Ann. Inst. Henri Poincaré,

Vol. 13, n° 1, 1996, p. 95-115 *Analyse non linéaire*

Solutions of elliptic equations with indefinite nonlinearities via Morse theory and linking

by

Stanley ALAMA*

McMaster Univ., Dept. of Math. & Stat., Hamilton Ont., L8S 4K1 Canada
alamas@icarus.math.mcmaster.ca

and

Manuel DEL PINO†

Univ. of Chicago, Dept. of Math., Chicago IL 60637
delpino@math.uchicago.edu

ABSTRACT. – We consider the Dirichlet problem for the equation $-\Delta u = \lambda u + h(x)f(u)$, with h changing sign. In particular, we study existence of nontrivial solutions in the case where f has superlinear growth, but is not assumed to be odd. Two different approaches are used: one involving Morse theory and one using min-max methods.

RÉSUMÉ. – Nous étudions le problème de Dirichlet pour l'équation $-\Delta u = \lambda u + h(x)f(u)$, où h est une fonction qui change de signe. En particulier, nous établissons l'existence de solutions non triviales quand f est surlinéaire, mais pas nécessairement impair. Nous nous servons de deux approches différentes, l'une basée sur la théorie de Morse, et l'autre sur les méthodes d'enlacement.

* Supported by an NSERC (Canada) Research Grant.

† Supported in part by NSF grant DMS 9200098 and by EC grant CI1*CT93-0323 CCE.

Annales de l'Institut Henri Poincaré - Analyse non linéaire - 0294-1449
Vol. 13/96/01/$ 4.00/

1. INTRODUCTION

In this paper we seek nontrivial solutions for:

$$(1.1)_\lambda \qquad \begin{cases} -\Delta u = \lambda u + h(x)f(u) \\ u|_{\partial\Omega} = 0 \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded open set with smooth boundary, $h \in C^\alpha(\overline{\Omega})$ changes sign in Ω, and f is a continuous function which satisfies certain superlinear, subcritical growth conditions. We mainly focus on the case $\lambda \geq \lambda_1(\Omega)$, where $\lambda_1(\Omega)$ denotes the smallest Dirichlet eigenvalue of $-\Delta$ in Ω.

The existence of positive solutions to $(1.1)_\lambda$ with indefinite h has already been established in various contexts. If the domain Ω is replaced by a compact manifold of dimension $N \geq 3$, the critical exponent case, $f(u) = u^{\frac{N+2}{N-2}}$, arises in the prescribed scalar curvature problem (*see* Kazdan & Warner [14]). For manifolds carrying scalar-flat metrics, sufficient conditions for the existence of positive solutions were given by Escobar & Schoen [12]. Ouyang [16] studied $(1.1)_\lambda$ on a compact manifold with homogeneous nonlinearities $f(u) = |u|^{p-2}u$, $p > 2$, via bifurcation analysis. Results for more general subcritical f were obtained by Alama & Tarantello [1] and by Berestycki, Capuzzo-Dolcetta & Nirenberg ([5], [7]). For instance, it is proven in [1] that if $f(u) \sim |u|^{q-2}u$ near zero, and

$$(1.2) \qquad \int_\Omega h(x) e_1^q \, dx < 0,$$

(where e_1 denotes the eigenfunction corresponding to $\lambda_1(\Omega)$), then there is a finite value $\Lambda > \lambda_1(\Omega)$ such that equation $(1.1)_\lambda$ admits a positive solution for $\lambda \in (\lambda_1(\Omega), \Lambda)$, but $(1.1)_\lambda$ has no positive solutions for any $\lambda > \Lambda$. If in addition f satisfies the estimate (1.7) given below, then $(1.1)_\lambda$ admits a second positive solution for $\lambda \in (\lambda_1(\Omega), \Lambda)$. In fact, condition (1.2) is an essential assumption when finding positive solutions of $(1.1)_\lambda$ with $\lambda \geq \lambda_1(\Omega)$, in the sense that it is necessary for their existence in case $f(u) = |u|^{p-2}u$. (*See* also Berestycki, Capuzzo-Dolcetta & Nirenberg [6], and Tehrani [19] for related results.) Without the sign condition (1.2) but assuming that f is odd, Alama & Tarantello [1] also prove the existence of infinitely many nontrivial solutions for (nearly) every $\lambda \in \mathbb{R}$.

Our objective is to find nontrivial (possibly changing-sign) solutions of $(1.1)_\lambda$, without imposing either a symmetry assumption on f or a sign condition on h (such as (1.2)). We conjecture that $(1.1)_\lambda$ admits a nontrivial solution for all $\lambda \in \mathbb{R}$, assuming only superlinear growth at zero and power growth at infinity (*see* (1.3) and (1.7) introduced below). In this paper we provide some progress in this direction. We introduce the usual action functional,

$$J_\lambda(u) = \int_\Omega \frac{1}{2}\left(|\nabla u|^2 - \lambda u^2\right) - h(x)F(u)\,dx$$

where $F(u) = \int_0^u f(t)\,dt$, and seek nontrivial critical points on $H_0^1(\Omega)$.

The main difficulty which arises in this problem lies in devising a min-max critical value for J_λ when $\lambda \geq \lambda_1(\Omega)$. Typically, one seeks a manifold whose boundary Γ links with a subset of $\{J_\lambda \geq \varepsilon > 0\}$, and for which $\sup_\Gamma J_\lambda < \varepsilon$. Then a positive critical value may be defined by a min-max over all surfaces having common boundary Γ. When $h(x) \geq 0$ these sets are easily constructed, using linear subspaces of eigenfunctions of the Laplacian, and (when f has subcritical growth) the equation admits a nontrivial solution for any $\lambda \in \mathbb{R}$ (cf. Rabinowitz [17]). If $h(x) \leq 0$ the results are quite different, but existence (and, in this case, nonexistence) theorems can likewise be derived from a judicious choice of linear subspaces (*see* [11] and [2]). When h changes sign both the quadratic and superquadratic terms of J_λ are indefinite, and it is not clear whether there exist linear subspaces on which both terms have the correct sign. The special case of odd f is simpler since it is not necessary to construct these linking sets explicitly. This is because the Krasnoselskii genus provides for a stronger intersection property of symmetric sets, based on the Borsuk-Ulam Theorem (*see* [17], [1]).

A second difficulty created by the indefiniteness of h is in verifying the Palais-Smale condition for the functional J_λ. As has already been remarked in [1], when h changes sign familiar inequality conditions relating F and $f = F'$ are not helpful in deriving estimates on Palais-Smale sequences. In this paper, we follow [1] in imposing sufficient conditions on h or f in order to ensure that the (PS) condition holds for I_λ. (*See* Proposition 2.6.)

