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Analyse non linéaire

A homotopy method for solving an equation of the type $-\Delta u = F(u)$

by (*)

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ABSTRACT. — We describe a homotopy algorithm for solving the equation $-\Delta u = F(u)$. To this end, we define a pseudo-inverse and a pseudo-determinant with sufficient regularity properties, for operators of Laplacian type.

RÉSUMÉ. — On décrit une méthode d'homotopie pour résoudre l'équation $-\Delta u = F(u)$. Dans ce but, on définit pour les opérateurs du type Laplacien un pseudo-inverse et un pseudo-déterminant munis des propriétés de régularité nécessaires.

In this paper, a homotopy algorithm is given to solve the following problem:

$$(1) \quad \begin{cases} -\Delta u = F(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

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where Ω is some bounded regular domain in R^n and $F \in C^2(R, R)$ a given function with compact support (*). More precisely, we define a homotopy continuation method as given recently in Alexander-Yorke [3], Chow and Mallet-Paret and Yorke [4], Eaves-Saigal [5], Kellog-Li-Yorke [7], Smale [10] and others.

All these methods have been elaborated in order to numerically solve finite dimensional problems of the type $g(x) = x$ or $g(x) = y$. In fact, any problem which can be shown to have a solution using topological degree, or a certain generalization thereof, fits into the general framework of homotopy continuation. Our aim is to generalize these methods to infinite dimensional problems whose resolution involves Leray-Schauder degree. Before expounding our results, let us briefly explain the finite dimensional method worked out in the preceding papers.

Let $g : R^N \to R^N$ be a C^2-map. Suppose we are searching for a u^* such that $g(u^*) = 0$. For this, define a C^2-homotopy $G : R^N \times R \to R^N$, such that $G(u, 1) = g(u)$, and assume we know some u_0 such that $G(u_0, 0) = 0$. The main idea of the method is that for « almost every » homotopy G, the set $\{ (u, \lambda), G(u, \lambda) = 0 \}$ defines a curve in R^N, $(u(s), \lambda(s))_{s \in R}$, passing through $(u_0, 0)$. This curve can be numerically computed until a point of interest ($\lambda = 1$) is encountered. One moves along the curve by solving a Cauchy problem as following:

$$(C) \quad \begin{cases} \dfrac{du}{ds} = (G'_u)^{\#} G'_\lambda(u, \lambda) \\[2mm] \dfrac{d\lambda}{ds} = -\det [G'_u(u, \lambda)] \\[2mm] (u(0), \quad \lambda(0)) = (u_0, 0) . \end{cases}$$

(If A is a regular N × N-matrix, we set $A^{\#} = (\det A) A^{-1}$, and we extend by continuity the mapping $A \to A^{\#}$ to all N × N-matrix).

Then the problem of numerical computation is driven back to a usual differential equation solver. Moreover, one usually obtains constructive proofs for existence theorems of the Brouwer type.

Let us now return to our problem. We have to solve $g(u) = 0$, with $g(u) = -\Delta u - F(u)$ and $u \in H^2(\Omega) \cap H^1_0(\Omega)$. Consider the following homotopy:

$$G : H^2(\Omega) \cap H^1_0(\Omega) \times R \to L^2(\Omega) ,$$

(*) The compact support assumption is not so restrictive. Indeed, let F be a more general function. In many cases (for instance under monotonicity assumptions on F), one can find by some maximum principle a L^∞ − bound b for the solutions of (1). Therefore, instead of F, we can consider à troncature of F with compact support $[-b, +b]$.

with $G(u, \lambda) = \Delta u + \lambda F(u) + (1 - \lambda)h$, where $h \in L^2(\Omega)$ is arbitrary. The associated problem is

$$(2) \quad \begin{cases} -\Delta u = \lambda F(u) + (1-\lambda)h & \text{in} \quad \Omega, \\ u = 0 \quad \text{on} \quad \partial\Omega. \end{cases}$$

In order to extend the finite dimensional method expounded above, the main difficulties are:

1) To obtain that the solution set $\{(u, \lambda)\}$ of (2) is a regular curve.

2) To extend in a constructive way definitions of $A^\#$ and det A to infinite dimensional operators of the Laplacian type.

3) To show that the method provides a solution of (1).

We now summarize our results in this way, and give the plan of this paper.

First section. — Using Smale's density theorem, we prove that for most h in $L^2(\Omega)$, the set E of solutions (u, λ) of (2) is a one-dimensional C^1-submanifold of $H^2(\Omega) \cap H_0^1(\Omega) \times R$ (see Theorem 1).

Second section. — Let h be as above, and $(u(s), \lambda(s))_{s\in R}$ be a smooth arc of solutions of (2). Then $G(u(s), \lambda(s)) = 0$, and therefore:

$$(3) \qquad G'_u(u(s)), \lambda(s))u'(s) + G'_\lambda(u(s), \lambda(s))\lambda'(s) = 0 .$$

Here
$$G'_u(u, \lambda): H^2 \cap H_0^1 \to L^2(\Omega)$$
$$v \to \Delta v + \lambda F'(u)v$$

is a perturbation of Δ.

We define maps $J: A \to A^\#$ and $\delta: A \to \delta(A)$ on a set of operators of the Laplacian type, verifying $AA^\# = \delta(A)\,\mathrm{Id}$, and $A^\#A = \delta(A)\,\mathrm{Id}$. These definitions are explicit, and they ensure that δ and J are regular enough to obtain classical solutions for (C).

This is the object of Theorem 2, and will be treated in a general functional framework.

Third Section. — Using the result of Section 1, and some compacity property of the solution set of (2), we prove that the algorithm (C) obtained in Section 2 accomplishes its task: it provides a t^* such that $\lambda(t^*) = 1$, and then $u(t^*)$ is a solution of (1). We show this in Theorems 3 and 4. Thus we obtain a constructive existence proof of a solution for Problem (1).

