$

Then set $\rho_1 = \dfrac{|\{x : v(x) > 0\}|^{1/n}}{C_n^{1/n}}$ we have

$$V(x) - v(x) = \begin{cases} K - h & \text{if } |x| < \rho_1 \\ w(x) + K & \text{if } \rho_1 \leq |x| \leq \bar{\rho} \\ 0 & \text{if } |x| > \bar{\rho} \end{cases}$$

which gives

$$0 \geq \int_{\Omega^\#} \left(-B|\nabla v|^{p-1} \frac{\partial v}{\partial |x|} + f^\#(x) \right) (V - v)\, dx$$

$$= C_n \int_0^R \left(-B|\nabla v|^{p-1} \frac{\partial v}{\partial r} + f^*(C_n r^n) \right) (V - v) r^{n-1}\, dr$$

$$\geq C_n (K - h) \int_0^{\rho_1} f^*(C_n r^n) r^{n-1}\, dr + C_n \int_{\rho_1}^{\bar\rho} f^*(C_n r^n)(V - v) r^{n-1}\, dr$$

$$= - C_n \int_{\rho_1}^{\bar\rho} \left(\int_0^r f^*(C_n \sigma^n) \sigma^{n-1}\, d\sigma \right) \frac{\partial w}{\partial r}\, dr$$

This quantity is strictly positive because for $r < \bar\rho$, by (2.3), we have

$$\int_0^r f^\#(\sigma)\sigma^{n-1}\, d\sigma \geq \int_0^r \exp(-B\sigma) f^\#(\sigma)\sigma^{n-1}\, d\sigma > 0.$$

Since the problem (2.2) has a unique spherically symmetric solution, the above arguments show that also the variational inequality (2.1) has a unique spherically symmetric solution.

Therefore we have proved the following

THEOREM 2.1. – *If $w(x) = w(|x|)$ is the solution of the Dirichlet problem* (2.2), *the variational inequality* (2.1) *has a unique spherically symmetric solution $v(x) = v^\#(x)$ given by $v(x) = 0$ if $f^\# \leq 0$; $v(x) = w(x)$ if $\int_0^R \exp(-Br) f^\#(r) r^{n-1}\, dr \geq 0$; and*

$$v(|x|) = \begin{cases} w(|x|) + K & \text{if } |x| < \bar\rho \\ 0 & \text{if } |x| \geq \bar\rho \end{cases}$$

where $-K = \min_{x\in\Omega^\#} w(x) = w(\bar\rho)$ in the remaining cases.

Now we will prove the pointwise comparison between the rearrangement of a solution u of the variational inequality (1.2) and the solution v of the symmetrized variational inequality (2.1).

THEOREM 2.2. – *If u is solution of the problem* (1.2) *with conditions* (i), (ii), (iii), *we have*

$$u^{\#}(x) \le v(x) \qquad \forall x \quad \text{a.e in}\, \Omega^{\#} \tag{2.4}$$

where $v(x) = v^{\#}(x)$ is the solution of the problem (2.1).

Proof. – The proof is based on techniques used, for example in [23], [1], [13], [6] in the case of equations and in [2], [5], [19] in the case of variational inequalities. If $u \equiv 0$ (2.4) is trivial, so we will suppose that u is not identically 0. Taking $h > 0$ and $t \in [0, \sup u[$, we choose as test function in (1.2) $U = u - \Phi_h$ where

$$\Phi_h(x) = \begin{cases} h & \text{if } u > t+h \\ (u(x)-t) & \text{if } t < u(x) \le t+h \\ 0 & \text{otherwise.} \end{cases} \tag{2.5}$$

We have

$$\frac{1}{h}\left(\int_\Omega a(x,u,\nabla u)\Phi_h\,dx + \int_\Omega H(x,\nabla u)\Phi_h\,dx + \int_\Omega g(x,u)\Phi_h\,dx\right) \le \frac{1}{h}\int_\Omega f(x)\Phi_h\,dx$$

and by (i), (ii), (iii), letting h go to 0, we obtain

$$-\frac{d}{dt}\int_{u>t}|\nabla u|^p\,dx \le B\int_{u>t}|\nabla u|^{p-1}\,dx + \int_{u>t} f(x)\,dx. \tag{2.6}$$

