

Echanges Annales

**The concentration-compactness principle
in the Calculus of Variations.
The Locally compact case, part 2**

by

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ABSTRACT. — In this paper (sequel of Part 1) we investigate further applications of the concentration-compactness principle to the solution of various minimization problems in unbounded domains. In particular we present here the solution of minimization problems associated with nonlinear field equations.

RÉSUMÉ. — Dans cette deuxième partie, nous examinons de nouvelles applications du principe de concentration-compacité à la résolution de divers problèmes de minimization dans des ouverts non bornés. En particulier nous résolvons des problèmes de minimisation associés aux équations de champ non linéaires.

Key-words: Concentration-compactness principle, minimization problem, unbounded domains, concentration function, nonlinear field equations, nonlinear Schrodinger equations, rotating stars, solitary waves, vortex rings.

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INTRODUCTION

In Part 1 ([24]) we introduced a general method for solving *minimization problems in unbounded domains*. In particular a general principle—

called *concentration-compactness principle*—was shown, indicating, roughly speaking, that for general classes of minimization problems with constraints in unbounded domains all minimizing sequences are relatively compact if and only if some *strict subadditivity inequalities* hold (while the large subadditivity inequalities always hold). These inequalities involve the value of the infimum. In particular the value of the infimum has to be compared with the value of the infimum of the « *problem at infinity* »—for more precise statements we refer to [24]—. This principle was derived, at least heuristically, by a method based upon the fact that, essentially, the loss of compactness may occur only if either the minimizing sequence slips to infinity, or the minimizing sequence breaks into at least two disjoint parts which are going infinitely far away from each other. Roughly speaking, this principle can be applied and thus leads to the solution of all minimizing problems with constraints in unbounded domains with a form of *local compactness* or in other words problems which if they were set in a bounded region would be treated by standard convexity-compactness methods.

As we indicated above the concentration-compactness principle is purely formal and has to be rigorously derived on each problem, following the general lines of the heuristic derivation we gave in Part. 1. In [24], we already explained how this can be done on two examples, namely the so-called rotating stars problem and the Choquard-Pekar problem. Here, we present other applications of the principle and of its method of proof.

In section I, we consider two minimization problems (closely related) which are motivated by the question of the existence of *standing waves* in *nonlinear Schrödinger equations* (see B. R. Suidam [39], W. Strauss [37], H. Berestycki and P. L. Lions [6]). We look for solutions of the problems:

$$(1) \quad \text{Inf} \left\{ \int_{\mathbb{R}^N} |\nabla u|^2 - F(x, u) dx / u \in H^1(\mathbb{R}^N), \quad |u|_{L^2(\mathbb{R}^N)}^2 = 1 \right\}$$

where F is a given nonlinearity like for example: $F(x, t) = |t|^p$;

$$(2) \quad \text{Inf} \left\{ \int_{\mathbb{R}^N} |\nabla u|^2 + V(x)u^2 dx / u \in H^1(\mathbb{R}^N), \quad \int_{\mathbb{R}^N} K(x) |u|^p dx = 1 \right\}$$

where K, V are given potentials and $p > 1$.

Finally in section I, we recall the results of T. Cazenave and P. L. Lions [14] which yield the *orbital stability* of the standing waves determined by (1), the proof being a direct application of the concentration-compactness principle.

Section II is devoted to the study of minimization problems associated

with the *nonlinear fields equations*. To give an example we completely solve the problem:

$$(3) \quad \text{Inf} \left\{ \int_{\mathbb{R}^N} |\nabla u|^2 dx \middle/ \int_{\mathbb{R}^N} F(x, u) = \lambda \right\}$$

where $F(x, t)$ is a given nonlinearity and λ is prescribed. In particular we will treat the so-called « zero-mass case », the case $N = 2$, the case of *systems* and *higher-order equations* where we prove the existence of a *ground state* without restrictions to spherically symmetric functions. A very special case of (3) is the case when F does not depend on x : this case was studied by many authors, Z. Nehari [34]; G. H. Ryder [36]; C. V. Coffman [17]; M. Berger [12]; W. Strauss [37]; Coleman, Glazer and Martin [18] and the most general results were obtained in H. Berestycki and P. L. Lions [6] [8]—let us mention that in [7] [8] was also treated the question of multiple solutions while here we consider only the ground state—if $N \geq 3$, and the case when $N=2$ was considered (independently of our study) in H. Berestycki, T. Gallouet, O. Kavian [5]. But in all these references the fact that F is independent of x is crucial since all authors reduce (3) to the same problem but with u spherically symmetric. And as it is explained in P. L. Lions [25] [26], the symmetry induces some form of compactness.

Finally sections III, IV and V are devoted to various other applications of the method:

- unconstrained problems: ex. Hartree equations (section III)
- Euler equations and minimization over manifolds (section III)
- problems with multiple constraints (section IV)
- problems in unbounded domains other than \mathbb{R}^N (section V)
- partial concentration-compactness method (section V) and applications to problems in strips, half-spaces (vortex rings, rotating stars...).

Finally let us mention that some of the results presented here were announced in P. L. Lions [27] [28].

I. EXISTENCE OF STANDING WAVES AND POHOZAEV PROBLEM

I.1. Standing waves in nonlinear Schrödinger equations.

Consider a nonlinear Schrödinger equation given by:

$$(4) \quad i \frac{\partial \phi}{\partial t} - \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial \phi}{\partial x_j} \right) + c(x)\phi = f(x, \phi) \quad \text{in } \mathbb{R}^N \times]0, \infty[$$

where a_{ij} , c are coefficients satisfying conditions detailed below and $f(x, t)$

is a nonlinearity satisfying: $f(x, z) = f(x, |z|) \frac{z}{|z|}$ for all $x \in \mathbb{R}^N, z \in \mathbb{C} - \{0\}$, and $f(x, 0) = 0$. A standing wave is a solution of (4) of the special form: $\phi(x, t) = e^{i\omega t} u(x)$ where $\omega \in \mathbb{R}$ and u is a scalar function (for example). Therefore we look for couples (ω, u) satisfying:

$$(5) \quad -\frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial u}{\partial x_j} \right) + c(x)u + \omega u = f(x, u) \quad \text{in } \mathbb{R}^N.$$

In addition it is natural to assume that ϕ and thus u are in $L^2(\mathbb{R}^N)$; and since (if a_{ij} is symmetric) it is well-known that for any smooth solution of (4) $|\phi(\cdot, t)|$ is constant for all $t \geq 0$, we look for couples (ω, u) in $\mathbb{R} \times L^2(\mathbb{R}^N)$ solving (5) and such that $\|u\|_{L^2}^2$ is prescribed. Therefore it is clear enough that we will find such a solution if we can solve the following minimization problem:

$$(6) \quad I_\lambda = \text{Inf} \{ \mathcal{E}(u)/u \in H^1(\mathbb{R}^N), \|u\|_{L^2(\mathbb{R}^N)}^2 = \lambda \}$$

where $\lambda > 0$ and \mathcal{E} is defined on H^1 by:

$$(7) \quad \mathcal{E}(u) = \int_{\mathbb{R}^N} \frac{1}{2} a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} + \frac{1}{2} c(x)u^2 - F(x, u) dx$$

$$\text{where } F(x, t) = \int_0^t f(x, s) ds.$$

In order to make the problem meaningful (and $I_\lambda > -\infty$) we will assume:

$$(8) \quad \begin{cases} a_{ij}(x) = a_{ji}(x) \in C_b(\mathbb{R}^N), & a_{ij}(x) \rightarrow \bar{a}_{ij} \text{ as } |x| \rightarrow \infty \\ \exists v > 0, \quad \forall (x, \xi) \in \mathbb{R}^N, & a_{ij}(x) \xi_i \xi_j \geq v |\xi|^2; \end{cases}$$

$$(9) \quad \begin{cases} c^+ \in L^1_{loc}; \quad \forall \delta > 0, \quad c^+ 1_{(c \geq \delta)} \in L^p \text{ with } \frac{N}{2} \vee 1 \leq p < \infty \\ c^- \in L^p + L^q \text{ with } \frac{N}{2} \vee 1 \leq p, q < \infty; \end{cases}$$

$$(10) \quad \begin{cases} f(x, t) \in C(\mathbb{R}^N \times \mathbb{R}), \quad f(x, t) \rightarrow \bar{f}(t) \text{ as } |x| \rightarrow \infty \\ \text{uniformly for } t \text{ bounded,} \\ \lim_{|t| \rightarrow 0} F(x, t)t^{-2} = 0, \quad \lim_{|t| \rightarrow \infty} F(x, t)|t|^{-l} = 0 \\ \text{uniformly in } x \in \mathbb{R}^N \end{cases}$$

$$\text{with } l = \frac{2N + 4}{N}, \quad \bar{F}(t) = \int_0^t \bar{f}(s) ds.$$

As we did in Part 1, we next introduce the problem at infinity:

$$(11) \quad I_\lambda^\infty = \text{inf} \{ \mathcal{E}^\infty(u)/u \in H^1(\mathbb{R}^N), \|u\|_{L^2}^2 = \lambda \}$$

with

$$\mathcal{E}^\infty(u) = \int_{\mathbb{R}^N} \frac{1}{2} \bar{a}_{ij} \frac{\partial u}{\partial x_j} \frac{\partial u}{\partial x_i} - \bar{F}(u) dx.$$

From the general arguments given in section I of Part 1, we immediately obtain the following inequalities:

$$I_\lambda \leq I_\alpha + I_{\lambda-\alpha}^\infty, \quad \forall \alpha \in [0, \lambda[$$

The following result then states that the concentration-compactness principle is indeed valid for problem (6):

THEOREM I.1. — *The strict subadditivity inequality*

$$(S.1) \quad I_\lambda < I_\alpha + I_{\lambda-\alpha}^\infty, \quad \forall \alpha \in [0, \lambda[$$

is a necessary and sufficient condition for the relative compactness in $H^1(\mathbb{R}^N)$ of all minimizing sequences of (6). In particular if (S.1) holds, there exists a minimum of (6).

THEOREM I.2. — *If a_{ij}, f are independent of x and $c \equiv 0$, then the strict subadditivity inequality:*

$$(S.2) \quad I_\lambda \leq I_\alpha + I_{\lambda-\alpha}^\infty, \quad \forall \alpha \in]0, \lambda[$$

is a necessary and sufficient condition for the relative compactness in $H^1(\mathbb{R}^N)$ up to a translation of all minimizing sequences of (6) (\equiv (11)). In particular if (S.2) holds, (6) has a minimum.

REMARK I.1. — We explain below how it is possible to check (S.1) and (S.2).

REMARK I.2. — It is possible to extend a little bit the assumption on the behavior of F as $|t| \rightarrow \infty$ by assuming:

$$\overline{\lim}_{|t| \rightarrow \infty} F(x, t) |t|^{-1} \leq M < C_0 \lambda^{2/N}, \quad \forall x \in \mathbb{R}^N$$

where C_0 is the « best constant » in the inequality:

$$\int_{\mathbb{R}^N} |u|^t dx \leq C_0 \left(\int_{\mathbb{R}^N} \frac{1}{2} a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial u}{\partial x_i} dx \right) \left(\int_{\mathbb{R}^N} u^2 dx \right) \quad \forall u \in H^1(\mathbb{R}^N)$$

this inequality is obtained combining Hölder and Sobolev inequalities. Let us also mention that if $F(x, t)t^{-2}$ converges as $t \rightarrow 0_+$ to some constant μ , we can apply the above results: indeed it is then enough to observe that subtracting μt^2 to $F(x, t)$ does not change neither problems (6)-(11) since $\|u\|_{L^2}^2 = \lambda$ nor the inequalities (S.1)-(S.2). It is also possible to extend the conditions on the behavior of F at 0 by assuming:

either

$$\lim_{t \rightarrow 0} F^+(x, t) |t|^{-2} = 0, \quad \lim_{t \rightarrow 0} F(x, t) |t|^{-2} > -\infty,$$

uniformly for $|x|$ large

or

$$\begin{cases} \lim_{t \rightarrow 0} F(x, t) |t|^{-2} = -\infty, & \text{uniformly for } |x| \text{ large} \\ \exists C > 0 \text{ such that for } |x| \text{ large, } 0 \leq s \leq t \leq 1 \\ F^-(x, s) \leq C[F^-(x, t) + |t|^2 + |t|^{2N/(N-2)}] \end{cases}$$

(if $N \leq 2$, $2N/(N-2)$ may be replaced by any $q > 2$).

REMARK I.3. — We could treat as well problems where the second-order operator is replaced by a higher order self-adjoint operator, or where the L^2 constraint is replaced by a L^p constraint, or where

$$a^{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j}$$

is replaced by a convex function $j(x, Du)$ with growth properties like $|Du|^p$ with $p \in]1, \infty[$ — then we need to work in $W^{1,p}(\mathbb{R}^N)$ instead of H^1 — ...

For example, if we consider the following nonlinear Schrödinger equation:

$$(4') \quad i \frac{\partial \phi}{\partial t} + (-\Delta)^m \phi = f(x, \phi)$$

with $m \geq 1$, then the associated minimization problem becomes:

$$(6') \quad I_\lambda = \text{Inf} \left\{ \int_{\mathbb{R}^N} |D^m u|^2 - F(x, u) dx / u \in H^m(\mathbb{R}^N), \quad |u|_{L^2}^2 = \lambda \right\}$$

and l is replaced by $l' = 2 + \frac{2m}{N}$. ■

We now discuss inequalities (S.1)-(S.2). We begin with (S.2), as we remarked in Part 1, it is enough to prove

$$I_{\theta\lambda}^\infty < \theta I_\lambda^\infty < 0, \quad \forall \theta > 1, \quad \forall \lambda > 0;$$

and this is the case if $I_\lambda < 0$ and \bar{F} satisfies:

$$(12) \quad \forall \theta > 1, \quad \exists \theta' > \theta, \quad \bar{F}(\theta t) \geq \theta' \bar{F}(t) \geq 0 \quad \forall t \in \mathbb{R}.$$

Indeed if $I_\lambda^\infty < 0$, then there exists $\delta > 0$ such that:

$$I_\lambda^\infty = \text{Inf} \left\{ \mathcal{E}^\infty(u) / u \in H^1(\mathbb{R}^N), \quad |u|_{L^2}^2 = \lambda, \quad \int_{\mathbb{R}^N} \bar{F}(u) dx \geq \delta \right\}$$

and thus

$$\begin{aligned}
 I_{\theta\lambda}^\infty &\leq \text{Inf} \left\{ \mathcal{E}^\infty(\sqrt{\theta}u)/u \in H^1(\mathbb{R}^N), |u|_{L^2}^2 = \lambda, \int_{\mathbb{R}^N} \bar{F}(u)dx \geq \delta \right\} \\
 &\leq \text{Inf} \left\{ \int_{\mathbb{R}^N} \frac{\theta}{2} \bar{a}_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} - \theta \bar{F}(u)dx / u \in H^1, |u|_{L^2}^2 = \lambda, \int_{\mathbb{R}^N} \bar{F}(u)dx \geq \delta \right\} \\
 &< \theta I_\lambda^\infty.
 \end{aligned}$$

Let us also point out that this inequality also holds if $I_\lambda^\infty < 0$ and $N = 1$: indeed

$$I_{\theta\lambda}^\infty = \text{Inf} \left\{ \mathcal{E}^\infty\left(u\left(\frac{\cdot}{\theta}\right)\right) / u \in H^1, |u|_{L^2}^2 = \lambda \right\}$$

and

$$\begin{aligned}
 \mathcal{E}^\infty\left(u\left(\frac{\cdot}{\theta}\right)\right) &= \frac{1}{\theta} \int_{\mathbb{R}^N} \frac{1}{2} |\nabla u|^2 dx - \theta \int_{\mathbb{R}^N} \bar{F}(u)dx \\
 &< \theta \mathcal{E}^\infty(u), \quad \text{if } u \equiv 0.
 \end{aligned}$$

Next, we claim that $I_\lambda^\infty < 0$ if either \bar{F} satisfies:

$$(13) \quad \lim_{|t| \rightarrow 0} \bar{F}(t) |t|^{-1} = +\infty$$

or if λ is large and \bar{F} satisfies:

$$(14) \quad \lim_{|t| \rightarrow \infty} \bar{F}(t) |t|^{-2} = +\infty.$$

Indeed, in the first case, observe that if $u \in \mathcal{D}(\mathbb{R}^N), |u|_{L^2}^2 = \lambda$ then

$$\mathcal{E}^\infty\left(\theta^{-N/2}u\left(\frac{\cdot}{\theta}\right)\right) = \frac{1}{\theta^2} \int_{\mathbb{R}^N} \frac{1}{2} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} dx - \theta^N \int_{\mathbb{R}^N} \bar{F}(\theta^{-N/2}u)dx$$

and we conclude using (13). In the second case one just considers $\sqrt{\theta}u$ with $u \in \mathcal{D}(\mathbb{R}^N), |u|_{L^2}^2 = 1$.

In conclusion (S.2) holds if we assume either (12) or $N = 1$, and either (13) or (14) and λ large. In a similar way (S.1) holds if (12) holds uniformly in $x \in \mathbb{R}^N$ and if $I_\lambda < I_\lambda^\infty$. This last inequality is in particular valid if we assume $I_\lambda^\infty < 0$ and:

$$(a_{ij}(x)) \leq (\bar{a}_{ij}), \quad c(x) \leq 0, \quad F(x, t) \geq \bar{F}(t) \quad \text{on } \mathbb{R}, \quad \forall x \in \mathbb{R}^N$$

and one of these inequalities is strict for some x .

We now turn to the proof of Theorem I.1-I.2: in view of the general arguments given in Part 1, we only have to prove that (S.1)-(S.2) are sufficient conditions for the compactness of minimizing sequences. Indeed

if u_n is a minimizing sequence of (6) or (11), we first claim that (u_n) is bounded in $H^1(\mathbb{R}^N)$. Indeed observe that in view of (10) for all $\varepsilon > 0$ we can find $c_\varepsilon > 0$ such that:

$$|F(x, t)| \leq \varepsilon [t^2 + |t|^l] + c_\varepsilon |t|^\alpha, \quad \forall t \in \mathbb{R}, \quad \forall x \in \mathbb{R}^N$$

where α is taken in $]2, l[$.