In order to deal with the first difficulty we use topological arguments in the spirit of earlier work by Hofer [13], Z. Q. Wang [20], and K. C. Chang [9]. Before stating our first two results, we introduce the following hypotheses and definitions: f satisfies

$$\lim_{u\to 0} \frac{f(u)}{u} = 0, \tag{1.3}$$

and for some p with $2 < p < 2^*$,

$$f(u) - |u|^{p-2}u = o(|u|) \qquad \text{as } |u| \to \infty. \tag{1.4}$$

In addition, we suppose that h has a "thick" zero-set,

$$\overline{\{x:\ h(x)>0\}}\cap\overline{\{x:\ h(x)<0\}}=\emptyset. \tag{1.5}$$

Finally, as in Ouyang [16] we define

$$\lambda_* = \inf\{\|\nabla v\|_2^2 :\ \|v\|_2 = 1,\ \int_\Omega h(x)|v|^p\,dx = 0\}.$$

Remark 1.1. – Straightforward arguments show that:

(a) $\lambda_* \geq \lambda_1(\Omega)$, with equality if and only if $\displaystyle\int_\Omega h(x)e_1^p\,dx = 0$.

(b) $\lambda_* < \lambda_1(\Omega_0)$, where we denote by

$$\Omega_0 = \{x \in \Omega:\ h(x) = 0\},$$

and by $\lambda_1(\Omega_0)$ the first Dirichlet eigenvalue of $-\Delta$ in Ω_0.

(c) One may find functions h for which λ_* is arbitrarily large. We provide an example to illustrate this fact in the Appendix (Section 4).

Under the above hypotheses we prove the following two existence and multiplicity results:

THEOREM 1.2. – *Assume* (1.3), (1.4), *and* (1.5) *hold,* $h \in C^\alpha(\overline{\Omega})$ *changes sign in* Ω, $\lambda < \lambda_*$, *and* λ *is not a Dirichlet eigenvalue of* $-\Delta$ *in* Ω. *Then* $(1.1)_\lambda$ *admits a nontrivial solution.*

Combining the topological information provided in the proof of Theorem 1.2 with previous results of Wang [20] and Alama & Tarantello [1], we may obtain some multiplicity results for $(1.1)_\lambda$:

THEOREM 1.3. – *Assume* (1.3), (1.4), *and* (1.5) *hold, and* $h \in C^\alpha(\overline{\Omega})$ *changes sign in* Ω.

i. If $\lambda < \lambda_1(\Omega)$, *then* $(1.1)_\lambda$ *admits at least* three *nontrivial solutions.*

ii. If

$$\lim_{u\to 0}\frac{f(u)}{|u|^{q-2}u} = \alpha > 0 \tag{1.6}$$

for some $q > 2$, *and* (1.2) *holds, then there exists* $\bar\lambda > \lambda_1(\Omega)$ *such that* $(1.1)_\lambda$ *admits at least* five *nontrivial solutions for* $\lambda \in (\lambda_1(\Omega), \bar\lambda)$.

Remark 1.4. – Note that when $\lambda < \lambda_1(\Omega)$ the functional J_λ exhibits a mountain-pass structure, and hence the first two solutions (one positive and one negative) found in part (a) of Theorem 1.3 may be obtained under the less stringent conditions of Proposition 2.6. Likewise, four of the solutions (two positive and two negative) claimed in part (b) of Theorem 1.3 may be derived via Theorem 2.7 of [1] under slightly weaker hypotheses.

Under the assumptions above, when $\lambda < \lambda_*$ we may explicitly compute the topology of negative sublevel sets of J_λ and argue indirectly using the Morse inequalities. The hypotheses (1.4) and (1.5) play a central but technical role in constructing a homotopy equivalence between these sublevel sets and an infinite dimensional sphere. Indeed, condition (1.5) was also introduced in a technical capacity in [1] (also [3]) in verifying the Palais-Smale condition for variational problems associated to indefinite semilinear problems. (*See* Proposition 2.6.) Rather different conditions on the zero-set of h were employed by Berestycki, Capuzzo-Dolcetta & Nirenberg [5] in studying positive solutions to $(1.1)_\lambda$. There they consider general f (satisfying (1.6) and (1.2)), but replace condition (1.5) with non-degeneracy conditions such as $\nabla h \neq 0$ on Ω_0. The methods of [5] are based on *a priori* estimates and fixed point arguments, and hence differ considerably from our variational approach to the theorems stated above. Indeed, the methods of [6] can not be expected to apply directly in our setting, since the multiplicity result of [1, Theorem 3.1] (in the symmetric case $f(-u) = -f(u)$) demonstrates that (changing-sign) solutions of $(1.1)_\lambda$ are not in general *a priori* bounded.

Denote by $\lambda_2(\Omega)$ the second (min-max) eigenvalue of $-\Delta$ in $H_0^1(\Omega)$. In Section 3 we present a different approach, based on linking, to treat $(1.1)_\lambda$ when $\lambda_1(\Omega) \leq \lambda < \lambda_2(\Omega)$. This approach has the advantage that it demands fewer hypotheses on f and h, although it may well be that $\lambda_* > \lambda_2(\Omega)$ for certain functions h. We prove:

THEOREM 1.5. – *Suppose* (1.3) *holds,*

$$f(u) - |u|^{p-2}u = O(|u|) \qquad \text{as } |u| \to \infty, \tag{1.7}$$

for some $2 < p < 2^*$, *and* $h \in C^\alpha(\overline{\Omega})$ *changes sign in* Ω. *If* $\lambda_1(\Omega) \leq \lambda < \lambda_2(\Omega)$ *and* λ *is not a Dirichlet eigenvalue of* $-\Delta$ *in* Ω_0, *then* $(1.1)_\lambda$ *admits a nontrivial solution.*

The requirement that λ not be a Dirichlet eigenvalue of $-\Delta$ in Ω_0 is related to the Palais-Smale condition (*see* Proposition 2.6).

Finally, we note that the each of the above results continues to hold if $-\Delta$ is replaced by any symmetric, uniformly elliptic operator with self-adjoint boundary condition, or if $(1.1)_\lambda$ is posed on a compact manifold.

2. THE CASE $\lambda < \lambda_*$

We define the Dirichlet Laplacian on any measurable subset $\omega \subset \Omega$ as the unique self-adjoint operator associated to the quadratic form $a(u) = \int_\Omega |\nabla u|^2 \, dx$ with form domain

$$H^1_D(\omega) = \{u \in H^1_0(\Omega) : \ u(x) = 0 \text{ for a.e. } x \in \Omega \setminus \omega\}.$$

When $\partial\omega$ is smooth $H^1_D(\omega)$ coincides with $H^1_0(\omega)$, and we obtain the classical Laplace operator with Dirichlet condition on $\partial\omega$. Throughout the paper we will denote the (Dirichlet) spectrum of $-\Delta$ on ω as $\sigma(\omega) = \{\lambda_i(\omega); \ i = 1, 2, \ldots\}$. Also, the letter c will be indiscriminately used to denote various constants whose exact value is irrelevant.