Isoperimetric inequality [11], Fleming-Rishel formula [14], and Schwartz inequality give (*see* also [1], [23])

$$\begin{aligned} nC_n^{1/n}\mu(t)^{1-1/n} &\le -\frac{d}{dt}\int_{u>t}|\nabla u| \\ &\le [-\mu'(t)]^{1/p'}\left(-\frac{d}{dt}\int_{u>t}|\nabla u|^p\right)^{1/p} \end{aligned} \tag{2.7}$$

We evaluate the first term on the right hand side of (2.6) using Höelder inequality and (2.7); we have

$$(2.8)\ \int_{u>t} |\nabla u|^{p-1} \le \int_t^{\infty} \left(-\frac{d}{ds} \int_{u>s} |\nabla u|^{p-1} \right) ds$$
$$\le \int_t^{\infty} (-\mu'(s))^{1/p} \left(-\frac{d}{ds} \int_{u>s} |\nabla u|^p \right)^{1/p'} ds$$
$$\le \frac{1}{nC_n^{1/n}} \int_t^{\infty} [-\mu'(s)] \left(-\frac{d}{ds} \int_{u>s} |\nabla u|^p \right) \mu(s)^{1/n-1}\, ds.$$

Since Hardy inequality gives

$$\int_{u>t} f(x)\, dx \le \int_0^{\mu(t)} f^*(s)\, ds$$

by (2.6), we have

$$(2.9)\qquad -\frac{d}{dt} \int_{u>t} |\nabla u|^p \le \frac{B}{nC_n^{1/n}} \int_t^{\infty} [-\mu'(s)] \left(-\frac{d}{ds} \int_{u>s} |\nabla u|^p \right)$$
$$\times\, \mu(s)^{1/n-1}\, ds + \int_0^{\mu(t)} f^*(s)\, ds$$

Now we put

$$\varphi(t) = \begin{cases} -\dfrac{d}{dt} \displaystyle\int_{u>t} |\nabla u|^p\, dx & \text{for } t \text{ s.t. } \mu'(t) \text{ exists} \\ 0 & \text{otherwise} \end{cases}$$

and then (2.9) becomes

$$\varphi(u^*(s)) \le \frac{B}{nC_n^{1/n}} \int_0^s \varphi(u^*(r)) r^{1/n-1}\, dr + \int_0^s f^*(r)\, dr$$

for a.e. $s \in [0, |u > 0|[$. With standard tools (*see* for example [1], [2], [25]) we obtain

$$(2.10)\qquad \varphi(u^*(s)) \le \exp\left(\frac{Bs^{1/n}}{C_n^{1/n}} \right) \int_0^s \exp\left(-\frac{Br^{1/n}}{C_n^{1/n}} \right) f^*(r)\, dr$$
$$\text{for a.e. } s \in [0, |u > 0|[$$

Using the definition of $\varphi(t)$, by (2.7) (*see* [25]) we have

$$\varphi(u^*(s)) \geq n^p C_n^{\frac{p}{n}} s^{(1-\frac{1}{n})p} \left(-\frac{d}{ds}u^*(s)\right)^{\frac{p}{p'}} \qquad \text{for a.e. } s \in [0, |u > 0|[$$

and then, by (2.10), we get

$$(2.11) \qquad -\frac{d}{ds}u^*(s) \leq \frac{s^{(1/n-1)p'}}{n^{p'} C_n^{p'/n}} \exp\left(\frac{Bs^{1/n}}{C_n^{1/n}(p-1)}\right) \times \left[\int_0^s \exp\left(-\frac{B\sigma^{1/n}}{C_n^{1/n}}\right) f^*(\sigma)\, d\sigma\right]^{\frac{1}{p-1}} \qquad \text{for a.e. } s \in [0, |u > 0|[$$

Now we consider the solution $v(x) = v^{\#}(x)$ of the problem (2.1). By theorem 2.1 setting $s = C_n |x|^n$ we have

$$(2.12) \qquad -\frac{d}{ds}v^*(s) = \frac{s^{(1/n-1)p'}}{n^{p'} C_n^{p'/n}} \exp\left(\frac{Bs^{1/n}}{C_n^{1/n}(p-1)}\right) \times \left[\int_0^s \exp\left(-\frac{B\sigma^{1/n}}{C_n^{1/n}}\right) f^*(\sigma)\, d\sigma\right]^{\frac{1}{p-1}} \qquad \forall s \in [0, |v > 0|[$$

