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Asymptotic behavior of solutions of a semilinear heat equation with subcritical nonlinearity

by

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ABSTRACT. – We consider the Cauchy problem for $u_t = \Delta u + u^p$ with $1 + 2/N < p$ and $(N-2)p < N+2$. We give a complete description of the asymptotic behavior of the positive solution.

RÉSUMÉ. – Nous considérons le problème de Cauchy pour $u_t = \Delta u + u^p$ avec $1+2/N < p$ et $(N-2)p < N+2$. On donne une description complète de comportement asymptotique de la solution positive.

1. INTRODUCTION AND MAIN RESULT

We study the asymptotic behavior of nonnegative solutions of the following Cauchy problem:

$$\text{(H)} \begin{cases} u_t = \Delta u + u^p & \text{in} \quad \mathbf{R}^N \times \mathbf{R}^+, \\ u(x,0) = u_0 & \text{in} \quad \mathbf{R}^N. \end{cases}$$

We assume $p > 1$ and $u_0 \geq 0$, $\not\equiv 0$ in $\mathbf{R}^N$. When $u_0 \in L^1 \cap L^\infty$, Problem (H) has a unique local classical solution (*see* [Kawa, Proposition 2.3]), which we denote by $u(x,t\,;u_0)$. We set

$$t_{\max}(u_0) := \sup\{T \in \mathbf{R}^+ \,;\, u(t\,;u_0) \in L^\infty((0,T)\,;\,L^\infty)\}.$$

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If $t_{\max}(u_0) < \infty$, then we say that $u(t;u_0)$ blows up in finite time. When $p \in (1, 1+2/N]$, it is well known (*see* e.g. [Kavi]) that all solutions of (H) blows up in finite time. In this paper we consider the next subcritical case:

$$1 + 2/N < p \quad \text{and} \quad (N-2)p < N+2. \tag{1.1}$$

In spite of the simple form of Problem (H), we need to transform the equation in order to obtain some important informations on the asymptotic behavior of solutions. Following [Kavi], we set

$$v(y, s; u_0) := (t+1)^{1/(p-1)} u(x, t;\, u_0), \tag{1.2}$$

$$x = (t+1)^{1/2} y \quad \text{and} \quad t = e^s - 1. \tag{1.3}$$

Then $v(y, s; u_0)$ satisfies

$$\text{(TH)} \begin{cases} v_s = \Delta v + \dfrac{y}{2} \cdot \nabla v + \dfrac{v}{p-1} + v^p & \text{in} \quad Q, \\ v(y, 0) = u_0 & \text{in} \quad \mathbf{R}^N. \end{cases}$$

By studying Problem (TH) Kavian [Kavi] showed

$$\|u(t;u_0)\|_\infty = O(t^{-1/(p-1)}) \quad \text{as} \quad t \to \infty, \tag{1.4}$$

provided $u_0 \in H^1_\rho$ and $t_{\max}(u_0) = \infty$. For the definition of H^1_ρ, *see* Notations just after this section. In this paper we will extend [Kavi] and clarify the structure of space of positive solutions of (H). Let $u_0 \in L^2_\rho \cap L^\infty$. Then our main result below shows that $u(t;u_0)$ is classified into one of the next three types:

Type (I): $t_{\max}(u_0) < \infty$, *i.e.* $u(t;u_0)$ blows up in finite time,

Type (II): $t_{\max}(u_0) = \infty$ and $\|u(t;u_0)\|_\infty \sim t^{-N/2}$ as $t \to \infty$,

Type (III): $t_{\max}(u_0) = \infty$ and $\|u(t;u_0)\|_\infty \sim t^{-1/(p-1)}$ as $t \to \infty$

and that the solution of Type (I) and the solution of Type (II) are stable and the solution of Type (III) is instable.

It is known (*see* e.g. [Kawa]) that if $E(u_0) < 0$ then $u(t;u_0)$ is of Type (I), where $E(u_0)$ is the 'energy' of u_0 defined by

$$E(u_0) := \frac{1}{2}\|\nabla u_0\|_2^2 - \frac{1}{p+1}\|u_0\|_{p+1}^{p+1}. \tag{1.5}$$

Fujita [F] showed that if u_0 is bounded by $\varepsilon \exp(-a|x|^2)$ then $u(t;u_0)$ is of Type (II), where $a > 0$ is a constant and $\varepsilon = \varepsilon(a) > 0$ is some small constant. In [Kawa] we gave a necessary and sufficient condition for the solution of (H) to be of Type (II) (*see* Proposition 3 in Section 2), which is one of crucial results to establish our main Theorem. Haraux and Weissler [HW] observed that (H) has a self-similar solution $w(x,t)$ of Type (III) constructed by

$$w(x,t) = t^{-1/(p-1)} f\left(\frac{x}{\sqrt{t}}\right), \tag{1.6}$$

where $f \in S$ and

$$S := \left\{ f \in H_\rho^1 \cap L^\infty \,;\, -\Delta f - \frac{y}{2} \cdot \nabla f = \frac{f}{p-1} + f^p \quad \text{and} \quad f > 0 \quad \text{in} \quad \mathbf{R}^{\mathrm{N}} \right\}. \tag{1.7}$$

Such a solution $w(x,t)$ is invariant by the similarity transformation:

$$w_\lambda(x,t) = \lambda^{2/(p-1)} w(\lambda x, \lambda^2 t), \tag{1.8}$$

namely, we have $w_\lambda(x,t) = w(x,t)$ for $\lambda > 0$.