We collect here the hypotheses for Theorem 1.2 and Theorem 1.3. First, f is superlinear at zero, *i.e.*, f satisfies (1.3). At infinity, we require f to be asymptotically homogeneous in the sense of (1.4) : for some $2 < p < 2^*$ we have,

$$\left\{ \begin{array}{l} \text{for all } \varepsilon > 0 \text{ there exists a constant } C = C(\varepsilon) \text{ so that} \\ \qquad |f(u) - |u|^{p-2}u| \leq \varepsilon |u| + C. \end{array} \right. \tag{2.1}$$

(As usual, we define $2^* = \frac{2N}{N-2}$ when $N \geq 3$ and $2^* = +\infty$ for $N = 1, 2$.) As consequences, we have:

$$F(u) - \frac{1}{p}|u|^p = o(u^2) \tag{2.2}$$

$$F(u) - \frac{1}{p}f(u)u = o(u^2) \tag{2.3}$$

as $|u| \to \infty$. In addition, we assume that:

$$\Omega_+ = \{x \in \Omega : \ h(x) > 0\} \neq \emptyset, \qquad \text{and} \qquad \Omega_- = \{x \in \Omega : \ h(x) < 0\} \neq \emptyset,$$

and we rewrite (1.5) as

$$\overline{\Omega_+} \cap \overline{\Omega_-} = \emptyset. \tag{2.4}$$

The first step in proving Theorem 1.2 and Theorem 1.3 is to develop the connection between λ_* and the topology of the sublevel sets of the functional J_λ.

LEMMA 2.1. – *Assume that f satisfies* (2.1) *and $\lambda < \lambda_*$. There exists a constant $K_1 > 0$ such that if $K \geq K_1$ and $J_\lambda(u) \leq -K$, then*

$$\int_\Omega h(x)|u|^p\,dx > 0.$$

Proof. – We suppose the contrary: for all $n > 0$ there exists u_n with $J_\lambda(u_n) \leq -n$ and $\int_\Omega h(x)|u|^p\,dx \leq 0$. First note that $\|u_n\|_2 \to \infty$. Indeed, $J_\lambda(u_n) \leq -n$ implies:

$$\begin{aligned}\frac{\lambda}{2}\|u_n\|_2^2 &\geq n - \frac{1}{p}\int_\Omega h(x)|u_n|^p\,dx + \left[\int_\Omega h(x)\Big(\frac{|u_n|^p}{p} - F(u_n)\Big)\,dx\right]\\ &\geq n - \varepsilon\|u_n\|_2^2 - C_\varepsilon\end{aligned}$$

using (2.3) Taking ε sufficiently small, we have $\|u_n\|_2 \to \infty$ as claimed.

Set $v_n = u_n/\|u_n\|_2$. We have

$$\frac{1}{2}\|\nabla v_n\|_2^2 - \frac{\lambda}{2} - \int_\Omega h(x)\frac{F(u_n)}{\|u_n\|_2^2}\,dx \leq 0. \tag{2.5}$$

Using (2.2), we see

$$\int_\Omega h(x)\frac{F(u_n)}{\|u_n\|_2^2}\,dx = \|u_n\|_2^{p-2}\int_\Omega h(x)|v_n|^p + o(1) \leq o(1),$$

since we are assuming that $\int_\Omega h(x)|u_n|^p\,dx \leq 0$. Hence, (2.5) yields

$$\frac{1}{2}\|\nabla v_n\|_2^2 \leq \frac{\lambda}{2} + o(1),$$

and $v_n \rightharpoonup v_0$ in $H_0^1(\Omega)$ with $\|v_0\|_2 = 1$ and

$$\|\nabla v_0\|_2^2 \leq \lambda. \tag{2.6}$$

Moreover, from (2.3), (2.5) we have

$$\begin{aligned}0 \geq \|u_n\|_2^{p-2}\int_\Omega h(x)|v_n|^p\,dx &= \int_\Omega h(x)\frac{F(u_n)}{\|u_n\|_2^2}\,dx + o(1)\\ &\geq \frac{1}{2}\big(\|\nabla v_n\|_2^2 - \lambda\|v_n\|_2^2\big) + o(1)\\ &\geq -\frac{\lambda}{2} + o(1).\end{aligned}$$

Hence,

$$\int_\Omega h(x)|v_0|^p\,dx = \lim_{n\to\infty}\int_\Omega h(x)|v_n|^p\,dx = 0.$$

But when $\lambda < \lambda_*$, (2.6) contradicts the definition of λ_*. ◇

We introduce the notation

$$J_\lambda^a = \{v \in H_0^1(\Omega) : \ J_\lambda(v) \le a\}.$$

Then, Lemma 2.1 implies that there exists a constant K_1 such that

$$J_\lambda^{-K} \subset A = \left\{v \in H_0^1(\Omega) : \ \int_\Omega h(x)|v|^p\,dx > 0\right\}$$

for all $K \ge K_1$. In fact, we will show that J_λ^{-K} is a *retract* of A for all K sufficiently large.

Note that if $u \in A$, then $J_\lambda(tu) \to -\infty$ as $t \to \infty$. We will now show that there is a continuous choice of $T = T(u)$ for which $J_\lambda(T(u)u) \le -K$ for any large K and for all $u \in A$. Then we may use this construction to define our retraction.

LEMMA 2.2. – *Suppose f satisfies* (2.1) *and $\lambda < \lambda_*$. Then there exists a constant $K_2 \ge K_1$ such that whenever $J_\lambda(u) \le -K_2$, then*

$$\left.\frac{\partial}{\partial t}\right|_{t=1} J_\lambda(tu) < 0$$

Proof. – From Lemma 2.1, if $K \ge K_1$ and $J_\lambda(u) \le -K$, then $u \in A$. Suppose (to obtain a contradiction) that for every $n \ge K_1$ there exists $u_n \in A$ with $J_\lambda(u_n) \le -n$ and

$$\left.\frac{\partial}{\partial t}\right|_{t=1} J_\lambda(tu_n) \ge 0.$$

Then,

$$\frac{1}{2}\|\nabla u_n\|_2^2 - \frac{\lambda}{2}\|u_n\|_2^2 - \int_\Omega h(x)F(u_n)\,dx \le -n \tag{2.7}$$

$$\|\nabla u_n\|_2^2 - \lambda\|u_n\|_2^2 - \int_\Omega h(x)f(u_n)u_n\,dx \ge 0 \tag{2.8}$$

As in Lemma 2.1, we claim that $\|u_n\|_2 \to \infty$. Indeed, if along some subsequence $\|u_n\|_2 \le C$, then

$$\left(\frac{1}{2}-\frac{1}{p}\right)(\|\nabla u_n\|_2^2 - \lambda\|u_n\|_2^2)$$
$$\le -n + \int_\Omega h(x)\left[F(u_n) - \frac{1}{p}f(u_n)u_n\right]dx \le C$$

(via hypothesis (2.3)). Hence, a subsequence $u_n \rightharpoonup u_0$ in $H_0^1(\Omega)$, and

$$J_\lambda(u_0) \le \liminf_{n\to\infty} J_\lambda(u_n) = -\infty,$$

which is impossible. Therefore $\|u_n\|_2 \to \infty$.