Now we will prove that

$$(2.13) \qquad -\frac{d}{ds}u^*(s) \leq -\frac{d}{ds}v^*(s) \qquad s \in \big[0, |u > 0|\big[.$$

Clearly, because of (2.11) and (2.12), (2.13) holds in $\big[0, \min\{|u > 0|, |v > 0|\}\big[$. We will show that $|u > 0| \leq |v > 0|$. If $|v > 0| = |\Omega|$ there is nothing to prove. If $|v > 0| < |\Omega|$ we suppose *ab absurdo* that $|u > 0| > |v > 0|$. If $|u > 0| > s > |v > 0|$, setting

$$\Psi(s) = \frac{s^{(1/n-1)p'}}{n^{p'} C_n^{p'/n}} \exp\left(\frac{Bs^{1/n}}{C_n^{1/n}(p-1)}\right)$$

by (2.11) we have

$$(2.14) \qquad \left(-\frac{d}{ds}u^*(s)\right)^{p-1} \leq \Psi(s)^{p-1}\left[\int_0^{|v>0|} \exp\left(-\frac{Bs^{1/n}}{C_n^{1/n}}\right) f^*(s)\, ds + \int_{|v>0|}^s \exp\left(-\frac{Bs^1/n}{C_n^{1/n}}\right) f^*(s)\, ds\right]$$

By theorem 2.1 we have $v^{*'}(|v>0|)=0$, that is

$$\int_0^{|v>0|} f^*(s)\exp\left(-\frac{Bs^{1/n}}{C_n^{1/n}}\right)ds=0 \tag{2.15}$$

Moreover since $f^*(s)$ is decreasing

$$\int_{|v>0|}^{s} f^*(s)\exp\left(-\frac{Bs^{1/n}}{C_n^{1/n}}\right)ds<0 \tag{2.16}$$

holds and then (2.14), (2.15), (2.16) give $(-u^{*'}(s))^{p-1}<0$ that is absurd. Thus we have $|u>0|\le|v>0|$ and integrating (2.13) between s and $|u>0|$ we obtain

$$u^*(s)\le v^*(s) \qquad \forall s\in[0,|u>0|] \tag{2.17}$$

that implies (2.4). ∎

The comparison result just proved gives an optimal upper bound for the measure of the set $\{x\in\Omega : u(x)>0\}$ or, that is the same, an optimal lower bound for the measure of the coincidence set of u. In fact theorems 2.1 and 2.2 give the following

THEOREM 2.3. – *If $u(x)$ and $v(x)$ are solutions respectively of problems* (1.2) *and* (2.1) *we have*

$$|u>0|\le|v>0|$$

More precisely

$$|v>0|=0 \qquad \text{if} \quad f^{\#}(x)<0,$$

$$|v>0|=|\Omega| \qquad \text{if} \quad \int_0^R \exp(-Br)f^{\#}(r)r^{n-1}\,dr\ge 0;$$

otherwise $|v>0|$ *is the unique solution of the equation*

$$F(s)=\int_0^s \exp(-Br)f^{\#}(r)r^{n-1}\,dr=0.$$

Techniques used to obtain the pointwise comparison between the solutions of the variational inequalities (1.2) and (2.1) allow us to establish a comparison between the L^p-norm of the gradients of u and v.

THEOREM 2.4. – *Under the same hypotheses of theorem* 2.1, *if* u *and* v *are solutions of the problems* (1.2) *and* (2.1) *we have*

$$\|\nabla u\|_p \leq \|\nabla v\|_p.$$

Proof. – Starting from (2.10) we obtain

$$-\frac{d}{dt}\int_{u>t}|\nabla u|^p \leq \exp\left(\frac{B\mu(t)^{1/n}}{C_n^{1/n}}\right)\int_0^{\mu(t)} f^*(r)\exp\left(-\frac{Br^{1/n}}{C_n^{1/n}}\right)dr$$

As we have already seen (*see* proposition 2.1) we have

$$\int_0^s f^*(\sigma)\exp\left(-\frac{B\sigma^{1/n}}{C_n^{1/n}}\right)d\sigma \geq 0 \qquad \forall s \leq |v>0|,$$

then, using (2.7) we get

$$-\frac{d}{dt}\int_{u>t}|\nabla u|^p \leq \frac{[-\mu'(t)]\,\mu(t)^{(1/n-1)p'}}{n^{p'}C_n^{p'/n}} \times \exp\left(\frac{p'B\mu(t)^{1/n}}{C_n^{1/n}}\right)\left[\int_0^{\mu(t)} f^*(s)\exp\left(-\frac{Bs^{1/n}}{C_n^{1/n}}\right)ds\right]^{p'}.$$