Now we will state our main result. Let $X := \{f \in L_\rho^2 \cap L^\infty \,;\, f \geq 0 \text{ in } \mathbf{R}^{\mathrm{N}}\}$ be a closed cone of the Banach space $L_\rho^2 \cap L^\infty$ with the norm $\|\cdot\| := |\cdot|_2 + \|\cdot\|_\infty$. We set

$$K := \{u_0 \in X \,;\, t_{\max}(u_0) = \infty\},$$

$$B := X - K = \{u_0 \in X \,;\, t_{\max}(u_0) < \infty\}.$$

We denote by $\mathrm{Int}\,(K)$ the interior of K in X and by ∂K the boundary of K in X.

THEOREM 1. – *We assume* (1.1) *Then we obtain the following:*

(i) *The set K is an unbounded, closed convex set in X and* $0 \in \mathrm{Int}\,(K)$.

(ii) *For any $u_0 \in X - \{0\}$ there exists a unique $\tau_0 \in \mathbf{R}^+$ such that*

$$\begin{cases} \tau_0 u_0 \in \partial K, \\ \tau u_0 \in \mathrm{Int}\,(K) \quad \text{if} \quad \tau \in (0, \tau_0), \\ \tau u_0 \in B \quad \text{if} \quad \tau \in (\tau_0, \infty). \end{cases} \tag{1.9}$$

Moreover, $G := \{u_0 \in X \,;\, \|u_0\| = 1\}$ and ∂K are homeomorphic by $P|_G$, where $P : X - \{0\} \to \partial K$ is the well-defined projection: $Pu_0 = \tau_0 u_0 \in \partial K$ in view of (1.9).

(iii) *If* $u_0 \in \operatorname{Int}(K) - \{0\}$, *then we have*

$$\|u(t;u_0)\|_q \sim t^{-(1-1/q)N/2} \quad \text{for} \quad q \in [1,\infty]. \tag{1.10}$$

More precisely, for $q \in [1,\infty]$

$$t^{(1-1/q)N/2}\|u(t;u_0) - m_\infty(4\pi t)^{-N/2}\exp\left(-\frac{|x|^2}{4t}\right)\|_q \to 0 \quad \text{as} \quad t \to \infty, \tag{1.11}$$

where $m_\infty = \sup_{t\geq 0} \|u(t)\|_1 \in \mathbf{R}^+$.

(iv) *If* $u_0 \in \partial K$ *then we have*

$$\|u(t;u_0)\|_q \sim t^{N/2q-1/(p-1)} \quad \text{for} \quad q \in [1,\infty]. \tag{1.12}$$

More precisely, we obtain $\omega(v(s;u_0)) \subset S$, *where* $\omega(v)$ *is* ω*-limit set of* v *in* $L^2_\rho \cap L^\infty$, i.e.

$$\omega(v(s;u_0)) = \bigcap_{t\geq 0} \overline{\{v(s;u_0)\,;\, s \geq t\}}^{L^2_\rho \cap L^\infty}. \tag{1.13}$$

(v) *If* $u_0 \in B$ *then we have*

$$E(u(t;u_0)) \to -\infty \quad \text{as} \quad t \to t_{\max}(u_0) - 0. \tag{1.14}$$

For the Dirichlet problem in bounded domains corresponding to (H) some similar results were established in [Li], [NST], [CL] and [G]. In this case, the solution blows up in finite time or the solution exists time-globally and converges whether to 0 or to nontrivial equilibria in L^∞ (thus in L^q for any $q \in [1,\infty]$) as $t \to \infty$. We remark that some methods used in their works play improtant roles in this paper by appropriate modifications. Recently, Lee and Ni [LN] and Wang [W] obtained some interesting necessary conditions and sufficient conditions for the solution of (H) to exist time-globally. In particular, they treat solutions with initial values $u_0(x)$ decaying slowly like $|x|^{-2/(p-1)}$ as $|x| \to \infty$.

In Section 2 we give some preliminary results in order to establish Theorem 1. In Section 3 we prove Theorem 1 and give some remarks.

Notations. – 1. $\mathbf{R}^+ := (0,\infty)$, $Q := \mathbf{R}^\mathbf{N} \times \mathbf{R}^+$, $Q(a,b) := \mathbf{R}^\mathbf{N} \times (a,b)$ and $Q[a,b) := \mathbf{R}^\mathbf{N} \times [a,b)$.