As before, we set $v_n = u_n/\|u_n\|_2$. Then (2.7), (2.8) yield:

$$\begin{aligned}\|\nabla v_n\|_2^2 - \lambda\|v_n\|_2^2 &\le \frac{C}{\|u_n\|_2}\int_\Omega h(x)\left[F(u_n) - \frac{1}{p}f(u_n)u_n\right]dx + o(1)\\ &\le o(1).\end{aligned}$$

Hence, $v_n \rightharpoonup v_0$ in $H_0^1(\Omega)$, and $\|\nabla v_0\|_2^2 \le \lambda$. Combining (2.7) and (2.8) in a different way we have

$$\int_\Omega h(x)\left[\frac{1}{2}f(u_n)u_n - F(u_n)\right]dx \le -n < 0,$$

and so (in view of (2.1) and (2.2)) we obtain:

$$\begin{aligned}0 &\le \left(\frac{1}{2} - \frac{1}{p}\right)\|u_n\|_2^{p-2}\int_\Omega h(x)|v_n|^p\,dx\\ &= \int_\Omega h(x)\frac{\frac{1}{2}f(u_n)u_n - F(u_n)}{\|u_n\|_2^2}\,dx + o(1) \le o(1).\end{aligned}$$

Since $v_n \to v_0$ strongly in $L^p(\Omega)$, we must have

$$\int_\Omega h(x)|v_0|^p\,dx = 0,$$

But $\|\nabla v_0\| \le \lambda < \lambda_*$, so this contradicts the definition of λ_*. ◇

We may now construct our retraction. Let $u \in A$. By Lemma 2.2, there is a *unique* $T = T(u)$ such that $J_\lambda(Tu) = -K_2$. Moreover, by applying the Implicit Function Theorem to the map

$$\begin{aligned}\mathcal{F} &: \mathbb{R} \times H_0^1(\Omega) \to \mathbb{R}\\ \mathcal{F}(t,u) &= J_\lambda(tu)\end{aligned}$$

at $(t,u) = (T(u),u)$ it follows that $T(u)$ is a continuous function on A. Set $\tilde{T}(u) = \max\{T(u),1\}$, also a continuous function on A. Then, define:

$$\begin{aligned}\eta(s,u) &= (1-s)u + s\tilde{T}(u)u\\ &= \left[1 + s(\tilde{T}(u) - 1)\right]u\end{aligned}$$

Clearly, $\eta(0,u) = u$ for all $u \in A$, and by Lemma 2.2, $\eta(1,u) = \tilde{T}(u)u \in J_\lambda^{-K_2}$ for all $u \in A$. Furthermore, Lemma 2.2 and the definition of $\tilde{T}(u)$ ensure that $\eta(s,u) = u$ for all $u \in J_\lambda^{-K_2}$. In conclusion, we have shown that $\eta: [0,1] \times A \to J_\lambda^{-K_2}$ is a strong deformation retraction:

LEMMA 2.3. – *Suppose f satisfies* (2.1) *and $\lambda < \lambda_*$. Then, $J_\lambda^{-K_2}$ is a strong retract of A. In particular, $J_\lambda^{-K_2}$ and A are homotopically equivalent.*

Now we examine the topology of the set A. Namely, we will show that A is homotopically equivalent to an infinite dimensional sphere (and hence is contractible.)

LEMMA 2.4. – *Suppose* (2.4) *holds. Then, $B = H_D^1(\Omega_+) \setminus \{0\}$ is a retract of A. (In particular, A and B are homotopically equivalent.)*

Proof. – The retraction will be constructed in two steps: first we use hypothesis (2.4) to truncate $u \in A$ to have support only in $\Omega_1 := \{x \in \Omega : \ h(x) \geq 0\}$. (This is *not* in itself a retraction.) Then we project the resulting function linearly into $H_D^1(\Omega_+)$.

By (2.4), there is a function $\psi \in C^\infty(\mathbb{R}^N)$ which satisfies:

$$\begin{cases} \psi(x) = 1 & \text{for all } x \in \overline{\Omega_+}, \\ \psi(x) = 0 & \text{for all } x \in \overline{\Omega_-}, \\ 0 \leq \psi(x) \leq 1 & \text{for all } x \in \Omega_0. \end{cases} \tag{2.9}$$

Consider also the projection operator $P: H_0^1(\Omega) \to H_D^1(\Omega_+)$. Then, define

$$\eta(s,u) = \begin{cases} (1-2s)u + 2s\psi u, & \text{if } 0 \leq s \leq \frac{1}{2}; \\ 2(1-s)\psi u + 2(s-\frac{1}{2})P(\psi u), & \text{if } \frac{1}{2} \leq s \leq 1. \end{cases}$$

Clearly, $\eta(0,u) = u$ if $u \in A$ and $\eta(s,u) = u$ for all $u \in H_D^1(\Omega_+)$ and for all $s \in [0,1]$. It remains only to show that $\eta(s,u) \in A$ for all $s \in [0,1]$, and that $\eta(1,u) \in H_D^1(\Omega_+) \setminus \{0\}$.

First, note that if $u \in A$ and $0 \leq s \leq \frac{1}{2}$,

$$\begin{aligned} \int_\Omega h(x)|\eta(s,u)|^p\,dx &= \int_{\Omega_+} h^+(x)|u|^p\,dx - (1-2s)^p \int_{\Omega_-} h^-(x)|u|^p\,dx \\ &\geq \int_\Omega h(x)|u|^p\,dx > 0. \end{aligned}$$

When $\frac{1}{2} < s \leq 1$, then clearly $\int_\Omega h(x)|\eta(s,u)|^p\,dx \geq 0$. If equality holds, then $\eta(s,u) = 0$ on Ω_+. But, in that case,

$$P(\psi u)(x) = -\left(\frac{1-s}{s-\frac{1}{2}}\right)\psi(x)u(x) \quad \text{on } \Omega_+.$$

This can only be the case if $P(\psi u) = 0$, and hence (from the definition of ψ in (2.9)) we have $u = 0$ on Ω_+. But this is impossible, since $u \in A$. Hence we may conclude that $\eta(s,u) \in A$ for all $s \in [0,1]$ and $u \in A$. A similar argument shows that

$$P(\psi u) = \eta(1,u) \in H_D^1(\Omega_+) \setminus \{0\}.$$

Hence, η is a retraction. $\diamondsuit$

Putting Lemma 2.4 and Lemma 2.3 together, we obtain:

PROPOSITION 2.5. – *Suppose f satisfies* (2.1) *and $\lambda < \lambda_*$. Let K_2 be chosen as in Lemma 2.2. Then $J_\lambda^{-K_2}$ is homotopically equivalent to an infinite dimensional sphere S^∞.*

At this point, we will use the topological information obtained in Lemmas 2.3 and 2.4 to infer the existence of nontrivial solutions of equation $(1.1)_\lambda$, via Morse Theory. An essential step is these arguments is the Palais-Smale condition for the functional J_λ. Although our hypotheses assume subcritical growth for J_λ at infinity, (PS) is a nontrivial issue in this problem. We will use the following version of (PS), derived in [1].

PROPOSITION 2.6. – *Suppose $\lambda \notin \sigma(\Omega_0)$. Then, J_λ satisfies (PS) provided either of the following two conditions hold:*

i. For some p, $2 < p < 2^*$, and constants $A, B > 0$ we have:

$$\left| f(u) - |u|^{p-2}u \right| \leq A|u| + B. \tag{2.10}$$

ii. Hypothesis (2.4) *holds, and f satisfies*

$$\lim_{|u|\to\infty} \frac{f(u)}{|u|^{p-2}u} = 1, \quad \text{and} \qquad |f'(u)| \leq A|u|^{p-2} + B$$

for some p, $2 < p < 2^*$, with $A, B > 0$ constants.