Now, integrating between 0 and $+\infty$ we have

$$\begin{aligned}\|\nabla u\|_p^p = \int_{u>0}|\nabla u|^p &\leq \frac{1}{n^{p'}C_n^{p'/n}}\int_0^{|u>0|} s^{(1/n-1)p'} \\ &\quad\times \exp\left(\frac{p'Bs^{1/n}}{C_n^{1/n}}\right)\left[\int_0^s f^*(\sigma)\exp\left(-\frac{B\sigma^{1/n}}{C_n^{1/n}}\right)d\sigma\right]^{p'} ds \\ &\leq \frac{1}{n^{p'}C_n^{p'/n}}\int_0^{|v>0|} s^{(1/n-1)p'} \\ &\quad\times \exp\left(\frac{p'Bs^{1/n}}{C_n^{1/n}}\right)\left[\int_0^s f^*(\sigma)\exp\left(-\frac{B\sigma^{1/n}}{C_n^{1/n}}\right)d\sigma\right]^{p'} ds\end{aligned}$$

By proposition 2.1 it is easy to show that the right hand side of this inequality is $\|\nabla v\|_p^p$. ■

3. *A PRIORI* ESTIMATES

The comparison theorems establish an optimal upper bound for the L^p-norm and the $W_0^{1,p}$-norm of a solution of the variational inequality (1.2) in terms of the norms of the solution of a suitable variational inequality. Now we will give these *a priori* estimates in terms of the norm of f in suitable Lorentz spaces (*see* also [24], [6], [12]). To this aim let us introduce the Lorentz spaces. Let φ be a measurable function; for $1 < p < \infty$, we put:

$$\|\varphi\|_{p,q} = \begin{cases} \left(\int_0^{+\infty} \left[\frac{1}{t} \int_0^t |\varphi|^*(s)\, ds \right]^q t^{q/p} \frac{dt}{t} \right)^{1/q} & \text{if } 0 < q < \infty \\ \sup_{t>0} t^{1/p-1} \int_0^t |\varphi|^*(s)\, ds & \text{if } q = \infty, \end{cases}$$

We say that φ belongs to the Lorentz space $L(p,q)$ if $\|\varphi\|_{p,q} < \infty$. It is well known that $L(p,q)$, $1 \leq q \leq \infty$, with this norm, is a Banach space. An important property of Lorentz spaces is that they are "intermediate" between L^p spaces. More precisely, $L(p,p)$ coincides with $L^p(\Omega)$ and, for $1 < p < \infty$ and $0 < q \leq \infty$, the following inclusions hold:

$$\begin{aligned} &L^p(\Omega) \subset L(r,q) \subset L^r(\Omega), && p > r,\ 0 < q < r, \\ &L^r(\Omega) \subset L(r,q) \subset L^p(\Omega), && p < r,\ r < q \leq \infty. \end{aligned}$$

Moreover we will denote by $L(1,q)$, $0 < q < \infty$, the space of function $\varphi \in L^1(\Omega)$ such that

$$\|\varphi\|_{1,q} = \left(\int_0^{|\Omega|} \left[\frac{1}{t} \int_0^t |\varphi|^*(s)\, ds \right]^q t^q \frac{dt}{t} \right)^{1/q} < +\infty.$$

For an exaustive treatment of Lorentz spaces we refer to [7]. Finally we recall a result wich will be useful in the following (*see* [15]).

PROPOSITION 3.1. – *Suppose that $q > 1$ and $K(s,t)$ is non-negative and homogeneus of degree -1 and*

$$\int_0^{+\infty} K(s,1) s^{-1/q}\, ds = \int_0^{+\infty} K(1,t) t^{-1/q'}\, dt = M$$

Then

$$\int_0^{+\infty} \left(\int_0^{+\infty} K(s,t) f(s)\, ds \right)^q dt \leq M^q \int_0^{+\infty} f^q(s)\, ds$$

If $K(s,t)$ is positive, there is inequality unless $f = 0$.

Now we can prove the estimate for the norm of the solution u of the problem (1.2) in $W_0^{1,p}(\Omega)$.