2. $L^p := L^p(\mathbf{R}^\mathbf{N})$ with the usual norm $\|\cdot\|_p := \left(\int_{\mathbf{R}^\mathbf{N}} |\cdot|^p\right)^{1/p}$. We denote $\|\cdot\|_{\infty,Q(a,b)} := \|\cdot\|_{L^\infty(Q(a,b))}$.

3. $\rho(x) := \exp(|x|^2/4)$.

4. $L^p_\rho := \left\{ f \in L^p\,;\, \int_{\mathbf{R}^N} |f|^p \rho < \infty \right\}$ is a weighted L^p-space with the norm $|\cdot|_p := \left(\int_{\mathbf{R}^N} |\cdot|^p \rho \right)^{1/p}$.

5. $\|\cdot\|$ denotes the norm of $L^2_\rho \cap L^\infty$, *i.e.* $\|\cdot\| := |\cdot|_2 + \|\cdot\|_\infty$.

6. $H^1_\rho := \{f \in H^1(\mathbf{R}^N)\,;\, \nabla f \in L^2_\rho\}$ is a Hilbert space with the inner product $(f,g)_\rho := \int_{\mathbf{R}^N} (\nabla f, \nabla g)\rho$ for $f,\ g \in H^1_\rho$.

7. $f(t) = O(g(t))$ means that $\limsup_{t\to\infty} |f(t)/g(t)| < \infty$ and $f(t) \sim g(t)$ that $0 < \liminf_{t\to\infty} |f(t)/g(t)| \le \limsup_{t\to\infty} |f(t)/g(t)| < \infty$.

2. PRELIMINARIES

In this section we give some preliminary results to prove Theorem 1.

We defined by (1.5) the energy $E(u)$ for Problem (H). We also define the energy $\hat{E}(v)$ for Problem (TH) by

$$\hat{E}(v) := \frac{1}{2}|\nabla v|_2^2 - \frac{1}{2(p-1)}|v|_2^2 - \frac{1}{p+1}|v|_{p+1}^{p+1}. \tag{2.1}$$

PROPOSITION 1. – (i) *Let* $u_0 \in X \cap H^1$. *If* $E(u_0) < 0$ *then* $u_0 \in B$.

(ii) *Let* $u_0 \in X \cap H^1_\rho$. *If* $\hat{E}(u_0) < 0$ *then* $u_0 \in B$.

Proof. – (i) This is well-known. *See* e.g. the proof of [Kawa, Proposition 3.1].

(ii) See the proof of [Kavi, Theorem (1.10)]. ■

PROPOSITION 2. – *We assume* (1.1). *Then the following hold.*

(i) *Let* $b \in \mathbf{R}^+$. *Then there exists some constant* $m \in \mathbf{R}^+$ *such that for any* $u_0 \in K$ *with* $\|u_0\|_2 + \|u_0\|_\infty \le b$ *we have* $\|u(t;u_0)\|_{\infty,Q} \le m$.

(ii) *The set* K *is closed in* X.

(iii) *Let* $u_0 \in B$. *Then we have* (1.14).

Proof. – Using Lemma 1 below, we can prove Proposition 2 essentially by the same argument as in [G]. Therefore, we leave it to the reader. ■

LEMMA 1. – *We assume* (1.1). *Let* $t_0 \in \mathbf{R}^+$ *and* u *be a classical solution of (H) on* $[0,T)$, $T > t_0$. *Assume that*

$$\int_0^T \|u_t\|_2^2\,dt \le l < \infty, \tag{2.2}$$

$$\|u\|_{\infty,Q(t_0,T)} = \|u\|_{\infty,Q(0,T)}. \tag{2.3}$$

Then there is some constant $a \in \mathbf{R}^+$ independant of u, u_0 and T (dependant of l and t_0) such that

$$\|u\|_{\infty,Q(0,T)} \leq a.$$

Proof. – Although the proof is similar to that of [G, Lemma], we will describe it for the sake of completeness. We proceed by a contradiction. Suppose that Lemma 1 does not hold. Then there is a sequence of solutions $u_n(x,t)$ of (H) on $[0,T_n)$, $T_n > t_0$ such that

$$\int_0^{T_n} \|u_{nt}\|_2^2 dt \leq l, \tag{2.4}$$

$$\|u_n\|_{\infty,Q(t_0,T_n)} = \|u_n\|_{\infty,Q(0,T_n)}, \tag{2.5}$$

and

$$\|u_n\|_{\infty,Q(0,T_n)} \to \infty \quad \text{as} \quad n \to \infty. \tag{2.6}$$