Then we argue as follows:

$$\begin{aligned} \int_{\mathbb{R}^N} |F(x, u_n)| dx &\leq \varepsilon \lambda + \varepsilon \int_{\mathbb{R}^N} |u_n|^l dx + c_\varepsilon \int_{\mathbb{R}^N} |u_n|^\alpha dx \\ &\leq \varepsilon \lambda + \varepsilon c_0 \lambda^{2/N} \int_{\mathbb{R}^N} a_{ij} \frac{\partial u_n}{\partial x_i} \frac{\partial u_n}{\partial x_j} dx + c_\varepsilon \|u_n\|_{L^2}^\beta \|\nabla u_n\|_{L^2}^\gamma \end{aligned}$$

for some constants $\beta > 0, 0 < \gamma < 2$;

and choosing $\varepsilon = \frac{1}{4} (c_0 \lambda^{2/N})^{-1}$, we obtain:

$$\int_{\mathbb{R}^N} |F(x, u_n)| dx \leq c + \frac{1}{4} \int_{\mathbb{R}^N} a_{ij} \frac{\partial u_n}{\partial x_i} \frac{\partial u_n}{\partial x_j} dx + c \|\nabla u_n\|_{L^2}^\gamma.$$

It is then easy to deduce from that bound that (u_n) is bounded in H^1 .

We then want to apply the scheme of proof given in section I of Part 1 with $\rho_n = u_n^2$ (Lemma I. 1 in [24]). Exactly as in the case of the Choquard-Pekar problem (section III of Part 1) we show that dichotomy cannot occur: here we use the form of Lemma I. 1 given by Lemma III. 1 of Part 1 and we observe that because of (13) there exists $c > 0$ such that:

$$|F(x, t)| \leq c [|t|^2 + |t|^l], \quad \forall t \in \mathbb{R}, \quad \forall x \in \mathbb{R}^N.$$

Next, we observe that vanishing cannot occur since if vanishing did occur, using the following Lemma I. 1 and the condition (13), we would

deduce: $\int_{\mathbb{R}^N} F(x, u_n) dx \xrightarrow{n} 0, \int_{\mathbb{R}^N} c u_n^2 dx \xrightarrow{n} 0$; therefore this would imply: $I^\lambda \geq 0$.

This would contradict (S. 1) or (S. 2) since in view of (13) for any $\mu > 0$:

$$\begin{aligned} I_\mu^\infty &\leq \mathcal{E}^\infty \left(\sigma^{-N/2} u \left(\frac{\cdot}{\sigma} \right) \right) \quad \text{with} \quad u \in \mathcal{D}(\mathbb{R}^N), \quad \|u\|_{L^2}^2 = \mu \\ I_\mu^\infty &\leq \frac{1}{\sigma^2} \int_{\mathbb{R}^N} |\nabla u|^2 dx - \sigma^N \int_{\mathbb{R}^N} F(x, \sigma^{-N/2} u) dx \end{aligned}$$

and when $\sigma \rightarrow \infty$, the right-hand side goes to 0 in view of (13).

LEMMA I.1. — Let $1 < p \leq \infty$, $1 \leq q < \infty$ with $q \neq \frac{Np}{N-p}$ if $p < N$. Assume that u_n is bounded in $L^q(\mathbb{R}^N)$, ∇u_n is bounded in $L^p(\mathbb{R}^N)$ and:

$$\sup_{y \in \mathbb{R}^N} \int_{y+B_R} |u_n|^q dx \xrightarrow{n} 0, \quad \text{for some } R > 0$$

Then $u_n \xrightarrow{n} 0$ in $L^\alpha(\mathbb{R}^N)$ for α between q and $\frac{Np}{N-p}$.

REMARKS I.4. — This kind of lemmas admits many extensions to more general functional spaces which are proved by the same method that follows.

Proof of Lemma I.1. — Let us prove this lemma with u_n bounded in $L^\infty(\mathbb{R}^N)$. Then clearly we have for all $\beta > \min\left(q, \frac{Np}{N-p}\right)$ ($\beta > q$ if $p \geq N$):

$$\sup_{y \in \mathbb{R}^N} \int_{y+B_R} |u_n|^\beta dx \xrightarrow{n} 0.$$

We introduce \bar{q} such that: $\bar{q} > q$, $\infty > (\bar{q}-1)p' > q$. By Hölder inequalities we see that:

$$\sup_{y \in \mathbb{R}^N} \int_{y+B_R} |u_n|^{\bar{q}-1} |\nabla u_n| dx \xrightarrow{n} 0.$$

Then by Sobolev embeddings, if $\gamma \in \left]1, \frac{N}{N-1}\right[$, there exists a constant c_0 independent of y such that:

$$\begin{aligned} \int_{y+B_R} |u_n|^{\bar{q}\gamma} dx &\leq \left(\int_{y+B_R} |u_n|^{\bar{q}} + \bar{q} |u_n|^{\bar{q}-1} |\nabla u_n| dx \right)^\gamma \\ &\leq \varepsilon_n^{\gamma-1} \int_{y+B_R} |u_n|^{\bar{q}} + \bar{q} |u_n|^{\bar{q}-1} |\nabla u_n| dx \end{aligned}$$

where $\varepsilon_n \xrightarrow{n} 0$. Then covering \mathbb{R}^N by balls of radius R in such a way that any point of \mathbb{R}^N is contained in at most m balls (where m is a prescribed integer), we deduce:

$$\int_{\mathbb{R}^N} |u_n|^{\bar{q}\gamma} dx \leq m \varepsilon_n^{\gamma-1} \int_{\mathbb{R}^N} |u_n|^{\bar{q}} + \bar{q} |u_n|^{\bar{q}-1} |\nabla u_n| dx \leq c \varepsilon_n^{\gamma-1}$$

and we conclude easily.

In the general case we observe that for any $C > 0$, $v_n = |u_n| \wedge C$ still satisfies the conditions of Lemma I.1 and therefore by the above proof

we obtain, for α between q and $\frac{Np}{N-p}$, $v_n \xrightarrow{n} 0$ in $L^\alpha(\mathbb{R}^N)$. Next if β lies between q and $\frac{Np}{N-p}$ and $\beta > \alpha$, we remark:

$$\begin{aligned} \int_{\mathbb{R}^N} |u_n|^\alpha dx &\leq \int_{\mathbb{R}^N} |v_n|^\alpha dx + \int_{\mathbb{R}^N} |u_n|^{\alpha 1_{\{|u_n| \geq C\}}} dx \\ &\leq \int_{\mathbb{R}^N} |v_n|^\alpha dx + \frac{1}{C^{\beta-\alpha}} \int_{\mathbb{R}^N} |u_n|^\beta dx \end{aligned}$$

thus:

$$\overline{\lim}_n \int_{\mathbb{R}^N} |u_n|^\alpha dx \leq \frac{K}{C^{\beta-\alpha}}, \quad \text{for all } C > 0$$

and we conclude letting $C \rightarrow +\infty$. ■

Since we ruled out the occurrence of dichotomy and vanishing (cf. Lemma I.1 in Part 1) we see that there exists y_n in \mathbb{R}^N such that:

$$\forall \varepsilon > 0, \quad \exists R < \infty, \quad \int_{y_n + B_R} |u_n|^2 dx \geq \lambda - \varepsilon.$$

Denoting by $\tilde{u}_n(\cdot) = u_n(\cdot + y_n)$, we see that \tilde{u}_n converges weakly in H^1 , a. e. and strongly in L^2 to some function $\tilde{u} \in H^1$ such that $\|\tilde{u}\|_{L^2}^2 = \lambda$. It is now easy to conclude (exactly as in section III of Part 1), remarking that (S.1) implies that $I_\lambda < I_\lambda^\infty$ and this yields the fact that (y_n) is bounded. ■

We conclude this section with some examples:

EXAMPLE 1. — $F(x, t) = K(x) |t|^p$ with $K \in C(\mathbb{R}^N)$, $K(x) \xrightarrow{|x| \rightarrow \infty} \bar{K}$ and $2 < p < l = 2 + \frac{4}{N}$. Of course Theorem I.1 applies to this situation and we have to check (S.1). Two cases are possible:

1) $\bar{K} \leq 0$. In this case, $I_\lambda^\infty = 0$ for all $\lambda > 0$ and thus (S.1) holds if and only if $I_\lambda < 0$.

2) $\bar{K} > 0$. In that case, we already know that $I_\mu^\infty < 0$ for all $\mu < 0$ and we check easily that (S.1) holds if and only if

$$I_\lambda < I_\lambda^\infty.$$

Let us finally point out that if $(a_{ij}(x)) \leq \bar{a}_{ij}$, $c(x) \leq 0$, $K(x) \geq \bar{K}$ on \mathbb{R}^N with one of the inequalities strict somewhere, then this strict inequality is satisfied; while if $(a_{ij}(x)) \geq \bar{a}_{ij}$, $c(x) \geq 0$, $K(x) \geq \bar{K}$, then clearly $I_\lambda = I_\lambda^\infty$, and a minimum does not exist (except if $a_{ij}(x) \equiv \bar{a}_{ij}$, $c(x) \equiv 0$, $K(x) \equiv \bar{K}$): indeed if u is a minimum, we have $\mathcal{E}^\infty(u) < \mathcal{E}(u) = I_\lambda = I_\lambda^\infty$, which contradicts the definition of I_λ^∞ .

EXAMPLE 2. — $F(x, t) = -K(x) |t|^p$ with $K \in C(\mathbb{R}^N)$, $K(x) \xrightarrow{|x| \rightarrow \infty} \bar{K} > 0$ and $0 < p < 2$. To simplify the discussion we take $a_{ij}(x) \equiv \bar{a}_{ij} = \delta_{ij}$, $c \neq 0$. Of course (10) is violated because of the behavior of F near 0, but F satisfies the conditions listed at the end of Remark I.2; and we may still apply Theorem I.1-I.2. We now have to discuss (S.1)-(S.2).

First of all $K(x) \equiv \bar{K}$, then for $\theta > 1$, $\lambda > 0$:

$$0 < I_{\theta\lambda} = \text{Inf} \left\{ \int_{\mathbb{R}^N} \theta |\nabla u|^2 + \theta^{p/2} \bar{K} |u|^p dx / u \in H^1(\mathbb{R}^N), \quad |u|_{L^2}^2 = \lambda \right\} < \theta I_\lambda$$

and thus (S.2) holds and the problem at infinity is solved.

In the same way if $K \geq 0$, then $I_{\theta\lambda} < \theta I_\lambda$ and thus (S.1) holds if and only if $I_\lambda < I_\lambda^\infty$.

I.2. Pohozaev problem.

If we consider the nonlinearity given in the example discussed at the end of the preceding section that is $F(x, u) = K(x) |u|^p$, we see that the equation (5) which determines standing waves in some nonlinear Schrödinger equations may now be written:

$$-\frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial u}{\partial x_j} \right) + V(x)u = pK(x) |u|^{p-2}u \quad \text{in } \mathbb{R}^N.$$

Clearly enough, in order to find a solution of this equation, one can solve the following minimization problem:

$$(15) \quad I_\lambda = \text{Inf} \left\{ \int_{\mathbb{R}^N} a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} + V(x)u^2 dx / u \in H^1(\mathbb{R}^N), \int_{\mathbb{R}^N} K(x) |u|^p dx = \lambda \right\}$$

where $\lambda > 0$. Indeed if u is a minimum of (15) then the above equation is satisfied up to a Lagrange multiplier which is taken care of by multiplying u by a convenient constant. We could treat as well much more general problems including ones where $K(x) |u|^p$ is replaced by a general nonlinearity: however in that case minima of (15) lead to solutions $(\lambda, u) \in \mathbb{R} \times H^1(\mathbb{R}^N)$ of the following nonlinear eigenvalue problem:

$$-\frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial u}{\partial x_j} \right) + V(x)u = \lambda f(x, u) \quad \text{in } \mathbb{R}^N, \quad u \neq 0.$$

Such minimization problems were studied by S. Pohozaev [35] in the case of a bounded region.

We will always assume (8), $2 < p < \frac{2N}{N-2}$ ($p < \infty$ if $N \leq 2$) and:

$$(16) \quad \left\{ \begin{array}{l} V^+ \in L^1_{loc}(\mathbb{R}^N), \quad V^- \in L^q(\mathbb{R}^N) \quad \text{with} \quad \frac{N}{2} \vee 1 \leq q < \infty, \\ V^+ \rightarrow \bar{V} > 0 \quad \text{as} \quad |x| \rightarrow \infty \\ \exists v > 0, \quad \forall u \in \mathcal{D}(\mathbb{R}^N), \quad \int_{\mathbb{R}^N} a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} + V(x)u^2 dx \geq v \|u\|_{H^1}^2 \\ K \in C(\mathbb{R}^N), \quad K^+ \neq 0, \quad K \rightarrow \bar{K} \quad \text{as} \quad |x| \rightarrow \infty \end{array} \right.$$

We may then introduce the problem at infinity:

$$(17) \quad I_\lambda^\infty = \text{Inf} \left\{ \int_{\mathbb{R}^N} \bar{a}_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} + \bar{V}u^2 dx / u \in H^1(\mathbb{R}^N), \quad \bar{K} \int_{\mathbb{R}^N} |u|^p dx = \lambda \right\}$$

if $\bar{K} > 0$; while if $\bar{K} \leq 0$ we set $I_\lambda^\infty = +\infty$.

By the general arguments we introduced in Part 1 [24], we have:

$$I_\lambda \leq I_\alpha + I_{\lambda-\alpha}^\infty, \quad \forall \alpha \in [0, \lambda];$$

in addition if $\bar{K} \leq 0$, the inequalities are obviously strict ones.

In addition, because of (16), we have: $0 < I_\lambda = \lambda^{2/p} I_1$; therefore:

$$I_\lambda < I_\alpha + I_{\lambda-\alpha} \leq I_\alpha + I_{\lambda-\alpha}^\infty, \quad I_\lambda^\infty < I_\alpha^\infty + I_{\lambda-\alpha}^\infty \quad \forall \alpha \in [0, \lambda].$$

Hence (S.2) holds, while (S.1) holds if and only if:

$$(18) \quad I_\lambda < I_\lambda^\infty.$$

This explains the:

THEOREM I.2. — *We assume (8), (16) and $2 < p < \frac{2N}{N-2}$ if $N \geq 3$, $2 < p < \infty$ if $N \leq 2$. Then all minimizing sequences of (15) are relatively compact in $H^1(\mathbb{R}^N)$ if and only if (18) holds. In the particular case when $a_{ij} \equiv \bar{a}_{ij}$, $V(x) \equiv \bar{V}$, $K(x) \equiv \bar{K} > 0$; all minimizing sequences of (17) are relatively compact in $H^1(\mathbb{R}^N)$ up to a translation.*

REMARK I.5. — As we did in section I.1, it is possible to give conditions on a_{ij} , V , K such that (18) holds. Of course (18) holds if $\bar{K} \leq 0$. Let us consider the case, for example, of $a_{ij} \equiv \bar{a}_{ij} = \delta_{ij}$, $V(x) \equiv \bar{V}$ and $\bar{K} > 0$. Then clearly (18) holds if $K(x) \geq \bar{K}$ and $K(x) \not\equiv \bar{K}$; while $K(x) \leq \bar{K}$ and $K \not\equiv \bar{K}$, not only (18) is false i. e. $I_\lambda = I_\lambda^\infty$ but there is no minimum in (15). It is interesting to recall (cf. [35] for example) that in a bounded domain there is always a minimum. Let us also point out that even if there is no minimum in (15), this does not mean there does not exist a solution of the corresponding Euler equation: indeed if we consider, for example,

K spherically symmetric such that $K(|x|) \leq \bar{K}$, $K \not\equiv \bar{K}$; then in view of the compactness arguments in [6] [25] [26], there is a minimum in (15) if we restrict the infimum to spherically symmetric functions (of course in that case the value of the infimum is changed giving a bigger value than I_λ).

REMARK I.6. — If $\bar{K} > 0$ and if (18) does not hold i. e. $I_\lambda = I_\lambda^\infty$, then the proof below actually implies that all minimizing sequences of (15) are relatively compact up to a translation in $H^1(\mathbb{R}^N)$.

Proof of Theorem I.2. — Let $(u_n)_n$ be a minimizing sequence of (15). In view of (16), (u_n) is bounded in $H^1(\mathbb{R}^N)$. We want to apply Lemma I.1 of Part 1 with $\rho_n = |u_n|^p$: at this point we would like to emphasize the fact that we have some flexibility in the choice of ρ_n , we would as well take $\rho_n = u_n^2$ or $\rho_n = |\nabla u_n|^2 + u_n^2 \dots$. To be more specific we have to remark that without loss of generality we may assume that

$$\tilde{\lambda}_n = \int_{\mathbb{R}^N} \rho_n dx \rightarrow \tilde{\lambda} > 0,$$

and we apply in fact Lemma I.1 of Part 1 to $\tilde{\rho}_n = \frac{1}{\tilde{\lambda}_n} \rho_n$.

We first need to rule out vanishing: indeed if we had

$$\sup_{y \in \mathbb{R}^N} \int_{y + B_R} |u_n|^p dx \xrightarrow{n} 0, \quad \forall R < \infty$$

then by Lemma I.1, $u_n \xrightarrow{n} 0$ in $L^\alpha(\mathbb{R}^N)$ for $p < \alpha < \frac{2N}{N-2}$. But u_n is bounded in $L^2(\mathbb{R}^N)$, therefore by Hölder inequalities: $u_n \xrightarrow{n} 0$ in L^p and this would contradict the constraint.