THEOREM 3.1. – *If* u *is a weak solution of variational inequality* (1.1) *with conditions* (i), (ii), (iii), $f_+ = \max\{f(x), 0\} \in L\left(\frac{np'}{n+p'}, p'\right)$ *if* $p \leq n$ *and* $f_+ \in L^1(\Omega)$ *if* $p > n$ *the following estimate holds*

$$\|\nabla u\|_p^p \leq C(n, B, |\Omega|, p)\, \|f_+\|^{p'}_{\frac{np'}{n+p'}, p'} \qquad p \leq n$$

(3.1)

$$\|\nabla u\|_p^p \leq C(n, B, |\Omega|, p)\, \frac{|\Omega|^{\frac{p-n}{n(p-1)}}}{\dfrac{p-n}{n(p-1)}}\, \|f_+\|_1^{p'} \qquad p > n$$

where

$$C(n, p, B, |\Omega|) = \frac{\exp\left(\dfrac{p' B |\Omega|^{1/n}}{C_n^{1/n}}\right)}{n^{p'} C_n^{p'/n}}. \tag{3.2}$$

Proof. – In theorem 2.4 we have obtained

$$\|\nabla u\|_p^p \leq \|\nabla v\|_p^p$$

$$= \frac{1}{n^{p'} C_n^{p'/n}} \int_0^{|v>0|} s^{(1/n-1)p'} \exp\left(\frac{p' B s^{1/n}}{C_n^{1/n}}\right)$$

$$\times \left[\int_0^s f^*(\sigma) \exp\left(-\frac{B \sigma^{1/n}}{C_n^{1/n}}\right) d\sigma\right]^{p'} ds$$

and then, defining C as in (3.2),

$$\|\nabla u\|_p^p \leq C \int_0^{|\Omega|} s^{(1/n-1)p'} \left(\int_0^s f_+^*(\sigma)\, d\sigma\right)^{p'} ds$$

which gives (3.1). ∎

The following theorems give *a priori* estimates for the L^q-norm of the solution of the problem (1.2).

THEOREM 3.2. – *If* $u(x)$ *is solution of* (1.2) *with conditions* (i), (ii), (iii) *and* $f_+ \in L\left(\frac{n}{p}, \frac{1}{p-1}\right)$ *with* $p \leq n$ *then the following estimate holds*

$$\|u\|_\infty \leq C_1(n,p,,B,|\Omega|)\, \|f_+\|^{\frac{1}{p-1}}_{\frac{n}{p},\frac{1}{p-1}}$$

where

$$C_1(n,p,B,|\Omega|) = \frac{\exp\left(\dfrac{B|\Omega|^{1/n}}{C_n^{1/n}(p-1)}\right)}{n^{p'} C_n^{p'/n}}. \tag{3.3}$$

Proof. – We start from (2.13). Using (2.12) and defining C_1 as in (3.3), we have

$$-\frac{d}{ds}u^*(s) \leq -\frac{d}{ds}v^*(s) \leq C_1\, s^{(1/n-1)p'} \left(\int_0^s f^*(\sigma)\, d\sigma\right)^{\frac{1}{p-1}}$$

Then, integrating between 0 and $|u > 0|$ we get

$$u^*(0) = \|u\|_\infty \leq C_1 \int_0^{|\Omega|} s^{(1/n-1)p'} \left[\int_0^s f_+^*(\sigma)\, d\sigma\right]^{\frac{1}{p-1}} ds$$

and then the assert. ∎

THEOREM 3.3. – *If* u *is solution of variational inequality* (1.1) *with conditions* (i), (ii), (iii) *and* $C_1(n,p,B,|\Omega|)$ *is defined as in* (3.3) *we have*

$$\|u\|_q \leq C_1(n,p,B,|\Omega|)\ \frac{q^2}{q-1}\ \|f_+\|_{r,\frac{q}{p-1}} \tag{3.4}$$

where $p < n$, $\frac{np'}{n+p'} \leq r < \frac{n}{p}$, $q = \frac{nr(p-1)}{n-rp}$.