(Recall that $\sigma(\Omega_0)$ denotes the collection of Dirichlet eigenvalues of $-\Delta$ in Ω_0.) For the reader's convenience, we provide a short proof of Proposition 2.6 assuming condition i. in Section 4.

Proof of Theorem 1.2. – By hypothesis (1.4) and the fact $\lambda_* < \lambda_1(\Omega_0)$ (*see* Remark 1.1) we may conclude from Proposition 2.6 that the Palais-Smale condition holds under the given assumptions. Denote by $H_k(X,Y)$ the kth relative homology group with integer coefficients. For v an isolated critical point with $J_\lambda(v) = c$ and U_v a neighborhood of v, set

$$C_k(J_\lambda, v) = H_k(J_\lambda^c \cap U_v, [J_\lambda^c \cap U_v] \setminus \{v\}), \quad k = 0, 1, \ldots,$$

the kth critical group at the critical point v. If J_λ has only finitely many critical points $u_1, \ldots, u_n$ with $a < J_\lambda(u_j) < b$, we define the Morse numbers of the pair $(J_\lambda^b, J_\lambda^a)$ by

$$M_k = M_k(J_\lambda^b, J_\lambda^a) = \sum_{j=1}^{n} \dim C_k(J_\lambda, u_j).$$

Then, if $\beta_k = \dim H_k(J_\lambda^b, J_\lambda^a)$ are the Betti numbers of the pair $(J_\lambda^b, J_\lambda^a)$, the Morse inequalities require that:

$$\sum_{k=0}^{\infty}(-1)^k M_k = \sum_{k=0}^{\infty}(-1)^k \beta_k \tag{2.11}$$

(For a derivation of (2.11) and other facts from infinite-dimensional Morse Theory see Mawhin & Willem [15] or Chang [9].)

To prove Theorem 1.2, we argue by contradiction and assume that 0 is the only critical point of J_λ. By hypothesis $\lambda \notin \sigma(\Omega)$, and hence the Morse numbers of the pair $(H_0^1(\Omega), J_\lambda^{-K_2})$ are

$$M_k = \dim C_k(J_\lambda, 0) = \delta_{k,m},$$

where m is the Morse index of the (non-degenerate) critical point 0. On the other hand, from Proposition 2.5 and Proposition 2.6, $J_\lambda^{-K_2}$ is contractible in $H_0^1(\Omega)$. In particular,

$$H_k(H_0^1(\Omega), J_\lambda^{-K_2}) = H_k(H_0^1(\Omega), S^\infty) = \{0\}, \qquad \text{for all } k = 0, 1, 2, \ldots$$

and so the Betti numbers of the pair $(H_0^1(\Omega), J_\lambda^{-K_2})$ all vanish, $\beta_k = 0$, $k = 1, 2, \ldots$. But this contradicts the Morse inequality (2.11), and hence J_λ must admit a nontrivial critical point. ◇

Remark 2.7. – Extending the proof of Theorem 1.2 to include the case where λ is an eigenvalue of $-\Delta$ in Ω depends on obtaining some additional information about the critical groups $C_k(J_\lambda, 0)$. For example, if f satisfies a estimate such as (1.6) we may determine $C_k(J_{\lambda_1}, 0)$ and obtain a solution at $\lambda = \lambda_1(\Omega)$ when $\int_\Omega h(x) e_1^q \, dx$ is nonzero. In general, however, one cannot expect that critical points with Morse index $m \geq 1$ have nontrivial critical groups: consider $\varphi(x, y) = y^3 - x^2$ for which all critical groups are trivial.

Remark 2.8. – Theorem 1.2 may also be proven via min-max arguments. If $\lambda < \lambda_1(\Omega)$ we may obtain two solutions by a straightforward application of the Mountain Pass Theorem (*see* Remark 1.1), so we assume that the

Morse index of 0 is $m \geq 1$. Since 0 is a nondegenerate critical point, we can decompose $H_0^1(\Omega) = X^- \oplus X^+$ by the eigenspaces of $-\Delta$, with $\dim X^- = m$, and

$$\sup_{X^- \cap S_\delta} J_\lambda \leq -\varepsilon < 0 \quad \text{and} \quad \inf_{X^+ \cap S_\delta} J_\lambda \geq +\varepsilon > 0$$

for some choice of $\varepsilon, \delta > 0$. Now, if J_λ has no critical points other than zero, there exists a deformation (obtained by a negative pseudogradient flow) $\eta : [0,1] \times H_0^1(\Omega) \to H_0^1(\Omega)$, such that $\eta(0,v) = v$, $J_\lambda(\eta(1,v)) < -K_2$ for all $v \in S_\delta \cap X^-$, $\eta(t,\cdot)$ is a homeomorphism for each t, and $J_\lambda(\eta(t,v)) \leq -\varepsilon < 0$ for all $(t,v) \in [0,1] \times (S_\delta \cap X^-)$. Moreover, note that $\{\eta(t,v) : t \in [0,1], v \in S_\delta \cap X^-\}$ defines a smooth manifold of dimension m in $H_0^1(\Omega)$. Define a surface Σ_1 by:

$$\Sigma_1 = [B_\delta \cap X^-] \cup \{\eta(t,v) : t \in [0,1], v \in S_\delta \cap X^-\}.$$

By Proposition 2.5, $\partial\Sigma_1 = \{\eta(1,v) : v \in S_\delta \cap X^-\}$ is contractible in $J_\lambda^{-K_2}$, so we may "close" Σ_1 to form a surface Σ with $\sup_\Sigma J_\lambda \leq 0$, and which links the sphere $S_\delta \cap X^+$. The linking theorem of Benci & Rabinowitz [4] would then give a nontrivial critical point of J_λ. The details are left to the interested reader.

Proof of Theorem 1.3. – To prove part *i.* we follow Wang [20] (*see* also Theorem III.2.3 of Chang [9].) When $\lambda < \lambda_1(\Omega) < \lambda_1(\Omega_0)$ Proposition 2.6 applies and hence the Palais-Smale condition is satisfied by J_λ. Moreover, when $\lambda < \lambda_1(\Omega)$ and $p < \frac{2N}{N-2}$, zero is a strict local minimum for J_λ, so $m_k(0) := \dim C_k(J_\lambda, 0) = \delta_{0,k}$.

The first two nontrivial solutions will be obtained via the Mountain-pass Theorem. Define

$$J_\pm(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \lambda(u_\pm)^2 - \int_\Omega h(x)F(u_\pm)\,dx,$$

where $u_+ = \max\{u(x), 0\} \geq 0$, $u_- = \min\{u(x), 0\} \leq 0$. Clearly there exists $v_0 \geq 0$ such that $J_\pm(\pm v_0) = J_\lambda(\pm v_0) < 0$. Since $\lambda < \lambda_1(\Omega) < \lambda_1(\Omega_0)$, (PS) holds for J_λ, and also for $J_\pm$. Applying familiar arguments we obtain critical points $u_+ > 0$ of J_+ and $u_- < 0$ for J_-, which are nontrivial solutions of $(1.1)_\lambda$ and critical points for J_λ.