We next rule out dichotomy: we denote by $Q_n(t)$ the concentration function of ρ_n

$$Q_n(t) = \sup_{y \in \mathbb{R}^N} \int_{y + B_t} \rho_n dx.$$

In view of the proof of Lemma I.1 of Part 1, we may assume:

$$Q_n(t) \rightarrow Q(t), \quad \forall t \geq 0, \quad \lim_{t \uparrow +\infty} \uparrow Q(t) = \tilde{\alpha} \in]0, \tilde{\lambda}[.$$

Let $\varepsilon > 0$, we choose R_0 such that: $Q(R_0) \geq \tilde{\alpha} - \varepsilon$. For $n \geq n_0$, we have.

$Q_n(R_0) = \int_{y_n + B_{R_0}} |u_n|^p dx \geq \tilde{\alpha} - 2\varepsilon$ for some y_n in \mathbb{R}^N . In addition there exists $R_n \xrightarrow{n} \infty$ such that $Q_n(R_n) \leq \alpha + \varepsilon$. Let $\xi, \varphi \in C_b^\infty$ satisfying: $0 \leq \xi, \varphi \leq 1$. $\text{Supp } \xi \subset B_2, \xi \equiv 1$ on $B_1, \varphi \equiv 0$ on $B_1, \varphi \equiv 1$ on $\mathbb{R}^N - B_2$; we denote by $\xi_n = \xi\left(\frac{\cdot - y_n}{R_1}\right), \varphi_n = \varphi\left(\frac{\cdot - y_n}{R_n}\right)$ where R_1 is determined below.

First of all we observe that for R_1 large enough (in particular larger than R_0) and for n large enough:

$$\left\{ \begin{aligned} & \left| \int_{\mathbb{R}^N} \xi_n^2 a_{ij}(x) \frac{\partial u_n}{\partial x_i} \frac{\partial u_n}{\partial x_j} dx - \int_{\mathbb{R}^N} a_{ij}(x) \frac{\partial}{\partial x_i} (\xi_n u_n) \frac{\partial}{\partial x_j} (\xi_n u_n) dx \right| \leq \varepsilon \\ & \left| \int_{\mathbb{R}^N} \varphi_n^2 a_{ij}(x) \frac{\partial u_n}{\partial x_i} \frac{\partial u_n}{\partial x_j} dx - \int_{\mathbb{R}^N} a_{ij}(x) \frac{\partial}{\partial x_i} (\varphi_n u_n) \frac{\partial}{\partial x_j} (\varphi_n u_n) dx \right| \leq \varepsilon. \end{aligned} \right.$$

Then if we set $u_n^1 = \xi_n u_n$, $u_n^2 = \varphi_n u_n$, we find for n large enough:

$$\left\{ \begin{aligned} & \|u_n - (u_n^1 + u_n^2)\|_{L^p} \leq 3\varepsilon, \quad \left| \lambda - \int_{\mathbb{R}^N} K |u_n^1|^p + K |u_n^2|^p dx \right| \leq C\varepsilon \\ & \int_{\mathbb{R}^N} a_{ij}(x) \frac{\partial u_n}{\partial x_i} \frac{\partial u_n}{\partial x_j} dx \geq \int_{\mathbb{R}^N} a_{ij}(x) \frac{\partial u_n^1}{\partial x_i} \frac{\partial u_n^1}{\partial x_j} dx + \int_{\mathbb{R}^N} a_{ij}(x) \frac{\partial u_n^2}{\partial x_i} \frac{\partial u_n^2}{\partial x_j} dx - 2\varepsilon \\ & \int_{\mathbb{R}^N} V^+(u_n)^2 dx \geq \int_{\mathbb{R}^N} V^+(u_n^1)^2 dx + \int_{\mathbb{R}^N} V^+(u_n^2)^2 dx. \end{aligned} \right.$$

In addition we deduce from the fact that u_n is bounded in H^1 and the assumptions on V^- the following inequalities:

$$\left\{ \begin{aligned} & \|u_n - (u_n^1 + u_n^2)\|_{L^\infty} \leq \delta_\alpha(\varepsilon) \rightarrow 0 \quad \text{as} \quad \varepsilon \rightarrow 0 \\ & \left| \int_{\mathbb{R}^N} V^-(u_n)^2 dx - \int_{\mathbb{R}^N} V^-(u_n^1)^2 + V^-(u_n^2)^2 dx \right| \leq \mu(\varepsilon) \rightarrow 0 \quad \text{as} \quad \varepsilon \rightarrow 0. \end{aligned} \right.$$

Without loss of generality, we may assume that we have:

$$\int_{\mathbb{R}^N} K |u_n^1|^p dx \xrightarrow{n} \lambda^1, \quad \int_{\mathbb{R}^N} K |u_n^2|^p dx \xrightarrow{n} \lambda^2$$

and thus: $|\lambda - (\lambda^1 + \lambda^2)| \leq C\varepsilon$.

If λ^1 (resp. λ^2) ≤ 0 , then remarking that in view of (16):

$$\begin{aligned} \exists \delta > 0, \quad & \int_{\mathbb{R}^N} a_{ij}(x) \frac{\partial u_n^1}{\partial x_i} \frac{\partial u_n^1}{\partial x_j} + V(u_n^1)^2 dx \geq \delta \\ & \left(\text{resp. } \int_{\mathbb{R}^N} a_{ij}(x) \frac{\partial u_n^2}{\partial x_i} \frac{\partial u_n^2}{\partial x_j} + V(u_n^2)^2 dx \geq \delta \right); \end{aligned}$$

we would have letting $n \rightarrow \infty$:

$$\begin{aligned} I_\lambda & \geq \liminf_n \int_{\mathbb{R}^N} a_{ij}(x) \frac{\partial u_n^1}{\partial x_i} \frac{\partial u_n^1}{\partial x_j} + V(u_n^1)^2 dx \\ & \quad + \liminf_n \int_{\mathbb{R}^N} a_{ij}(x) \frac{\partial u_n^2}{\partial x_i} \frac{\partial u_n^2}{\partial x_j} + V(u_n^2)^2 dx \\ & \geq \delta + I_{\lambda_2} \quad (\text{resp. } \delta + I_{\lambda_1}). \end{aligned}$$

But λ_2 (resp. λ_1) $\geq \lambda - C\varepsilon$, and if we let $\varepsilon \rightarrow 0$, we would reach a contradiction. Therefore we may assume that $\lambda^1, \lambda^2 > 0$.

Next, two cases are possible: first, $|y_n|$ (or a subsequence) goes to infinity. Then since $\text{Supp } u_n^1 \subset y_n + B_{R_1}$, we have:

$$\left\{ \begin{array}{l} \left| \int_{\mathbb{R}^N} a_{ij}(x) \frac{\partial u_n^1}{\partial x_i} \frac{\partial u_n^1}{\partial x_j} + V(u_n^1)^2 dx - \int_{\mathbb{R}^N} \bar{a}_{ij} \frac{\partial u_n^1}{\partial x_i} \frac{\partial u_n^1}{\partial x_j} + \bar{V}(u_n^1)^2 dx \right| \xrightarrow{n} 0 \\ \int_{\mathbb{R}^N} \bar{K} |u_n^1|^p dx \xrightarrow{n} \lambda_1. \end{array} \right.$$

If on the other hand $|y_n|$ remains bounded, since $\text{Supp } u_n^2 \subset \mathbb{R}^N - (y_n + B_{R_n})$ and $R_n \xrightarrow{n} \infty$, we may replace u_n^1 by u_n^2 in the above limits.

In both cases we find: either $I_\lambda \geq I_{\lambda_1}^\infty + I_{\lambda_2}$ or $I_\lambda \geq I_{\lambda_2}^\infty + I_{\lambda_1}$. Recalling that $|\lambda - (\lambda_1 + \lambda_2)| \leq C\varepsilon$ and sending $\varepsilon \rightarrow 0$, we see that these inequalities contradict the strict subadditivity condition (S.1). This contradiction rules out the possibility of dichotomy.

Therefore, we deduce that there exists y_n in \mathbb{R}^N such that:

$$\forall \varepsilon > 0, \exists R < \infty, \int_{y_n + B_R} |u_n|^p dx \geq \tilde{\lambda}_n - \varepsilon.$$

This implies that $u_n(y_n + \cdot)$ converges strongly in $L^p(\mathbb{R}^N)$ and weakly in $H^1(\mathbb{R}^N)$ to some $u \in H^1(\mathbb{R}^N)$ satisfying: $\int_{\mathbb{R}^N} K(x) |u|^p = \lambda$ if $|y_n|$ is bounded, or $\bar{K} \int_{\mathbb{R}^N} |u|^p = \lambda$ if $|y_n|$ is unbounded (up to a subsequence). It is then easy to conclude. ■

I.3. Orbital stability of standing waves in nonlinear Schrödinger equations.

Let us consider the following example of nonlinear Schrödinger equation:

$$i \frac{\partial \phi}{\partial t} - \Delta \phi = K(x) |\phi|^{p-2} \phi \quad \text{in } \mathbb{R}^N \times]0, \infty[, \quad \phi|_{t=0} = \phi_0$$

where $\phi_0 \in H^1(\mathbb{R}^N)$, $\phi \in C([0, T]; H^1(\mathbb{R}^N))$ ($\forall T < \infty$), $2 < p < 2 + \frac{4}{N}$ and

where $K \in C(\mathbb{R}^N)$, $K(x) \rightarrow \bar{K}$ as $|x| \rightarrow \infty$. We saw in section I.1 how it is possible to find some standing waves with prescribed L^2 norm via a minimization problem. We want to recall here results of T. Cazenave and P. L. Lions [14] which show the relevance of knowing that all

minimizing sequences converge compared with an existence result only. If $\lambda > 0$ is given, we recall the associated minimization problem (6):

$$I_\lambda = \text{Inf} \left\{ \int_{\mathbb{R}^N} \frac{1}{2} |\nabla u|^2 - \frac{1}{p} K |u|^p dx / u \in H^1(\mathbb{R}^N), \quad |u|_{L^2(\mathbb{R}^N)}^2 = \lambda \right\}.$$

In view of Theorem I.1 and various remarks made in section I.1, we see that if $I_\lambda < I_\lambda^\infty$, then all minimizing sequences are relatively compact in $H^1(\mathbb{R}^N)$. We then denote by S the following set:

$$S = \{ \phi = e^{i\theta} u \quad \text{with} \quad \theta \in \mathbb{R}, \quad u \in H^1(\mathbb{R}^N), \quad |u|_{L^2}^2 = \lambda, \quad \mathcal{E}(u) = I_\lambda \}$$

this set consists of the orbits of the standing waves determined by the minima of I_λ .

The main result of [14] then states – still assuming that $I_\lambda < I_\lambda^\infty$ – that S is orbitally stable that is: for all $\varepsilon > 0$, there exists $\delta > 0$ such that if $\inf_{\phi \in S} \|\phi - \phi_0\|_{H^1(\mathbb{R}^N)} \leq \delta$ and if $\phi(t)$ denotes the solution in $C([0, \infty[; H^1(\mathbb{R}^N))$ of the nonlinear Schrödinger equation corresponding to the initial condition ϕ_0 then we have for all $t \geq 0$:

$$\inf_{\phi \in S} \|\phi(t) - \phi\|_{H^1(\mathbb{R}^N)} \leq \varepsilon.$$

In the special case when $K(x) \equiv \bar{K} > 0$, in view of Theorem I.2 and the form of the nonlinearity, (S.2) is satisfied and thus all minimizing sequences are relatively compact up to a translation in $H^1(\mathbb{R}^N)$; in that case in [14] the same stability result is proved.

The proof of the orbital stability of S is a simple application of Theorems I.1-I.2 and of the conservation laws of the nonlinear Schrödinger equations: indeed multiplying by ϕ and taking the imaginary part we find for all $t \geq 0$

$$|\phi(t)|_{L^2(\mathbb{R}^N)}^2 = |\phi_0|_{L^2(\mathbb{R}^N)}^2;$$

while multiplying by $\frac{\partial \phi}{\partial t}$ and taking the real part we obtain

$$\int_{\mathbb{R}^N} \frac{1}{2} |\nabla \phi(t)|^2 - \frac{1}{p} K |\phi(t)|^p dx = \int_{\mathbb{R}^N} \frac{1}{2} |\nabla \phi_0|^2 - \frac{1}{p} K |\phi_0|^p dx$$

Then roughly speaking if ϕ_0 is near S, then the last functional above is near I_λ for all $t \geq 0$ and $|\phi(t)|_{L^2}^2$ is near λ for all $t \geq 0$. But Theorem I.1 then yields that $\phi(t)$ is also (for all $t \geq 0$) near S !

II. NONLINEAR FIELDS EQUATIONS

II.1. The case of positive mass.

Throughout this section we will study various variational problems associated with nonlinear fields equation. We will begin by the scalar

case (sections II. 1-4) and then we will consider extensions in sections II. 5-6. In the scalar case the equation may be described as:

$$-\frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial u}{\partial x_j} \right) = f(x, u) \quad \text{in } \mathbb{R}^N, \quad u(x) \xrightarrow{|x| \rightarrow \infty} 0, \quad u \neq 0$$

where $f(x, 0) = 0$ and $a_{ij}(x), f(x, t)$ satisfy various conditions listed below. One way to obtain solutions of this equation is to consider the following minimization problems:

$$(19) \quad I_\lambda = \text{Inf} \{ \mathcal{E}(u)/u \in H^1(\mathbb{R}^N), \quad J(u) = \lambda \},$$

where $\lambda > 0$, and the functionals \mathcal{E}, J are given by:

$$\begin{aligned} \mathcal{E}(u) &= \int_{\mathbb{R}^N} a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} dx \\ J(u) &= \int_{\mathbb{R}^N} F(x, u) dx, \quad \text{with} \quad F(x, t) = \int_0^t f(x, s) ds. \end{aligned}$$

Of course if we solve (19) for some λ (or for each λ) we do not solve exactly the above equation but instead we obtain $(\theta, u) \in \mathbb{R}^+ \times H^1(\mathbb{R}^N)$ solution of:

$$-\frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial u}{\partial x_j} \right) = \theta f(x, u) \quad \text{in } \mathbb{R}^N, \quad u \neq 0.$$

However, in the case of particular interest when a_{ij}, f are independent of x , the Lagrange multiplier θ may be eliminated by a scale change: indeed consider $u \left(\frac{\cdot}{\sqrt{\theta}} \right) = \tilde{u}$, \tilde{u} is now a solution of the equation. In addition as it was remarked in Coleman, Glazer and Martin [18], the solution thus found enjoys minimum properties (recalled below)—such minimal solutions are called *ground-state solutions*.

We will always assume that $a_{ij}(x)$ satisfies (8) and that f, F satisfy at least:

$$(10') \quad \left\{ \begin{array}{l} f(x, t) \in C(\mathbb{R}^N \times \mathbb{R}), \quad f(x, t) \rightarrow \bar{f}(t) \quad \text{as } |x| \rightarrow \infty \\ \hspace{15em} \text{uniformly for } t \text{ bounded} \\ \lim_{|t| \rightarrow \infty} F(x, t) |t|^{-l} = 0 \quad \text{uniformly in } x \in \mathbb{R}^N, \quad \text{with } l = \frac{2N}{N-2} \\ \exists u \in \mathcal{D}(\mathbb{R}^N), \quad \int_{\mathbb{R}^N} F(x, u) dx \geq \lambda \end{array} \right.$$

(we will consider here only the case $N \geq 3$).

The so-called positive-mass case corresponds to the following additional assumption:

$$(20) \quad \begin{aligned} &\exists \delta > 0, \quad \{f(x, t) - \bar{f}(t)\} t^{-1} \rightarrow 0 \quad \text{as } |x| \rightarrow \infty, \\ &\hspace{15em} \text{uniformly for } |t| \leq \delta. \\ &-\infty < \liminf_{t \rightarrow 0} f(x, t)t^{-1} \leq \overline{\lim}_{t \rightarrow 0} f(x, t)t^{-1} < 0, \quad \text{uniformly for } x \text{ large.} \end{aligned}$$

We will denote by $\bar{F}(t) = \int_0^t \bar{f}(s) ds$; and (as usual) we introduce the problem at infinity:

$$(21) \quad I_\lambda^\infty = \text{Inf} \{ \mathcal{E}^\infty(u) / u \in H^1(\mathbb{R}^N), J^\infty(u) = \lambda \}$$

where
$$\mathcal{E}^\infty(u) = \int_{\mathbb{R}^N} \bar{a}_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} dx, \quad J^\infty(u) = \int_{\mathbb{R}^N} \bar{F}(u) dx.$$

If for some $\mu > 0$, the constraint set $\{u \in H^1(\mathbb{R}^N), J^\infty(u) = \mu\}$ is empty, we set $I_\mu^\infty = +\infty$.

We may now state our main results:

THEOREM II.1. — *Let $N \geq 3$, we assume (8), (10') and (20). Then the condition*

$$(S.1) \quad I_\lambda < I_\alpha + I_{\lambda-\alpha}^\infty, \quad \forall \alpha \in [0, \lambda[$$

is necessary and sufficient for the relative compactness in $H^1(\mathbb{R}^N)$ of all minimizing sequences of (19).

THEOREM II.2. — *Let $N \geq 3$, and assume that a_{ij}, f are independent of x and that (10'), (20) hold for \bar{F} . Then all minimizing sequences of (19) (= (21)) are relatively compact in $H^1(\mathbb{R}^N)$ up to a translation.*

REMARK II.1. — If $f(x, t) \equiv \bar{f}(t)$ for all x , the condition (10') is then equivalent (see H. Berestycki and P. L. Lions [6]) to:

$$\lim_{|t| \rightarrow \infty} \bar{F}(t) |t|^{-1} = 0; \quad \exists \zeta \in \mathbb{R}, \quad \bar{F}(\zeta) > 0.$$

In addition in [37] [6], the restriction on the growth of \bar{F} is shown to be in general necessary.

