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Higher dimensional problems with volume constraints—Existence and Γ-convergence

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We study variational problems with volume constraints (also called level set constraints) of the form

$$\text{Minimize } E(u) := \int_\Omega f(u, \nabla u)\,\mathrm{d}x,$$
$$|\{x \in \Omega : u(x) = a\}| = \alpha, \quad |\{x \in \Omega : u(x) = b\}| = \beta,$$

on $\Omega \subset \mathbb{R}^n$, where $u \in H^1(\Omega)$ and $\alpha + \beta < |\Omega|$. The volume constraints force a phase transition between the areas on which $u = a$ and $u = b$.

We give some sharp existence results for the decoupled homogeneous and isotropic case $f(u, \nabla u) = \psi(|\nabla u|) + \theta(u)$ under the assumption of p-polynomial growth and strict convexity of ψ. We observe an interesting interaction between p and the regularity of the lower order term which is necessary to obtain existence, and find a connection to the theory of dead cores. Moreover, we obtain some existence results for the vector-valued analogue with constraints on $|u|$.

In the second part of this article we derive the Γ-limit of the functional E for a general class of functions f in the case of vanishing transition layers, i.e. when $\alpha + \beta \to |\Omega|$. The limit problem we obtain is a nonlocal free boundary problem.

1. Introduction

We consider variational problems with level set constraints of the type

$$\begin{aligned} &\text{Minimize } E(u) := \int_\Omega f(u(x), \nabla u(x))\,\mathrm{d}x, \\ &|\{x \in \Omega : u(x) = a\}| = \alpha, \qquad |\{x \in \Omega : u(x) = b\}| = \beta, \end{aligned} \tag{1.1}$$

where $u \in H^1(\Omega)$ and $\alpha + \beta < |\Omega|$. The difficulty of this problem is the special structure of its constraints: A sequence of functions satisfying these constraints can have a limit which fails to satisfy the constraints.

Such minimization problems but with only one volume constraint have been studied by various authors (see e.g. [3]). In the last years problems with two or more constraints have caught attention [4, 19, 18, 14, 13, 17, 7, 15]. This interest was partially motivated by physical problems related to immiscible fluids [11] and mixtures of micromagnetic materials [1, 2]. We will briefly discuss this relation below.

Problems with two (or more) volume constraints have a very different nature than problems with only one volume constraint: In the case of one volume constraint, only additional boundary

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conditions or the design of the energy can induce transitions of the solution between different values. Two or more volume constraints, on the other hand, force transitions of the solution by their very nature. Ambrosio, Marcellini, Fonseca and Tartar [4] studied this class of problems for the first time and proved an existence result for the problem of two (or more) level set constraints with an energy density $f = f(|\nabla u|)$. Moreover, they derived the Γ-limit for a vanishing transition layer in the special case $f = |\nabla u|^2$. It turned out that unlike usual variational problems, lower order terms pose serious difficulties for the analysis and can lead, even in very easy examples, to nonexistence [14, 13]. However, under certain regularity assumptions on the energy density the existence results were extended to energy functionals depending on ∇u and u [14]. For the special case of one space dimension a more or less complete analysis of existence, uniqueness, local minimizers and the Γ-limit has been given in [13]. It turned out that there is a strong link between existence and the regularity of the lower order term. One of the goals of this paper is to investigate this link in higher dimensional problems. We prove an existence result for a special class of energies under minimal regularity assumptions. The proof is based on the use of a maximum principle for solutions of elliptic equations recently established by Pucci and Serrin [16] and draws some interesting connection to the theories of dead cores. We also consider extensions to vector-valued problems of the form

$$\begin{aligned} &\text{Minimize } E(u) := \int_\Omega [\psi(|\nabla u|) + \theta(|u|)]\,\mathrm{d}x, \\ &|\{x \in \Omega : |u(x)| = a\}| = \alpha, \qquad |\{x \in \Omega : |u(x)| = b\}| = \beta. \end{aligned} \tag{1.2}$$

Such problems are related to the analysis of mixtures of micromagnetic materials [1, 2], where the vector-valued variable u describes the magnetization. The absolute value of u is prescribed on every material by the so-called *magnetic saturation* m^s, but only the amount of each material, and not its location, is given, so that the constraints of the minimization problem read as

$$\left|\{x \in \Omega : |u(x)| = m_i^s\}\right| = \alpha_i, \tag{1.3}$$

with fixed numbers α_i. The three main difficulties when studying existence for this problem are

- the volume constraint,
- that u is vector-valued, and
- that the energy functional involves nonlocal terms.

The first of these problems has helped to motivate the recent interest in volume constraints. In the first part of this article we extend some previous existence results (Theorems 2.1 and 2.2). The second difficulty has also been addressed previously, but only with a simplified constraint that prescribed u, rather than $|u|$. We study in this article the full constraint involving $|u|$ and provide an existence result for (1.2) (see Theorem 2.5).

In the second part of this paper we study the Γ-limit of (1.2) for general energy densities f as the two phases α and β tend to saturate the whole domain. This limit is of physical interest as it corresponds to the case where the transition layer between the phases is negligibly small—as is the case, e.g., in immiscible fluids. It turns out that the limit problem that we obtain is nonlocal, hence a standard extension of the Γ-limit in the one-dimensional problem (see [13]) by a simple slicing argument is not possible. Instead our proof has to rely on methods from geometric measure theory.

2. Sharp existence results

In this section we present some new existence results partially extending the sharp results of [13] to the higher dimensional case. As in [13] we consider for simplicity only decoupled functionals of the form $f(u, \nabla u) = \psi(|\nabla u|) + \theta(u)$, where ψ is strictly convex and takes its minimum at zero. We define

$$H(t) := \int_0^{\psi'(t)} (\psi')^{-1}(w)\,\mathrm{d}w,$$

where $(\psi')^{-1}$ denotes the inverse of ψ', which is well-defined since ψ' is strictly increasing. H is by definition strictly increasing, hence its inverse H^{-1} is well-defined. We prove the following result:

THEOREM 2.1 (Existence) Let $\delta > 0$. Let Ω be a bounded open set in $\mathbb{R}^n$, $\alpha, \beta > 0$ with $\alpha + \beta < |\Omega|$ and $\theta \in C^{0,1}((0,1), \mathbb{R}_{\geqslant 0})$. Assume the existence of Lipschitz continuous functions θ_1 and θ_2 with $\theta_1' \geqslant \theta'$ on $[0, \delta)$ and $\theta_2' \leqslant \theta'$ on $(1-\delta, 1]$. Moreover, let θ_1 be strictly convex on $(0, \delta)$ and θ_2 be strictly convex on $(1-\delta, 1)$, and let θ_1 and θ_2 satisfy the integrability conditions

$$\int_0^{\delta} \frac{\mathrm{d}u}{H^{-1}(\theta_1)} = +\infty, \tag{2.1}$$

$$\int_{1-\delta}^{1} \frac{\mathrm{d}u}{H^{-1}(\theta_2)} = +\infty. \tag{2.2}$$

Let ψ be locally Lipschitz continuous and strictly convex with $\psi(0) = \psi'(0) = 0$ and $C_1 t^p \leqslant \psi(t) \leqslant C_2 t^p$ for some constants $C_1, C_2 > 0$. Then the volume-constrained minimization problem

$$\begin{aligned} &\text{Minimize} \int_\Omega [\psi(|\nabla u|) + \theta(u(x))]\,\mathrm{d}x, \\ &|\{x \in \Omega : u(x) = 0\}| = \alpha, \qquad |\{x \in \Omega : u(x) = 1\}| = \beta, \end{aligned} \tag{2.3}$$

admits a solution $u \in W^{1,p}(\Omega, [0,1])$.