SECTION 1

THEOREM 1. — Assume the following property:

(P) | 0 is a regular value of $\Delta . + F(.)$, i. e. for every solution $u \in H^2 \cap H_0^1(\Omega)$ of $\Delta u + F(u) = 0$, the linear operator $\begin{cases} v \to \Delta v + F'(u)\cdot v \\ H^2 \cap H_0^1 \to L^2 \end{cases}$ is onto.

Then there exists a residual subset R of $L^2(\Omega)$, such that, for h in R, the set

$$E=\{ (u, \lambda)\in H^2 \cap H_0^1(\Omega)\times R,\ G(u, \lambda)=\Delta u+\lambda F(u)+(1-\lambda)h=0 \}$$

is a one-dimensional C^1-submanifold of $H^2 \cap H_0^1(\Omega) \times R$.

In order to prove Theorem 1, assume first the next proposition:

PROPOSITION 1. — Suppose that, for every (u, λ) in E,

$$G'(u, \lambda) : H^2 \cap H_0^1(\Omega) \times R \to L^2(\Omega)$$

is an onto linear map. Then E is a one-dimensional C^1-submanifold of $H^2 \cap H_0^1(\Omega) \times R$.

Proof of Theorem 1. — Let G'_u and G'_λ be the partial derivatives of G:

$$\begin{aligned} G'_u(u, \lambda) : H^2 \cap H_0^1(\Omega) &\to L^2(\Omega) \\ v &\to G'_u(u, \lambda)v = \Delta v + \lambda F'(u)v . \\ G'_\lambda(u, \lambda) : \qquad R &\to L^2(\Omega) \\ \mu &\to G'_\lambda(u, \lambda)\mu = \mu(F(u) - h). \end{aligned}$$

Thus we have: $G'(u, \lambda) = (G'_u(u, \lambda), G'_\lambda(u, \lambda))$ and

$$G'(u, \lambda)(v, \mu)=\Delta v+\lambda F'(u)v+\mu(F(u)-h) \qquad \text{for} \qquad v\in H^2 \cap H_0^1(\Omega) \times R .$$

LEMMA 1. — $G'_u(u, \lambda)$, as an operator from $L^2(\Omega)$ to $L^2(\Omega)$, is self adjoint with compact resolvent, and therefore:

i) $\quad \operatorname{Im} G'_u(u, \lambda)$ is closed in $L^2(\Omega)$

$\quad \dim \operatorname{Ker} G'_u(u, \lambda)=\operatorname{codim} \operatorname{Im} G'_u(u, \lambda) < +\infty$

ii) $\quad \operatorname{Ker} G'_u(u, \lambda) = (\operatorname{Im} G'_u(u, \lambda))^\perp$

Remark. — $G'_u(u, \lambda)$ is a Fredholm operator with index 0.

Lemma 1 is an immediate consequence of a perturbation theorem of Kato [5] (th. 3.17, p. 214).

In order to prove Theorem 1, it is sufficient, by Proposition 1, to show that for almost every h in $L^2(\Omega)$, the map $G'(u, \lambda)$ is surjective for (u, λ) in $E = \{ (u. \lambda), G(u, \lambda) = 0 \}$.

Define the auxiliar map:

$$\begin{aligned} \Psi : H^2 \cap H_0^1(\Omega) \times (R\backslash \{ 1 \}) = X &\to L^2(\Omega) = Y \\ (u, \lambda) &\to \Psi(u, \lambda) = \frac{\Delta u + \lambda F(u)}{\lambda - 1} \end{aligned}$$

and apply to Ψ Smale's density theorem (Abraham-Robbin [*1*]).

Density theorem.

Let X and Y be C^r-manifolds, with X Lindelöff (every open cover of X has a countable subcover), and $\Psi : X \to Y$ a C^r-Fredholm map.

Suppose that $r > \max(0, \text{index } \Psi'(x))$ for every x in X. Then the set of regular values of Ψ, $R_\Psi = \{ y \in Y, \forall x \in X, (y = \Psi(x) \Rightarrow \Psi'(x) \text{ is surjective}) \}$ is a residual subset of Y.

Recall that a map $\Psi C^1 : X \to Y$ is said to be Fredholm if, for every $x \in X$, $\Psi'(x)$ is a linear Fredholm operator, i. e.:

i) Ker $\Psi'(x)$ is finite-dimensional

ii) Im $\Psi'(x)$ is closed and finite codimensional.

We define the index of $\Psi'(x)$ to be:

$$\text{Ind } \Psi'(x) = \dim \text{Ker } \Psi'(x) - \text{codim Im } \Psi'(x).$$

Let us first admit that Smale's theorem applies to Ψ with $r = 2$. Then, if $h \in R_\Psi$, $\Psi'(u, \lambda)$ is surjective for every (u, λ) such that

$$\Psi(u, \lambda) = h \quad (\Leftrightarrow G(u, \lambda) = 0, \lambda \neq 1).$$

But, for such a (u, λ), we have:

$$\Psi'(u, \lambda) = (\Psi'_u(u, \lambda), \Psi'_\lambda(u, \lambda)) = \left(\frac{\Delta + \lambda F'(u)}{\lambda - 1}, \frac{F(u)(\lambda - 1) - (\Delta u + \lambda F(u))}{(\lambda - 1)^2} \right)$$

$$= \frac{1}{\lambda - 1} (\Delta + F'(u), F(u) - h) = \frac{1}{\lambda - 1} G'(u, \lambda).$$

Therefore, if $h \in R_\Psi$, $(u, \lambda) \in E$ and if $\lambda \neq 1$, $G'(u, \lambda)$ is surjective. According to Property (P) this result still holds for $\lambda = 1$. Then applying Proposition 1 concludes the proof of theorem 1.