Proof. – By theorem 2.2 we have $\|u\|_q \leq \|v\|_q$ where v is solution of (2.1). Then, using (2.12) and defining $C_1(n,p,B,|\Omega|)$ as in (3.3), we have

$$\|u\|_q^q \leq C_1^q \int_0^{|\Omega|} \left[\int_s^{|\Omega|} t^{(\frac{1}{n}-1)p'} \left(\int_0^t f_+^*(\sigma)\, d\sigma\right)^{\frac{1}{p-1}} dt\right]^q ds$$

$$\leq C_1^q \int_0^{+\infty} \left[\int_0^{+\infty} \frac{t^{(\frac{1}{n}-1)p'+1}}{\max(s,t)} \left(\int_0^t f_+^*(\sigma)\, d\sigma\right)^{\frac{1}{p-1}} dt\right]^q ds$$

then, using proposition 3.1 with $K(s,t) = \dfrac{1}{\max(s,t)}$, we obtain

$$\|u\|_q^q \le C_1^q M^q \int_0^{+\infty} t^{[(\frac{1}{n}-1)p'+1]q} \left(\int_0^t f_+^*(\sigma)\, d\sigma \right)^{\frac{q}{p-1}} dt \tag{3.5}$$

where

$$M = \int_0^{+\infty} \frac{s^{-1/q}}{\max(s,1)}\, ds = \int_0^{+\infty} \frac{t^{-1/q'}}{\max(1,t)}\, dt = \frac{q^2}{q-1}.$$

Then by (3.5), (3.4) easily follows. ∎

Proceeding as in the theorem 3.3 we can obtain also estimates of the norm of u in Lorentz spaces (*see* also [24], [6]).

4. AN EXISTENCE RESULT

In this section we will use the *a priori* estimate obtained in § 2 to obtain an existence result for the variational inequality (1.2).

THEOREM 4.1. – *Let hypotheses* (i), (ii), (iii) *be satisfied and let the following conditions hold*

(iv) $|a(x,\eta,\xi)| \le k(x) + |\eta|^{p-1} + |\xi|^{p-1} \qquad k(x) \in L^{p'}(\Omega)$

(v) $\big(a(x,\eta,\xi_1) - a(x,\eta,\xi_2)\big)\,(\xi_1 - \xi_2) > 0 \qquad \xi_1 \ne \xi_2$

(vi) $g(x,\eta) \le c_0(x)|\eta|^\delta \qquad \delta \in [1, p^* - 1]$ *if* $p < n$, $\delta \in [1, +\infty[$ *if* $p \ge n$, $c_0(x) \in L^\infty(\Omega)$

If $f_+ \in L\left(\dfrac{np'}{n+p'}, p'\right)$, *if* $p \le n$, *and* $f_+ \in L^1(\Omega)$ *if* $p > n$, *then a function* $u \in W_0^{1,p}(\Omega)$ *solution of* (1.2) *exists.*

We remark that the condition (vi) on δ is given just to guarantee that the formulation of the problem (1.2) makes sense.

Proof. – We consider the operator $Lu = Au + H + g$ defined as in (1.1). We will prove that it is pseudomonotone (*see* [16] for $p < n$, $\delta < p^* - 1$). We take a sequence u_k weakly convergent to u in $W_0^{1,p}(\Omega)$ and we suppose

$$\limsup_{k\to\infty} < Lu_k, u_k - u > \le 0. \tag{4.1}$$

Firstly we observe that by Rellich-Kondrachov theorem there exists a subsequence (still indicated with u_k) such that

$$u_k \to u \qquad \text{a.e.} \quad \text{in}\,\Omega.$$

Moreover, since $g(x,u)$ is a Carathéodory function,

$$g(x,u_k) \to g(x,u) \qquad \text{a.e.} \quad \text{in}\,\Omega$$

If $A \subset \Omega$ by (v), if $p < n$, we have, $\forall v \in W_0^{1,p}(\Omega)$,

$$\begin{aligned}\int_A |g(x,u_k)v|\,dx &\leq \|c_0(x)\|_\infty \int_A |u_k^\delta v|\,dx \\ &\leq \|c_0(x)\|_\infty \|u_k\|_{\delta p^{*\prime}}^\delta \|v\|_{L^{p^*}(A)}\end{aligned}$$

and then Vitali theorem gives

$$g(x,u_k)v \to g(x,u)v \qquad \text{in} \quad L^1(\Omega) \qquad \forall v \in W_0^{1,p}(\Omega) \tag{4.2}$$