REMARK II.2. — Condition (S.1) can be quite difficult to check. There are a few simple situations when it is possible to check it: first, if $\bar{F}(t) \leq 0$ then $I_\mu^\infty = +\infty$ for all $\mu > 0$ and (S.1) obviously holds. Next, if (for example) $F(x, t) = -c(x)t^2 + F_0(x, t)$ with $M \geq c(x) \geq m > 0, F_0 \geq 0$ satisfying:

$$\begin{aligned} &F_0(x, t) |t|^{-p} \text{ is nondecreasing (resp. nonincreasing) for } t \geq 0 \\ &(\text{resp. for } t \leq 0) \end{aligned}$$

for some $p > 2$; then we claim that (S.1) reduces to:

$$I_\lambda < I_\lambda^\infty$$

which can be discussed as we did many times.

Indeed we show that: $I_{\theta\mu} < \theta I_\mu$ if $\mu \in]0, \lambda[$, $1 < \theta \leq \lambda/\mu$; then this yields:

$$I_\lambda < I_\alpha + I_{\lambda-\alpha} \leq I_\alpha + I_{\lambda-\alpha}^\infty, \quad \forall \alpha \in]0, \lambda[.$$

To show the above strict inequality, we first observe that there exists $\delta > 0$ such that:

$$I_\mu = \text{Inf} \left\{ \mathcal{E}(u)/u \in H^1(\mathbb{R}^N), \quad J(u) = \mu, \quad \int_{\mathbb{R}^N} F_0(x, u) dx \geq \delta \right\}.$$

But if u satisfies: $J(u) = \mu$ and $\int_{\mathbb{R}^N} F_0(x, u) dx \geq \delta$; then $v = \sqrt{\theta}u$ satisfies:

$$\int_{\mathbb{R}^N} F_0(x, v) dx \geq \theta^{p/2} \delta$$

and

$$\begin{aligned} J(v) &= \int_{\mathbb{R}^N} -\theta c(x)u^2 + F_0(x, \sqrt{\theta}u) dx \geq \int_{\mathbb{R}^N} -\theta c(x)u^2 + \theta^{p/2} F_0(x, u) dx \\ &\geq \bar{\mu} > \theta\mu. \end{aligned}$$

Hence:

$$\theta I_\mu \geq \text{Inf} \left\{ \mathcal{E}(u)/u \in H^1(\mathbb{R}^N), \quad J(u) \geq \bar{\mu}, \quad \int_{\mathbb{R}^N} F_0(x, u) dx \geq \delta' > 0 \right\}.$$

Next we remark that if u satisfies: $J(u) \geq \bar{\mu} > \theta\mu$ (and $\int_{\mathbb{R}^N} F_0(x, u) dx \geq \delta' > 0$)

then there exists $\tau \in [0, 1[$ such that τu satisfies $J(\tau u) = \theta\mu$. If we show that τ is bounded away from 1 uniformly for such u , we will have shown: $\theta I_\mu > I_{\theta\mu}$. Let us argue by contradiction: if there exist $\tau_n \xrightarrow{n} 1$, $u_n \in H^1(\mathbb{R}^N)$

with $J(u_n) \geq \bar{\mu}$, $\int_{\mathbb{R}^N} F_0(x, u_n) dx \geq \delta' > 0$, $J(\tau_n u_n) = \theta\mu$ and $\mathcal{E}(u_n) \leq K$, then we would have (∇u_n) bounded in L^2 and thus

$$\begin{aligned} \int_{\mathbb{R}^N} c(x)u_n^2 &\leq \bar{\mu} + \int_{\mathbb{R}^N} F_0(x, u_n) dx \\ &\leq \bar{\mu} + \int_{\mathbb{R}^N} \varepsilon |u_n|^2 + C_\varepsilon |u_n|^{2N/(N-2)} dx \\ &\leq \bar{\mu} + \varepsilon \|u_n\|_{L^2}^2 + C'_\varepsilon. \end{aligned}$$

Thus (u_n) is bounded in $H^1(\mathbb{R}^N)$. Next in view of (10'), there exists $\omega_n \xrightarrow{n} 0$

such that: $|F_0(x, t) - F_0(x, \tau_n t)| \leq \omega_n$ if $|t| \leq 1/\delta$; for any given $\delta > 0$. Therefore:

$$\begin{aligned} \int_{\mathbb{R}^N} |F_0(x, u_n) - F_0(x, \tau_n u_n)| dx &\leq \omega_n \text{ meas } \left\{ \delta \leq |u_n| \leq \frac{1}{\delta} \right\} + \\ &+ \int_{\mathbb{R}^N} |F_0(x, u_n) - F_0(x, \tau_n u_n)| \left\{ 1_{(|u_n| \leq \delta)} + 1_{(|u_n| \geq \frac{1}{\delta})} \right\} dx \\ &\leq \omega_n \text{ meas } \left\{ \delta \leq |u_n| \leq 1/\delta \right\} + \varepsilon(\delta) \int_{\mathbb{R}^N} |u_n|^2 + |u_n|^{\frac{2N}{N-2}} dx \end{aligned}$$

with $\varepsilon(\delta) \rightarrow 0$ as $\delta \rightarrow 0_+$. And we finally get:

$$\int_{\mathbb{R}^N} |F_0(x, u_n) - F_0(x, \tau_n u_n)| dx \leq \frac{\omega_n}{\delta^2} \int_{\mathbb{R}^N} u_n^2 dx + K\varepsilon(\delta)$$

and if we let $n \rightarrow \infty$ and then $\delta \rightarrow 0$, we obtain: $J(u_n) - J(\tau_n u_n) \xrightarrow{n} 0$, contradicting the choice of τ_n and u_n . The contradiction proves our claim. ■

REMARK II.3. — To explain Theorem II.2, we just need to point out that if a_{ij}, f are independent of x , then (S.2) automatically holds since:

$$\begin{aligned} I_\lambda &= \text{Inf } \left\{ \mathcal{E}(u)/J(u) = \lambda \right\} \\ &= \text{Inf } \left\{ \mathcal{E} \left(u \left(\frac{\cdot}{\lambda^{1/N}} \right) \right) / J(u) = 1 \right\} \\ &= \lambda^{\frac{N-2}{N}} I_1. \end{aligned}$$

Let us also remark that such a scaling argument could be used in order to check (S.1) with some convenient assumptions on the x -dependence of a_{ij}, f .

REMARK II.4. — It is possible to extend a little bit assumption (20) by assuming instead:

$$(20') \quad \begin{cases} \overline{\lim}_{t \rightarrow 0} F(x, t) |t|^{-2} < 0, & \text{uniformly for } |x| \text{ large} \\ \exists C > 0, \forall 0 \leq s \leq t \leq 1, F^-(x, s) \leq CF^-(x, t) & \text{for } |x| \text{ large,} \end{cases}$$

in addition we need some technical condition for the behavior of F as $|x| \rightarrow \infty$ that we skip here.

REMARK II.5. — Let us remark that $I_\lambda = I_\lambda^\infty = 0$ if $\lambda \leq 0$. Indeed we have $0 \leq I_\lambda \leq I_\lambda^\infty$; and if $\lambda = 0$ we may choose $u \equiv 0$, while if $\lambda < 0$ there exists $(u_n)_n$ in $H^1(\mathbb{R}^N)$ such that: $\nabla u_n \xrightarrow{n} 0$ in $L^2(\mathbb{R}^N)$, $|u_n|_{L^2} = 1$. Thus $\bar{F}^+(u_n) \xrightarrow{L^1} 0$ since $\bar{F}^+(t) \leq \varepsilon[t^2 + |t|^{2N/(N-2)}] + C_\varepsilon t^\alpha$ with $2 < \alpha < \frac{2N}{N-2}$,

and $\int_{\mathbb{R}^N} \bar{F}(u_n) dx = -\alpha_n \xrightarrow{n} -\alpha < 0$. If we consider $\tilde{u}_n = u_n \left(\frac{\cdot}{\theta_n} \right)$ with $\theta_n = \left(\frac{-\lambda}{\alpha_n} \right)^{1/N}$, $J^\infty(\tilde{u}_n) = \lambda$ and $\mathcal{E}^\infty(u_n) \xrightarrow{n} 0$. ■

In the introduction we gave many references concerning the case when a_{ij}, f are independent of x , the most general results being obtained in H. Berestycki and P. L. Lions [6] [7]: in Theorem II.2 we not only recover the results of [6] [7] but in addition we obtain the compactness (up to a translation) of all minimizing sequences, and we saw in section I.3 that this additional information may be useful.

Theorems II.1-II.2 are proved in the next section. We want to conclude this section by a few considerations concerning the case when a_{ij}, f are independent of x and to simplify the presentation we will assume that $a_{ij}(x) = \bar{a}_{ij} = \delta_{ij}$. The remarks that follow are taken from [6] [7]. In view of Theorem II.2, we obtain a minimum u of (21)-(19): of course if $F = \bar{F}$ is even, then $(-u)$ is also a minimum and u^* —the Schwarz symmetrization of u —is also a minimum. In addition with very little additional assumption on f , one can prove (see B. Gidas, W. H. Ni and L. Nirenberg [22], or A. Alvino, P. L. Lions and G. Trombetti [1]) that any minimum is, up to a change of sign and a translation, radial decreasing and thus positive.

We now want to describe some important property of any minimum u of (21)-(19). As we remarked before, for some $\sigma > 0$, $\bar{u}(\cdot) = u \left(\frac{\cdot}{\sigma} \right)$ is a solution of

$$(22) \quad -\Delta \bar{u} = f(\bar{u}) \quad \text{in } \mathbb{R}^N, \quad \bar{u} \in H^1(\mathbb{R}^N), \quad \bar{u} \not\equiv 0$$

(here we need to assume that: $\lim_{|t| \rightarrow \infty} |f(t)| |t|^{-\frac{N+2}{N-2}} = 0$).

The scaling invariance of the problem implies some conservation law—often called Pohozaev identity [35]—for any solution v of (22):

$$(23) \quad \frac{N-2}{2N} \int_{\mathbb{R}^N} |\nabla v|^2 dx = \int_{\mathbb{R}^N} F(v) dx.$$

Heuristically this identity may be obtained as follows: if v solves (22), v is a critical point of the action S :

$$S(\varphi) = \int_{\mathbb{R}^N} \frac{1}{2} |\nabla \varphi|^2 - F(\varphi) dx.$$

Then we have: $\frac{d}{d\sigma} S \left(v \left(\frac{\cdot}{\sigma} \right) \right) \Big|_{\sigma=1} = 0$; and we obtain (23). A rigorous deri-

vation is given in (23). As a consequence of (23) we see that for any solution of (22):

$$S(v) = \frac{1}{N} \int_{\mathbb{R}^N} |\nabla v|^2 dx = \frac{2}{N-2} \int_{\mathbb{R}^N} F(v) dx > 0.$$

In addition:

$$\int_{\mathbb{R}^N} |\nabla \bar{u}|^2 dx = \sigma^{N-2} I_\lambda, \quad \int_{\mathbb{R}^N} F(\bar{u}) dx = \sigma^N \lambda$$

and, using (23), we deduce: $\sigma = \lambda^{-1/2} \left(\frac{N-2}{2N} I_\lambda \right)^{1/2}$.

On the other hand if v solves (22) and $\mu = \int_{\mathbb{R}^N} F(v) dx$, by the scaling invariance of (21) (cf. Remark II.3) we deduce:

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla v|^2 dx &\geq \int_{\mathbb{R}^N} \left| \nabla u \left(\frac{\cdot}{\theta} \right) \right|^2 dx, \quad \text{where } \theta = (\mu/\lambda)^{1/N}; \\ &\geq \theta^{N-2} \left(\int_{\mathbb{R}^N} |\nabla u|^2 dx \right) = \theta^{N-2} I_\lambda \geq \\ &\geq \mu^{\frac{N-2}{N}} \lambda^{-\frac{N-2}{N}} \sigma^{-(N-2)} \left(\int_{\mathbb{R}^N} |\nabla \bar{u}|^2 dx \right) \\ &\geq \mu^{\frac{N-2}{N}} \left(\int_{\mathbb{R}^N} |\nabla \bar{u}|^2 dx \right) \left(\int_{\mathbb{R}^N} F(\bar{u}) dx \right)^{-\frac{N-2}{N}} \\ &\geq \left(\int_{\mathbb{R}^N} |\nabla v|^2 dx \right)^{\frac{N-2}{N}} \left(\int_{\mathbb{R}^N} |\nabla \bar{u}|^2 dx \right)^{2/N} \end{aligned}$$

or
$$\int_{\mathbb{R}^N} |\nabla v|^2 dx \geq \int_{\mathbb{R}^N} |\nabla \bar{u}|^2 dx.$$

And using again (23), this finally yields: $S(u) \geq S(\bar{u}) > 0$.

In conclusion, we just showed that any solution of (22) obtained via the minimization problem (21) realizes the infimum of the action S among all solutions of (22)—such a solution is called a ground state.

REMARK II.6. — We want to mention the following open question: if \bar{F} is not even, is the ground state spherically symmetric and with constant sign?

REMARK II.7. — The analogues of Theorems II.1-II.2 hold for the following minimization problems (« dual of (19)-(21) »);

$$\begin{aligned} \text{Inf } \{ -J(u)/u \in H^1(\mathbb{R}^N), \quad \mathcal{E}(u) = \lambda \} \\ \text{Inf } \{ -J^\infty(u)/u \in H^1(\mathbb{R}^N), \quad \mathcal{E}^\infty(u) = \lambda \}. \end{aligned}$$

Let us also remark that we could treat as well functionals \mathcal{E} of the form:

$$\mathcal{E}(u) = \int_{\mathbb{R}^N} a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} + c(x)u^2 dx$$

with convenient assumptions.

II.2. **Proof of Theorems II.1.-II.2.**

We begin with a few preliminary observations: first of all we already know that:

$$0 < I_\lambda \leq I_\alpha + I_{\lambda-\alpha}^\infty, \quad \forall \alpha \in [0, \lambda[.$$

It is also easy to check that I_μ, I_μ^∞ are continuous functions of $\mu \in [0, \lambda]$ — here and below, we assume to simplify the presentation that there exists $\zeta \in \mathbb{R}$ such that $\bar{F}(\zeta) > 0$ and this ensures that $I_\lambda^\infty < \infty$, for all λ . Let us also observe that if $\mu \geq \lambda$ then $I_\mu \geq I_\lambda$: indeed if we take $u \in H^1(\mathbb{R}^N)$ satisfying

$$J(u) = \mu, \quad \mathcal{E}(u) \leq I_\mu + \varepsilon;$$

then we can find $\tau \in]0, 1[$ such that $J(\tau u) = \lambda$, therefore

$$I_\lambda \leq \mathcal{E}(\tau u) = \tau^2 \mathcal{E}(u) < I_\mu + \varepsilon.$$

Finally, in view of the general arguments of Part 1, we see that we only have to prove in Theorem II.1 the fact that (S.1) is a sufficient condition for compactness.

Let (u_n) be a minimizing sequence in (19) or (21), then obviously (∇u_n) is bounded in $L^2(\mathbb{R}^N)$ and by Sobolev inequalities (u_n) is bounded in $L^{2N/(N-2)}(\mathbb{R}^N)$. In view of (10'), we observe that for all $\varepsilon > 0$ there exists C_ε such that:

$$(24) \quad F^+(x, t) \leq \varepsilon [t^2 + |t|^{\frac{2N}{N-2}}] + C_\varepsilon |t|^\alpha \quad \text{on } \mathbb{R}, \quad \text{for } |x| \geq R_0$$

$$(25) \quad F^+(x, t) \leq \varepsilon t^2 + C_\varepsilon |t|^{\frac{2N}{N-2}} \quad \text{on } \mathbb{R}, \quad \text{for } |x| \geq R_0$$

$$(25') \quad F^-(x, t) \leq \varepsilon |t|^{2N/(N-2)} + C_\varepsilon |t|^2 \quad \text{on } \mathbb{R}, \quad \text{for } |x| \geq R_0$$

$$(25'') \quad |t|^2 \leq C [F^-(x, t) + |t|^{2N/(N-2)}] \quad \text{on } \mathbb{R}, \quad \text{for } |x| \geq R_0$$

for some $\alpha \in]2, 2N/(N-2)[$, and for some $R_0 \geq 0$.

Hence using (25') we find:

$$\begin{aligned} \int_{\mathbb{R}^N} u_n^2 dx &\leq C + C \int_{\mathbb{R}^N} F^-(x, u_n) dx + C \left(\int_{B_{R_0}} |u_n|^{\frac{2N}{N-2}} dx \right)^{\frac{N-2}{2N}} \\ &\leq C + C \left[-\lambda + \int_{\mathbb{R}^N} F^+(x, u_n) dx \right] \\ &\leq C + C \int_{|x| \geq R_0} F^+(x, u_n) dx \end{aligned}$$

$$\begin{aligned} \text{using (24')} \quad &\leq C + C \int_{|x| \geq R_0} \varepsilon u_n^2 + C_\varepsilon |u_n|^{2N/(N-2)} dx \\ &\leq C_\varepsilon + C\varepsilon \int_{\mathbb{R}^N} u_n^2 dx. \end{aligned}$$

And choosing ε small enough, we obtain that (u_n) is bounded in $H^1(\mathbb{R}^N)$.

We are next going to apply Lemma I. 1 of Part 1 [24] with $\rho_n = |\nabla u_n|^2 + u_n^2$. Observe that without loss of generality we may assume that

$$\int_{\mathbb{R}^N} \rho_n dx = \tilde{\lambda}_n \xrightarrow{n} \tilde{\lambda} \geq 0,$$

and that $\tilde{\lambda} = 0$ would imply easily: $J(u_n) \xrightarrow{n} 0$, contradicting the constraint. Hence $\tilde{\lambda} > 0$ and we may apply Lemma I. 1 of Part 1 [24] to ρ_n (or to $\rho_n/\tilde{\lambda}_n$). If vanishing occurs, applying Lemme I. 1 (section I), we see that this would imply: $u_n \xrightarrow{n} 0$ in $L^\alpha(\mathbb{R}^N)$ for $2 < \alpha < \frac{2N}{N-2}$. Then in view of (24), this would yield: $F^+(x, u_n) \xrightarrow{n} 0$ in $L^1(\mathbb{R}^N)$, contradicting the constraint.