We have now to verify the hypothesis of Smale's theorem:

a) The map Ψ is Fredholm, and index $\Psi'(x) \leq 1$ for every x in X. Indeed,

$$\Psi'(u, \lambda) = (\Psi'_u(u, \lambda), 0) + (0, \Psi'_\lambda(u, \lambda))$$

$$= \frac{1}{\lambda - 1} (\Delta + \lambda F'(u), 0) + \frac{1}{\lambda - 1} \left(0, F(u) + \frac{\Delta u + \lambda F(u)}{1 - \lambda} \right)$$

with $(\Delta + \lambda F'(u), 0)(v, \mu) = \Delta v + \lambda F'(u) v$ for $(v, \mu) \in H^1_0 \cap H^2(\Omega) \times R$

and $\left(0, F(u) + \dfrac{\Delta u + \lambda F(u)}{1 - \lambda} \right)(v, \mu) = \left[F(u) + \dfrac{1}{1 - \lambda} (\Delta u + \lambda F(u)) \right] \mu.$

Now, by Lemma 1, $\Delta + \lambda F'(u) = G'_u(u, \lambda)$ is a Fredholm operator with null index and:

$$\text{Ker } (\Delta + \lambda F'(u), 0) = \text{Ker } (\Delta + \lambda F'(u)) \times R$$
$$\text{Im } (\Delta + \lambda F'(u), 0) = \text{Im } (\Delta + \lambda F'(u)).$$

Thus $T = (\Delta + \lambda F'(u), 0)$ is a Fredholm operator with index 1.

Moreover, it is well known (Lang [*6*], p. 202) that, if T is Fredholm and A a compact linear map, then T + A is Fredholm and index (T + A) = index T.

Now $A = \left(0, F(u) + \frac{\Delta u + \lambda F(u)}{1-\lambda}\right)$ if of finite rank and then compact. We conclude that $\Psi'(u, \lambda)$ is a Fredholm operator with index 1.

b) Ψ is $C^2 : X \to Y$.

Using Sobolev embedding, it is easy to see that

$$F : \left\{ \begin{array}{l} H^2(\Omega) \to L^2(\Omega) \\ u \to F \circ u \end{array} \right. \quad \text{is} \quad C^2. \quad \text{Then } \Psi \text{ is } C^2 : X \to Y.$$

Proof of Proposition 1. — We are going to use two lemmas.

LEMMA 2. — The following relations are equivalent:

i) $\dim \operatorname{Ker} G'_u(u, \lambda) = 1$ and $G'_\lambda(u, \lambda) \notin \operatorname{Im} G'_u(u, \lambda)$

ii) $\dim \operatorname{Ker} G'_u(u, \lambda) = 1$ and $\dim \operatorname{Ker} G'(u, \lambda) = 1$.

(A point (u, λ) which verifies one of these assertions is said to be a turning point).

The proof is obvious.

LEMMA 3. — $G'(u, \lambda)$ is surjective if and only if:

$$\dim \operatorname{Ker} G'(u, \lambda) = 1 .$$

Proof. — Assume $G'(u, \lambda)$ is surjective. Let us consider two cases:

a) $G'_u(u, \lambda)$ is surjective:

Since $G'_u(u, \lambda)$ is Fredholm with index 0, we have:

$$\dim \operatorname{Ker} G'_u(u, \lambda) = 0 .$$

This implies :

$$G'(u, \lambda)(v, \mu) = 0 \Rightarrow v = -(G'_u(u, \lambda))^{-1}(\mu G'_\lambda(u, \lambda)),$$

and therefore $\operatorname{Ker} G'(u, \lambda) = R((G'_u(u, \lambda))^{-1} G'_\lambda(u, \lambda), -1)$ is a one dimensional subspace of $H^2 \cap H^1_0(\Omega) \times R$.

b) $G'_u(u, \lambda)$ is not surjective:

Then $G'_\lambda(u, \lambda) \notin \operatorname{Im} G'_u(u, \lambda)$ and since $\dim \operatorname{Ker} G'_u(u, \lambda) = \operatorname{codim} \operatorname{Im} G'_u(u, \lambda)$ (Lemma 1), we have $\dim \operatorname{Ker} G'_u(u, \lambda) = 1$. From Lemma 2, we obtain:

$$\dim \operatorname{Ker} G'(u, \lambda) = 1 .$$

The converse is easy to check in the same way.

Now we can achieve the proof of Proposition 1:

By Lemma 3, $\dim \operatorname{Ker} G'(u, \lambda) = 1$ for every (u, λ) in E. We claim that for every (u, λ) in E there exists a C^1-chart from a neighbourhood of (u, λ) to R.

We examine two cases:

a) $\dim \operatorname{Ker} G'_u(u_0, \lambda_0) = 0$.

Thus we have codim Im $G'_u(u_0, \lambda_0) = 0$. So $G'_u(u_0, \lambda_0)$ is an isomorphism from $H^2 \cap H_0^1(\Omega)$ to $L^2(\Omega)$.

It follows from the Implicit Function Theorem that there exist a neighbourhood I of λ_0 in R, a neighbourhood V of (u_0, λ_0) in $H^2 \cap H_0^1(\Omega) \times R$ and a C^1-function $\varphi : I \to H^2 \cap H_0^1(\Omega)$ such that:

$$\begin{cases} G(u, \lambda) = 0 \\ (u, \lambda) \in V \end{cases} \Leftrightarrow \begin{cases} (u, \lambda) = (\varphi(\lambda), \lambda) \\ \lambda \in I \end{cases}$$

This defines a local chart of E at (u_0, λ_0).

b) dim Ker $G'_u(u_0, \lambda_0) = 1$. (Then (u_0, λ_0) is a turning point).

Write now for u in $L^2(\Omega)$: $u = u_1 + u_2$ with $u_1 \in \text{Ker } G'_u(u_0, \lambda_0)$ and $u_2 \in \text{Im } G'_u(u_0, \lambda_0)$. In particular: $u_0 = u_{1,0} + u_{2,0}$.

By Lemma 2, $G'_\lambda(u_0, \lambda_0) \notin \text{Im } G'_u(u_0, \lambda_0)$: so the restriction of $G'(u_0, \lambda_0)$ to Im $G'(u_0, \lambda_0) \times R$ is an isomorphism onto $L^2(\Omega)$. By using the Inverse Mapping Theorem, we easily deduce that the mapping χ defined by:

$$(u, \lambda) = (u_1 + u_2, \lambda) \to (u_1, G(u, \lambda)) = \chi(u, \lambda)$$

is a diffeomorphism from a neighbourhood V of (u_0, λ_0) on a neighbourhood W of $(u_{1,0}, G(u_0, \lambda_0))$. Thus we have:

$$\begin{cases} G(u, \lambda) = 0 \\ (u, \lambda) \in V \end{cases} \Leftrightarrow \begin{cases} (u, \lambda) = \chi^{-1}(u_1, 0) \\ (u_1, 0) \in W \end{cases}$$

This provides a local chart of E at (u_0, λ_0).