An immediate consequence of this result is the following existence theorem which gives easier sufficient conditions on θ for the special case of quadratic growth:

THEOREM 2.2 (Existence) Let Ω be a bounded open set in $\mathbb{R}^n$, $\alpha, \beta > 0$ with $\alpha + \beta < |\Omega|$ and $\theta \in C^{0,1}((0,1), \mathbb{R}_{\geqslant 0})$, locally $C^{1,1}$ at 0 and 1 with $\theta'(0) \leqslant 0$ and $\theta'(1) \geqslant 0$. Let ψ be locally Lipschitz continuous and strictly convex with $\psi(0) = \psi'(0) = 0$ and quadratic growth. Then the volume-constrained minimization problem

$$\begin{aligned} &\text{Minimize} \int_\Omega [\psi(|\nabla u|) + \theta(u(x))]\,\mathrm{d}x, \\ &|\{x \in \Omega : u(x) = 0\}| = \alpha, \qquad |\{x \in \Omega : u(x) = 1\}| = \beta, \end{aligned} \tag{2.4}$$

admits a solution $u \in W^{1,2}(\Omega, [0,1])$.

Theorem 2.2 is sharp in the following sense: If $\theta \notin C^{1,1}$ locally, but instead is in $C^{1,\alpha}$ for some $\alpha < 1$, there are cases of nonexistence.

Before we prove these results, we would like to mention the connections to earlier results for the one-dimensional case. A sharp characterization of functions that yield existence for a volume-constrained problem of the form (2.3) was given in [13]. Theorem 2.1 comes close to this, but its conditions are slightly stronger:

The integrability condition for θ is the same as in the one-dimensional case (see [13]) but we have to assume the existence of the functions θ_1 and θ_2 since we need the local convexity condition in order to apply a maximum principle (see below). However, this condition is not very strong, as can be seen in Theorem 2.2: Without the sign condition on θ' it seems possible that a local minimum of θ at 0 or 1 leads to nonexistence if the domain Ω is chosen appropriately. This was not possible in the one-dimensional case, where only global minimum of θ at 0 or 1 could lead to nonexistence: if there is a global minimum at $\lambda \in (0, 1)$, then it is easy to see that a solution exists (for an illustration of the main idea see Fig. 1, where $\Omega = [0, 1]$) [13, 17].

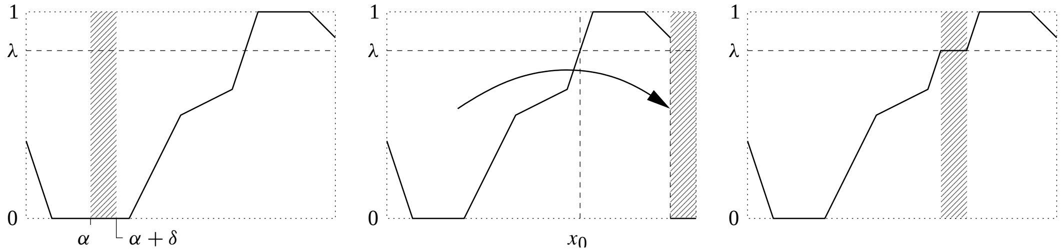

FIG. 1. Left: Assume the solution of the relaxed problem is zero on the interval $[\alpha, \alpha + \delta]$, hence violating the constraint. Middle: Shift the interval $[\alpha, \alpha + \delta]$ to the right. Right: Insert a constant piece with the optimal value λ at x_0 such that the resulting function is continuous and in $W^{1,2}$.

In the higher dimensional case we cannot argue like this, since the shape of the domain might make it energetically preferable to violate the constraint, even when the absolute minimum of θ is in (0, 1); for a hypothetical situation in which such a phenomenon may occur, see Fig. 2. This motivates us to believe that the additional condition on θ is not a mere technical problem, but in fact needed.

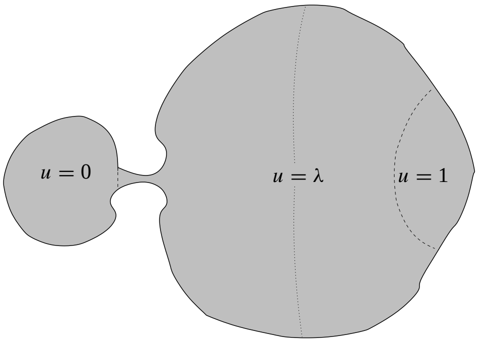

FIG. 2. An example of a domain Ω where it might be energetically preferable to violate the constraint, in order to concentrate most of the energy into the small “bottle neck”. Increasing the area where $u = \lambda$ would therefore probably not reduce the energy.

The second major difference is that we now consider only functions with values in [0, 1]. In the one-dimensional case this was not necessary, we only had to assume that θ (defined on $\mathbb{R}$ rather than on [0, 1]) has a minimum in [0, 1]. In higher dimensional situations it is not at all clear that this condition would be sufficient, since again the shape of Ω might make it attractive to violate the constraint. However, it is possible to give slightly more general conditions than above:

REMARK 2.3 If $\theta \in C(\mathbb{R}, \mathbb{R}_{\geqslant 0})$ satisfies $\theta(z) \geqslant \theta(0)$ for $z < 0$ and $\theta(z) \leqslant \theta(1)$ for $z > 1$ then any solution $u \in W^{1,p}((0,1), \mathbb{R})$ of the minimization problem

$$\begin{aligned} &\text{Minimize } \int_\Omega [\psi(|\nabla u(x)|) + \theta(u(x))]\,\mathrm{d}x, \\ &|\{x \in \Omega : u(x) = 0\}| = \alpha, \qquad |\{x \in \Omega : u(x) = 1\}| = \beta, \end{aligned}$$

satisfies $u \in [0, 1]$.

Proof. Assuming the contrary, the function $v(x) := \min(\max(u(x), 0), 1)$ would have lower energy than u. □

Proof of Theorem 2.1. Our proof relies on a maximum principle for the Euler–Lagrange equation associated to 2.3, which corresponds to a recent result by Pucci and Serrin [16].

First we extend θ to a function $\tilde{\theta}$ by

$$\tilde{\theta}(z) := \begin{cases} z^2 - z\theta'(0) + \theta(0), & z < 0, \\ \theta(z), & 0 \leqslant z \leqslant 1, \\ (z-1)^2 + (z-1)\theta'(1) + \theta(1), & z > 1. \end{cases}$$

By standard variational methods the relaxed problem

$$\begin{aligned} &\text{Minimize } \int_\Omega [\psi(|\nabla u(x)|) + \tilde{\theta}(u(x))]\,\mathrm{d}x, \\ &|\{x \in \Omega : u(x) = 0\}| \geqslant \alpha, \qquad |\{x \in \Omega : u(x) = 1\}| \geqslant \beta, \end{aligned}$$

admits a solution $u \in W^{1,p}(\Omega, \mathbb{R})$. By a general regularity result of Mosconi and Tilli [14, Theorem 3.3] the function u is continuous, and by Remark 2.3, which can be applied also to the relaxed case, u takes only values in [0, 1].