Let $(x_n, t_n) \in Q(t_0, T_n)$ be a sequence such that

$$|u_n(x_n,t_n)| \geq \frac{1}{2}\|u_n\|_{\infty,Q(0,T_n)}. \tag{2.7}$$

We choose a sequence $\lambda_n > 0$ such that

$$\lambda_n^{2/(p-1)}|u_n(x_n,t_n)| = 1. \tag{2.8}$$

We remark that λ_n satisfies that $\lambda_n \to 0$ as $n \to \infty$. We define the function v_n by

$$v_n(x,t) = \lambda_n^{2/(p-1)} u_n(x_n + \lambda_n x, t_n + \lambda_n^2 t).$$

We can easily verify that v_n is a solution of (H) in $Q_n := Q(-t_n/\lambda_n^2, (T_n - t_n)/\lambda_n^2)$. In view of (2.7) and (2.8) we have

$$v_n(0,0) = 1, \tag{2.9}$$

$$\|v_n\|_{\infty,Q_n} = \lambda_n^{2/(p-1)}\|u_n\|_{\infty,Q(0,T_n)} \leq 2\lambda_n^{2/(p-1)}|u_n(x_n,t_n)| = 2.$$

Since $\{v_n\}$ are uniformly bounded, $\{v_n\}$ are equi-continuous on every compact subset of $Q(-\infty, 0]$ (*see* [D] or [S]). Thus, there is a subsequence (still denoted v_n) and a function $v(x,t)$ such that

$$v_n \to v \quad \text{in} \quad L^\infty(D) \quad \text{as} \quad n \to \infty, \tag{2.10}$$

where D is any compact subset of $Q(-\infty, 0]$. The function v is a solution of (H) in the sense of distribution and is bounded in $Q(-\infty, 0]$. Therefore, v is a classical solution of (H). It follows from (H) that

$$\text{(2.11)} \quad \int_{-t_0/\lambda_n^2}^{0} \|v_{nt}\|_2^2 dt = \lambda_n^{4p/(p-1)-(N+2)} \int_{t_n-t_0}^{t_n} \|u_{nt}\|_2^2 dt \leq l\lambda_n^{4p/(p-1)-(N+2)} \to 0 \quad \text{as} \quad n \to \infty.$$

By (2.9), (2.10) and (2.11), we obtain

$$\text{(2.12)} \qquad v(0,0) = 1 \quad \text{and} \quad v_t \equiv 0 \quad \text{in} \quad Q(-\infty, 0].$$

Thus, v is a nontrivial equlibrium solution of (H). This contradicts a Liouville theorem in [GS]. The proof is complete. ■

PROPOSITION 3. – *Let $p > 1 + 2/N$ and $p_0 := N(p-1)/2$. Assume that $u_0 \in L^1 \cap L^\infty$, $u_0 \geq 0$, $\not\equiv 0$ in $\mathbf{R}^N$ and $|x|u_0(x) \in L^1$. Then the following* (2.13) *and* (2.14) *are equivalent:*

$$\text{(2.13)} \qquad t_{\max}(u_0) = \infty \quad \text{and} \quad \|u(t\,; u_0)\|_\infty \sim t^{-N/2},$$

$$\text{(2.14)} \qquad \inf\{\, \|u(t\,; u_0)\|_{p_0}\,;\, t \in [0, t_{\max}(u_0)\,)\,\} < \delta_0,$$

where $\delta_0 > 0$ is a constant depending only on N and p. If (2.13) *holds then $u(t\,; u_0)$ satisfies*

$$\text{(2.15)} \quad t^{(1-1/q)N/2} \left\| u(t\,; u_0) - m_\infty (4\pi t)^{-N/2} \exp\left(-\frac{|x|^2}{4t}\right) \right\|_q \to 0 \quad \text{as} \quad t \to \infty$$

for any $q \in [1, \infty]$, where

$$0 < m_\infty := \sup_{t \geq 0} \|u(t)\|_1 = \int_{\mathbf{R}^N} u_0 dx + \int_0^\infty dt \int_{\mathbf{R}^N} u(t)^p dx < \infty.$$

Proof. – The equivalence of (2.13) and (2.14) follows from [Kawa, Corollary 1.1], and (2.15) with $q = \infty$ from [Kawa, Theorem 4.1]. Using [EZ, Lemma 3], we can prove (2.15) with $q = 1$ in the same way as in the proof of (2.15) with $q = \infty$. By linear interpolation we obtain (2.15) for $q \in (1, \infty)$. ■

PROPOSITION 4. – *Assume* $p > 1 + 2/N$. *We set*

$$W := \{u_0 \in X\,;\, t_{\max}(u_0) = \infty \quad \text{and} \quad \|u(t\,;u_0)\|_\infty \sim t^{-N/2}\}.$$

Then W *is open in* X.