SECTION 2

A. DEFINITION OF A PSEUDO-INVERSE AND A PSEUDO-DETERMINANT

Let H be a Hilbert space and V a closed subspace of H. Consider the set $\mathscr{A}$ of self adjoint operators $A : D(A) \subset H \to H$ with compact resolvent, bounded from below spectrum, and $D(A) = V$. For every A in $\mathscr{A}$, V is a Hilbert space under the graph norm: $\| x \|_H + \| Ax \|_H$. Note that if A and B are two elements of $\mathscr{A}$, the associated graph norms are equivalent.

THEOREM 2. — There exist (and we construct explicitly) a map J: $D(J) = \mathscr{A} \subset \mathscr{L}(V, H) \to \mathscr{L}(H, V)$, that we note $J(A) = A^\#$, and a map:

$$\delta : D(\delta) = \mathscr{A} \subset \mathscr{L}(V, H) \to R, \qquad \delta : A \to \delta(A),$$

such that:

1) $$AA^\# = \delta(A)\, \text{Id}_H$$

2) $$A^{\#}A = \delta(A)\,\mathrm{Id}_V$$

3) *i*) $$\mathrm{Ker}\,A \neq \{0\} \Leftrightarrow \delta(A) = 0$$

ii) If $\mathrm{Ker}\,A = \{0\}$, $\mathrm{sgn}\,\delta(A) = (-1)^p$, where p is the total multiplicity of the negative eigenvalues.

4) *i*) δ is locally Lipschitz from $\mathscr{A}$ to R.

ii) J is locally Lipschitz on the subset of the elements of $\mathscr{A}$ such that $\dim \mathrm{Ker}\,A \leq 1$.

Remarks. — 1) The preceding properties allow us to call $A^{\#}$ pseudo-inverse of A, and $\delta(A)$ pseudo-determinant. Note that if $V = H = R^N$, $\delta(A) = \det A$ and $A^{\#}$ is the matrix defined in Introduction.

2) It is possible to generalize the property 4 (*ii*) in the following way:

4 (ii) bis: J is locally Lipschitz from $\mathscr{A}$ to $\mathscr{L}(H, V)$.

The proof of this result is somewhat tedious and we shall omit it here.

Proof of theorem 2. — Since A is self-adjoint with compact resolvent, it admits an orthonormal basis of eigenvectors $(e_1, e_2, \ldots, e_n, \ldots)$ associated with the eigenvalues: $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \leq \ldots$, multiple eigenvalues being counted repeatedly.

Relatively to this basis, we write A as an infinite matrix:

$$A = \begin{pmatrix} \lambda_1 & & & & \\ & \ddots & & & \\ & & \lambda_n & & \\ & & & \ddots & \end{pmatrix}.$$

Now, set $N = \sup\{ i/\lambda_i \leq 1 \}$,
and:

$$A^{\#} = \begin{pmatrix} \prod\limits_{\substack{i=1\\ i\neq 1}}^{N} \lambda_i & & & & & & \\ & \prod\limits_{\substack{i=1\\ i\neq 2}}^{N} \lambda_i & & & 0 & & \\ & & \cdot & & & & \\ & & & \prod\limits_{\substack{i=1\\ i\neq N}}^{N} \lambda_i & & & \\ & & & & \ddots & & \\ & 0 & & & & \dfrac{\delta(A)}{\lambda_{N+1}} & \\ & & & & & & \dfrac{\delta(A)}{\lambda_{N+2}} \quad \ddots \end{pmatrix}$$

where $\delta(A) = \prod\limits_{i=1}^{N} \lambda_i$

Clearly this definition does not depend on the chosen basis of eigenvectors. Note that if A is an isomorphism, we have simply: $A^{\#} = \delta(A)A^{-1}$. A trivial computation provides immediately properties 1), 2), 3). Let us show now property 4) (*i*). We first list some technical tools:

LEMMA 4. — Let $A \in \mathscr{A}$, and $(\lambda_n)_{n\in\mathbb{N}}$, $(e_n)_{n\in\mathbb{N}}$ defined as above, then:

$$\lambda_n = \inf_{\substack{\dim F = n \\ F \subset V}} \Big(\sup_{\substack{x \in F \\ \|x\|_H = 1}} (Ax, x)_H \Big).$$

Proof. — Let F_n be the subspace generated by $(e_1, \ldots, e_n)$; clearly we have:

$$\sup_{\substack{x \in F_n \\ \|x\|_H = 1}} (Ax, x)_H = \lambda_n .$$

Let now F be an arbitrary n-dimensional subspace of V. Since

$$\dim F \cap F_{n-1}^{\perp} \geq 1 ,$$

one can choose $x \in F \cap F_{n-1}^{\perp}$ verifying $\| x \|_H = 1$.

Thus: $x = \sum_{i \geq n} x_i e_i$ and then $(Ax, x)_H = \sum_{i \geq n} \lambda_i x_i^2 \geq \lambda_n$.

LEMMA 5. — Let $A_0 \in \mathscr{A}$. Define on V the norm $\| x \|_V = \| x \|_H + \| A_0 x \|_H$, and on $\mathscr{L}(V, H)$ the corresponding norm $\| \cdot \|_{V,H}$. Then for every pair of elements of $\mathscr{A}$, A and B, which verify:

$$\| A - A_0 \|_{V,H} \leq \frac{1}{2} \qquad \text{and} \qquad \| B - A_0 \|_{V,H} \leq \frac{1}{2},$$

one has:

$$\lambda_n \leq \mu_n + \| A - B \|_{V,H} \quad (2 + 2 \sup (| \mu_1 |, | \mu_n |)) \tag{4}$$

$$\mu_n \leq \lambda_n + \| A - B \|_{V,H} \quad (2 + 2 \sup (| \lambda_1 |, | \lambda_n |)) . \tag{5}$$

Here λ_n and μ_n are the n^{th} eigenvalues of A and B respectively, multiple eigenvalues being counted repeatedly.