Now assume that u does *not* solve problem (2.3). Then either $|\{x \in \Omega : u(x) = 0\}| > \alpha$ or $|\{x \in \Omega : u(x) = 1\}| > \beta$. We consider the first case, i.e. $|\{u = 0\}| := |\{x \in \Omega : u(x) = 0\}| = \alpha + \varepsilon$ with $\varepsilon > 0$. Now choose $\eta > 0$ such that an n-dimensional ball with radius η has volume less than ε, i.e. $|B(0, \eta)| < \varepsilon$. Take $x \in \Omega$ such that

$$\begin{aligned} B(x, \eta) \cap \{u = 0\} &\neq \emptyset, \\ B(x, \eta) \cap \{u \in (0, \delta)\} &\neq \emptyset, \\ B(x, \eta) \cap \{u \geqslant \delta\} &= \emptyset. \end{aligned} \tag{2.5}$$

(This is possible for η small enough since u is continuous and hence $\{u \in (0, \delta)\}$ is open.)

Now consider variations $u + t\varphi$ with $\varphi \in C_0^\infty(B(x, \eta))$. Since u is a minimizer of the relaxed problem it satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega [\psi(\nabla u + t\nabla\varphi) + \theta(u + t\varphi)]\,\mathrm{d}x \bigg|_{t=0} = 0.$$

This leads to the Euler–Lagrange equality

$$\operatorname{div} A(|\nabla u|)\nabla u - \theta'(u) = 0, \tag{2.6}$$

where $A(|\nabla u|) := \psi'(|\nabla u|)/|\nabla u|$.

By the integrability conditions (2.1)–(2.2) and the local convexity of θ_1 and θ_2 we deduce that $\theta_1'(0) \leqslant 0$ and $\theta_2'(1) \geqslant 0$. We consider the first of these inequalities and distinguish the two cases where $\theta_1'(0) < 0$ and $\theta_1'(0) = 0$:

CASE 1: $\theta_1'(0) = 0$. We can apply the regularity theory for degenerate elliptic equations of p-Laplacian type (see e.g. [10, 8.9]) to (2.6) to deduce that the solution u has C^1-regularity. Moreover, $\theta' \leqslant \theta_1'$ on $[0, \delta)$, hence we can apply the maximum principle in [16, Theorem 1] with θ_1 on the domain $B(x, \eta)$. This gives $u = 0$ on all of $B(x, \eta)$, contradicting (2.5).

CASE 2: $\theta_1'(0) < 0$. Choose $\eta > 0$ such that $|B(0, \eta)| < \varepsilon$. Take $x \in \Omega$ such that

$$|B(x, \eta) \cap \{u = 0\}| > 0, \qquad B(x, \eta) \cap \{u = 1\} = \emptyset. \tag{2.7}$$

On the set $B(x, \eta) \cap \{u = 0\}$ we have $\theta'(u) = \theta'(0) \leqslant \theta_1'(0) < 0$. But since on the same set $\operatorname{div} A(|\nabla u|)\nabla u = 0$, we get a contradiction to the Euler–Lagrange equality (2.6).

Hence we have proved in both cases that $|\{u = 0\}| = \alpha$. Using the function θ_2 we can prove in the same way that $|\{u = 1\}| = \beta$. Thus we have proved existence for the original problem (1.1). □

Theorem 2.2 is now an easy consequence:

Proof of Theorem 2.2. Let $L := \mathrm{Lip}_{(0,\delta)\cup(1-\delta,1)}\, \theta'$. Choose $\theta_1(z) := \theta(z) - \theta(0) + L z^2$, $\theta_2(z) := \theta(z) - \theta(1) + L(1 - z)^2$ and $\delta > 0$ sufficiently small. These functions satisfy the conditions of Theorem 2.1: First, both functions are strictly convex, because their derivatives are strictly monotone. Moreover, they satisfy the integrability conditions (2.1) resp. (2.2). We prove this for θ_1, the proof for θ_2 is symmetric:

Due to the quadratic growth of ψ and the condition $\psi(0) = \psi'(0) = 0$ we have $\psi'(t) \geqslant C_1 t$ for a certain constant $C_1 > 0$. This implies a bound on $(\psi')^{-1}$, namely $(\psi')^{-1}(w) \geqslant w/C_1$. Applying this to the definition of H gives

$$H(t) = \int_0^{\psi'(t)} (\psi')^{-1}(w)\, \mathrm{d}w \geqslant \int_0^{\psi'(t)} \frac{w}{C_1}\, \mathrm{d}w \geqslant \frac{1}{2} C_1 t^2.$$

Using this estimate for the inverse function of H we deduce

$$H^{-1}(t) \leqslant \frac{2}{C_1}\sqrt{t}.$$

Hence

$$\int_0^\delta \frac{\mathrm{d}u}{H^{-1}(\theta_1(u))} \geqslant \int_0^\delta \frac{C_1}{2\sqrt{\theta_1(u)}}\, \mathrm{d}u,$$

and it is therefore sufficient to prove that the latter is infinite. To see this we first have to distinguish three cases:

CASE 1: $\theta(u) \leqslant \theta(0)$ on $(0, \delta)$. Here we have

$$\sqrt{\theta_1(u)} = \sqrt{\theta(u) - \theta(0) + Lu^2} \leqslant C_3 |u|,$$

where $C_3 := \sqrt{L}$.

CASE 2: $\theta(u) > \theta(0)$ on $(0, \delta)$. Here we use the estimate

$$\sqrt{\theta_1(u)} = \sqrt{\theta(u) - \theta(0) + Lu^2} \leqslant \sqrt{\theta(u) - \theta(0)} + \sqrt{L}|u|,$$

and because of the regularity of θ and the assumption that $\theta'(0) \leqslant 0$,

$$\sqrt{\theta(u) - \theta(0)} \leqslant C_2|u|$$

with some constant $C_2 > 0$.

Combining both we get again $\sqrt{\theta_1(u)} \leqslant C_3|u|$, this time with $C_3 := C_2 + \sqrt{L} > 0$.

CASE 3: remaining situations. This case can be excluded provided we choose $\delta > 0$ sufficiently small, since locally $\theta \in C^{1,1}$.

Using the estimates proved above we obtain (2.1), since

$$\int_0^\delta \frac{C_1}{2\sqrt{\theta_1(u)}}\,\mathrm{d}u \geqslant \int_0^\delta \frac{C_1}{2C_3|u|}\,\mathrm{d}u = +\infty.$$

Thus Theorem 2.1 can be applied, and a solution u exists. ☐

It is remarkable that the necessary regularity for θ depends on the growth properties of ψ. In other words, the growth of the leading order term prescribes the necessary regularity for the lower order term! This is not only a technical problem of the proof, very much to the contrary: Theorem 2.1 is sharp, i.e. there are counterexamples to existence if one of the integrability conditions (2.1)–(2.2) is violated—even if $\theta \in C^\infty$, although in the case of quadratic growth $\theta \in C^{1,1}$ is sufficient as we have seen in Theorem 2.2. The following corollary provides such an example. It can be proved by copying the methods used in [13]. (The function H^{-1} of Theorem 2.1 becomes in this case simply $\sqrt[4]{\cdot}$.)

COROLLARY 2.4 The one-dimensional volume constrained minimization problem

$$\begin{aligned} &\text{Minimize } \int_0^1 [|u'|^4 + 256\,|u|^2]\,\mathrm{d}x, \\ &|\{x \in (0,1) : u(x) = 0\}| = \alpha, \quad |\{x \in (0,1) : u(x) = 1\}| = \beta, \end{aligned}$$

with $\alpha = \beta = 1/10$ does not admit a solution.