Next, if dichotomy occurs, then arguing exactly as in section I.2 we would find, for all $\varepsilon > 0$, (y_n) in \mathbb{R}^N , $R_n \xrightarrow{n} +\infty$, u_n^1, u_n^2 bounded in $H^1(\mathbb{R}^N)$ such that:

$$\begin{cases} \text{Supp}(u_n^1) \subset y_n + B_{R_0} \text{ for some } R_0 < \infty, \text{ Supp}(u_n^2) \subset \mathbb{R}^N - (y_n + B_{R_n}) \\ \|u_n - (u_n^1 + u_n^2)\|_{H^1} \leq \varepsilon, \quad \|u_n^1\|_{H^1}^2 \xrightarrow{n} \tilde{\lambda}_1, \quad \|u_n^2\|_{H^1}^2 \xrightarrow{n} \tilde{\lambda}_2 > 0; \end{cases}$$

of course $u_n^1, u_n^2, R_0, \tilde{\lambda}_1, \tilde{\lambda}$ depend on ε , but we know that:

$$\tilde{\lambda}_1 = \tilde{\lambda}_1(\varepsilon), \quad \tilde{\lambda}_2 = \tilde{\lambda}_2(\varepsilon) \geq v > 0.$$

Clearly this yields:

$$\int_{\mathbb{R}^N} |F(x, u_n) - \{F(x, u_n^1) + F(x, u_n^2)\}| dx \leq \mu(\varepsilon) \rightarrow 0 \quad \text{as} \quad \varepsilon \rightarrow 0.$$

Without loss of generality we may assume that:

$$\int_{\mathbb{R}^N} F(x, u_n^i) dx \xrightarrow{n} \lambda_i = \lambda_i(\varepsilon) \quad \text{for} \quad i = 1, 2.$$

If $\lambda_1(\varepsilon) \leq 0$ (or $\lambda_2(\varepsilon) \leq 0$) then either (taking subsequences if necessary) $\lambda_1(\varepsilon) \leq \lambda_1 < 0$ or $\lambda_1(\varepsilon) \xrightarrow{\varepsilon} 0$. In the first case we argue as follows:

$$\begin{aligned} I_\lambda &= \lim_n \mathcal{E}(u_n) \geq \lim_n \mathcal{E}(u_n^1) + \lim_n \mathcal{E}(u_n^2) - C\varepsilon \\ &\geq \delta + I_{\lambda_2} - C\varepsilon \end{aligned}$$

and $\lambda_2 \geq \lambda - \bar{\lambda}_1 - \mu(\varepsilon) \geq \bar{\lambda}_2 > \lambda$ for ε small enough; this yields a contradiction in view of the following:

LEMMA II.1. — Under the assumptions of Theorem II.1, if $\lambda < \mu$ then $I_\lambda < I_\mu$.

Proof of the lemma. — If $I_\mu = +\infty$, there is nothing to prove. If $I_\mu < \infty$, let (u_n) be a minimizing sequence for I_μ : we know that (u_n) is bounded in $H^1(\mathbb{R}^N)$. There exists $(\tau_n)_n \in]0, 1[$ such that, $J(\tau_n u_n) = \lambda$. We will argue by contradiction; hence we assume that $\tau_n \xrightarrow[n]{\rightarrow} 1$. Let $\delta > 0$, there exists $\omega_n \xrightarrow[n]{\rightarrow} 0$ such that: $|F(x, t) - F(x, \tau_n t)| \leq \omega_n$ on \mathbb{R}^N if $|t| \leq 1/\delta$.

Therefore we have:

$$\begin{aligned} \int_{\mathbb{R}^N} |F(x, u_n) - F(x, \tau_n u_n)| dx &\leq \omega_n \text{ meas } \{ \delta \leq |u_n| \leq 1/\delta \} \\ &+ \int_{\mathbb{R}^N} |F(x, u_n) - F(x, \tau_n u_n)| 1_{|u_n| \leq \delta} dx + \\ &+ \int_{\mathbb{R}^N} |F(x, u_n) - F(x, \tau_n u_n)| 1_{|u_n| \geq (1/\delta)} dx \\ &\leq \int_{\mathbb{R}^N} |F(x, u_n) - F(x, \tau_n u_n)| 1_{|u_n| \leq \delta} dx + \frac{\omega_n}{\delta^2} \|u_n\|_{L^2}^2 + \varepsilon(\delta) \int_{\mathbb{R}^N} |u_n|^{\frac{2N}{N-2}}. \end{aligned}$$

To conclude we observe that for $0 \leq t \leq \delta$:

$$|F(x, t) - F(x, \tau_n t)| \left| \int_{\tau_n t}^t f(x, s) \right| \leq Ct^2(1 - \tau_n^2), \quad \text{for } |x| \geq R_0$$

and this enables us to get: $\int_{\mathbb{R}^N} |F(x, u_n) - F(x, \tau_n u_n)| dx \xrightarrow[n]{\rightarrow} 0$, and the contradiction proves the lemma. ■

In the second case that is if $\lambda_1(\varepsilon) \rightarrow 0$ (or if $\lambda_2(\varepsilon) \rightarrow 0$) as $\varepsilon \rightarrow 0$; we observe that $\mathcal{E}(u_n^1)$ is bounded away from 0 independently of n and ε : indeed if it were not the case, we would have: $F^+(x, u_n^1) \rightarrow 0$ in L^1 , $\nabla u_n \rightarrow 0$ in L^2 .

But since $\int_{\mathbb{R}^N} F(x, u_n^1) dx \xrightarrow[n]{\rightarrow} \lambda_1(\varepsilon) \xrightarrow[\varepsilon]{\rightarrow} 0$, this would yield: $F^-(x, u_n^1) \rightarrow 0$ in L^1 and thus $u_n^1 \rightarrow 0$ in L^2 . And we would reach a contradiction.

In conclusion, we see that we may assume that $\lambda_1 (= \lambda_1(\varepsilon))$ and $\lambda_2 (= \lambda_2(\varepsilon))$ are both positive. If (y_n) is unbounded, then we show as usual that this implies

$$I_\lambda \geq \liminf_n \mathcal{E}(u_n^1) + \liminf_n \mathcal{E}(u_n^2) - C\varepsilon \geq I_{\lambda_1}^\infty + I_{\lambda_2} - C\varepsilon$$

while $|\lambda - (\lambda_1 + \lambda_2)| \leq \mu(\varepsilon)$. If (y_n) is bounded, a similar inequality holds with λ_1 replaced by λ_2 . In both cases, letting $\varepsilon \rightarrow 0$, we reach a contradiction with (S.1) which proves that dichotomy cannot occur.

Therefore we have proved the existence of (y_n) in \mathbb{R}^N such that:

$$\forall \varepsilon > 0, \quad \exists R < \infty, \quad \int_{|x-y_n| \geq R} |\nabla u_n|^2 + u_n^2 dx \leq \varepsilon.$$

Now if $\xi \in C_b^\infty(\mathbb{R}^N)$, $0 \leq \xi \leq 1$ on \mathbb{R}^N , $\xi \equiv 1$ on $\mathbb{R}^N - B_2$, $\xi \equiv 0$ on B_1 ; we see that we have, denoting by $\xi_n = \xi((y_n + \cdot)/R)$:

$$\int_{\mathbb{R}^N} |\nabla(\xi_n u_n)|^2 + \xi_n^2 u_n^2 \leq \frac{2 \|u_n\|_{L^2}^2}{R^2} + 2\varepsilon \leq 4\varepsilon$$

choosing R large enough. By Sobolev inequalities this yields:

$$\int_{\mathbb{R}^N} |\xi_n u_n|^{2N/(N-2)} dx \leq C\varepsilon^{N/(N-2)}.$$

Therefore we have shown:

$$\forall \varepsilon > 0, \quad \exists R < \infty, \quad \int_{|x-y_n| \geq R} |\nabla u_n|^2 + u_n^2 + |u_n|^{\frac{2N}{N-2}} dx \leq \varepsilon.$$

Then in the case of Theorem II. 1, if y_n is unbounded we easily show that this would imply: $I_\lambda \geq I_\lambda^\infty$, and we get a contradiction with (S. 1). Hence (y_n) is bounded and u_n converges weakly in H^1 , a. e. and strongly in L^p for $2 \leq p < 2N/(N-2)$ to some $u \in H^1(\mathbb{R}^N)$ which satisfies obviously: $J(u) = \lambda$, $\mathcal{E}(u) \leq \liminf_n \mathcal{E}(u_n) = I_\lambda$. Hence u is a minimum and u_n converges strongly in H^1 to u . We argue in a similar way in the case of Theorem II. 2.

II.3. The zero-mass case.

We want now to relax assumption (20): we will begin with the translation invariant case that is (21). We will now assume

$$(26) \quad \lim_{t \rightarrow 0} \{ t f(t) \}^+ |t|^{-\frac{2N}{N-2}} = 0, \quad \lim_{|t| \rightarrow \infty} |F(t)| |t|^{-\frac{2N}{N-2}} = 0$$

where we set $f = \bar{f}$, $F = \bar{F}$, and we will also denote $a_{ij} = \bar{a}_{ij}$. Of course we assume:

$$(27) \quad \exists v > 0, \quad \forall \xi \in \mathbb{R}^N, \quad a_{ij} \xi_i \xi_j \geq v |\xi|^2.$$

We have now to precise a little bit the functional space in which we need to work: let H be the Hilbert space which is the closure of $\mathcal{D}(\mathbb{R}^N)$ for the scalar product $((u, v)) = \int_{\mathbb{R}^N} \nabla u \cdot \nabla v dx$. Because of the Sobolev inequalities we have: $H \subset L^{2N/(N-2)}(\mathbb{R}^N)$. Then we want to solve:

$$(21') \quad I_\lambda^\infty = \text{Inf} \left\{ \int_{\mathbb{R}^N} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} dx / u \in H, \quad F(u) \in L^1(\mathbb{R}^N), \int_{\mathbb{R}^N} F(u) dx = \lambda \right\}.$$

By the scaling argument of Remark II.3 above, we know:

$$0 < I_\lambda^\infty = \lambda^{\frac{N-2}{N}} I_1^\infty$$

and (S.2) still holds. We then obtain the:

THEOREM II.3. — *Let $N \geq 3$, we assume (26)-(27) and that $F(\zeta) > 0$ for some ζ . Then all minimizing sequences u_n of (21') are relatively compact in H up to a translation and $F(u_n)$ is relatively compact in L^1 up to a translation.*

REMARK II.8. — The fact that a minimum exists was proved in H. Berestycki and P. L. Lions [6] by the use of symmetrization and of the compactness induced by spherical symmetry.

Sketch of proof of Theorem II.3. — We first introduce a function f_1 as follows:

$$f_1 = f^+ \text{ if } 0 \leq t \leq 1, \quad = f \text{ if } |t| > 1, \quad = -f^- \text{ if } -1 \leq t \leq 0.$$

We then set: $F_1(t) = \int_0^t f_1(s) ds$ if $|t| \leq 1$, $F_1(t) = F^+(t) + F_1(1) - F^+(1)$ if $|t| \geq 1$ and $F_2(t) = F_1(t) - F(t)$ on \mathbb{R} .

Obviously F_1, F_2 are nondecreasing on $[0, 1]$ and nonincreasing on $[-1, 0]$; in addition we have: $0 \leq F^+(t) \leq F_1(t)$, $0 \leq F^-(t) \leq F_2(t)$ on \mathbb{R} , and

$$\begin{aligned} \lim_{|t| \rightarrow 0} F_1(t) |t|^{-2N/(N-2)} &= 0, & \lim_{|t| \rightarrow \infty} F_1(t) |t|^{-2N/(N-2)} &= 0, \\ \lim_{|t| \rightarrow \infty} F_2(t) |t|^{-2N/(N-2)} &= 0. \end{aligned}$$

We are going to apply the concentration-compactness method with $\rho_n = |\nabla u_n|^2 + |u_n|^{2N/(N-2)} + F_2(u_n)$ where u_n is a minimizing sequence. First of all we need some *a priori* bounds on u_n : of course u_n is bounded in H and thus in $L^{2N/(N-2)}$. But clearly there exists $C > 0$ such that:

$$0 \leq F_1(t) \leq C |t|^{2N/(N-2)},$$

and thus $F_1(u_n)$ is bounded in L^1 . Therefore $F_2(u_n)$ is bounded in L^1 since $F(u_n) = F_1(u_n) - F_2(u_n)$.

To rule out vanishing we use the following:

LEMMA II.2. — *Let (u_n) be a bounded sequence in H and let $G \in C(\mathbb{R})$ satisfy:*

$$\lim_{|t| \rightarrow 0} G(t) |t|^{-2N/(N-2)} = 0, \quad \lim_{|t| \rightarrow \infty} G(t) |t|^{-2N/(N-2)} = 0.$$

Assume that for some $R < \infty$, we have:

$$\sup_{y \in \mathbb{R}^N} \int_{y + B_R} |\nabla u_n|^2 + |u_n|^{\frac{2N}{N-2}} dx \xrightarrow{n} 0$$

Then $G(u_n) \xrightarrow{n} 0$ in $L^1(\mathbb{R}^N)$.

Proof of Lemma II.2. — We first assume that (u_n) is bounded in $L^\infty(\mathbb{R}^N)$. Then, in view of Lemma I.1, we deduce: $u_n \xrightarrow{n} 0$ in L^q for $\frac{2N}{N-2} < q < \infty$. To conclude we then just remark that for all $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that:

$$|G(t)| \leq \varepsilon |t|^{\frac{2N}{N-2}} + C_\varepsilon |t|^q \quad \text{on } \mathbb{R}$$

for some $q > \frac{2N}{N-2}$.

Next, if (u_n) is only bounded in H , we observe that for all $M > 0$ the function $v_n = \max(\min(u_n, M), -M)$ satisfies the same assumptions than u_n and is now bounded in L^∞ . Therefore by the above proof:

$$G(v_n) \xrightarrow{n} 0 \quad \text{in } L^1(\mathbb{R}^N).$$

To conclude, we observe:

$$\begin{aligned} \int_{\mathbb{R}^N} |G(u_n)| dx &\leq \int_{\mathbb{R}^N} |G(v_n)| dx + \int_{\mathbb{R}^N} |G(u_n)| 1_{(|u_n| \geq M)} dx \\ &\leq \int_{\mathbb{R}^N} |G(v_n)| dx + \varepsilon(M) \int_{\mathbb{R}^N} |u_n|^{2N/(N-2)} dx \end{aligned}$$

with $\varepsilon(M) \rightarrow 0$ as $M \rightarrow +\infty$.

To rule out dichotomy, we just have to observe that with the notations of the preceding sections we have:

$$\int_{\mathbb{R}^N} (|\nabla(\xi_n u_n)|^2 - \xi_n^2 |\nabla u_n|^2) dx \leq C \left| \int_{\mathbb{R}^N} |\nabla \xi_n|^2 u_n^2 dx \right|^{1/2} + C \int_{\mathbb{R}^N} |\nabla \xi_n|^2 u_n^2 dx;$$

and

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla \xi_n|^2 u_n^2 dx &= \int_{R \leq |x-y_n| \leq 2R} |\nabla \xi_n|^2 u_n^2 dx \\ &\leq \left(\int_{\mathbb{R}^N} |\nabla \xi_n|^N \right)^{2/N} \left(\int_{R \leq |x-y_n| \leq 2R} |u_n|^{2N/(N-2)} dx \right)^{N/(N-2)} \\ &\leq C \left(\int_{R \leq |x-y_n| \leq 2R} |u_n|^{2N/(N-2)} dx \right)^{N/(N-2)} \leq C\varepsilon \end{aligned}$$

if we choose $R_0 \leq R \leq R_n/2$ (for n large enough).

The remainder of the proof is then totally similar to the proofs made in the preceding sections. ■

We now consider without proof the x -dependent case: we will always assume (8), (10') and

$$(28) \quad \lim_{t \rightarrow 0} \{ tf(x, t) \}^+ |t|^{-\frac{2N}{N-2}} = 0, \quad \text{uniformly for } |x| \text{ large.}$$

We need some additional technical assumptions (which might not be necessary):

$$(29) \quad |F^-(x, t) - \bar{F}^-(t)| \leq \varepsilon(\mathbf{R})(\bar{F}^-(t) + |t|^{2N/(N-2)}) \quad \text{for } |t| \leq 1, \\ \text{for } |x| \geq \mathbf{R}$$

where $\varepsilon(\mathbf{R}) \rightarrow 0$ if $\mathbf{R} \rightarrow +\infty$.

We need to define precisely problem (19) as follows:

$$(19') \quad I_\lambda = \text{Inf} \left\{ \int_{\mathbb{R}^N} a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} dx / u \in \mathbf{H}, F(x, u) \in L^1, \int_{\mathbb{R}^N} F(x, u) dx = \lambda \right\}.$$

THEOREM II.4. — *Let $N \geq 3$, we assume (8), (10') and (28)-(29). Then the condition*

$$(S.1) \quad I_\lambda < I_\alpha + I_{\lambda-\alpha}^\infty, \quad \forall \alpha \in [0, \lambda[$$

is necessary and sufficient for the relative compactness in \mathbf{H} of all minimizing sequences of (19'). In addition if (S.1) holds, $F(x, u_n)$ is relatively compact in L^1 for all minimizing sequences u_n and thus in particular there exists a minimum in (19').

II.4. The case $N = 2$.