To obtain the third solution, we must determine the critical groups of $u_\pm$. By Corollary 8.5 in [15], if these are the only nontrivial solutions of $(1.1)_\lambda$ we have:

$$\dim C_k(J_\pm, u_\pm) = \delta_{1k}.$$

Denote by $\tilde{J}, \tilde{J}_\pm$ the restrictions of J_λ and $J_\pm$ (respectively) to $C_0^1(\overline{\Omega})$. By Theorem 1 in [10] (*see* also Corollary III.1.2 in [9]), the critical groups of $u_\pm$ are the same in $H_0^1(\Omega)$ as in the dense subspace $C_0^1(\overline{\Omega})$:

$$C_k(J_\lambda, u_\pm) = C_k(\tilde{J}, u_\pm) \quad \text{and} \quad C_k(J_\pm, u_\pm) = C_k(\tilde{J}_\pm, u_\pm).$$

On the other hand, $\tilde{J} = \tilde{J}_\pm$ when restricted to a small C_0^1 neighborhood of $u_\pm$, so $C_k(\tilde{J}, u_\pm) = C_k(\tilde{J}_\pm, u_\pm)$, and hence

$$m_k(u_\pm) := \dim C_k(J_\lambda, u_\pm) = \delta_{1k}.$$

Now, if 0, $u_\pm$ were the only critical points of J_λ, we would have

$$M_k = m_k(0) + m_k(u_+) + m_k(u_-) = \begin{cases} 1, & \text{if } k = 0, \\ 2, & \text{if } k = 1, \\ 0, & \text{if } k \geq 2. \end{cases}$$

Since $\lambda < \lambda_1(\Omega) \leq \lambda_*$, by Proposition 2.5 we again have Betti numbers $\beta_k = 0$, $k = 0, 1, 2, \ldots$, and the Morse inequality (2.11) is violated. This proves i.

Finally, we prove ii. Assume that (1.6) and (1.2) hold, and $\lambda > \lambda_1(\Omega)$. In [1] it is proven that there exists $\Lambda_+ > \lambda_1(\Omega)$ such that $(1.1)_\lambda$ admits a pair of positive solutions for each $\lambda \in (\lambda_1(\Omega), \Lambda_+)$. By the exact same arguments there exists $\Lambda_- > \lambda_1(\Omega)$ such that $(1.1)_\lambda$ admits a pair of negative solutions for each $\lambda \in (\lambda_1(\Omega), \Lambda_-)$. In order to obtain the fifth nontrivial solution we must determine the critical groups for each of these four solutions, and so we briefly review their derivation in [1].

Let $\bar{\lambda} = \min\{\lambda_*, \Lambda_+, \Lambda_-, \lambda_2(\Omega)\}$. Recall from Remark 1.1 that hypothesis (1.2) ensures that $\lambda_* > \lambda_1(\Omega)$, and hence $\bar{\lambda} > \lambda_1(\Omega)$. Moreover we have $\bar{\lambda} \leq \lambda_* < \lambda_1(\Omega_0)$, so (PS) holds for J_λ when $\lambda < \bar{\lambda}$. Furthermore, if $\lambda \in (\lambda_1(\Omega), \bar{\lambda})$, we have

$$m_k(0) = \dim C_k(J_\lambda, 0) = \delta_{1k}.$$

It is easy to see that for all $\lambda > \lambda_1(\Omega)$ there exists $t_0 = t_0(\lambda) > 0$ such that for $0 < t \leq t_0$, $\underline{v} = te_1$ is a subsolution and $\overline{w} = -te_1$ is a supersolution of $(1.1)_\lambda$. In addition, a positive solution $\overline{v}$ of $(1.1)_\mu$ with $\mu > \lambda$ is a supersolution for $(1.1)_\lambda$, while a negative solution $\underline{w}$ of $(1.1)_\mu$ with $\mu > \lambda$ is a subsolution for $(1.1)_\lambda$. By choosing $t > 0$ sufficiently small so that $0 < \underline{v} \leq \overline{v}$ and $\underline{w} \leq \overline{w} < 0$ we may determine a positive solution $u_\lambda^+ > 0$ and a negative solution $u_\lambda^- < 0$ to $(1.1)_\lambda$ via minimization,

$$J_\lambda(u_\lambda^+) = \inf\{J_\lambda(u) : \ 0 < \underline{v} \leq u \leq \overline{v}\}$$
$$J_\lambda(u_\lambda^-) = \inf\{J_\lambda(u) : \ \underline{w} \leq u \leq \overline{w} < 0\}.$$

(*See* [18].) Since $u_\lambda^\pm$ represent minimizers of J_λ in the $C_0^1(\overline{\Omega})$ topology, by a result of Brezis & Nirenberg [8] they are also minimizers in $H_0^1(\Omega)$. We assume that both $u_\lambda^\pm$ are isolated minima, (otherwise, we obtain infinitely many solutions), and hence

$$m_k(u_\lambda^\pm) = \dim C_k(J_\lambda, u_\lambda^\pm) = \delta_{0k}.$$

(*See* also [10].)

To obtain the second pair of single-sign solutions, we appeal to the Mountain-pass Theorem. Define functionals

$$\begin{aligned} I_\pm(v) =& \frac{1}{2}\int_\Omega |\nabla v|^2 \\ &- \lambda(v_\pm)^2 - \int_\Omega h(x)\left[F(u_\lambda^\pm + v_\pm) - F(u_\lambda^\pm) - f(u_\lambda^\pm)v_\pm\right] dx \end{aligned}$$

where (as before) $v_+ = \max\{v(x), 0\} \geq 0$, $v_- = \min\{v(x), 0\} \leq 0$. Note that if $v \geq 0$ or $v \leq 0$, then $I_\pm(v) = J_\lambda(u_\lambda^\pm + v) - J_\lambda(u_\lambda^\pm)$. Simple calculations show that: $v = 0$ is a strict local minimum for $I_\pm$; there exists $v_0 \geq 0$ such that $I_\pm(\pm t v_0) \to -\infty$ as $t \to +\infty$; and $I_\pm$ satisfy the Palais-Smale condition. (*See* Section 2 of [1] for the detailed computations.) Applying the Mountain-pass Theorem to $I_\pm$ we obtain nontrivial critical points $v^+ \geq 0$, $v^- \leq 0$ which give rise to solutions of $(1.1)_\lambda$, (and hence critical points of J_λ,) $w_\lambda^+ = u_\lambda^+ + v^+ > u_\lambda^+ > 0$ and $w_\lambda^- = u_\lambda^- + v^- < u_\lambda^- < 0$.