The results obtained so far can partially be extended to vector-valued problems of the form

$$\begin{aligned} &\text{Minimize } E(u) := \int_\Omega [\psi(|\nabla u|) + \theta(u)]\,\mathrm{d}x, \\ &|\{x \in \Omega : |u(x)| = a\}| = \alpha, \quad |\{x \in \Omega : |u(x)| = b\}| = \beta, \end{aligned} \tag{2.8}$$

where now $u \colon \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$. In fact we have the following theorem:

THEOREM 2.5 (Vector-valued case) Let Ω be a bounded open set in $\mathbb{R}^n$, $p > 1$, $\alpha, \beta > 0$ with $\alpha + \beta < |\Omega|$, and let ψ be locally Lipschitz continuous and strictly convex with $\psi(0) = \psi'(0) = 0$ and $C_1 t^p \leqslant \psi(t) \leqslant C_2 t^p$ for some constants $C_1, C_2 > 0$. Let $\theta \in C^{0,1}(\mathbb{R}^m, \mathbb{R}_{\geqslant 0})$ satisfy one of the following conditions:

(i) The function θ is *isotropic*, i.e. there exists $\tilde{\theta}$ such that $\theta(P) = \tilde{\theta}(|P|)$ for all $P \in \mathbb{R}^m$ with $a \leqslant |P| \leqslant b$.
(ii) There exists $\nu \in \mathbb{R}^m$ with $|\nu| = 1$ such that $\theta(P) \geqslant \theta(|P| \cdot \nu) =: \tilde{\theta}(|P|)$ for all $P \in \mathbb{R}^m$ with $a \leqslant |P| \leqslant b$.

Moreover, let $\tilde{\theta}$ satisfy the analogous conditions of either Theorem 2.1 or Theorem 2.2. Then there exists a solution $u \in W^{1,p}(\Omega, \mathbb{R}^m)$ which solves the vector-valued minimization problem (2.8).

Proof. First, we remark that the conditions (i) and (ii) are not equivalent, for instance (in polar coordinates) $\theta(r, \alpha) := |r|^p(2 + \sin\alpha)$ satisfies (ii), but not (i). However, condition (i) is obviously a special case of (ii). Hence we assume (ii) is satisfied. We also remark that ψ is increasing on $\mathbb{R}_+$, since it is convex and $\psi'(0) = 0$.

The existence of a solution to the relaxed problem follows as in the scalar case. We denote this solution by v. Now we define $w := |v| \cdot \nu$. From the isotropy of ψ and condition (ii) we derive that the energy of w cannot be larger than the energy of v: first, we see that

$$|\nabla v|^2 = |\nabla w|^2 + \left|\frac{\partial}{\partial\tau} v\right|^2 \geqslant |\nabla w|^2,$$

where $\partial/\partial\tau$ denotes the partial derivatives orthogonal to the radial direction. From this estimate using the monotonicity of ψ we obtain $\psi(|\nabla v|) \geqslant \psi(|\nabla w|)$. Finally, condition (ii) ensures that $\theta(v) \geqslant \theta(|v| \cdot \nu) = \theta(w)$. Thus, we have proved that the energy of w is not larger than the energy of v. (This trick is due to B. Dacorogna and I. Fonseca, personal communication.) We have thus reduced the problem to the scalar case. An application of Theorem 2.1 or 2.2, respectively, concludes the proof. □

We would like to mention that the general vector-valued situation with the constraint as given in (2.8) is much harder. One reason for this is that the solution does not have to be constant on the constraint volumes. Another reason is that continuity for the solutions to the relaxed problem has so far only been obtained for the scalar case using methods that are difficult to apply to the vectorial situation.

3. The Γ-limit of vanishing transition layers

In this section we compute the limiting functional for the case where the transition layer vanishes. We perform this computation in two steps: first, we assume that the energy function is isotropic and homogeneous; then we extend our result to a large class of nonisotropic functions. The key idea of the second step will be to transform the problem into an isotropic problem, therefore we start with this case. The main method we use is Γ-convergence. A Γ-limit is essentially a limit of functionals that is chosen in such a way that sequences of minimizers for the approximating functionals converge to a minimizer of the limit functional. For a precise definition and further background on this topic we refer the reader to the books by Braides or Dal Maso [6, 8].

3.1 *The isotropic and homogeneous case*

To study the Γ-convergence for the case where $|\Omega| - \alpha - \beta \to 0$ we need the following lemma on the one-dimensional problem, which can be found in [13, Lemma 4.1].

LEMMA 3.1 Let $f : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous function satisfying the following conditions:

(i) for every u, $f(u, \cdot)$ is convex and increasing;

(ii) there exist $c > 0$ and $p > 1$ such that

$$\frac{1}{c}|\xi|^p - c \leqslant f(u, \xi) \leqslant c(|\xi|^p + 1),$$

for all $u, \xi \in \mathbb{R}$.

Then the function P defined for every $t > 0$ by

$$P(t) := \min\left\{\int_0^t f(v, v')\,\mathrm{d}x : v \in W^{1,1}(0, t),\ v(0) = 0,\ v(t) = 1\right\} \tag{3.1}$$

is convex. Moreover, the function $\varphi(t) := tP(1/t)$ is increasing and convex.

Let $\Omega \subset \mathbb{R}^N$ be an bounded open set. For fixed $\alpha, \beta \in (0, |\Omega|)$, we define the following functional:

$$F_{\alpha,\beta} := \begin{cases} \dfrac{1}{\gamma}\displaystyle\int_\Omega f(u, \gamma|\nabla u|)\,\mathrm{d}x & \text{if } u \in \mathcal{A}_{\alpha,\beta}, \\ +\infty & \text{elsewhere in } L^1(\Omega), \end{cases}$$

where $\gamma := |\Omega| - (\alpha + \beta)$ and

$$\mathcal{A}_{\alpha,\beta} := \{u \in W^{1,p}(\Omega) : |\{u = 0\}| \geqslant \alpha \text{ and } |\{u = 1\}| \geqslant \beta\}.$$

This constraint is the relaxed version of the original constraint in (1.2). Therefore the Γ-limit of this functional will coincide with the Γ-limit of the original problem.

THEOREM 3.2 Let $f : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous function satisfying the conditions of Lemma 3.1 and

$$f(0, 0) = f(1, 0) = 0.$$

Let $\bar{\alpha} \in (0, |\Omega|)$. Then

$$\Gamma(L^1)\text{-}\lim_{\substack{\alpha\to\bar{\alpha}\\ \beta\to|\Omega|-\bar{\alpha}}} F_{\alpha,\beta} = G_{\bar{\alpha}},$$

with $G_{\bar{\alpha}}$ given by

$$G_{\bar{\alpha}} := \begin{cases} \varphi(\mathrm{Per}\{u = 0\}) & \text{if } u \in BV(\Omega, \{0, 1\}) \text{ and } |\{u = 0\}| = \bar{\alpha}, \\ +\infty & \text{elsewhere in } L^1(\Omega), \end{cases}$$

where φ is defined as in Lemma 3.1.

Estimate of the Γ-limit from above. To derive a Γ-limit it is necessary to check two conditions: an estimate from above (for a suitably constructed approximating sequence) and an estimate from below (for arbitrary approximating sequences of functions). We will start with the estimate from above (the “limsup-inequality”). Our general idea is to define the approximating sequence as minimizers of the one-dimensional problem applied to the distance from the boundary. In other words, we construct equal-sized layers around the boundary on which we define the approximating sequence as a function of the distance to Γ. An application of the coarea formula for the case where the boundary of the level sets is smooth provides us then with the desired estimate. Finally, we prove the case of general level sets by an approximation argument.