Proof. – We fix $u_0 \in W$. Let $p_0 = N(p-1)/2$ (> 1). It suffices to prove

$$(2.16) \qquad \exists\delta = \delta(N,p) > 0\,;\, u_1 \in X \text{ and } \|u_1 - u_0\|_{p_0} < \delta \Longrightarrow u_1 \in W.$$

In view of the comparison principle we may assume without loss of generality that $u_1 \geq u_0$ in $\mathbf{R^N}$. We set $w(x,t) = u(x,t;u_1) - u(x,t;u_0)$ (≥ 0 in $\mathbf{R^N}$). Then, w satisfies

$$(2.17) \qquad \begin{aligned} w_t &= \Delta w + [w + u(t;u_0)]^p - u(t;u_0)^p \\ &= \Delta w + pw\int_0^1 [sw + u(t\,;u_0)]^{p-1} ds \\ &\leq \Delta w + 2^{p-1}pw[w^{p-1} + \|u(t\,;u_0)\|_\infty^{p-1}]. \end{aligned}$$

We set $f(t) = 2^{p-1}p\|u(t\,;u_0)\|_\infty^{p-1}$ and

$$w(x,t) = W(x,t)\exp\left[\int_0^t f(s)ds\right].$$

Then, $f(t) \in L^1(0,\infty)$. The function W satisfies

$$W_t \leq \Delta W + C_1 W^p.$$

Here, $C_1 = 2^{p-1}p\exp\left[(p-1)\int_0^\infty f(s)ds\right] \in \mathbf{R}^+$. By Proposition 3, there exist $\delta = \delta(N,p) > 0$ such that if $\|W(0)\|_{p_0} = \|u_1 - u_0\|_{p_0} < \delta$ then we have

$$t_{\max}(u_1) = \infty \quad \text{and} \quad \|W(t)\|_\infty = O(t^{-N/2}).$$

Therefore, we obtain $\|u(t;u_1)\|_\infty \sim t^{-N/2}$. Hence, (2.16) holds. ■

LEMMA 2. – *Let* $f,\ g \in S$. *If* $f \leq g$ *in* $\mathbf{R^N}$ *then* $f = g$ *in* $\mathbf{R^N}$.

Proof. – Our proof is very close to that of [Li, Lemma 2.2]. By integration by parts we find

$$\int_{\mathbf{R^N}} g^p f\rho = \int (\nabla f\cdot\nabla g)\rho - \frac{1}{p-1}\int fg\rho = \int f^p g\rho,$$

which leads to

$$\int_{\mathbf{R^N}} \rho fg(g^{p-1} - f^{p-1}) = 0.$$

This yields $f \equiv g$ in $\mathbf{R^N}$. ■

PROPOSITION 5. – *We assume* (1.1). *Let* $u_0 \in X$ *and* $t_{\max}(u_0) = \infty$. *Then* $u(t\,;u_0)$ *satisfies whether*

$$\|u(t\,;u_0)\|_\infty \sim t^{-1/(p-1)} \tag{2.18}$$

or

$$\|u(t\,;u_0)\|_\infty \sim t^{-N/2}. \tag{2.19}$$

Moreover, if (2.18) *holds then we have*

$$\omega(v(s\,;u_0)) \subset S, \tag{2.20}$$

and if (2.19) *holds then we have*

$$\omega(v(s\,;u_0)) = \{0\}. \tag{2.21}$$

Proof. – We can verify that

$$\|u(t\,;u_0)\|_q = (t+1)^{(1/q-1/p_0)N/2}\|v(s\,;u_0)\|_q \quad \text{for} \quad q \in [1,\infty], \tag{2.22}$$

$$\frac{d}{ds}\hat{E}(v(s\,;u_0)) = -|v_s(s\,;\,u_0)|_2^2. \tag{2.23}$$

Kavian [Kavi, Theorem (1.13)] showed

$$\|u(t\,;u_0)\|_\infty = O(t^{-1/(p-1)}) \quad (\Longleftrightarrow \|v(s\,;u_0)\|_\infty = O(1)\,) \tag{2.24}$$

and

$$\omega(v(s\,;u_0)) \subset S \cup \{0\}. \tag{2.25}$$

He proved (2.24) by using (2.23) and the method in [CL]. Once we obtain (2.24), we can derive (2.25) from the smoothing effect: $v(s\,;u_0) \in L^\infty(\,[\tau,\infty)\,;\,H^1_\rho \cap C^1(\mathbf{R}^\mathbf{N})\,)$ for $\tau > 0$ and the compactness of the embedding: $H^1_\rho \cap C^1(\mathbf{R}^\mathbf{N}) \subset L^2_\rho \cap L^\infty$. We remark that the method in [G] is also applicable to deduce (2.24). Indeed, using Lemma 3 below, we can prove (2.24) by the same argument in the proof of Proposition 2, (i).

Next, we will show that if (2.18) does not hold then (2.19) holds. Let $u(t\,;u_0)$ do not satisfy (2.18). Then we have

$$\liminf_{t\to\infty} t^{1/(p-1)}\|u(t\,;u_0)\|_\infty = 0,$$

Or equivalently,

$$\liminf_{s\to\infty} \|v(s\,;u_0)\|_\infty = 0.$$

Therefore, we deduce that

$$0 \in \omega(v(s\,;u_0)), \tag{2.26}$$

which leads to

$$\liminf_{t\to\infty} \|u(t\,;u_0)\|_{p_0} = \liminf_{s\to\infty} \|v(s\,;u_0)\|_{p_0} = 0. \tag{2.27}$$

By Proposition 3 we obtain (2.19). Now, we see that (2.19), (2.21) and (2.26) are equivalent. Thus, (2.18) and (2.20) are also equivalent. ■

LEMMA 3. – *We assume* (1.1). *Let* $s_0 \in \mathbf{R}^+$ *and* v *be a classical solution of (TH) on* $[0,T)$, $T > s_0$. *Assume that*

$$\int_0^T |v_s|_2^2\, ds \le l < \infty,$$

$$\|v\|_{\infty,Q(s_0,T)} = \|v\|_{\infty,Q(0,T)}.$$

Then there is some constant $a \in \mathbf{R}^+$ *independant of* v, u_0 *and* T *(dependant of* l *and* s_0*) such that*

$$\|v\|_{\infty,Q(0,T)} \le a.$$

Proof. – Since the proof is essentially the same as that of Lemma 1, we leave it to the reader.