Proof. — For every x in V, we have:

$$\| A_0 x \|_H \leq \| Ax \|_H + \| A - A_0 \|_{V,H} \| x \|_V ,$$

then
$$\| A_0 x \|_H \leq \| Ax \|_H + \frac{1}{2} (\| x \|_H + \| A_0 x \|_H)$$

and therefore:

$$\| A_0 x \|_H \leq 2 \| Ax \|_H + \| x \|_H . \tag{6}$$

On the same way:

$$\| A_0 x \|_H \leq 2 \| Bx \|_H + \| x \|_H . \tag{7}$$

Let E_n (resp. F_n) be the subspace of V generated by the n first eigenvectors of an orthonormal eigenvectors basis for A (resp. B).
Then, for every $x \in V$ with $\| x \|_H = 1$, $(Ax - Bx, x)_H \leq \| A - B \|_{V,H} \| x \|_V$.
Hence $(Ax, x)_H \leq (Bx, x)_H + \| A - B \|_{V,H}(1 + \| A_0 x \|_H)$. Therefore:

$$\sup_{\substack{x \in F_n \\ \|x\|_H = 1}} (Ax, x)_H \leq \sup_{\substack{x \in F_n \\ \|x\|_H = 1}} (Bx, x)_H + \| A - B \|_{V,H} \sup_{\substack{x \in F_n \\ \|x\|_H = 1}} (1 + \| A_0 x \|_H).$$

Recall that by Lemma 4:

$$\lambda_n = \sup_{\substack{x \in F_n \\ \|x\|_H = 1}} (Ax, x)_H$$

and

$$\mu_n = \sup_{\substack{x \in F_n \\ \|x\|_H = 1}} (Bx, x)_H$$

Moreover, according to (7):

$$\sup_{\substack{x \in F_n \\ \|x\|_H = 1}} \| A_0 x \|_H \leq 2 \sup_{\substack{x \in F_n \\ \|x\|_H = 1}} \| Bx \|_H + 1 \leq 2 \sup(|\mu_1|, |\mu_n|) + 1.$$

This provides relation (4). In order to check (5), we exchange A and B.

LEMMA 6. — Note λ_n the map $A \to \lambda_n(A)$ which associates to A its n^{th} eigenvalue, multiple eigenvalues being counted repeatedly.
Then $\lambda_n : \mathscr{A} \subset \mathscr{L}(V, H) \to R$ is locally Lipschitz.

Proof. — 1) Fix an element $A_0 \in \mathscr{A}$, with eigenvalues $\lambda_1^0, \ldots, \lambda_n^0 \ldots$. We first prove that the eigenvalues μ_1 and μ_n of an operator B in $\mathscr{A}$ are bounded if $\| B - A_0 \|_{V,H} \leq \frac{1}{4}$.
Indeed, applying (4) to A_0 and B provides:

$$\lambda_1^0 \leq \mu_1 + \frac{1}{2}(1 + |\mu_1|)$$

and then:

$$\mu_1 \geq 2\lambda_1^0 - 1 \qquad (\text{if} \quad \mu_1 \leq 0),$$

and

$$\mu_1 \geq \frac{1}{3}(2\lambda_1^0 - 1) \qquad (\text{if} \quad \mu_1 \geq 0).$$

Similarly we obtain by (7):

$$\mu_n \leq \lambda_n^0 + \frac{1}{2}(1 + \sup(|\lambda_1^0|, |\lambda_n^0|)).$$

2) Let us consider now two operators A and B in $\mathscr{A}$ such that:

$$\| A - A_0 \|_{V,H} \leq \frac{1}{4} \qquad \text{and} \qquad \| B - A_0 \|_{V,H} \leq \frac{1}{4}.$$

From lemma 5 we deduce the following inequality:

$$|\lambda_n - \mu_n| \leq \| A - B \|_{V,H}(2 + \sup(|\lambda_1|, |\lambda_n|, |\mu_1|, |\mu_n|)).$$

Using the result of paragraph 1) achieves the proof.

We are now able to prove property 4 (*i*) of Theorem 2:

Notice that $\delta(A)$ may be written:

$$\delta(A) = \prod_{i=1}^{\infty} \theta(\lambda_i(A)) \qquad \text{where} \qquad \begin{cases} \theta(\lambda) = \lambda & \text{if} \quad \lambda \leq 1 \\ \theta(\lambda) = 1 & \text{if} \quad \lambda \geq 1 \end{cases}$$

Set $\lambda^0_{N+1} = \lambda_{N+1}(A_0) = \inf \{ \lambda^0_n(A),\ n \in \mathbb{N}^*,\ \lambda^0_n > 1 \}$.

By Lemma 6, if $\| A - A_0 \|_{V,H}$ is small enough, we have $\lambda_{N+1}(A) > 1$, and then:

$$\delta(A) = \prod_{i=1}^{N} \theta(\lambda_i(A)).$$

The function δ, being locally the product of N Lipschitz functions, is still locally Lipschitz.

Proof of property 4 (ii) of Theorem 2. — Let A_0 be an element of $\mathscr{A}$ such that dim Ker $A_0 \leq 1$. Two eventualities are to consider:

1) $$\operatorname{Ker} A_0 = \{0\}.$$

Since Isom (V, H) is open, there exists an open neighbourhood W of A_0 in $\mathscr{A}$ such that:

$$A \in W \Rightarrow \operatorname{Ker} A = \{0\}.$$

Thus, from the definition of $A^\#$, we have:

$$A \in W \Rightarrow A^\# = \delta(A)A^{-1}.$$

Upon applying property 4 (*i*) and reducing W if necessary, it follows that δ: $A \to \delta(A)$, $A \to A^{-1}$, and so $J: A \to A^\#$ are Lipschitz on W.