Let $\alpha_n \to \bar{\alpha}$ and $\beta_n \to |\Omega| - \bar{\alpha}$. Define $\gamma_n := |\Omega| - \alpha_n - \beta_n$. Let us first assume that $\Gamma := \partial^*(\{u = 0\})$ is smooth. We write $d(x) := \operatorname{dist}(x, \Gamma)$ and define $\varepsilon(\gamma_n)$ such that

$$\int_0^{\varepsilon(\gamma_n)} \mathcal{H}^{N-1}(\{x \in \Omega : d(x) = t\})\,\mathrm{d}t = \gamma_n.$$

Let v_n be the minimizer of

$$P\left(\frac{\varepsilon(\gamma_n)}{\gamma_n}\right) := \min\left\{\int_0^{\varepsilon(\gamma_n)/\gamma_n} f(u, |u'|)\,\mathrm{d}x : u(0) = 0,\ u(\varepsilon(\gamma_n)/\gamma_n) = 1\right\}.$$

Define $u_n(x) := v_n(d(x)/\gamma_n)$. By using this definition and the coarea formula we get

$$\begin{aligned} F_{\alpha_n,\beta_n} &:= \frac{1}{\gamma_n}\int_\Omega f(u_n, \gamma_n \nabla u_n)\,\mathrm{d}x \\ &= \frac{1}{\gamma_n}\int_0^{\varepsilon(\gamma_n)} \bar{f}(v_n(t/\gamma_n), |v_n'(t/\gamma_n)|)\,\mathcal{H}^{N-1}(\{x \in \Omega, d(x) = t\})\,\mathrm{d}t. \end{aligned}$$

Now we use the fact that

$$\lim_{t\to 0} \mathcal{H}^{N-1}(\{x \in \Omega : d(x) = t\}) = \mathcal{H}^{N-1}(\Gamma) \tag{3.2}$$

(see [12, Lemma 4]). From this and a change of variable we get, for every $\delta > 0$ and for n large enough,

$$\begin{aligned} F_{\alpha_n,\beta_n} &\leqslant \frac{1}{\gamma_n}(1+\delta)\,\mathcal{H}^{N-1}(\Gamma)\int_0^{\varepsilon(\gamma_n)} f(v_n(t/\gamma_n), |v_n'(t/\gamma_n)|)\,\mathrm{d}t \\ &= (1+\delta)\,\mathcal{H}^{N-1}(\Gamma)\int_0^{\varepsilon(\gamma_n)/\gamma_n} f(v_n(s), |v_n'(s)|)\,\mathrm{d}s. \end{aligned}$$

By (3.1) and since $\gamma_n/\varepsilon(\gamma_n) \to \mathcal{H}^{N-1}(\Gamma)$ (which follows from (3.2)), we get

$$\limsup_{n\to\infty} F_{\alpha_n,\beta_n} \leqslant \lim_{n\to\infty}(1+\delta)\,P(\varepsilon(\gamma_n)/\gamma_n)\,\mathcal{H}^{N-1}(\Gamma) = (1+\delta)\varphi(\mathcal{H}^{N-1}(\Gamma)). \tag{3.3}$$

Since $\delta > 0$ is arbitrarily small the Γ-limsup inequality is proved for Γ smooth. The general case follows from a standard density argument based upon the following lemma [4, Lemma 4.3]:

LEMMA 3.3 Let $E \subset \Omega$ be a set with finite perimeter such that $0 < \mathcal{L}^N(E) < \mathcal{L}^N(\Omega)$. There exists a sequence of bounded open sets $D_n \subset \mathbb{R}^N$ with smooth boundaries such that $\mathcal{L}^N(E) = \mathcal{L}^N(D_n \cap \Omega)$, χ_{D_n} converges to χ_E in $L^2(\Omega)$, and

$$\lim_{n\to\infty} \mathcal{H}^{N-1}(\partial D_n \cap \Omega) = \mathcal{H}^{N-1}(\partial^*(E \cap \Omega)).$$

Estimate of the Γ-limit from below. The general plan to prove the limit from below (the "liminf-inequality") is to approximate the boundary of the level sets by piecewise smooth manifolds (which can be done thanks to a theorem by De Giorgi). On each of these smooth pieces we can then estimate the energy of the transition layer by a one-dimensional transition.

We start with the following measure-theoretical result whose proof is based upon a standard recovering argument and De Giorgi's structure theorem. We will sketch the argument for the reader's convenience and illustrate it in Fig. 3.

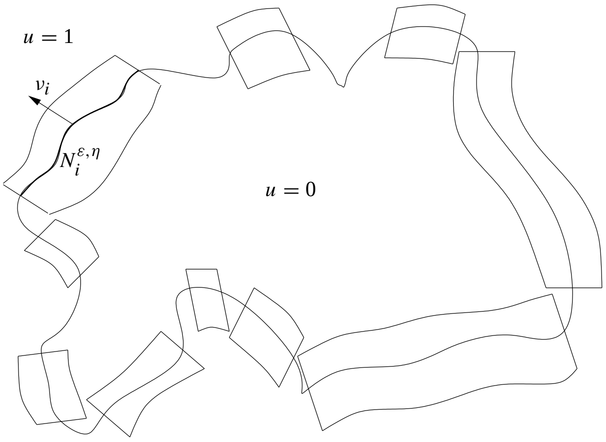

FIG. 3. Illustration of the sets $N_i^{\varepsilon,\eta}$, partially covering $\partial^*\{u = 0\}$.

LEMMA 3.4 Let $u \in BV(\Omega, \{0, 1\})$ and set $\Gamma := \partial^*\{u = 0\}$. Then for every $\varepsilon, \eta > 0$ we can find a decomposition of Γ of the form

$$\Gamma = \bigcup_{i=1}^{k_{\varepsilon,\eta}} N_i^{\varepsilon,\eta} \cup M_{\varepsilon,\eta}$$

with the following properties:

(i) $\mathcal{H}^{N-1}(M_{\varepsilon,\eta}) < \varepsilon$;
(ii) $N_i^{\varepsilon,\eta} \cap N_j^{\varepsilon,\eta} = \emptyset$ if $i \neq j$;
(iii) for every $i \in \{1, \dots, k_{\varepsilon,\eta}\}$ the set $N_i^{\varepsilon,\eta}$ is a compact subset of a C^1-manifold; more precisely, there exists $\nu_i \in S^{n-1}$ such that $N_i^{\varepsilon,\eta}$ is contained in the graph of a C^1-function g_i defined on the plane Π_{ν_i} orthogonal to ν_i;
(iv) for every $x \in N_i^{\varepsilon,\eta}$ we have $|\nu(x) - \nu_i| < \eta$.

Proof. We recall first that by De Giorgi's structure theorem (see e.g. [9]) the reduced boundary $\partial^*\{u = 0\}$ is $(n-1)$-rectifiable and so, in particular, we can find a decomposition of the form

$$\Gamma = \bigcup_{i=1}^{k_\varepsilon} N_i^\varepsilon \cup M_\varepsilon,$$

where $\mathcal{H}^{N-1}(M_\varepsilon) < \varepsilon/2$ and the N_i^ε have the following properties:

(i) $N_i^\varepsilon \cap N_j^\varepsilon = \emptyset$ if $i \neq j$;
(ii) for every $i \in \{1, \dots, k_\varepsilon\}$ the set N_i^ε is a compact subset of the graph of a C^1-function.