Moreover (vi) and Fatou lemma give

$$\liminf_k \int_\Omega g(x,u_k)u_k\,dx \geq \int_\Omega g(x,u)u\,dx \tag{4.3}$$

and then by (4.1)

$$\limsup_k < Au_k, u_k - u > + < H(u_k,\nabla u_k), u_k - u > \leq 0.$$

Since the operator $Au + H$ is pseudomonotone (*see* [16]), we have

$$\liminf_k < Au_k + H(u_k,\nabla u_k), u_k - v > \geq < Au + H(u,\nabla u), u - v > \qquad \forall v \in W_0^{1,p}$$

This means that (4.2) and (4.3) gives

$$\begin{aligned}&\liminf_k < Au_k + H(u_k,\nabla u_k), u_k - v > + < g(x,u_k), u_k - v > \\ &\geq < Au + H(u,\nabla u), u - v > + < g(x,u), u - v > \qquad \forall v \in W_0^{1,p}\end{aligned}$$

that is the pseudomonotonicity of Lu.

Now, following classical tecniques, we consider the problem

$$(4.4) \qquad \int_\Omega a(x, u_k, \nabla u_k)(\nabla U - \nabla u_k) + H(x, \nabla u_k)(U - u_k) + g(x, u_k)(U - u_k) \geq \int_\Omega f(U - u_k) \qquad \forall U \in E_k$$

where $E_k = \{v \in W_0^{1,p}(\Omega) : \|v\|_{W_0^{1,p}} \leq k,\, v \geq 0\}$. We have that E_k is bounded, closed and convex, then (*see* [16], Th. 8.1, p. 245) there exists a function $u_k \in E_k$ that is solution of the problem (4.4). Moreover the function $U = u_k - \Phi_h$ where

$$\Phi_h(x) = \begin{cases} h & \text{if } u_k(x) > t + h \\ u_k(x) - t & \text{if } t < u_k(x) \leq t + h \\ 0 & \text{otherwise} \end{cases}$$

is in E_k. This means that we can repeat the proof of theorem 3.1 to get that the functions u_k are bounded in $W_0^{1,p}(\Omega)$ uniformly with respect to k, therefore we can find k such that $\|u_k\|_{W_0^{1,p}} < k$. Then, arguing as in [21] (*see* theorem 2.5) we can say that u_k is solution of (1.2). In fact $u_k(x) \geq 0$. Moreover for all $v \in W_0^{1,p}(\Omega)$, $\quad v \geq 0$, there exist a function $w \in E_k$ and $\varepsilon > 0$ such that

$$w - u_k = \varepsilon(v - u_k).$$

Therefore from (4.4)

$$\varepsilon < Lu_k, v - u_k > = < Lu_k, w - u_k > \geq < f, w - u_k > = \varepsilon < f, v - u_k > \quad \blacksquare$$

5. EXTENSIONS

In this section we will show that the results of the previous sections can be obtained also if in the variational inequality (1.2) we substitute the hypotheses (ii) with the hypotheses

(ii′) $|H(x,\xi)| \leq b(x)|\xi|^{p-1} \qquad b(x) \in L^r(\Omega),\ r \geq p,\ r > n.$

We consider a function $B(s)$ such that

$$(5.1) \qquad \int_{u>s} |b(x)|^p\, dx = \int_0^{\mu(s)} B^p(\sigma)\, d\sigma$$

According with a lemma in [3] the function $B^p(s)$ is weak limit of functions which have the same rearrangement of $|b(x)|^p$. Moreover let $v(x)$ be the solution of the variational inequality

$$(5.2)\quad \begin{cases} \int_{\Omega^\#} |\nabla v|^{p-2} v_{x_i}(V-v)_{x_i} + B(C_n|x|^n)|\nabla v|^{p-2} v_{x_i}\frac{x_i}{|x|}(V-v) \\ \geq \int_{\Omega^\#} f^\#(V-v) \\ \qquad\qquad \forall V \in W_0^{1,p}(\Omega^\#), \qquad V, v \geq 0 \end{cases}$$

with $B(x)$ defined as in (5.1). Arguing as in theorem 2.1 it is possible to prove that this problem has a unique spherically symmetric solution $v(x) = v^\#(x)$. Moreover the following theorem holds

THEOREM 5.1. – *If* u *is solution of the problem* (1.2) *with conditions* (i), (ii′), (iii), *we have*

$$u^\#(x) \leq v(x) \qquad \forall x \quad \text{a.e} \quad \text{in}\, \Omega^\#$$

and

$$\|\nabla u\|_p \leq \|\nabla v\|_p$$

where $v(x) = v^\#(x)$ *is the solution of the problem* (5.2).