3. PROOF OF THEOREM 1 AND REMARKS

Proof of Theorem 1. – Let W be the open set in X defined in the statement of Proposition 4.

(i) We already proved the closedness of K (*see* Proposition 2). By the same argument as in [Li], we can verify that K is convex. The unboundedness of K follows from Proposition 3. Indeed, we can easily find $u_0^n \in X$ such that $\|u_0^n\|_{p_0} < \delta_0$ and $\|u_0^n\|_\infty > n$ for $n \in \mathbf{N}$. By Proposition 3, $\{u_0^n\}$ is an unbounded sequence in K. We can see $0 \in \operatorname{Int}(K)$ also in view of Proposition 3.

(ii) We fix any $u_0 \in X - \{0\}$. We set $L = \{\tau \in \mathbf{R}^+ ; \tau u_0 \in W\}$ and $M = \{\tau \in \mathbf{R}^+ ; \tau u_0 \in B\}$. The sets L and M are open connected sets with $L \neq \phi$ and $M \neq \phi$. Therefore, $\mathbf{R}^+ - (L \cup M) \neq \phi$. Set $\tau_0 = \min\{\mathbf{R}^+ - (L \cup M)\}$ and $\tau_1 = \max\{\mathbf{R}^+ - (L \cup M)\}$. By the definition we have $\tau_1 u_0 \in \partial K$, $\tau u_0 \in W$ if $\tau < \tau_0$ and $\tau u_0 \in B$ if $\tau > \tau_1$. We will show that $\tau_0 = \tau_1$. Since $\tau_1 \tau_0^{-1} v(s\,;\,\tau_0 u_0)$ is a subsolution of (TH) with the initial value $\tau_1 u_0$, we obtain

$$v(y,s\,;\tau_1 u_0) \geq \frac{\tau_1}{\tau_0} v(y,s\,;\tau_0 u_0) \quad \text{in} \quad Q. \tag{3.1}$$

Therefore, there exist $f \in \omega(v(s;\tau_1 u_0))$ and $g \in \omega(v(s;\tau_0 u_0))$ such that

$$f \geq \frac{\tau_1}{\tau_0} g \quad \text{in} \quad \mathbf{R}^\mathbf{N}. \tag{3.2}$$

By Proposition 5,

$$\omega(v(s;\tau_0 u_0)) \cup \omega(v(s;\tau_1 u_0)) \subset S. \tag{3.3}$$

It follows from (3.2), (3.3) and Lemma 2 that $f = g$ and $\tau_0 = \tau_1$. Thus we have proved (1.9).

Now we know that the map $P|_G : G \to \partial K$ is one to one and onto. Cleary, $(P|_G)^{-1}$ is continuous. Thus, it suffices to show that $P|_G$ is continuous. Let $\{x_n\}_{n\in\mathbf{N}} \subset G$ be a sequence in X such that $x_n \to x_0$ in X as $n \to \infty$ for some $x_0 \in G$. We will prove

$$Px_n \to Px_0 \quad \text{in} \quad X \quad \text{as} \quad n \to \infty. \tag{3.4}$$

We fix a number $\lambda > 1$. We can easily check

$$\lambda \|Px_0\| x_n \to \lambda Px_0 \quad \text{in} \quad X \quad \text{as} \quad n \to \infty. \tag{3.5}$$

Since $\lambda Px_0 \in B$ and B is open, $\lambda\|Px_0\|x_n \in B$ for sufficiently large n. Thus we obtain

$$\sup_{n\in\mathbf{N}} \|Px_n\| < \infty, \tag{3.6}$$

By (3.6) there exist a subsequence (still denoted $\{Px_n\}_n$) and a number $a > 0$ such that

$$\|Px_n\| \to a \quad \text{as} \quad n \to \infty. \tag{3.7}$$

It follows that

$$(3.8)\quad \|Px_n - ax_0\| \le \|\, Px_n - \|Px_n\|x_0 \,\| + \|\, \|Px_n\|x_0 - ax_0 \,\| \\ = \|Px_n\|\, \|x_n - x_0\| + |\, \|Px_n\| - a \,| \to 0 \quad \text{as} \quad n \to \infty.$$

Therefore, we deduce that $ax_0 \in \partial K$ and $a = \|Px_0\|$. Since a is a unique constant independant of the way to choose a subsequence, we obtain (3.4).