Using the compactness of N_i^ε, we can find a $\delta > 0$ (independent of i) such that for every $x, y \in N_i^\varepsilon$ with $|x - y| \leqslant \delta$ we have $|\nu(x) - \nu(y)| < \min\{\eta, \sqrt{2}\}$. For $x \in \Gamma \setminus M_\varepsilon$ and for $0 \leqslant s \leqslant \delta$ we can consider the set $A(x, s) := B(x, s) \cap N_i^\varepsilon$, where i is the (unique) index such that $x \in N_i^\varepsilon$. The family $\{A(x, s) : x \in \Gamma,\ 0 \leqslant s \leqslant \delta\}$ forms a fine covering of $\Gamma \setminus M_\varepsilon$. Therefore we can apply the Besicovitch theorem to extract a finite subfamily $\{A(x_i, s_i)\}_{i=1}^{k_{\varepsilon,\eta}}$ of pairwise disjoint sets such that

$$\mathcal{H}^{N-1}\Big((\Gamma \setminus M_\varepsilon) \setminus \bigcup_{i=1}^{k_{\varepsilon,\eta}} A(x_i, s_i)\Big) < \frac{\varepsilon}{2}.$$

Setting $N_i^{\varepsilon,\eta} := A(x_i, s_i)$ and $\nu_i = \nu(x_i)$, we see that the family $\{N_i^{\varepsilon,\eta}\}_{i=1}^{k_{\varepsilon,\eta}}$ meets all the requirements. ☐

We are now in a position to prove the Γ-liminf inequality. Suppose that $u_n \to u$ in $L^1(\Omega)$ and a.e., where $u \in BV(\Omega, \{0, 1\})$. We only need to consider functions with finite energy, thus $u_n \in W^{1,p}(\Omega)$. We may assume without loss of generality that $F_{\alpha_n,\beta_n}(u_n)$ admits a finite limit. If u_n takes values outside [0, 1], we can apply the following truncation argument: define $S := \{x \in \Omega : u_n(x) > 1\}$ and $v := \max\{u, 1\}$, then the energy on S is zero (due to the assumption that $f(0, 0) = f(1, 0) = 0$) and hence minimal (since f maps to $\mathbb{R}_+$). We then repeat the same argument for zero instead of one. In the following we can therefore assume that $0 \leqslant u_n \leqslant 1$ a.e.

We fix $\varepsilon > 0$ and we find $\eta = \eta(\varepsilon) > 0$ such that

$$\nu_1, \nu_2 \in S^{n-1},\ |\nu_1 - \nu_2| < \eta \ \Rightarrow\ \langle \nu_1, \nu_2 \rangle > 1 - \varepsilon. \tag{3.4}$$

We can now find a decomposition of Γ of the form

$$\Gamma = \bigcup_{i=1}^{k_{\varepsilon,\eta}} N_i^{\varepsilon,\eta} \cup M_{\varepsilon,\eta}$$

with the properties stated in the previous lemma.

CLAIM. There exist $\Gamma' \subset \Gamma \setminus M_{\varepsilon,\eta}$ and a subsequence u_n (not relabelled) such that

(i) $\mathcal{H}^{N-1}((\Gamma \setminus M_{\varepsilon,\eta}) \setminus \Gamma') < \varepsilon$;
(ii) for every n large enough there exist two positive functions s_n and t_n such that for $x \in N_i^{\varepsilon,\eta} \cap \Gamma'$ we have $u_n(x + t_n(x)\nu_i) = 0$ and either

$$u_n(x + (t_n(x) + \gamma_n s_n(x))\nu_i) = 1 \quad \text{or} \quad u_n(x + (t_n(x) - \gamma_n s_n(x))\nu_i) = 1;$$

(iii) $\int_{\Gamma'} s_n \, d\mathcal{H}^{N-1} \leqslant 1/(1 - \varepsilon)$ for every $n \geqslant \bar{n}$;
(iv) $\gamma_n s_n \to 0$ uniformly in Γ'.

Set (see Fig. 4)

$$\Gamma_{0,n} := \{x \in \Omega : u_n(x) = 0\} \quad \text{and} \quad \Gamma_{1,n} := \{x \in \Omega : u_n(x) = 1\}.$$

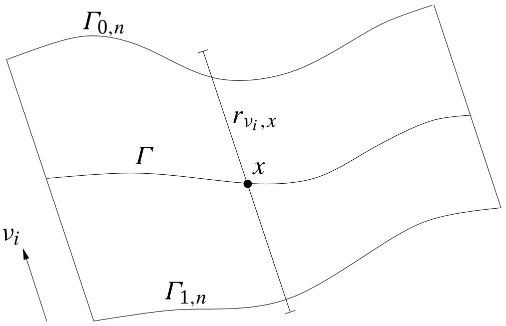

FIG. 4. A set $N_i^{\varepsilon,\eta}$ with ν_i and $r_{\nu_i,x}$.

Fix $\tau = \tau(\varepsilon, \eta) > 0$ so small that the sets

$$N_i^{\varepsilon,\eta,\tau} := \{x + t\nu_i : x \in N_i^{\varepsilon,\eta},\ t \in (-2\tau, 2\tau)\}, \qquad i = 1, \dots, k_{\varepsilon,\eta}, \tag{3.5}$$

are pairwise disjoint. We denote by $r_{\nu_i,x}$ the straight line segment parallel to ν_i with center at x and length 2τ. Let $G_{i,n} \subset N_i^{\varepsilon,\eta}$ be the set on which $\Gamma_{0,n} \cap r_{\nu_i,x}$ and $\Gamma_{1,n} \cap r_{\nu_i,x}$ are both nonempty. On $G_{i,n}$ we define

$$s_n(x) := \frac{1}{\gamma_n} \operatorname{dist}(\Gamma_{0,n} \cap r_{\nu_i,x}, \Gamma_{1,n} \cap r_{\nu_i,x}).$$

From this definition it is clear that we can define a function t_n such that

$$u_n(x + t_n(x)\nu_i) = 0$$

and either

$$u_n(x + (t_n(x) + s_n(x))\nu_i) = 1 \quad \text{or} \quad u_n(x + (t_n(x) - s_n(x))\nu_i) = 1.$$

For simplicity we will discuss only the first case. (Due to the symmetry of f the latter case can be handled in the same way.)

The closedness of $\Gamma_{0,n}$ and $\Gamma_{1,n}$ and the smoothness of $N_i^{\varepsilon,\eta}$ imply that $G_{i,n}$ is closed and hence measurable, and s_n is also measurable over $G_{i,n} \setminus M_{\varepsilon,\eta}$. Property (ii) is satisfied by construction almost everywhere in $G_{i,n} \setminus M_{\varepsilon,\eta}$. Using the fact that $u_n \to u$, we obtain

$$\lim_{n\to\infty} \Big| \Gamma \setminus \bigcup_i G_{i,n} \Big| = 0.$$

Denoting by π_{ν_i} the orthogonal projection on Π_{ν_i}, we have

$$\begin{aligned} \gamma_n &\geqslant |\{x \in \Omega : 0 < u_n(x) < 1\}| \geqslant \sum_{i=1}^{k_{\varepsilon,\eta}} \int_{\pi_{\nu_i}(N_i^{\varepsilon,\eta} \setminus Q^{\varepsilon,\eta})} \gamma_n s_n \,\mathrm{d}\mathcal{H}^{N-1} \\ &= \sum_{i=1}^{k_{\varepsilon,\eta}} \int_{N_i^{\varepsilon,\eta} \setminus Q^{\varepsilon,\eta}} \gamma_n s_n \langle \nu(x), \nu_i \rangle \,\mathrm{d}\mathcal{H}^{N-1} \geqslant (1-\varepsilon) \int_{\Gamma \setminus (M_{\varepsilon,\eta} \cup Q^{\varepsilon,\eta})} \gamma_n s_n \,\mathrm{d}\mathcal{H}^{N-1}, \end{aligned} \tag{3.6}$$

where the last inequality is a consequence of (3.4). This proves (iii).