To explain the difficulties and the results, we begin by the translation invariant case and we take $f = f \in C(\mathbb{R})$ ($f(0) = 0$), a_{ij} satisfying (27) and we denote by $F(t) = \int_0^t f(s)$. Then for $\lambda \in \mathbb{R}$, we consider the minimization problem

$$I_\lambda = \text{Inf} \left\{ \int_{\mathbb{R}^2} a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial u}{\partial x_i} dx / u \in \mathcal{D}(\mathbb{R}^2), \int_{\mathbb{R}^2} F(u) dx = \lambda \right\}.$$

First of all, we remark that $I_\lambda = I_\mu$ if $\lambda, \mu > 0$ or if $\lambda, \mu < 0$: indeed if we replace u by $u\left(\frac{\cdot}{\sigma}\right)$ (with $\sigma > 0$) then $\mathcal{E}^\infty(u) = \mathcal{E}^\infty\left(u\left(\frac{\cdot}{\sigma}\right)\right)$ while

$$\int_{\mathbb{R}^N} F\left(u\left(\frac{\cdot}{\sigma}\right)\right) dx = \sigma^2 \int_{\mathbb{R}^2} F(u) dx.$$

Next, if there is a minimum of I_λ satisfying: $\forall u \in L^2(\mathbb{R}^2), F(u) \in L^1(\mathbb{R}^2)$; then the Euler equation implies via (23)

$$\int_{\mathbb{R}^2} F(u) dx = 0.$$

Therefore the only λ for which we can hope to find a solution is $\lambda = 0$. And since we want to avoid the trivial absolute minimum 0, we finally have to consider

$$(30) \quad I = \text{Inf} \left\{ \int_{\mathbb{R}^2} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} dx / u \in K_0 \right\}$$

where $K_0 = \left\{ u \in L^1_{loc}(\mathbb{R}^2); \forall t > 0, |u| 1_{(|u| \geq t)} \in L^1; \forall u \in L^2; F(u) \in L^1 \text{ and } \int_{\mathbb{R}^2} F(u) dx = 0 \right\}$.

We need the complicated form of K_0 in order to be sure to deal with functions which « vanish at infinity »: indeed functions in $L^1_{loc}(\mathbb{R}^2)$ such that $F(u) \in L^1(\mathbb{R}^2), \forall u \in L^2(\mathbb{R}^2)$ need not to vanish at infinity (they may even be unbounded). Observe also that if $u \in K_0$ then for all $q \geq 1$.

$$\int_{\mathbb{R}^2} |u|^q 1_{(|u| \geq t)} dx < \infty, \quad \forall t > 0.$$

Indeed let us denote by $H(s) = (s-t)^+$. Then $v = H(|u|)$ satisfies $\nabla v \in L^1(\mathbb{R}^2), v \in L^1(\mathbb{R}^2)$. Therefore by Sobolev inequalities

$$\left(\int_{\mathbb{R}^2} |v|^q dx \right)^{1/q} \leq \|\nabla v\|_{L^{q'}(\mathbb{R}^2)}^{1/q'} \|v\|_{L^{q'}(\mathbb{R}^2)}^{1/q}$$

and thus

$$\begin{aligned} \left(\int_{\mathbb{R}^2} |u|^q 1_{(|u| \geq t)} dx \right)^{1/q} &\leq \|v\|_{L^q} + t \text{meas}(|u| \geq t)^{1/q} \\ &\leq \|v\|_{L^q} + t^{1/q'} \left| \int_{\mathbb{R}^2} |u| 1_{(|u| \geq t)} dx \right|^{1/q} \end{aligned}$$

We will assume

$$(31) \quad \exists \delta > 0, \quad t f(t) < 0 \quad \text{if } 0 < |t| \leq \delta; \quad \exists \zeta \in \mathbb{R}, \quad F(\zeta) > 0$$

$$(32) \quad \lim_{|t| \rightarrow \infty} F(t) |t|^{-\gamma} = 0, \quad \text{for some } \gamma > 0.$$

THEOREM II.5. — *Under assumptions (31), (32), any minimizing sequence of (30) is relatively compact up to a translation and a scale change i. e. for any minimizing sequence (u_n) there exist $(y_n) \in \mathbb{R}^2, (\sigma_n) \in \mathbb{R}^+$ such that the minimizing sequence $\tilde{u}_n = u_n((\cdot - y_n)/\sigma_n^{-1})$ satisfies:*

$$\left\{ \begin{array}{l} \nabla \tilde{u}_n \text{ is relatively compact in } L^2(\mathbb{R}^2), F(\tilde{u}_n) \text{ is relatively compact in } L^1(\mathbb{R}^2) \\ \text{and } \tilde{u}_n 1_{(|\tilde{u}_n| \geq t)} \text{ are relatively compact in } L^q(\mathbb{R}^2) \text{ for all } 1 \leq q < \infty \text{ and} \\ \text{for all } t > 0. \end{array} \right.$$

REMARK II.9. — Independently of this work, H. Berestycki, T. Gallouet and O. Kavian [5] proved the existence of a minimum in the positive mass case and with the use of Schwarz symmetrization. Of course in the positive mass case, the compactness of $F(\tilde{u}_n)$ in L^1 implies the compactness of \tilde{u}_n in L^2 and thus in $H^1(\mathbb{R}^2)$.

REMARK II.10. — Since in two dimensions, because of (23), the action S reduces to $\frac{1}{2} \int_{\mathbb{R}^2} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} dx$ for solutions of the fields equation; it is clear that all minima of (30) are ground states.

REMARK II.11. — We could as well treat the case when (32) is replaced by:

$$\lim_{|t| \rightarrow \infty} F(t)e^{-|t|^\gamma} = 0, \quad \text{for some } 0 < \gamma < 2.$$

We treat here only the case of (32) to simplify the presentation.

REMARK II.12. — If we are interested in positive solutions of the fields equation (Euler equation) then we may assume without loss of generality that F is even, and using the maximum principle we see that we can treat in fact the case of f satisfying (31) and

$$(32) \quad \begin{cases} \text{either } \exists \zeta > \zeta_0, & f(\zeta) \leq 0 \\ \text{or } \lim_{|t| \rightarrow \infty} F(t)e^{-|t|^\gamma} = 0, & \text{for some } 0 < \gamma < 2. \end{cases}$$

where we denote by $\zeta_0 = \inf (\zeta > 0, F(\zeta) > 0) > 0$. ■

Sketch of proof of Theorem II.1. — We begin with a few preliminaries. We introduce a function F_2 satisfying:

$$\begin{cases} F_2 \in C^1(\mathbb{R}), & F_2 \text{ is increasing for } t \geq 0 \text{ and decreasing for } t \leq 0 \\ F_2(t) = -F(t) \text{ if } |t| \leq \delta, & \lim_{|t| \rightarrow \infty} F_2(t)|t|^{-\gamma} = 1, \quad F_2 \geq |F| \text{ on } \mathbb{R} \end{cases}$$

with the same γ as in (32) (we may always assume that $\gamma > 1$).

In view of the properties of elements of K_0 , we see that for any $u \in K_0$, $F_2(u) \in L^1$.

Let (u_n) be a minimizing sequence of (30), ∇u_n is bounded in L^2 . We choose σ_n such that, if $\tilde{u}_n = u_n \left(\frac{\cdot}{\sigma} \right)$, $\|F_2(\tilde{u}_n)\|_{L^1} \in [\alpha, \beta]$ for some $0 < \alpha < \beta < \infty$

(for example, take $\sigma_n = \theta_n^{-1/2}$ with $\theta_n = \int_{\mathbb{R}^2} F_2(u_n) dx$). We want to prove the compactness up to a translation of the new minimizing sequence \tilde{u}_n that, to simplify notations, we still denote by u_n . If we set $F_1 = F_2 + F$, F_1 is

nonnegative and vanishes on $[-\delta, +\delta]$. We want to prove that $F_1(u_n)$ is bounded in L^1 : to this end we just need to observe that:

$$0 \leq F_1(u_n) \leq F_2(u_n) + |F(u_n)| \leq 2F_2(u_n)$$

Next we claim that for all $t > 0$ and for all $q \geq 1$, there exists C independent of n such that:

$$\int_{\mathbb{R}^2} |u_n|^q 1_{(|u_n| \geq t)} dx \leq C, \quad \text{meas } (|u_n| \geq t) \leq C.$$

Indeed observe that we have:

$$\text{meas } (|u_n| \geq t) \leq \frac{1}{F_2(t)} \int_{\mathbb{R}^2} F_2(u_n) dx \leq C.$$

And for all $t > 0$, there exists $\nu > 0$ such that: $F_2(s) \geq \nu(|s| - t)^+$. This yields easily:

$$\int_{\mathbb{R}^2} |u_n| 1_{(|u_n| \geq t)} dx \leq C.$$

And by an argument given above (after the definition of K_0) we conclude:

$$\forall q \geq 1, \quad \forall t > 0, \quad \exists C = C(q, t) > 0, \quad \int_{\mathbb{R}^2} |u_n|^q 1_{(|u_n| \geq t)} dx \leq C.$$

We are going to apply the concentration-compactness method with $\rho_n = |\nabla u_n|^2 + F_2(u_n)$. In view of the choice of (σ_n, \tilde{u}_n) , we may assume without loss of generality that: $\int_{\mathbb{R}^2} \rho_n dx \xrightarrow{n} \lambda > 0$. If vanishing occurs, we remark that $v_n = \max[\min(u_n, M), -M]$ (for $0 < M < \infty$) satisfies for some $R < \infty$:

$$\sup_{y \in \mathbb{R}^2} \int_{y + B_R} |\nabla u_n|^2 + F_2(v_n) dx \xrightarrow{n} 0.$$

In addition we have:

$$\begin{aligned} 0 < \alpha &\leq \int_{\mathbb{R}^2} F_2(u_n) dx = \int_{\mathbb{R}^2} F_1(u_n) dx \\ &\leq \int_{\mathbb{R}^2} F_1(v_n) dx + \int_{\mathbb{R}^2} F_1(u_n) 1_{(|u_n| \geq M)} dx \\ &\leq \int_{\mathbb{R}^2} F_1(v_n) dx + C \int_{\mathbb{R}^2} |u_n|^{\gamma} 1_{(|u_n| \geq M)} dx \\ &\leq \int_{\mathbb{R}^2} F_1(v_n) dx + C \int_{\mathbb{R}^2} \frac{|u_n|^{\gamma+1}}{M} 1_{(|u_n| \geq M)} dx \end{aligned}$$

and if we show that $\int_{\mathbb{R}^2} F_1(v_n) dx \xrightarrow{n} 0$, we will reach a contradiction which rules out vanishing. To prove this claim we recall that $F_1 = 0$ if $|t| \leq \delta$; and we take $\delta' \in]0, \delta[$, we set $H(t) = (|t| - \delta')^+$. Remarking that $w_n = H(v_n)$ is bounded in $L^1(\mathbb{R}^2)$ and that ∇w_n is bounded in $L^2(\mathbb{R}^2)$, we obtain by Sobolev inequalities:

$$\int_{y+B_R} w_n^p dx \leq C \left\{ \left(\int_{y+B_R} |\nabla w_n|^2 dx \right)^{p/2} + \left(\int_{y+B_R} w_n dx \right)^p \right\},$$

for all $p > 2$, where C is independent of $y \in \mathbb{R}^2$.

Then we observe that: $H(t) \leq CF_2(t)$ for some C and thus:

$$\int_{y+B_R} w_n^p dx \leq \varepsilon_n \int_{y+B_R} |\nabla v_n|^2 + F_2(v_n) dx$$

where $\varepsilon_n \xrightarrow{n} 0$. Covering \mathbb{R}^2 with balls of radius R , we deduce that w_n converges to 0 in $L^p(\mathbb{R}^2)$ for all $p > 2$; in particular for $p = \gamma$. And we conclude observing that, $F_1 \leq CH^\gamma$, for some $C > 0$.

We then rule out dichotomy exactly as we did before, let us only explain the main two new points. We first explain how to use the cut-off functions: with the notations of the preceding sections, we see that we have to bound:

$$\begin{aligned} \int_{\mathbb{R}^2} |\nabla \xi_n|^2 u_n^2 dx &\leq \delta^2 \int_{\mathbb{R}^2} |\nabla \xi_n|^2 dx + \int_{\mathbb{R}^2} |\nabla \xi_n|^2 u_n^2 1_{|u_n| \geq \delta} dx \quad \text{for all } \delta > 0, \\ &\leq \left(\int_{\mathbb{R}^2} |\nabla \xi|^2 dx \right) \delta^2 + \left(\int_{\mathbb{R}^2} |\nabla \xi_n|^4 \right)^{1/2} \left(\int_{\mathbb{R}^2} |u_n|^4 1_{|u_n| \geq \delta} dx \right)^{1/2} \\ &\leq C\delta^2 + \frac{C(\delta)}{R} \end{aligned}$$

and choosing δ small and then R large, we see that we can now follow the arguments of the preceding section.

Next, we have to prove that for any $\alpha > 0$

$$I < \text{Inf} \left\{ \int_{\mathbb{R}^2} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} dx / u \in K_\alpha \right\}$$

where

$$\begin{aligned} K_\alpha = \{ u \in L^1_{\text{loc}}(\mathbb{R}^2), \quad \forall u \in L^2, \quad |u| 1_{|u| \geq t} \in L^1, \quad \forall t > 0, \quad F(u) \in L^1, \\ \int_{\mathbb{R}^2} F_2(u) dx \leq C, \quad \int_{\mathbb{R}^2} F(u) dx = \alpha \} \quad \text{for some } C > 0. \end{aligned}$$

To prove this strict inequality we argue as follows: we first observe that if $u \in K_\alpha$ then there exists $\theta \in]0, 1[$ such that: $\int_{\mathbb{R}^2} F(\theta u) dx = 0$. If we

show that θ is bounded away from 1 uniformly for $u \in K_\alpha$, then the strict inequality is proved. If there exist $u_n \in K_\alpha, \theta_n \rightarrow 1$ such that

$$\int_{\mathbb{R}^2} F(\theta_n u_n) dx = 0,$$

then it is easy to show that:

$$\int_{\mathbb{R}^2} F_1(u_n) - F_1(\theta_n u_n) dx \rightarrow 0.$$

Then

$$\begin{aligned} \alpha &= \int_{\mathbb{R}^2} F(u_n) dx = \int_{\mathbb{R}^2} F_1(u_n) - F_2(u_n) dx \\ &= \lim_n \int_{\mathbb{R}^2} F_1(\theta_n u_n) - F_2(u_n) dx \\ &= \lim_n \int_{\mathbb{R}^2} F_2(\theta_n u_n) - F_2(u_n) dx \end{aligned}$$

and this is not possible since: $F_2(\theta t) \leq F_2(t)$ for $0 \leq \theta \leq 1, t \in \mathbb{R}$.

We thus have proved the existence of y_n in \mathbb{R}^2 such that:

$$\forall \varepsilon > 0, \exists R < \infty, \int_{|x-y_n| \geq R} |\nabla u_n|^2 + F_2(u_n) dx \leq \varepsilon.$$

We then denote by $\tilde{u}_n = u_n(\cdot + y_n)$, extracting a subsequence if necessary we may assume that \tilde{u}_n converges a. e. to some $u \in L^1_{loc}(\mathbb{R}^2), \nabla \tilde{u}_n$ converges weakly in L^2 to ∇u and that $F_2(\tilde{u}_n)$ converges in L^1 to $F_2(u)$. It is then easy to conclude. ■

We now turn to the case of nonlinearities depending on x : in that case the situation is even more complicated. To simplify the presentation, we will only consider a typical situation where:

$$\begin{aligned} \mathcal{E}(u) &= \int_{\mathbb{R}^2} a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} dx, \\ J(u) &= \int_{\mathbb{R}^2} K(x) |u|^q - c(x) |u|^p dx \end{aligned}$$

where $(a_{ij}(x))$ satisfies (8), where $1 \leq p < q < \infty$ and where K, c satisfy:

$$(33) \quad \exists 0 < v \leq \mu < \infty, \quad v \leq K(x), c(x) \leq \mu \quad \text{on } \mathbb{R}^N; \quad K(x) \rightarrow K^\infty, \\ c(x) \rightarrow c^\infty \quad \text{as } |x| \rightarrow \infty.$$

We then consider for $\lambda \in \mathbb{R}$ the minimization problem:

$$I_\lambda = \text{Inf} \{ \mathcal{E}(u) / \nabla u \in L^2(\mathbb{R}^2), \quad u \in L^p(\mathbb{R}^2), \quad J(u) = \lambda \}$$

(notice that if $u \in L^p(\mathbb{R}^2), \nabla u \in L^2(\mathbb{R}^2)$ then by Sobolev embeddings $u \in L^\alpha(\mathbb{R}^2)$ for all $\alpha \geq p$).

For $\lambda < 0$, we claim that $I_\lambda = 0$: indeed there exist (u_n) satisfying:

$$\|u_n\|_{L^p(\mathbb{R}^2)} = 1, \quad \|\nabla u_n\|_{L^2} + \|u_n\|_{L^q} \rightarrow 0.$$

Then we may assume that $\theta_n = \int_{\mathbb{R}^2} c(x)|u_n|^p - K(x)|u_n|^q dx \rightarrow \theta > 0$ hence we can find $\lambda_n > 0$ converging to $(\lambda/\theta)^{1/p}$ such that $J(\lambda_n u_n) = \lambda$ and $\mathcal{E}(\lambda_n u_n) \rightarrow 0$.

We will consider the case $\lambda = 0$ later on and the case $\lambda > 0$ is solved by the

THEOREM II.6. — *Under assumptions (8), (33) and if $\lambda > 0$; then we always have: $I_\lambda \leq I_0^\infty$, where I_0^∞ is given by:*

$$I_0^\infty = \text{Inf} \{ \xi^\infty(u) / \nabla u \in L^2(\mathbb{R}^2), u \in L^p(\mathbb{R}^2), J^\infty(u) = 0, u \equiv 0 \}$$

and

$$\xi^\infty(u) = \int_{\mathbb{R}^2} \bar{a}_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} dx, \quad J^\infty(u) = \int_{\mathbb{R}^2} K^\infty |u|^q - c^\infty |u|^p dx.$$

If $I_\lambda < I_0^\infty$, then every minimizing sequence (u_n) satisfies: $u_n, \nabla u_n$ are relatively compact in (resp.) L^p, L^2 .

If $I_\lambda = I_0^\infty$, there exists a minimizing sequence which is not bounded in $L^p(\mathbb{R}^2)$ (or in $L^q(\mathbb{R}^2)$); in addition all minimizing sequences are either relatively compact up to a translation as above, or satisfy for some $\sigma_n \rightarrow 0$:

$\tilde{u}_n(\cdot) = u_n\left(\frac{\cdot}{\sigma_n}\right)$ is relatively compact up to a translation as above and all limit points of \tilde{u}_n are minima of the problem I_0^∞ .