2) $$\operatorname{Dim} \operatorname{Ker} A_0 = 1.$$

Note: $\lambda^0_{i_0-1}$ the greatest strictly negative eigenvalue of A_0,
$\lambda^0_{i_0}$ its null eigenvalue,
and $\lambda^0_{i_0+1}$ its smallest strictly positive eigenvalue.

Let γ be the circle with centre 0 and radius

$$\rho = \inf\left(\frac{|\lambda^0_{i_0} - 1|}{2}, \lambda^0_{i_0+1}\right) \qquad \text{oriented in the direct sense.}$$

Let $W_\eta = \{ A \in \mathscr{A},\ \| A - A_0 \|_{V,H} \leq \eta \}$.

By Lemma 6 and inequalities (4) and (5), there readily exists a real η such that every A in W_η verifies:

i) $\lambda_{i_0}(A)$ is the unique eigenvalue of A enclosed by γ;

ii) dist (γ, spectrum of A) $\geq \frac{\rho}{2}$.

Consider, for A in W_η, the orthogonal projection Q(A) on the eigenspace associated to $\lambda_{i_0}(A)$. Thus we have:

$$Q(A) = \frac{1}{2i\Pi}\int_\gamma (z - A)^{-1}dz \qquad \text{(see Kato [5])}.$$

We wish to prove the mapping A $\rightarrow$ Q(A) is Lipschitz from

$$W_\eta \subset \mathscr{L}(V, H) \rightarrow \mathscr{L}(H, V).$$

For this, let A and B be two elements of W_η. We have:

$$\| Q(A) - Q(B) \|_{H,V} \leq \frac{1}{2\Pi}\int_\gamma \| (z-A)^{-1} - (z-B)^{-1} \|_{H,V} \, | dz |,$$

then:

$$\| Q(A) - Q(B) \|_{H,V} \leq \frac{1}{2\Pi}\int_\gamma \| (z-A)^{-1} \|_{H,V} \| A - B \|_{V,H} \| (z-B)^{-1} \|_{H,V}.$$

Therefore $\| Q(A) - Q(B) \|_{H,V} \leq \frac{C}{\rho^2} \| A - B \|_{V,H}$.

Now, setting $\lambda_i = \lambda_i(A)$, $A^\#$ may be written in the following way:

$$A^\# = \left(\prod_{i \neq i_0} \theta(\lambda_i) - \frac{\delta(A)}{1 + \lambda_{i_0}}\right) Q(A) + \delta(A)(A + Q(A))^{-1}.$$

Indeed, writing this formula relatively to the basis of eigenvectors yields the relation:

$$A^\# = \begin{pmatrix} 0 & & & 0 \\ & \ddots & & \\ & & \prod_{i \neq i_0} \theta(\lambda_i) - \frac{\delta(A)}{1 + \lambda_{i_0}} & \\ & & & \ddots \\ 0 & & & 0 \end{pmatrix} + \delta(A) \begin{pmatrix} \frac{1}{\lambda_1} & & & & & \\ & \ddots & & & & \\ & & \frac{1}{\lambda_{i_0 - 1}} & & & \\ & & & \frac{1}{1 + \lambda_{i_0}} & & \\ & & & & \frac{1}{\lambda_{i_0 + 1}} & \\ 0 & & & & & \ddots \end{pmatrix}$$

which is obvious.

Reducing W_η if necessary, the mappings A $\rightarrow \prod_{i \neq i_0} \theta(\lambda_i)$, A $\rightarrow \delta(A)$, A $\rightarrow \lambda_{i_0}(A)$, A $\rightarrow$ Q(A) and A $\rightarrow (A + Q(A))^{-1}$ are clearly Lipschitz on W_η. So is the mapping A $\rightarrow A^\#$. This achieves the proof of Theorem 2.

B. PARAMETRIZING BY DIFFERENTIAL EQUATION (4) THE COMPONENT OF MANIFOLD E WHICH CONTAINS $(u_0, 0)$

For every (u, λ) in $V \times R = H^2 \cap H_0^1(\Omega) \times R$, $-G'_u(u, \lambda) = -\Delta. - \lambda F'(u)$. is a self adjoint with compact resolvent operator. Its spectrum is bounded from below, its domain is $V = H^2 \cap H_0^1(\Omega)$ and it ranges in $H = L^2(\Omega)$. Upon applying Theorem 2 to this operator, we can define differential equation (8) in V:

$$\begin{cases} \dfrac{du}{ds}(s) = [G'_u(u(s), \lambda(s))]^{\#} G'_\lambda(u(s), \lambda(s)) \\ \dfrac{d\lambda}{ds}(s) = \delta(-G'_u(u(s), \lambda(s))) \\ (u(0), \lambda(0)) = (u_0, 0). \end{cases}$$

Readily for every solution $(u(s), \lambda(s))_{s \in [0,T[}$ of (8):

$$G(u(s), \lambda(s)) = G(u_0, 0) = 0,$$

and then: $(u(s), \lambda(s))_{s \in [0,T[} \subset E$.

We claim that differential equation (8) is locally Lipschitz on an open U containing E.

Indeed, for every (u, λ) in E: $\dim \operatorname{Ker} (G'_u(u, \lambda)) \leq 1$.

Referring to Lemma 6, there exists a neighbourhood W of $G'_u(u, \lambda)$ in $\mathscr{L}(V, H)$ such that for every A in $W \cap \mathscr{A}$: $\dim \operatorname{Ker} A \leq 1$.