Using (3.6) and Egorov's theorem we can find a subsequence u_n and $\Gamma' \subset \Gamma \setminus M_{\varepsilon,\eta}$ with all the required properties.

Now set

$$U_n := \bigcup_{i=1}^{k_{\varepsilon,\eta}} \{x + t\nu_i : x \in N_i^{\varepsilon,\eta} \cap \Gamma',\ t \in (t_n(x), t_n(x) + \gamma_n s_n(x))\},$$

and choose $n \in \mathbb{N}$ so large that $U_n \subset \bigcup_{i=1}^{k_{\varepsilon,\eta}} N_i^{\varepsilon,\eta,\tau}$ (see (3.5)) with τ chosen as before. Then, using Fubini's theorem and the monotonicity of f we can estimate

$$\begin{aligned}
\frac{1}{\gamma_n}\int_\Omega f(u_n, \gamma_n \nabla u_n)\,\mathrm{d}x \ &\geqslant\ \frac{1}{\gamma_n}\int_{U_n} f(u_n, \gamma_n \nabla u_n)\,\mathrm{d}x \\
&\geqslant \sum_{i=1}^{k_{\varepsilon,\eta}} \int_{\pi_{\nu_i}(N_i^{\varepsilon,\eta}\cap\Gamma')} \frac{1}{\gamma_n}\Bigg(\int_{t_n(g_i(y))}^{t_n(g_i(y))+\gamma_n s_n(g_i(y))} f(u_n(g_i(y)+t\nu_i), \\
&\qquad\qquad \gamma_n \partial_{\nu_i} u_n(g_i(y)+t\nu_i))\,\mathrm{d}t\Bigg)\,\mathrm{d}\mathcal{H}^{N-1}(y) \\
&= \sum_{i=1}^{k_{\varepsilon,\eta}} \int_{\pi_{\nu_i}(N_i^{\varepsilon,\eta}\cap\Gamma')} \int_0^{s_n(g_i(y))} f(v_n^y(t), (v_n^y)'(t))\,\mathrm{d}t\,\mathrm{d}\mathcal{H}^{N-1}(y) =: I, \qquad (3.7)
\end{aligned}$$

where we set $v_n^y(t) := u_n(g_i(y) + t_n(g_i(y)) + \gamma_n t\nu_i)$ (g_i is the function appearing in (iii) of Lemma 3.4). Recalling the definition of $P(t)$ and (3.4) we can continue our estimate as follows:

$$\begin{aligned}
I &\geqslant \sum_{i=1}^{k_{\varepsilon,\eta}} \int_{\pi_{\nu_i}(N_i^{\varepsilon,\eta}\cap\Gamma')} P(s_n(g_i(y)))\,\mathrm{d}\mathcal{H}^{N-1}(y) = \sum_{i=1}^{k_{\varepsilon,\eta}} \int_{N_i^{\varepsilon,\eta}\cap\Gamma'} P(s_n(z))\langle\nu(z), \nu_i\rangle\,\mathrm{d}\mathcal{H}^{N-1}(z) \\
&\geqslant (1-\varepsilon)\int_{\Gamma'} P(s_n(z))\,\mathrm{d}\mathcal{H}^{N-1}(z);
\end{aligned}$$

using the convexity and monotonicity of P (see Lemma 3.1) and property (iii) of the previous claim we get

$$\begin{aligned}
I &\geqslant (1-\varepsilon)\mathcal{H}^{N-1}(\Gamma')P\bigg(\frac{1}{\mathcal{H}^{N-1}(\Gamma')}\int_{\Gamma'} s_n(z)\,\mathrm{d}\mathcal{H}^{N-1}(z)\bigg) \\
&\geqslant (1-\varepsilon)\mathcal{H}^{N-1}(\Gamma')P\bigg(\frac{1}{(1-\varepsilon)\mathcal{H}^{N-1}(\Gamma')}\bigg). \qquad (3.8)
\end{aligned}$$

Since ε is arbitrarily small and the measure of Γ' is arbitrarily close to the measure of Γ, by combining (3.7) and (3.8) we complete the proof of the Γ-liminf inequality. ☐

3.2 *Anisotropic energies*

In this section we extend the result from the previous section to a class of anisotropic functionals where the energy density g is given by

$$g(u, \xi) := f(u, \psi(\xi)) \qquad (3.9)$$

where ψ is a norm given by

$$\psi(\xi) := \sqrt{\langle L\xi, \xi\rangle} \tag{3.10}$$

with $L\colon \mathbb{R}^n \to \mathbb{R}^n$ a symmetric positive definite linear operator.

For fixed $\alpha, \beta \in (0, |\Omega|)$, we define the functional

$$F_{\alpha,\beta} := \begin{cases} \dfrac{1}{\gamma}\displaystyle\int_\Omega g(u, \gamma\nabla u)\,\mathrm{d}x & \text{if } u \in \mathcal{A}_{\alpha,\beta}, \\ +\infty & \text{elsewhere in } L^1(\Omega), \end{cases}$$

where γ and $\mathcal{A}_{\alpha,\beta}$ are defined as above.

The main result of this section is the following Γ-convergence theorem:

THEOREM 3.5 Let f satisfy the same conditions as in the previous section, and let ψ be as in (3.10). Let $\bar{\alpha} \in (0, |\Omega|)$. Then

$$\Gamma(L^1)\text{-}\lim_{\substack{\alpha\to\bar{\alpha}\\ \beta\to|\Omega|-\bar{\alpha}}} F_{\alpha,\beta} = G_{\bar{\alpha}},$$

with $G_{\bar{\alpha}}$ given by

$$G_{\bar{\alpha}} := \begin{cases} \varphi\Big((\det L^{-1/2})\displaystyle\int_{\partial^*\{u=0\}} \psi(\nu(x))\,\mathrm{d}\mathcal{H}^{N-1}(x)\Big) & \text{if } u \in BV(\Omega, \{0,1\}) \text{ and } |\{u=0\}| = \bar{\alpha}, \\ +\infty & \text{elsewhere in } L^1(\Omega), \end{cases}$$

where $\varphi\colon (0, \infty) \to \mathbb{R}$ is a monotone function defined by (3.16) below.

Proof. The key idea of the proof is a change of variables that transforms the problem back to the class of isotropic problems. To perform this change of variables we need the following lemma that is a consequence of the so-called generalized area formula (see Theorem 2.91 in [5]). For the reader's convenience we give here a simple direct proof based on the divergence theorem.