Sketch of the proof of Theorem II.6. — Let us first explain why $I_\lambda \leq I_0^\infty$: let u_0 be a minimum of I_0^∞ (use Theorem II.5), $u_0 \in L^p \cap L^q(\mathbb{R}^2), \nabla u_0 \in L^2(\mathbb{R}^2)$ and using the Euler equation we see that $u_0 \in C_b^1(\mathbb{R}^2)$. Define $u_n(\cdot) = u_0(\theta_n \cdot)$ where $\theta_n \rightarrow 0$ is determined below, then

$$\begin{aligned} \mathcal{E}(u_n) &= \int_{\mathbb{R}^2} a_{ij}(x) \frac{\partial u_n}{\partial x_i} \frac{\partial u_n}{\partial x_j} dx = \int_{\mathbb{R}^2} a_{ij}\left(\frac{x}{\theta_n}\right) \frac{\partial u_0}{\partial x_i} \frac{\partial u_0}{\partial x_j} dx \\ J(u_n) &= \int_{\mathbb{R}^2} K(x) |u_n|^q - c(x) |u_n|^p dx = \frac{1}{\theta_n^2} \int_{\mathbb{R}^2} K\left(\frac{x}{\theta_n}\right) |u_0|^q - c\left(\frac{x}{\theta_n}\right) |u_0|^p dx \end{aligned}$$

On the other hand

$$\int_{\mathbb{R}^2} \left| a_{ij}\left(\frac{x}{\theta_n}\right) \frac{\partial u_0}{\partial x_i} \frac{\partial u_0}{\partial x_j} - \bar{a}_{ij} \frac{\partial u_0}{\partial x_i} \frac{\partial u_0}{\partial x_j} \right| dx \leq \delta_\epsilon^n + C \int_{B_\epsilon} \left| a_{ij}\left(\frac{x}{\theta_n}\right) - \bar{a}_{ij} \right| dx \leq \delta_\epsilon^n + C\epsilon^2$$

with $\delta_\varepsilon^n \rightarrow 0$, for all $\varepsilon > 0$. In a similar way

$$\int_{\mathbb{R}^2} K\left(\frac{x}{\theta_n}\right) |u_0|^q - c\left(\frac{x}{\theta_n}\right) |u_0|^p dx = \mu_n \xrightarrow{n} 0.$$

We may then choose $\theta_n \xrightarrow{n} 0$, $\lambda_n \xrightarrow{n} 1$ such that: $J(\lambda_n u_n) = \lambda$ and clearly $\mathcal{E}(\lambda_n u_n) \xrightarrow{n} I_0^\infty$. Next, if $(u_n)_n$ is a minimizing sequence of problem I_λ and if

$$\theta_n = \int_{\mathbb{R}^2} |u_n|^p dx \xrightarrow{n} \infty$$

(or a subsequence), then defining $\tilde{u}_n = u_n\left(\frac{\cdot}{\sigma_n}\right)$ with $\sigma_n = \theta_n^{-1/2}$ we deduce:

$$\begin{aligned} I_\lambda &= \lim_n \mathcal{E}(u_n) = \lim_n \int_{\mathbb{R}^2} a_{ij}\left(\frac{x}{\sigma_n}\right) \frac{\partial \tilde{u}_n}{\partial x_i} \frac{\partial \tilde{u}_n}{\partial x_j} dx \\ &\geq \lim_n \int_{\mathbb{R}^2 - B_{\varepsilon_n}} a_{ij}\left(\frac{x}{\sigma_n}\right) \frac{\partial \tilde{u}_n}{\partial x_i} \frac{\partial \tilde{u}_n}{\partial x_j} dx \\ &\geq \lim_n \int_{\mathbb{R}^2 - B_{\varepsilon_n}} \bar{a}_{ij} \frac{\partial \tilde{u}_n}{\partial x_i} \frac{\partial \tilde{u}_n}{\partial x_j} dx \end{aligned}$$

for some sequence $\varepsilon_n \xrightarrow{n} 0$.

We then consider: $\xi_n = \xi\left(\frac{\cdot}{2\varepsilon_n}\right)$ with $\xi \equiv 0$ in B_1 , $0 \leq \xi \leq 1$, $\xi \in C_b^\infty(\mathbb{R}^2)$, $\xi \equiv 1$ if $|x| \geq 2$. Then

$$\begin{aligned} \int_{\mathbb{R}^2 - B_{\varepsilon_n}} \bar{a}_{ij} \frac{\partial \tilde{u}_n}{\partial x_i} \frac{\partial \tilde{u}_n}{\partial x_j} dx &\geq \int_{\mathbb{R}^2} \xi_n^2 \bar{a}_{ij} \frac{\partial \tilde{u}_n}{\partial x_i} \frac{\partial \tilde{u}_n}{\partial x_j} dx \\ &\geq \mathcal{E}^\infty(\xi_n \tilde{u}_n) - C \left| \int_{\mathbb{R}^2} |\nabla \xi_n|^2 \tilde{u}_n^2 dx \right|^{1/2} - C \left| \int_{\mathbb{R}^2} |\nabla \xi_n|^2 \tilde{u}_n^2 dx \right|. \end{aligned}$$

On the other hand if $\alpha \geq p/2$

$$\int_{\mathbb{R}^2} |\nabla \xi_n|^2 \tilde{u}_n^2 dx \leq \left| \int_{\mathbb{R}^2} |\nabla \xi_n|^{2\alpha'} \right|^{1/\alpha'} \left| \int_{\mathbb{R}^2} \tilde{u}_n^{2\alpha} \right|^{1/\alpha} \leq Cn^{-2/\alpha}.$$

Therefore:

$$I_\lambda \geq \lim_n \mathcal{E}^\infty(\xi_n \tilde{u}_n).$$

On the other hand we have:

$$\lambda = J(u_n) = \frac{1}{\sigma_n^2} \int_{\mathbb{R}^2} K\left(\frac{x}{\sigma_n}\right) |\tilde{u}_n|^q - c\left(\frac{x}{\sigma_n}\right) |\tilde{u}_n|^p dx$$

and thus: $J^\infty(\tilde{u}_n) \rightarrow 0$. Next, we remark that:

$$\int_{\mathbb{R}^2} \left| |\tilde{u}_n|^q - \xi_n^q |\tilde{u}_n|^q \right| dx \leq \left| \int_{\mathbb{R}^2} |1 - \xi_n^q|^2 dx \right|^{1/2} \left| \int_{\mathbb{R}^2} |\tilde{u}_n|^{2q} \right|^{1/2} \leq C\varepsilon_n$$

and similarly:

$$\int_{\mathbb{R}^2} \|\tilde{u}_n\|^p - \xi_n^p \|\tilde{u}_n\|^p dx \leq C\varepsilon_n.$$

Therefore: $J^\infty(\xi_n \tilde{u}_n) \rightarrow 0$, and it is easy to find $k_n \rightarrow \frac{1}{n}$ such that $J^\infty(k_n \xi_n \tilde{u}_n) = 0$. This yields $I_\lambda \geq I_0^\infty$ and thus $I_\lambda = I_0^\infty$.

In particular if $I_\lambda < I_0^\infty$, (u_n) remains bounded in L^p . Since (∇u_n) is bounded in L^2 , we deduce that (u_n) is bounded in L^α for $p \leq \alpha < \infty$. We may then apply the concentration-compactness method with $\rho_n = |\nabla u_n|^2 + |u_n|^p$ and we conclude easily remarking that we have:

$$0 < I_{\theta\lambda} < \theta I_\lambda, \quad \text{for all } \lambda > 0, \theta > 1.$$

Indeed if u satisfies: $\nabla u \in L^2$, $J(u) = \lambda$, $u \in L^p$; then first of all it is easy to see that $|\nabla u|_{L^2}$ remains bounded away from 0. Indeed if this was not the case, remarking that:

$$\lambda + v \int_{\mathbb{R}^2} |u_n|^p dx \leq \mu \int_{\mathbb{R}^2} |u_n|^q dx \leq \mu |\nabla u_n|_{L^2}^{2-p} \int_{\mathbb{R}^2} |u_n|^p dx$$

we would get a contradiction. Next, we claim that there exists $\mu = \mu(u) > 1$ such that: $J(\mu u) = \theta \lambda$ and $\mu \leq \theta^{1/q}$. Indeed observe that

$$\theta \lambda = \int_{\mathbb{R}^2} K |\mu u|^q - c |\mu u|^p dx \geq \mu^q \int_{\mathbb{R}^2} K |u|^q - c |u|^p dx = \mu^q \lambda.$$

Then:

$$I_{\theta\lambda} \leq \inf \{ \mu(u)^2 \mathcal{E}(u) / u \in L^p, \nabla u \in L^2, J(u) = \lambda \} \leq \theta^{2/q} I_\lambda < \theta I_\lambda. \quad \blacksquare$$

We now conclude this section by a few considerations on the case $\lambda = 0$:

$$I_0 = \inf \{ \mathcal{E}(u) / \nabla u \in L^2(\mathbb{R}^2), u \in L^p(\mathbb{R}^2), u \neq 0, J(u) \equiv 0 \}.$$

We claim that we have: $I_0 \leq \inf_{x \in \mathbb{R}^2} I^x$,

where

$$I^x = \min \left\{ \int_{\mathbb{R}^2} a_{ij}(x) \frac{\partial u}{\partial y_i} \frac{\partial u}{\partial y_j} dy / \nabla u \in L^2(\mathbb{R}^2), u \in L^p(\mathbb{R}^2), u \neq 0, \int_{\mathbb{R}^2} K(x) |u|^q(y) - c(x) |u|^p(y) dy = 0 \right\}.$$

In addition we can show that if $I_0 < \inf_{x \in \mathbb{R}^2} I^x$, then all minimizing sequences are relatively compact, but we do not know if this strict inequality is satisfied (by modifying artificially \mathcal{E} , we can treat similar problems where the above strict inequality is satisfied). We now show the above

large inequality: let $x_0 \in \mathbb{R}^2$ and let u_0 be a minimum — which exists by Theorem II.5 — of I^{x_0} ; we then set $u_n(\cdot) = u_0\left(\frac{\cdot - x_0}{\theta_n}\right)$ where $\theta_n \rightarrow 0$.

$$\begin{aligned} \mathcal{E}(u_n) &= \int_{\mathbb{R}^2} a_{ij}(x) \frac{\partial u_n}{\partial x_i} \frac{\partial u_n}{\partial x_j} dx \\ &= \int_{\mathbb{R}^2} a_{ij}(x_0 + \theta_n x) \frac{\partial u_0}{\partial x_i}(x) \frac{\partial u_0}{\partial x_j}(x) dx \\ &\geq \int_{\mathbb{B}_R} a_{ij}(x_0) \frac{\partial u_0}{\partial x_i} \frac{\partial u_0}{\partial x_j} dx + \int_{\mathbb{B}_R} a_{ij}(x_0 + \theta_n x) - a_{ij}(x_0) \frac{\partial u_0}{\partial x_i} \frac{\partial u_0}{\partial x_j} dx \end{aligned}$$

and thus: $\liminf_n \mathcal{E}(u_n) \geq I^{x_0}$. On the other hand we have:

$$J(u_n) = \int_{\mathbb{R}^2} K(x) |u_n|^q - c(x) |u_n|^p dx = \theta_n^2 \int_{\mathbb{R}^2} K(x_0 + \theta_n x) |u_0|^q - c(x_0 + \theta_n x) |u_0|^p dx,$$

clearly:

$$\int_{\mathbb{R}^2} K(x_0 + \theta_n x) |u_0|^q - c(x_0 + \theta_n x) |u_0|^p dx \rightarrow 0$$

and we can find $k_n \rightarrow 1$, such that:

$$\int_{\mathbb{R}^2} K(x_0 + \theta_n x) k_n^q |u_0|^q - c(x_0 + \theta_n x) k_n^p |u_0|^p dx = 0.$$

This proves $I_0 \leq I^{x_0}$. Of course if there exists $x_0 \in \mathbb{R}^2$ such that $I_0 = I^{x_0}$ the above proof shows that there exists a minimizing sequence which is not relatively compact.

II.5. Extensions and variants.

In this section we will state without proof some results concerning variants and extensions of the previous problems namely problems with higher order derivatives, systems, different powers or in integral form. In order to restrict the length of this section, we will consider here only problems that are translations invariant but it will be clear enough that we could treat as well problems with functionals « depending on x ». In addition, in order to keep the ideas clear, we will not try to obtain the greatest generality.

We begin with systems of the form:

$$(34) \quad \begin{cases} -a_{ij}^1 \partial_{ij} u^1 = f_1(u^1, \dots, u^m) & \text{in } \mathbb{R}^N, \\ -a_{ij}^m \partial_{ij} u^m = f_m(u^1, \dots, u^m) & \text{in } \mathbb{R}^N, \end{cases}$$

where $\partial_i, \partial_{ij}$ denote $\frac{\partial}{\partial x_i}, \frac{\partial^2}{\partial x_i \partial x_j}$ and $a_{ij}^1, \dots, a_{ij}^m$ are symmetric $N \times N$ matrices satisfying:

$$(35) \quad \exists v > 0, \quad (a_{ij}^k) \geq vI_N \quad \text{for} \quad 1 \leq k \leq m.$$

and m is a given integer ≥ 1 . Our main assumption will be that the system is potential that is: $f_i \in C(\mathbb{R}^m), f_i(0, \dots, 0) = 0$ and there exists $F \in C^1(\mathbb{R}^m)$ such that:

$$\frac{\partial F}{\partial t_i}(t_1, \dots, t_m) = f_i(t_1, \dots, t_m) \quad \text{on} \quad \mathbb{R}^m, \quad F(0, \dots, 0) = 0.$$

It can be useful to point out that the system may turn out to be in potential form after some change of variables: the simplest example being the multiplication of u^i by $\theta^i \in \mathbb{R} - \{0\}$.

As was remarked in H. Berestycki and P. L. Lions [6] [9] [10]; in order to find a solution of (34), we may consider the minimization problem:

$$(36) \quad I_\lambda = \text{Inf} \left\{ \int_{\mathbb{R}^N} \sum_{k=1}^m a_{ij}^k \partial_i u^k \partial_j u^k dx / \nabla u^k \in L^2(\mathbb{R}^N), \quad u^k \in L^{2N/(N-2)}(\mathbb{R}^N), \right. \\ \left. F(u^1, \dots, u^m) \in L^1(\mathbb{R}^N), \quad \int_{\mathbb{R}^N} F(u^1, \dots, u^m) dx = \lambda \right\}$$

where $\lambda > 0$ and $N \geq 3$ (to simplify).

Then if (u^1, \dots, u^m) is a solution of this minimization problem, in general there exists $\theta \in \mathbb{R}$ such that:

$$- a_{ij}^k \partial_i u^k = \theta f_k(u^1, \dots, u^m) \quad \text{in} \quad \mathbb{R}^N, \quad \text{for} \quad 1 \leq k \leq m.$$

Now the analogue of Pohozaev identity (23) holds and yields:

$$\frac{N-2}{2N} \sum_k \int_{\mathbb{R}^N} a_{ij}^k \partial_i u^k \partial_j u^k dx = \theta \int_{\mathbb{R}^N} F(u^1, \dots, u^m) dx = \theta \lambda$$

thus $\theta > 0$, and considering $\bar{u}^k(\cdot) = u^k\left(\frac{\cdot}{\theta^{1/2}}\right)$ for $1 \leq k \leq m$, we find a solution of (34) and in addition by the same argument than the one made in section II.1 concerning the solutions obtained in Theorem II.2, any minimum in (34) yields a ground state of the system (34) i. e. $\bar{u} = (\bar{u}^1, \dots, \bar{u}^m)$ satisfies:

$$0 < S(\bar{u}) \leq S(v) \quad \text{for all } v \text{ solution of (34)}$$

with $\nabla v \in L^2(\mathbb{R}^N), v \in L^{2N/(N-2)}(\mathbb{R}^N), F(v) \in L^1(\mathbb{R}^N)$ and

$$S(w) = \frac{1}{2} \sum_k \int_{\mathbb{R}^N} a_{ij}^k \partial_i w^k \partial_j w^k dx - \int_{\mathbb{R}^N} F(w) dx.$$

We now state the assumptions we need on F , f_i -very much similar to those used in section II.3:

$$(37) \quad \lim_{|t| \rightarrow 0} \{t_k f_k(t)\}^+ |t|^{-\frac{2N}{N-2}} = 0 \quad \text{for } 1 \leq k \leq m,$$

$$\lim_{|t| \rightarrow \infty} |F(t)| |t|^{-2N/(N-2)} = 0;$$

$$(38) \quad \exists t \in \mathbb{R}^m, \quad F(t) > 0.$$

And we obtain the

THEOREM II.7. — *Under assumptions (35), (37), (38), every minimizing sequence $u^n = (u_1^n, \dots, u_m^n)$ satisfies: there exist $(y_n) \in \mathbb{R}^N$ such that $(\nabla \tilde{u}^n)$, $F(\tilde{u}^n)$ are relatively compact in L^2 , L^1 respectively where $\tilde{u}^n(\cdot) = u^n(\cdot + y_n)$. In particular there exists a minimum in (36).*

Observing that in view of easy scaling arguments $I_\lambda = \lambda^{(N-2)/N} I_1$ with $I_1 > 0$, the proof is very much the same as the one of Theorem II.3; we apply the concentration-compactness method on

$$\rho_n = \sum_k (|\nabla u_n^k|^2 + |u_n^k|^{2N/(N-2)}) + F_2(u_n)$$

where F_2 is built in a similar way as in the proof of Theorem II.3.

REMARK II.12. — In [9] [10] the existence of a minimum among spherically symmetric functions (u^1, \dots, u^m) is proved when $a_{ij}^k = \delta_{ij}$ for all k by methods using the « compactness of spherically symmetric functions ». An interesting open question in this particular case is the symmetry of the ground state solutions of (34) which is proved to exist by Theorem II.7.