Now we repeat the analysis of part i. to calculate the corresponding critical groups. Restricted to a $C_0^1(\overline{\Omega})$ neighborhood of $v_\lambda^\pm$, we have $I_\pm(v) = J_\lambda(u_\lambda^\pm + v) - J_\lambda(u_\lambda^\pm)$. Therefore we may apply Corollary 8.5 of [15] and Theorem 1 of [10] to obtain

$$m_k(w_\lambda^\pm) := \dim C_k(J_\lambda, w_\lambda^\pm) = \dim C_k(I_\pm, v^\pm) = \delta_{1k}.$$

Hence, if 0, $u_\lambda^\pm$, $w_\lambda^\pm$ were the only solutions of $(1.1)_\lambda$, we would have Morse numbers

$$\begin{aligned} M_k &= m_k(0) + m_k(u_\lambda^+) + m_k(u_\lambda^-) + m_k(w_\lambda^+) + m_k(w_\lambda^-) \\ &= \begin{cases} 2, & \text{if } k = 0, \\ 3, & \text{if } k = 1, \\ 0, & \text{if } k \geq 2. \end{cases} \end{aligned}$$

Again, this contradicts the Morse inequality (2.11) over $(H_0^1(\Omega), J_\lambda^{-K_2})$, and so there must exist another nontrivial solution. $\diamondsuit$

3. A LINKING APPROACH

We prove the following, which includes Theorem 1.5:

THEOREM 3.1. – *Suppose f satisfies the hypotheses of Proposition* 2.6 *and* $\lambda_1(\Omega) \leq \lambda < \lambda_2(\Omega)$. *Then* $(1.1)_\lambda$ *possesses at least one nontrivial solution.*

Define the linear subspace $V = (\text{span}\,\langle e_1\rangle)^\perp$, and let $w_1, w_2 \in C_0^\infty(\Omega_+)$ be fixed functions with $\text{supp}\, w_1$, $\text{supp}\, w_2$ disjoint. Choose $R > 0$ sufficiently large so that

$$J_\lambda(\pm R w_1) < 0 \quad \text{and} \quad J_\lambda(\pm R w_1) < - \max_{t\in[0,\infty)} J_\lambda(t w_2)$$

Since f has superlinear growth at 0, there exists a radius $\rho > 0$ and constant $\alpha > 0$ with

$$J(u) \geq \alpha > 0 \quad \text{for } u \in V \cap S_\rho.$$

Introduce the modified functionals

$$J_\pm(u) = \int_\Omega \frac{1}{2}\left[|\nabla u|^2 - \lambda(u_\pm)^2\right] - h(x)F(u_\pm)\,dx.$$

Standard arguments show that a critical point of $J_\pm$ corresponds to a positive (respectively, negative) solution of $(1.1)_\lambda$. Set

$$\beta_\pm = \inf_{\gamma\in\Gamma_\pm} \sup_{t\in[0,1]} J_\pm(\gamma(t))$$
$$\Gamma_\pm = \{\gamma \in C([0,1]; H_0^1(\Omega)) : \ \gamma(0) = 0,\ \gamma(1) = \pm R w_1\}.$$

Clearly, $\beta_\pm \geq 0$. If either $\beta_\pm > 0$, then we obtain a positive (respectively, negative) solution to $(1.1)_\lambda$ with critical value $\beta_\pm > 0$, and hence a nontrivial solution to $(1.1)_\lambda$.

Suppose that both $\beta_\pm = 0$. Then there exist curves $\gamma_\pm \in \Gamma_\pm$ with

$$J_\pm(\gamma_\pm(t)) \leq \frac{\alpha}{2}.$$

Without loss, we may suppose that $\gamma_\pm$ are simple curves, and that $\gamma_+(t) \geq 0$, $\gamma_-(t) \leq 0$ for all $t \in [0,1]$, with $\gamma_\pm(t) \equiv 0$ if and only if $t = 0$. Since V is characterized by

$$v \in V \quad \text{if and only if} \quad \int_\Omega v e_1 = 0,$$

we have $\gamma_\pm(t) \in B_\rho \cap V$ if and only if $t = 0$.

Finally, we connect Rw_1 to $-Rw_1$ by a path $\gamma_0 \subset H^1_D(\Omega_+)$ with

$$J_\lambda(\gamma_0(\cdot)) < 0 \quad \text{and} \quad \|\gamma_0\|_{H^1_D(\Omega_+)} > \rho.$$

To do this, choose $T > 0$ such that

$$J_\lambda(Tw_2) < 0 \quad \text{and} \quad J_\lambda(Tw_2) < -\max_{t\in[-R,R]} J_\lambda(tw_1).$$

Then, let

$$\gamma_0(t) = \begin{cases} w + 3tTw_2, & \text{when } t \in \left[0, \frac{1}{3}\right], \\ Tw_2 + 3(1-2t)w, & \text{when } t \in \left[\frac{1}{3}, \frac{2}{3}\right], \\ -w + (3-3t)Tw_2, & \text{when } t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

Since w_1, w_2 have disjoint supports, $J_\lambda(\alpha w + \beta w_2) = J_\lambda(\alpha w) + J_\lambda(\beta w_2)$, and the choice of R and T ensure that $J_\lambda(\gamma_0(t)) < 0$ for all $t \in [0, 1]$.

Let C be the closed curve obtained by joining γ_+ to γ_0 to γ_-. By construction, C intersects $S_\rho \cap V$ only at 0. Parametrize C by a one-to-one map $\psi : S^1 \to C \subset H^1_0(\Omega)$ with $\psi(e^{i0}) = 0$, and set

$$\Sigma = \{\sigma \in C(D^2; H^1_0(\Omega)) : \sigma|_{S^1} = \psi\}$$

(where we write D^2 to represent the unit disk in $\mathbb{R}^2$ with boundary $\partial D^2 = S^1$). Then, we have

$$J_\lambda|_C \le \frac{\alpha}{2} \quad \text{and} \quad J_\lambda|_{S_\rho \cap V} \ge \alpha.$$

Let

$$b = \inf_{\sigma\in\Sigma} \sup_{\xi\in D^2} J_\lambda(\sigma(\xi)).$$

If we can show that $S_\rho \cap V$ links with C, then we may conclude that $b \ge \alpha > 0$ is a critical value of J_λ with nontrivial critical point.

LEMMA 3.2. – *For all $\sigma \in \Sigma$,*

$$\sigma(B^2) \cap S_\rho \cap V \neq \emptyset$$

Proof. – Assume the contrary: there exist $\sigma \in \Sigma$ such that $\sigma(B^2) \cap S_\rho \cap V = \emptyset$. We define a family of maps,

$$F_t : (B_\rho \cap V) \times S^1 \to H^1_0(\Omega)$$

$$F_t(z, \xi) = z - \sigma(t\xi), \quad \text{for } t \in [0, 1],\ z \in B_\rho \cap V,\ \xi \in S^1.$$

Note that (by assumption above), if $(z, \xi) \in \partial((B_\rho \cap V) \times S^1) = (S_\rho \cap V) \times S^1$, then $F_t(z, \xi) \neq 0$. Hence $\deg(F_t, (B_\rho \cap V) \times S^1, 0)$ is a constant for all $t \in [0,1]$. By reparametrizing D^2 if necessary, we may assume that $\sigma(0) \notin B_\rho \cap V$. Hence, $F_0(z,\xi) = z - \sigma(0) \neq 0$ for all $z \in B_\rho \cap V$, and hence

$$\deg(F_t, (B_\rho \cap V) \times S^1, 0) = 0 \quad \text{for all } t \in [0,1].$$