But the mapping $(u, \lambda) \to G'_u(u, \lambda)$ is continuous from $V \times R$ to $\mathscr{L}(V, H)$. Then by Lemma 6 there exists a ball $B_{u,\lambda}$ in $V \times R$ with centre (u, λ) such that for every (v, μ) in $B_{u,\lambda}$ we still have:

$$\dim \operatorname{Ker} (G'_u(v, \mu)) \leq 1.$$

$$\text{Set now: } U = \bigcup_{(u,\lambda) \in E} B_{u,\lambda}.$$

Thus, the following mappings are locally Lipschitz:

$$\begin{cases} (u, \lambda) \to G'_u(u, \lambda) \\ U \subset V \times \mathbb{R} \to \mathscr{L}(V, H) \end{cases} \qquad \begin{cases} (u, \lambda) \to G'_\lambda(u, \lambda) \\ U \subset V \times \mathbb{R} \to \mathscr{L}(H, V) \end{cases}$$

$$\begin{cases} A \to A^{\#} \\ G'_u(U) \subset \mathscr{L}(V, H) \to \mathscr{L}(H, V) \end{cases} \qquad \begin{cases} A \to \delta(A) \\ G'_u(U) \subset \mathscr{L}(V, H) \to \mathbb{R}. \end{cases}$$

Equation (8) is therefore locally Lipschitz on U. Then the branch of E containing $(u_0, 0)$ can be partially parametrized by the maximal solution $(u(s), \lambda(s))_{s\in[0,T[}$ of (8).

SECTION 3

THE CONTINUATION METHOD DEFINED ABOVE PROVIDES A POINT $(u(t^*), \lambda(t^*))$ SUCH THAT $\lambda(t^*) = 1$ (so $u(t^*)$ IS A SOLUTION OF (1))

THEOREM 3. — Under the assumptions of Theorem 1, there exists a residual set R of $L^2(\Omega)$ such that for every h in R the maximal solution $(u(s), \lambda(s))_{s\in[0,T[}$ of the differential equation (8) associated with h verifies:

$$\exists t^* < T, \qquad \lambda(t^*) = 1 \qquad \text{and} \qquad \begin{cases} -\Delta(u(t^*)) = F(u(t^*)) \\ u(t^*) = H^2 \cap H_0^1(\Omega) \end{cases}$$

Proof. — Let R be the residual set whose existence is ensured by Theorem 1. Fix h in R. Thus E, defined as in the Introduction is a one-dimensional C^1-submanifold of $H^2 \cap H_0^1(\Omega) \times \mathbb{R}$.

Following a classical way of the homotopy method, we wish to prove successively that:

A. For $s > 0$ small enough, $\lambda(s) > 0$.

B. Solution $(u(s), \lambda(s))$ for $s > 0$ does not « recross » the hyperplane $H^2 \times H_0^1(\Omega) \times \{0\}$.

C. Trajectory $(u(s), \lambda(s))_{s\in[0,T[}$ cannot be entirely enclosed in

$$H^2 \cap H_0^1(\Omega) \times [0, 1].$$

Theorem 3 follows immediately from A., B., C.

Proof of A. — Since all the eigenvalues of Laplacian are strictly positive, we obtain by Theorem 2 (3 (*ii*)):

$$\frac{d\lambda}{ds}(0) = \delta(-G'_u(u_0, 0)) = \delta(-\Delta) > 0.$$

Proof of B. — Set $t = \inf\{ s \in]0, T[, \lambda(s) = 0 \}$. Thus by A., $t > 0$, and $\lambda(s) \geq 0$ for $s \leq t$. Therefore, if $t < +\infty$, $\lambda'(t) \leq 0$.

But $\lambda'(t) = \delta(-G'_u(u(t), 0)) = \delta(-\Delta) > 0$.

This is a contradiction.

Proof of C. — First of all, prove the following assertions:

ASSERTION 1. — The set $D = E \cap (H^2 \cap H_0^1(\Omega) \times [0, 1])$ is compact in $H^2 \cap H_0^1(\Omega) \times \mathbb{R}$.

Indeed, for every (u, λ) in D,

$$-\Delta u = \lambda F(u) + (1 - \lambda)h .$$

Thus $\| \nabla u \|_{L^2}^2 \leq 2 \int F(u)u dx + \| h \|_{L^2} \| u \|_{L^2}$.

Therefore, since Ω is bounded,

$$\| \nabla u \|_{L^2}^2 \leq C \| F \|_{L^\infty} \| u \|_{L^2} + \| h \|_{L^2} \| u \|_{L^2}$$

for some constant C. Using Friedrichs-Poincaré's inequality (Adams [2]), it follows that:

$$\| \nabla u \|_{L^2} \leq C .$$

Thus D is bounded in $H_0^1(\Omega) \times \mathbb{R}$, and then relatively compact in $L^2(\Omega) \times \mathbb{R}$. Let now $(u_n, \lambda_n)_{n\in\mathbb{N}}$ be a sequence in D. Then there exists a subsequence which we still note $(u_n, \lambda_n)_{n\in\mathbb{N}}$, that converges in $L^2(\Omega) \times \mathbb{R}$ to some (u, λ) in $L^2(\Omega) \times \mathbb{R}$. Thus we have:

$$-\Delta u_n = \lambda_n F(u_n) + (1-\lambda_n)h \to \lambda F(u) + (1-\lambda)h \text{ in } L^2(\Omega) \quad u_n \to u \text{ in } L^2(\Omega).$$

Since $(-\Delta)$ is a closed operator: $L^2(\Omega \to L^2(\Omega)$, $u \in H^2 \cap H_0^1(\Omega)$, and $-\Delta u = \lambda F(u) + (1-\lambda)h$. (Then, $(u, \lambda) \in D$).

Now we have:

$$u_n \to u \quad \text{in} \quad L^2(\Omega),$$
$$\Delta u_n \to \Delta u \quad \text{in} \quad L^2(\Omega),$$

and therefore:

$$u_n \to u \quad \text{in} \quad H^2 \cap H_0^1(\Omega).$$

ASSERTION 2. — Set, for $(u, \lambda) \in E$,

$$K(u, \lambda) = [(G'_u(u, \lambda))^\# G'_\lambda(u, \lambda), \delta(-G'_u(u, \lambda))] .$$

$K(u, \lambda)$ is the second member of (8). Then $K(u, \lambda)$ never vanishes for (u, λ) in E.