LEMMA 3.6 Let $L : \mathbb{R}^N \to \mathbb{R}^N$ be a symmetric positive definite linear mapping. Let Γ be an $(N-1)$-rectifiable set. Then for every $\mathcal{H}^{N-1}$-measurable set $A \subset \Gamma$ we have

$$\mathcal{H}^{N-1}(L(A)) = (\det L)\int_A |L^{-1}\nu(y)|\,\mathrm{d}\mathcal{H}^{N-1}(y).$$

Proof. Using the definition of a rectifiable set we can assume without loss of generality that Γ is a C^1-manifold (cf. Lemma 3.4). We consider the pull-back measure $L^\sharp\mathcal{H}^{N-1}$ defined on $\mathbb{R}^N$ as

$$L^\sharp\mathcal{H}^{N-1} : B \mapsto \mathcal{H}^{N-1}(L(B)).$$

It is easy to see that its restriction to Γ, denoted by $L^\sharp\mathcal{H}^{N-1}\lfloor\Gamma$, is absolutely continuous with respect to $\mathcal{H}^{N-1}\lfloor\Gamma$. We claim that for all $x_0 \in \Gamma$,

$$\frac{\mathrm{d}(L^\sharp\mathcal{H}^{N-1}\lfloor\Gamma)}{\mathrm{d}(\mathcal{H}^{N-1}\lfloor\Gamma)}(x_0) = (\det L)|L^{-1}\nu(x_0)|. \tag{3.11}$$

Let $r > 0$ be so small that $B(x_0, r) \setminus \Gamma$ has two connected components B_+ and B_-. Define $D := B(x_0, r) \cap \Gamma$. Define

$$\Phi := \{\eta \in C_0^1(L(B), \mathbb{R}^N) : \|\eta\|_\infty \leqslant 1\}.$$

Using the divergence theorem we see that

$$\mathcal{H}^{N-1}(L(D)) = \sup_{\eta\in\Phi} \int_{L(B_+)} \operatorname{div}\eta \,\mathrm{d}x. \tag{3.12}$$

Given η as above, for every $y \in B$ we set

$$\hat{\eta}(y) := \eta(Ly).$$

Note that for every $x \in L(B)$ we have

$$\operatorname{div}\eta(x) = \operatorname{div}(L^{-1}\hat{\eta})(L^{-1}x). \tag{3.13}$$

Therefore, using (3.12) and (3.13), we can compute

$$\begin{aligned}
\mathcal{H}^{N-1}(L(D)) &= \sup_{\eta\in\Phi} \int_{L(B)} \operatorname{div}(L^{-1}\hat{\eta})(L^{-1}x)\,\mathrm{d}x = \sup_{\eta\in\Phi}(\det L) \int_B \operatorname{div}(L^{-1}\hat{\eta})(y)\,\mathrm{d}y \\
&= \sup_{\eta\in\Phi}(\det L) \int_D \langle L^{-1}\hat{\eta}(y), \nu(y)\rangle \,\mathrm{d}\mathcal{H}^{N-1} = \sup_{\eta\in\Phi}(\det L) \int_D \langle \hat{\eta}(y), L^{-1}\nu(y)\rangle \,\mathrm{d}\mathcal{H}^{N-1} \\
&= \sup_{\substack{\eta\in C^1(B,\mathbb{R}^N)\\ \|\eta\|_\infty \leqslant 1}} (\det L) \int_D \langle \eta(y), L^{-1}\nu(y)\rangle \,\mathrm{d}\mathcal{H}^{N-1} = (\det L) \int_D |L^{-1}\nu(y)| d\mathcal{H}^{N-1}(y),
\end{aligned}$$

where the last equality follows by taking $\eta_n := L^{-1}\nu/|L^{-1}\nu|$ as maximizing sequence on $D_n \subset\subset D$, with D_n increasing to D. This concludes the proof of (3.11) and therefore of the lemma. □

We now prove the Γ-liminf inequality. Let $\alpha_n \to \bar{\alpha}$ and $\beta_n \to |\Omega| - \bar{\alpha}$. Define as before $\gamma_n := |\Omega| - \alpha_n - \beta_n$ and $\Gamma := \partial^*(\{u = 0\})$. Suppose that $u_n \to u$ in $L^1(\Omega)$ and a.e. where $u \in BV(\Omega, \{0, 1\})$. We may assume without loss of generality that $F_{\alpha_n,\beta_n}(u_n)$ admits a finite limit.

We now change variables by setting, for every $y \in L^{-1/2}\Omega$,

$$v_n(y) := u_n(L^{1/2}y) \quad \text{and} \quad v(y) := u(L^{1/2}y).$$

Note that $v_n \to v$ in L^1 and that for $x \in \Omega$,

$$\nabla v_n(L^{-1/2}x) = L^{1/2}\nabla u_n(x),$$

which yields

$$|\nabla v_n(L^{-1/2}x)| = \sqrt{\langle L\nabla u_n(x), \nabla u_n(x)\rangle}.$$

Thus we have

$$\begin{aligned}
\frac{1}{\gamma_n}\int_\Omega g(u_n, \gamma_n\nabla u_n)\,\mathrm{d}x &= \frac{1}{\gamma_n}\int_\Omega f(v_n(L^{-1/2}x), \gamma_n|\nabla v_n(L^{-1/2}x)|)\,\mathrm{d}x \\
&= \frac{\det L^{1/2}}{\gamma_n} \int_{L^{-1/2}\Omega} f(v_n, \gamma_n|\nabla v_n|)\,\mathrm{d}y.
\end{aligned} \tag{3.14}$$

Let us now define

$$h(u, \xi) := f(u, (\det L^{1/2})\xi).$$

Since f is isotropic, we can write h as a function of u and $|\xi|$, and we define

$$\tilde{P}(t) := \inf\left\{\int_0^t h(u, u')\,\mathrm{d}s : u \in H^1(0,t),\ u(0) = 0,\ u(t) = 1\right\}.$$

Since the measure of the transition layer of v_n is given by

$$\tilde{\gamma}_n := \frac{\gamma_n}{\det L^{1/2}}$$

and since f is isotropic we can use the results of the previous section to estimate

$$\begin{aligned}
\liminf_{n\to\infty} \frac{\det L^{1/2}}{\gamma_n} \int_{L^{-1/2}\Omega} & f(v_n, \gamma_n|\nabla v_n|)\,\mathrm{d}y \\
&= \liminf_{n\to\infty} \frac{1}{\tilde{\gamma}_n} \int_{L^{-1/2}\Omega} h(v_n, \tilde{\gamma}_n|\nabla v_n|)\,\mathrm{d}y \\
&\geqslant \mathcal{H}^{N-1}(L^{-1/2}\Gamma)\tilde{P}\left(\frac{1}{\mathcal{H}^{N-1}\left(L^{-1/2}\Gamma\right)}\right) \\
&= (\det L^{-1/2}) \int_\Gamma |L^{1/2}\nu|\,\mathrm{d}\mathcal{H}^{N-1}\,\tilde{P}\left(\frac{1}{(\det L^{-1/2})\int_\Gamma |L^{1/2}\nu|\,\mathrm{d}\mathcal{H}^{N-1}}\right),
\end{aligned} \tag{3.15}$$

where in the last equality we have used Lemma 3.6. Defining

$$\varphi(t) := t\tilde{P}(1/t), \tag{3.16}$$

we deduce finally

$$\liminf_{n\to\infty} \frac{1}{\gamma_n} \int_\Omega g(u_n, \gamma_n \nabla u_n)\,\mathrm{d}x \geqslant \varphi\left((\det L^{-1/2}) \int_\Gamma \psi(\nu(x))\,\mathrm{d}\mathcal{H}^{N-1}(x)\right).$$

This concludes the proof of the Γ-liminf inequality. The Γ-limsup inequality can be proved in an analogous way. ☐

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