(iii) By the proof of (ii) we can derive that Int $(K) = W$. Thus we have (1.11) by Proposition 3. The estimate (1.10) follows from (1.11).

(iv) Let $u_0 \in \partial K$. By the proof of Proposition 5 we have

$$(3.9)\qquad \|v(s\,;u_0)\|_q \sim 1$$

for $q = p_0$ and $q = \infty$. Since $v(s\,;u_0)$ is bounded in X for $s \ge 0$, we have

$$\|v(s\,;u_0)\|_1 = O(1).$$

Therefore, (3.9) actually holds for any $q \in [1, \infty]$. Combining (2.22) and (3.9), we deduce (1.12). We obtain from (3.9) and Proposition 5 that $\omega(v(s\,;u_0)) \subset S$.

(v) We have already obtained (1.14) (*see* Proposition 2). ■

Finally we give two remarks concerning Theorem 1.

Remark 1. – We observe that the Haraux-Weissler self-similar solution $w(t)$ given in (1.6) satisfies $\|w(t)\|_q \to \infty$ as $t \to \infty$ for $q \in [1, N(p-1)/2)$. This fact also leads to the unboundedness of K in X.

Remark 2. – With respect to (iv) of Theorem 1 we have the following result:

PROPOSITION. – *Assume* (1.1) *and* $p \in \mathbf{N}$. *Then for any* $u_0 \in \partial K$ *the set* $\omega(v(s\,;u_0)) \subset S$ *consists of only one element,* i.e. $\omega(v) = \{\varphi\}$, *where* φ *is an element of* S.

Outline of the proof of Proposition. – We will apply the method by Simon [Si]. Let $\varphi \in \omega(v(s\,;u_0))$ for $u_0 \in \partial K$. We will derive $\omega(v) = \{\varphi\}$. We set $\mathcal{E}(u) := \hat{E}(u + \varphi)$ (*see* (2.1) for the definition of $\hat{E}$) and $w(s) := v(s\,;u_0) - \varphi$. Then $w(s)$ satisfies

$$(3.10)\qquad w_s = \mathcal{M}(w),$$

$$(3.11)\qquad \frac{d}{ds}\mathcal{E}(w(s)) = -|w_s|_2^2.$$

where we set

$$(3.12)\quad \begin{aligned}\mathcal{M}(w) :=&\Delta(w+\varphi) + \frac{y}{2}\cdot\nabla(w+\varphi) + \frac{w+\varphi}{p-1} + (w+\varphi)^p \\ =&\Delta w + \frac{y}{2}\cdot\nabla w + \frac{w}{p-1} + (w+\varphi)^p - \varphi^p.\end{aligned}$$

Let $H^2_\rho := \{f \in H^1_\rho\,;\, \nabla f \in H^1_\rho\}$. The space H^2_ρ is a Hilbert space with the norm $|f|_\rho := \left(\sum_{i,j=1}^N |\partial^2 f/\partial y_i \partial y_j|_2^2\right)^{1/2}$ for $f \in H^2_\rho$. Since $p \in \mathbf{N}$ and $H^2_\rho \hookrightarrow L^{2p}_\rho$ (*cf.* [Kavi, Lemma 2.1]), the map $\mathcal{M} : H^2_\rho \to L^2_\rho$ is analytic. We set $L := d\mathcal{M}(0)$. We have

$$(3.13)\qquad Lw = \Delta w + \frac{y}{2}\cdot\nabla w + \frac{w}{p-1} + p\varphi^{p-1}w.$$

We define $A : L^2_\rho \to L^2_\rho$ by

$$(3.14)\qquad Aw := \Delta w + \frac{y}{2}\cdot\nabla w$$

with $D(A) = H^2_\rho$. We know (*see* [Kavi, Lemma 2.1]) that $-A$ is a positive self-adjoint operator with compact inverse. Since $\varphi \in L^\infty$, there exists a complete ortho-normal system $\{\psi_j\}_{j=1}^\infty$ for L^2_ρ which consists of eigenfunctions of the operator L. We denote by Π the orthogonal projection of L^2_ρ onto the (finite-dimensional) subspace $\{\psi \in H^2_\rho\,;\, L\psi = 0\}$. It follows that the map $\mathcal{L} : H^2_\rho \to L^2_\rho$ defined by

$$(3.15)\qquad \mathcal{L}u := \Pi u + Lu$$

is a one to one and onto map. We define $\mathcal{N} : H^2_\rho \to L^2_\rho$ by

$$(3.16)\qquad \mathcal{N}(u) := \Pi u + \mathcal{M}(u).$$

Then, $\mathcal{N}$ is analytic with $d\mathcal{N}(0) = \mathcal{L}$. Therefore, we obtain from the same argumentation as in [Si, Section 2] that there are constants $\theta \in (0, 1/2)$ and $\sigma \in \mathbf{R}^+$ such that if $u \in H^2_\rho$ with $|u|_\rho \le \sigma$ then

$$(3.17)\qquad |\mathcal{M}(u)|_2 \ge |\mathcal{E}(u) - \mathcal{E}(0)|^{1-\theta}.$$