Indeed, dim Ker $G'_u(u, \lambda) \leq 1$. Consider two cases:

a) Dim Ker $G'_u(u, \lambda) = \{ 0 \}$. Then, by Theorem 2 (3 (*i*)), $\delta(G'_u(u, \lambda)) \neq 0$.

b) Dim Ker $G'_u(u, \lambda) = 1$. Let λ_{i_0} be the single null eigenvalue of $G'_u(u, \lambda)$.

Then $\prod_{\substack{i \neq i_0 \\ i=1}}^{N} \lambda_i(G'_u(u, \lambda)) \neq 0$, and therefore $(G'_u(u, \lambda))^\# \neq 0$

(See the definition of $J : A \to A^\#$).

Assume, by contradiction, the trajectory $(u(s), \lambda(s))_{s\in[0,T[}$ is contained in D. Then D being compact, there exists a sequence $(s_n)_{n\in\mathbb{N}}$ such that:

$$s_n \to T \quad \text{as} \quad n \to \infty$$
$$(u(s_n), \lambda(s_n)) \to (u^*, \lambda^*) \quad \text{for some} \quad (u^*, \lambda^*) \quad \text{in} \quad D .$$

Thus, by assertion 2 $K(u^*, \lambda^*) \neq 0$, and Theorem 4 below provides an immediate contradiction and achieves the proof of Theorem 3.

THEOREM 4. — Let H be a Hilbert space, and E a one-dimensional closed C^0-submanifold of H. Let K be a locally Lipschitz mapping from some open set $U \supset E$ to H. Assume the maximal solution $(y(t))_{t \in [0,T[}$ of the differential system

$$(9) \qquad \begin{cases} y'(t) = K(y(t)) \\ y(0) = y_0 \in E \end{cases} \qquad \text{remains in E and is not periodic.}$$

Then every adherent point y^* of $y(t)$ as $t \to T$ is a stationary point of (9) (i. e. $K(y^*) = 0$).

Proof. — Assume, by contradiction, that for some sequence $(s_n)_{n \in \mathbb{N}}$ converging to T one has:

$$y(s_n) \to y^* \quad \text{and} \quad K(y^*) \neq 0 .$$

Clearly, we can suppose that $(s_n)_{n \in \mathbb{N}}$ is an increasing sequence. Note that, since E is closed, $y^* \in E$.

STEP 1. — Define an open ball B in H such that $\overline{B} \subset U$, with centre y^* and radius r small enough to ensure that the following conditions are realized:

$$a) \qquad (K(y), K(y^*)) \geq \frac{1}{2} \| K(y^*) \|^2, \qquad \forall y \in \overline{B} .$$

b) There exists $\hat{t} \in [0, T[$ such that $y(\hat{t}) \notin \overline{B}$. (Indeed, the trajectory is not stationary).

c) There is an homeomorphism $h : B \cap E \to]0, 1[$. (h is a local chart of E).

STEP 2. — Since $y(\hat{t}) \notin \overline{B}$, we can choose s_n such that $y(s_n) \in B$ and $s_n > \hat{t}$. Now consider the maximal interval containing s_n, $I =]t_0, t_1[$, such that $y(t) \in B$ for every t in I. I is open since, at every point of H, there exists a local solution of (9).

Moreover: $\hat{t} < t_0 < s_n < t_1$.

STEP 3. — We claim that $t_1 < T$, i. e. $y(t)$ « leaves » B for some $t > t_1$.

If not, the whole trajectory $(y(t))_{t \in [t_0, T[}$ would be enclosed in B. Apply now a classical property of the locally Lipschitz differential equations: since $y(t)$ does not explode as $t \to T$, we would have: $T = +\infty$. But, by *c*):

$$\frac{d}{dt}(y(t), K(y^*)) = (K(y), K(y^*)) \geq \frac{1}{2} \| K(y^*) \|^2$$

and then

$$(y(t), \mathrm{K}(y^*)) \geq (y(t_0), \mathrm{K}(y^*)) + \frac{t - t_0}{2} \| \mathrm{K}(y^*) \|^2 .$$

Thus $\| y(t) \| \to +\infty$, therefore $y(t)$ would leave B, which contradicts our assumption.

STEP 4. — We now prove that y: $]t_0, t_1[\ \to \mathrm{E} \cap \mathrm{B}$ is onto, i. e. $h \circ y$: $]t_0, t_1[\ \to \ ]0, 1[$ is onto. First remark that since the solution $y(t)$ of (9) is not periodic, the mapping $t \to y(t)$ is one to one. Thus the map $h \circ y$: $]t_0, t_1[\ \to \ ]0, 1[$ is one to one, continuous and therefore monotone. Then it has a limit λ_0 as $t \to t_0^+$, and a limit λ_1 as $t \to t_1^-$.

Necessarily $\lambda_0 = 0$. If not, as $t \to t_0$, $h(y(t))$ would remain in a compact interval $[\lambda_0, \lambda_0 + \varepsilon]$.

Then $y(t)$ would remain in the compact $h^{-1}([\lambda_0, \lambda_0 + \varepsilon])$ and would admit some adherent point in this compact as $t \to t_0$.

We would obtain: $y(t_0) \in h^{-1}([\lambda_0, \lambda_0 + \varepsilon]) \subset \mathrm{B}$. This contradicts the definition of t_0. In the same way, we can prove $\lambda_1 = 1$.

STEP 5. — Let us show now that $y(t)$ « returns » in B for some $t > t_1$. Thus it will « pass again » by some point of the trajectory, and this contradicts the nonperiodicity assumption.

Let s_p be some element of the sequence $(s_n)_{n \in \mathbb{N}}$ such that $s_p > t_1$ and $y(s_p) \in \mathrm{B} \cap \mathrm{E}$. Such a s_p exists by Step 3.

From Step 1 *c*), there exists τ in $]0, 1[$ such that: $y(s_p) = h^{-1}(\tau)$ and then: $h \circ y(s_p) = \tau \in \ ]0, 1[$. But, by Step 4, we can find t_2 in $]t_0, t_1[$ such that $h \circ y(t_2) = \tau$.

Thus $y(s_p) = y(t_2)$ with $s_p > t_2$. This achieves the proof.

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