Remark. – We observe that in this case the "symmetrized" problem depends not only on the data of the problem (1.2), but also on its solution $u(x)$.

Proof. – Proceeding as in the proof of theorem 2.1 and using the function $B(s)$ defined in (5.1), instead of (2.8) we obtain

$$\int_{u>t} b(x)|\nabla u|^{p-1}\,dx$$

$$\leq \int_t^\infty \left(-\frac{d}{ds}\int_{u>s}|b(x)|^p\right)^{1/p}\left(-\frac{d}{ds}\int_{u>s}|\nabla u|^p\right)^{1/p'} ds$$

$$\leq \int_t^\infty B(\mu(s))[-\mu'(s)]^{1/p}\left(-\frac{d}{ds}\int_{u>s}|\nabla u|^p\right)^{1/p'} ds$$

$$\leq \frac{1}{nC_n^{1/n}}\int_t^\infty B(\mu(s))[-\mu'(s)]\left(-\frac{d}{ds}\int_{u>s}|\nabla u|^p\right)\mu(s)^{1/n-1}\,ds.$$

Arguing as in theorems 2.2 and 2.4 we obtain the desired result. ■

As far as the *a priori* estimates concern, using the function $B(s)$ defined in (5.1), we can obtain results analogous to theorem 3.1 and 3.2. For example we get the following

THEOREM 5.2. – *If u is a weak solution of variational inequality* (1.1) *with conditions* (i), (ii′), (iii), $f_+ = \max\{f(x), 0\} \in L\left(\frac{np'}{n+p'}, p'\right)$ *if* $p \le n$ *and* $f_+ \in L^1(\Omega)$ *if* $p > n$ *the following estimate holds*

$$(5.4)\qquad \begin{aligned} &\|\nabla u\|_p^p \le C(n, \|b\|_r, |\Omega|, p)\, \|f_+\|^{p'}_{\frac{np'}{n+p'}, p'} && p \le n \\ &\|\nabla u\|_p^p \le C(n, \|b\|_r, |\Omega|, p)\, \frac{|\Omega|^{\frac{p-n}{n(p-1)}}}{\frac{p-n}{n(p-1)}}\, \|f_+\|_1^{\frac{1}{p-1}} && p > n \end{aligned}$$

where

$$C(n, \|b\|_r, |\Omega|, p) = \frac{1}{n^{p'} C_n^{p'/n}} \exp\left(\frac{p'\|B\|_r |\Omega|^{\frac{r-n}{rn}}}{C_n^{1/n}} \left(\frac{r-n}{n(r-1)}\right)^{1/r-1}\right).$$

Proof. – By the previous comparison theorem we have obtained

$$\|\nabla u\|_p^p \le \|\nabla v\|_p^p =$$

$$\le \frac{1}{n^{p'} C_n^{p'/n}} \int_0^{|u>0|} s^{(1/n-1)p'} \exp\left(\frac{p'}{nC_n^{1/n}} \int_0^s \rho^{1/n-1} B(\rho)\, d\rho\right)$$

$$\times \left[\int_0^s f^*(\sigma) \exp\left(-\frac{1}{nC_n^{1/n}} \int_0^\sigma \rho^{1/n-1} B(\rho)\, d\rho\right) d\sigma\right]^{p'} ds$$

and then Höelder inequality gives

$$\|\nabla u\|_p^p \le \frac{\exp\left(\frac{p'\|B\|_r |\Omega|^{\frac{r-n}{nr}}}{nC_n^{1/n}} \left(\frac{r-n}{n(r-1)}\right)^{\frac{1}{r}-1}\right)}{n^{p'} C_n^{p'/n}}$$

$$\times \int_0^{|\Omega|} s^{(1/n-1)p'} \left(\int_0^s f_+^*(\sigma)\, d\sigma\right)^{p'} ds \quad ■$$

Using theorem 5.2 and proceeding as in theorem 4.1 also the existence result can be stated

THEOREM 5.3. – *Let hypotheses* (i), (ii′), (iii), (iv), (v),(vi) *be satisfied. If* $f \in L\left(\frac{np'}{n+p'}, p'\right)$, *if* $p \leq n$, *and* $f \in L^1(\Omega)$ *if* $p > n$, *then a function* $u \in W_0^{1,p}(\Omega)$ *solution of* (1.2) *exists.*

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