Let $|w(s)|_\rho < \sigma$ for $s \in [s_1, s_2]$. Then, by (3.11) and (3.17),

$$(3.18)\quad \begin{aligned}\frac{d}{ds}\{\mathcal{E}(w(s)) - \mathcal{E}(0)\}^\theta &= \theta\{\mathcal{E}(w(s)) - \mathcal{E}(0)\}^{\theta-1}\cdot(-|w_s|_2^2) \\ &= -\theta\{\mathcal{E}(w(s)) - \mathcal{E}(0)\}^{\theta-1}\cdot|w_s|_2\,|\mathcal{M}(w(s))|_2 \\ &\le -\theta|w_s|_2 \qquad \text{for}\quad s \in [s_1, s_2].\end{aligned}$$

It follows that

$$(3.19)\quad |w(s_2)-w(s_1)|_2 \le \theta^{-1}\int_{s_1}^{s_2} |w_s(s)|_2 ds \le \theta^{-1}\{\mathcal{E}(w(s_1))-\mathcal{E}(0)\}^{\theta}.$$

Since $\mathcal{M}(0) = 0$, we can verify that there exist constants $C_j \in \mathbf{R}^+$ $(1 \le j \le 3)$ such that for $s,\ \tau > 0$

$$(3.20)\qquad |w(s+\tau)|_2 \le \exp(C_1\tau)|w(s)|_2,$$

$$(3.21)\qquad |w(s+\tau)|_\rho \le C_2\left(1+\frac{1}{\tau}\right)\exp(C_3\tau)|w(s)|_2.$$

By (3.19), (3.20), (3.21) and the assumption: $0 \in \omega(w(s))$, we obtain that $w(s) \to 0$ in L^2_ρ (and also in H^2_ρ) as $s \to \infty$. Hence, $\omega(v(s)) = \{\varphi\}$. ■

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REFERENCES

[CL] T. Cazenave and P. L. Lions, Solutions globales d'équations de la chaleur semi linéaires, *Comm. Partial Differ. Equations*, Vol. **9**, 1984, pp. 955-978.

[D] E. DiBenedetto, Continuity of weak solutions to a general porous medium equation, *Indiana Univ. Math. J.*, Vol. **32**, 1983, pp. 83-118.

[EZ] Escobedo and Zuazua, Lare time behavior for convection - diffusion equations in $\mathbf{R}^\mathbf{N}$, *J. Func. Anal.*, Vol. **100**, 1991, pp. 119-161.

[F] H. Fujita, On the blowing up of solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$, *J. Fac. Sci. Univ. Tokyo Sect. 1*, Vol. **13**, 1966, pp. 109-124.

[G] Y. Giga, A bound for global solutions of semilinear heat equations, *Comm. Math. Phys.*, Vol. **103**, 1986, pp. 415-421.

[GS] B. Gidas and J. Spruck, A priori bounds for positive solutions of nonlinear elliptic equations, *Comm. P. D. E.*, Vol. **6**, 1981, pp. 883-901.

[HW] A. Haraux and F. B. Weissler, Non-uniqueness for a semilinear initial value problem, *Indiana Univ. Math. J.*, Vol. **31**, 1982, pp. 167-189.

[Kavi] O. KAVIAN, Remarks on the large time behavior of a nonlinear diffusion equation, *Annal. Institut Henri Poincaré - Analyse non linéaire*, Vol. **4**, 1987, pp. 423-452.

[Kawa] T. KAWANAGO, Existence and behavior of solutions for $u_t = \Delta(u^m) + u^l$, *Preprint.*

[Le] H. A. LEVINE, The role of critical exponents in blowup theorems, *SIAM Review*, Vol. **32**, 1990, pp. 262-288.

[Li] P. L. LIONS, Asymptotic behavior of some nonlinear heat equations, *Physica 5D*, 1982, pp. 293-306.

[LN] T. Y. LEE and W. M. NI, Global existence, large time behavior and life span of solutions of semilinear parabolic Cauchy problem, *Trans. Amer. Math. Soc.*, Vol. **333**, 1992, pp. 365-371.

[NST] W. M. NI, P. E. SACKS and J. TAVATZIS, On the asymptotic behavior of solutions of certain quasilinear equation of parabolic type, *J. Differ. Equations*, Vol. **54**, 1984, pp. 97-120.

[S] P. E. SACKS, Continuity of solutions of a singular parabolic equation, *Nonlinear Anal.*, Vol. **7**, 1983, pp. 387-404.

[Si] L. SIMON, Asymptotics for a class of non-linear evolution equations, with applications to geometric problem, *Annals Math.*, Vol. **118**, 1983, pp. 525-571.

[W] X. WANG, On the Cauchy problem for reaction - diffusion equations, *Trans. Amer. Math. Soc.*, Vol. **337**, 1993, pp. 549-590.

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