On the other hand, when $t = 1$, $F_1(z,\xi) = z - \sigma(\xi) = z - \psi(\xi)$. By the construction of the curve C, $F_1(z, e^{i\theta}) = 0$ has the unique solution $z = 0$ and $\theta = 0$. By the excision property of the degree,

$$\deg(F_1, (B_\rho \cap V) \times S^1, 0) = \deg(F_1(z, e^{i\theta}), (B_\rho \cap V) \times (-\varepsilon, \varepsilon), 0).$$

To calculate this degree, deform the arc $\psi(e^{i\theta})$, $\theta \in (-\varepsilon, \varepsilon)$ to the straight segment, θe_1. Then, we have

$$\begin{aligned} &\deg(F_1(z, e^{i\theta}), (B_\rho \cap V) \times (-\varepsilon, \varepsilon), 0) \\ &\quad = \deg(z - \theta e_1, (B_\rho \cap V) \times (-\varepsilon, \varepsilon), 0) = -1, \end{aligned}$$

a contradiction. $\diamondsuit$

In conclusion, $S_\rho \cap V$ links with C, so by the linking theorem of Benci & Rabinowitz [4], $b > 0$ is a critical value of J_λ. This completes the proof of Theorem 3.1.

4. APPENDIX

First, we show by an example that, depending on the function h, the value λ_* could be arbitrarily large. Without loss of generality, suppose that $B_\varepsilon(0) \subset \Omega$, for some $\varepsilon > 0$. Define a sequence $\{h_n\}$ of smooth functions on $\mathbb{R}^N$ with $-1 \leq h(x) \leq 1$ and

$$h_n(x) = \begin{cases} -1, & \text{when } |x| \leq \dfrac{1}{4}\dfrac{\varepsilon}{n}, \\ 0, & \text{when } \dfrac{1}{2}\dfrac{\varepsilon}{n} \leq |x| \leq \dfrac{3}{4}\dfrac{\varepsilon}{n}, \\ 1, & \text{when } \dfrac{\varepsilon}{n} \leq |x|. \end{cases}$$

Clearly, $\displaystyle\int_\Omega h_n(x) e_1^p \, dx > 0$ for all n sufficiently large, and hence the corresponding values $\lambda_* = \lambda_*^{(n)} > \lambda_1(\Omega)$. Suppose $\lambda_*^{(n)} \leq M$ for all n. Each minimization problem is attained at some $v_n \in H_0^1(\Omega)$ with $\|\nabla v_n\|_2^2 = \lambda_*^{(n)} \leq M$, $\|v_n\|_2 = 1$, and $\int_\Omega h_n |v_n|^p \, dx = 0$. For some subsequence we have $v_n \rightharpoonup v_0$ in $H_0^1(\Omega)$ with $\|v_0\|_2 = 1$. Since $h_n \to 1$ in $L^t(\Omega)$ for all $t < \infty$ and $p < \frac{2N}{N-2}$, we have $\int_\Omega |v_0|^p\, dx = 0$, a contradiction.

We conclude with a proof of the Palais-Smale condition (Theorem 2.6 under the hypothesis $i.$, which suffices for Theorems 1.2, 1.3, and 1.5. Note that (2.10) implies in addition,

$$F(u) - \qquad = O(u^2) \tag{4.1}$$

$$F(u) - \frac{1}{p} f(u)u = O(u^2), \tag{4.2}$$

as $|u| \to \infty$.

Let $\{u_n\}$ be a (PS) sequence,

$$J_\lambda(u_n) = \int_\Omega \frac{1}{2}(|\nabla u_n|^2 - \lambda u_n^2) - h(x)F(u_n)\, dx \leq C \tag{4.3}$$

$$\begin{aligned} J'_\lambda(u)\varphi &= \int_\Omega \nabla u_n \cdot \nabla \varphi - \lambda u_n \varphi - h(x) f(u_n)\varphi\, dx \\ &= \int_\Omega \nabla z_n \cdot \nabla \varphi\, dx \end{aligned} \tag{4.4}$$

where $z_n \to 0$ in $H_0^1(\Omega)$ and $\varphi \in H_0^1(\Omega)$ is any fixed function. First, we claim $\|u_n\|_2 \leq C$. Suppose the contrary, and set $v_n = u_n/\|u_n\|_2$. From (4.3), (4.4), and hypothesis (4.2) we have,

$$\begin{aligned} \left(\frac{1}{2} - \frac{1}{p}\right)(\|\nabla v_n\|_2^2 - \lambda) &\leq \frac{1}{\|u_n\|_2^2} \int_\Omega h(x)\left[F(u_n) - \frac{1}{p} f(u_n)u_n\right] dx \\ &\quad + o(1)\|\nabla v_n\|_2 \\ &= O(1) + o(1)\|\nabla v_n\|_2 \end{aligned}$$

Hence, v_n is bounded in $H_0^1(\Omega)$ and a subsequence (which we still denote by v_n) converges weakly $v_n \rightharpoonup v_0$ in $H_0^1(\Omega)$. From (4.4) and (2.10) and for each fixed $\varphi \in H_0^1(\Omega)$ we obtain:

$$\begin{aligned} \|u_n\|_2^{p-2} \int_\Omega h(x)|v_n|^{p-2} v_n \varphi\, dx &= \frac{1}{\|u_n\|} \int_\Omega h(x)|u_n|^{p-2} u_n \varphi\, dx \\ &= \frac{1}{\|u_n\|} \int_\Omega h(x) f(u_n)\varphi\, dx + O(1) \\ &= \int_\Omega [\nabla v_n \cdot \nabla \varphi - \lambda v_n \varphi]\, dx + O(1) \\ &= \int_\Omega [\nabla v_0 \cdot \nabla \varphi - \lambda v_0 \varphi]\, dx + O(1) = O(1). \end{aligned}$$

In particular

$$\int_{\Omega} h(x)|v_0|^{p-2} v_0 \varphi \, dx = 0$$

for all $\varphi \in H_0^1(\Omega)$. Hence, $v_0 \in H_D^1(\Omega_0)$ and $||v_0||_2 = 1$, so $v_0 \not\equiv 0$. Finally, taking $\varphi \in H_D^1(\Omega_0)$ we apply (4.4) again to obtain:

$$\int_{\Omega_0} \nabla v_0 \cdot \nabla \varphi - \lambda v_0 \varphi \, dx = 0$$

which yields a contradiction unless $\lambda \in \sigma(\Omega_0)$. We conclude that $||u_n||_2 \leq C$.

Now, combining (4.3) and (4.4) and using (4.2) we have:

$$\begin{aligned}\left(\frac{1}{2} - \frac{1}{p}\right)\left(||\nabla u_n||_2^2 - \lambda ||u_n||_2^2\right) &\leq c + \int_{\Omega} h(x)\left[F(u_n) - \frac{1}{p} f(u_n) u_n\right] dx \\ &\quad + o(1) ||\nabla u_n||_2 \\ &\leq c ||u_n||_2^2 + c + o(1) ||\nabla u_n||_2.\end{aligned}$$

Therefore, $||\nabla u_n||_2$ is uniformly bounded, and the subcritical growth of F ensures that there is a strongly convergent subsequence. ◇

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(Manuscript received April 25, 1994;
Revised version received March 23, 1995;
Final version, March 1995.)