We now turn to higher order equations like for example:

$$(39) \quad \Delta^2 u = f(u) \quad \text{in } \mathbb{R}^N$$

where $f \in C(\mathbb{R})$, $f(0) = 0$ and we look for a solution $u \neq 0$ that goes to 0 at ∞ .

To simplify we will assume $N \geq 5$ (the case $N = 4$ is very much similar to the case $N = 2$, section II.4) and:

$$(40) \quad \lim_{|t| \rightarrow 0} \{t f(t)\}^+ |t|^{-\frac{2N}{N-4}} = 0, \quad \lim_{|t| \rightarrow \infty} F(t) |t|^{-\frac{2N}{N-4}} = 0$$

$$(41) \quad \exists \zeta \in \mathbb{R}, \quad F(\zeta) > 0$$

where $F(t) = \int_0^t f(s) ds$. We introduce the minimization problem:

$$(42) \quad I_\lambda = \text{Inf} \left\{ \int_{\mathbb{R}^2} |D^2 u|^2 dx / D^2 u \in L^2(\mathbb{R}^N), \quad u \in L^{2N/(N-4)}(\mathbb{R}^N), \right. \\ \left. F(u) \in L^1(\mathbb{R}^N), \quad \int_{\mathbb{R}^N} F(u) dx = \lambda \right\}$$

where $\lambda > 0$. Observe that by an easy integration by parts we have:

$$\int_{\mathbb{R}^2} |D^2u|^2 dx = \int_{\mathbb{R}^2} |\Delta u|^2 dx.$$

By scaling arguments, we find that $I_\lambda = \lambda^{\frac{N-4}{N}} I_1$ with $I_1 > 0$ and that any minimum of (42) yields a solution of (39) which is a ground state. Applying the concentration-compactness method to

$$\rho_n = |D^2u_n|^2 + |Du_n|^{\frac{2N}{N-2}} + |u_n|^{\frac{2N}{N-4}} + F_2(u_n),$$

we obtain the:

THEOREM II.8. — *Under assumptions (40)-(41), any minimizing sequence u_n satisfies: there exists a sequence (y_n) in \mathbb{R}^N such that $D^2\tilde{u}_n, F(\tilde{u}_n), \tilde{u}_n$ are relatively compact in $L^2, L^1, L^{2N/(N-4)}$ respectively where $\tilde{u}_n(\cdot) = u_n(\cdot + y_n)$. In particular there exists a minimum in (42).*

REMARK II.13. — Exactly as in remark II.12, an interesting open question concerns the symmetry of ground states of equation (39) or of minima of (42) (since up to a scale change they coincide). The only very partial answer we have is in the case when F is increasing for $t \geq 0$ and even. In this case we can prove that any minimum u is necessarily positive, spherically symmetric decreasing and $(-\Delta u)$ is also positive, spherically symmetric, decreasing. Indeed if u is a minimum we consider v solution of

$$-\Delta v = (-\Delta u)^* \quad \text{in } \mathbb{R}^N, \quad v \in L^{2N/(N-4)}(\mathbb{R}^N), \quad \nabla v \in L^{2N/(N-2)}(\mathbb{R}^N);$$

where f^* denotes the Schwarz symmetrization of f .

Then by an easy adaptation of the main result of G. Talenti [40], we see that we have: $u^* \leq v$ in \mathbb{R}^N . In addition in view of the results of A. Alvino, P. L. Lions and G. Trombetti [1], we see that:

$$\int_{\mathbb{R}^N} F(u) dx = \int_{\mathbb{R}^N} F(u^*) dx < \int_{\mathbb{R}^N} F(v) dx$$

or there exists $y_0 \in \mathbb{R}^N$ such that: $u(\cdot + y_0) = u^*, -\Delta u(\cdot + y_0) = (-\Delta u)^*$. Therefore, if we argue by contradiction, we find v such that.

$$\begin{cases} \int_{\mathbb{R}^2} |\Delta v|^2 dx = \int_{\mathbb{R}^N} |D^2v|^2 dx = I_\lambda \\ \int_{\mathbb{R}^2} F(v) dx > \lambda. \end{cases}$$

and there exists $\theta < 1$ such that $\tilde{v} = v\left(\frac{\cdot}{\sigma}\right)$ satisfies: $\int_{\mathbb{R}^N} F(\tilde{v})dx = \lambda$ and $\int_{\mathbb{R}^N} |D^2 \tilde{v}|^2 dx = \theta^{N-4} I_\lambda < I_\lambda$. This contradiction proves our claim.

REMARK II. 14. — We could treat as well problems where we replace $|D^2 u|^2$ or $|Du|^2$ by $|D^m u|^p$ with $m \geq 1, 1 < p < \infty$. Then we need to assume $N > mp$ and $\frac{2N}{N-4}$ is then replaced by $\frac{Np}{N-mp}$.

We now conclude this section by looking at another variant of the above minimization problems:

$$(43) \quad I_\lambda = \text{Inf} \left\{ \frac{1}{2} \iint_{\mathbb{R}^N \times \mathbb{R}^N} K(x-y)u(x)u(y)dx dy / F(u) \in L^1(\mathbb{R}^N), \int_{\mathbb{R}^N} F(u)dx = \lambda \right\}$$

where K, F satisfy:

$$(44) \quad K \in M^p(\mathbb{R}^N), \quad \text{for some } 1 < p < \infty \quad \text{or} \quad K \in L^1(\mathbb{R}^N)$$

$$(45) \quad \begin{cases} \exists C > 0, F(t) \geq C|t|^\alpha; & F(0) = 0, F \in C(\mathbb{R}); \\ \lim_{t \rightarrow 0} F(t)|t|^{-\alpha} = +\infty; & \lim_{|t| \rightarrow \infty} F(t)|t|^{-\alpha} = +\infty. \\ \forall \theta > 1, \exists C(\theta) > 1, F(\theta t) \leq C(\theta)F(t) \quad \text{on } \mathbb{R} \end{cases}$$

with $\alpha = 2p(2p - 1)^{-1}$.

EXAMPLE. — An example of particular interest is the case $K(x) = \frac{-1}{|x|^\mu}$ with $0 < \mu < N$. Then of course $p = \frac{N}{\mu}$. Observe that in this case, by scaling arguments we see that:

- i) $I_\lambda = \lambda^{2-\mu/N} I_1 < 0$ and thus $I_\lambda < I_\alpha + I_{\lambda-\alpha} \quad \forall \lambda > 0, \forall \alpha \in]0, \lambda[$.
- ii) If u is a minimum of (43), then u solves for some $\theta > 0$:

$$\frac{1}{|x|^\mu} * u = \theta f(u) \quad \text{in } \mathbb{R}^N$$

if $F \in C^1$ and $F' = f$. And $\tilde{u}(\cdot) = \tilde{u}\left(\frac{\cdot}{\theta^\beta}\right)$ with $\beta = \frac{1}{N-\mu}$ is a ground state solution of the equation:

$$\frac{1}{|x|^\mu} * \tilde{u} = f(\tilde{u}) \quad \text{in } \mathbb{R}^N.$$

iii) Notice that if for example $\mu = N - 2$, then the above equation is equivalent to

$$-\Delta(f(\tilde{u})) = C_N \tilde{u} \quad \text{in } \mathbb{R}^N;$$

and if f is invertible, $\tilde{v} = f(\tilde{u})$ solves some scalar field equation. ■

THEOREM II.9. — *Under assumptions (44), (45), the condition:*

$$(S.2) \quad I_\lambda < I_\alpha + I_{\lambda-\alpha}$$

is necessary and sufficient for the relative compactness in $L^\alpha(\mathbb{R}^N)$ of all minimizing sequences up to a translation. In addition if (S.2) holds then any minimizing sequence $(u_n)_n$ satisfies: there exists a sequence (y_n) in \mathbb{R}^N such that $\tilde{u}_n, F(\tilde{u}_n)$ are relatively compact in L^α, L^1 respectively, where we denote by $\tilde{u}_n(\cdot) = u_n(\cdot + y_n)$. In particular if (S.2) holds, (43) has a minimum.

REMARK II.15. — If $K \in M^p + M^q$ with $1 \leq p < q < \infty$, then denoting by $\alpha = \frac{2p}{2p-1} > \beta = \frac{2q}{2q-1}$, we need to assume

$$F(t) \geq C \{ |t|^\alpha + |t|^\beta \}, \quad \lim_{t \rightarrow 0} \frac{F(t)}{|t|^\beta} = +\infty, \quad \lim_{|t| \rightarrow \infty} \frac{F(t)}{|t|^\alpha} = +\infty.$$

While if $K \in M^p \cap M^q$ (with $1 \leq p < q < \infty$), we just need to assume:

$$F(t) \geq C \text{Min} \{ |t|^\alpha, |t|^\beta \}, \quad \lim_{t \rightarrow 0} \frac{F(t)}{|t|^\alpha} = +\infty, \quad \lim_{|t| \rightarrow \infty} \frac{F(t)}{|t|^\beta} = +\infty.$$

III. UNCONSTRAINED PROBLEMS

III.1. Free minimization problems.

We begin by a few heuristic considerations: let $\mathcal{E}(u)$ be a functional defined on a functional space H which commutes with translations and such that $\mathcal{E}(0) = 0$. Take for example:

$$\mathcal{E}(u) = \int_{\mathbb{R}^N} j(Au(x)) dx, \quad \forall u \in H$$

where $j \in C(\mathbb{R}^m)$ and Au is a nonlinear operator defined on H , commuting with translations, with range in a functional space with vector-valued functions with m components (example: $Au = (\nabla u, F(u), u^2 * K)$) where $A0 = 0, j(0) = 0$. Then we denote by:

$$I = \text{Inf} \{ \mathcal{E}(u)/u \in H \}.$$

In general, we remark that either $I = -\infty$ or $I = 0$ and 0 is a global minimum. Indeed if $I < 0$, then let $u \in H$ be such that:

$$I < \mathcal{E}(u) < I/2 < 0$$

then clearly if we consider the new test function \tilde{u} where « we add to u the same u but translated to infinity » we find:

$$\mathcal{E}(\tilde{u}) \simeq 2\mathcal{E}(u) < I$$

and our claim is proved.

Next, we consider more general situations where \mathcal{E} is no more translation invariant like:

$$\mathcal{E}(u) = \int_{\mathbb{R}^N} j(x, Au(x))dx, \quad \forall u \in H;$$

where $j(x, q) \rightarrow j^\infty(q)$ as $|x| \rightarrow \infty$, for $q \in \mathbb{R}^m$.

We then denote by:

$$I = \text{Inf} \{ \mathcal{E}(u)/u \in H \}, \quad I^\infty = \text{Inf} \{ \mathcal{E}^\infty(u)/u \in H \}$$

with $\mathcal{E}^\infty(u) = \int_{\mathbb{R}^N} j^\infty(Au(x))dx$. Clearly, we have: $I \leq I^\infty$.

Therefore if we want I to be finite, we need to assume that $I^\infty = \mathcal{E}^\infty(0)$. If it is the case, the question of the existence of a minimum and of the compactness of minimizing sequences reduces to the obtention of *a priori* estimates in H on minimizing sequences, if we assume some form of local compactness (that would imply the solvability of the problem if \mathbb{R}^N is replaced by a bounded region): a typical example of such situation is given in P. L. Lions [29] and concerns Hartree theory.

III.2. The artificial constraint method.

In [15] [16], C. V. Coffman considered a general method to reduce the question of the existence of a non-trivial critical point to the solution of a minimization problem on a manifold. More precisely if \mathcal{E} is a C^1 functional on a Banach space H such that: $\mathcal{E}'(0) = 0$; then we look for nontrivial solutions of the equation

$$(46) \quad \mathcal{E}'(u) = 0, \quad u \neq 0, \quad u \in H.$$

Remarking that solutions of (46) always satisfy some identities like for example:

$$0 = J(u) = \langle \mathcal{E}'(u), u \rangle,$$

it is tempting to try to minimize \mathcal{E} on the set $M = \{ u \in H / J(u) = 0 \}$ i. e.

$$(47) \quad \text{Inf} \{ \mathcal{E}(u)/u \in H, \quad u \neq 0, \quad J(u) = 0 \}.$$

Now if 0 is « isolated in M » and if \mathcal{E} satisfies for example:

$$\mathcal{E} \in C^2; \quad \langle \mathcal{E}''(u)u, u \rangle - \langle \mathcal{E}'(u), u \rangle > 0 \quad \forall u \in H - \{0\};$$

then one checks easily that a minimum of (47) is indeed a solution of (46). In [15] [16] C. V. Coffman applied this general idea to solve various semilinear elliptic equations or integral equations in bounded regions.

Here, we want to combine this general idea with the concentration-compactness method. We will treat only one example:

$$(48) \quad -\frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial u}{\partial x_j} \right) + c(x)u = f(x, u) \quad \text{in } \mathbb{R}^N, \quad u \neq 0$$

where u satisfies: $\forall u \in L^2(\mathbb{R}^N), u \in L^{2N/(N-2)}(\mathbb{R}^N)$ for example. We will always assume that $(a_{ij}(x))$ satisfy condition (8).

We will just treat the case corresponding to « positive mass » i. e.:

$$(49) \quad c(x) \in C(\mathbb{R}^N), \quad c(x) > 0 \quad \text{on } \mathbb{R}^N, \quad c(x) \rightarrow \bar{c} > 0 \quad \text{as } |x| \rightarrow \infty;$$

$$(50) \quad \begin{cases} f \in C^{0,1}(\mathbb{R}^N \times \mathbb{R}), \quad \frac{\partial f}{\partial t} \in BUC(\mathbb{R}^N \times [-R, +R]) \quad \text{for all } R < \infty \\ \exists \theta \in]0, 1[, \quad 0 \leq \frac{f(x, t)}{t} \leq \theta \frac{\partial f}{\partial t}(x, t) \quad \text{on } \mathbb{R}^N \times \mathbb{R}; \end{cases}$$

$$(51) \quad \lim_{|t| \rightarrow \infty} f(x, t) |t|^{-\frac{N+2}{N-2}} = 0, \quad \lim_{|t| \rightarrow \infty} |f(x, t)| > 0, \quad \text{uniformly in } x \in \mathbb{R}^N;$$

(if $N \leq 2$, $(N + 2)/(N - 2)$ may be replaced by any finite power),

$$(52) \quad \begin{cases} f(x, t) \rightarrow \bar{f}(t) \quad \text{as } |x| \rightarrow \infty, \quad \text{uniformly for } t \text{ bounded} \\ \exists \delta > 0, \quad |f(x, t) - \bar{f}(t)| \leq \varepsilon(\mathbf{R}) |t|^2 \quad \text{for } |x| \geq \mathbf{R}, \quad |t| \leq \delta. \end{cases}$$

with $\varepsilon(\mathbf{R}) \rightarrow 0$ as $\mathbf{R} \rightarrow +\infty$.

We then set: $F(x, t) = \int_0^t f(x, s) ds, \quad \bar{F}(t) = \int_0^t \bar{f}(s) ds,$ and

$$\mathcal{E}(u) = \int_{\mathbb{R}^N} \frac{1}{2} a_{ij}(x) \partial_i u \partial_j u + \frac{1}{2} c(x)u^2 - F(x, u) dx, \quad \forall u \in H^1(\mathbb{R}^N)$$

$$\mathcal{E}^\infty(u) = \int_{\mathbb{R}^N} \frac{1}{2} \bar{a}_{ij} \partial_i u \partial_j u + \frac{1}{2} \bar{c}u^2 - \bar{F}(u) dx, \quad \forall u \in H^1(\mathbb{R}^N)$$

$$J(u) = \int_{\mathbb{R}^N} a_{ij}(x) \partial_i u \partial_j u + c(x)u^2 - f(x, u) u dx, \quad \forall u \in H^1(\mathbb{R}^N)$$

$$J^\infty(u) = \int_{\mathbb{R}^N} \bar{a}_{ij} \partial_i u \partial_j u + \bar{c}u^2 - \bar{f}(u) u dx, \quad \forall u \in H^1(\mathbb{R}^N).$$

And we finally consider:

$$(53) \quad I = \text{Inf} \{ \mathcal{E}(u)/u \in H^1(\mathbb{R}^N), \quad u \neq 0, \quad J(u) = 0 \}$$

$$(54) \quad I^\infty = \text{Inf} \{ \mathcal{E}^\infty(u)/u \in H^1(\mathbb{R}^N), \quad u \neq 0, \quad J^\infty(u) = 0 \}.$$

We have the:

THEOREM III.1. — *Let $N \geq 1$. Under assumptions (8), (49)-(52); every minimizing sequence of (53) is relatively compact in $H^1(\mathbb{R}^N)$ if and only if $I < I^\infty$. If this condition holds, then there exists a minimum in (53) and any such minimum is a solution of (48) in $H^1(\mathbb{R}^N)$.*

In the particular case when $a_{ij}(x) \equiv \bar{a}_{ij}$, $c(x) \equiv \bar{c}$, $f(x, t) \equiv \bar{f}(t)$, then every minimizing sequence of (54) is relatively compact in $H^1(\mathbb{R}^N)$ up to a translation; hence there exists a minimum of (54) and any such minimum is a solution of (48) in $H^1(\mathbb{R}^N)$.

In the zero-mass case, we take $c \equiv 0$ and we assume in addition to (50), (51):

$$(55) \quad \lim_{t \rightarrow 0} f(x, t) |t|^{-\frac{N+2}{N-2}} = 0, \quad \text{uniformly in } x \in \mathbb{R}^N$$

$$(52') \quad \begin{cases} f(x, t) \rightarrow \bar{f}(t) \text{ as } |x| \rightarrow \infty, & \text{uniformly for } t \text{ bounded} \\ \exists \delta > 0, |f(x, t) - \bar{f}(t)| \leq \varepsilon(R) |t|^{-\frac{N+2}{N-2}} & \text{for } |x| \geq R, |t| \leq \delta \end{cases}$$

with $\varepsilon(R) \rightarrow 0$ as $R \rightarrow \infty$. We now replace $H^1(\mathbb{R}^N)$ by the space

$$H = \{ u \in L^{2N/(N-2)}(\mathbb{R}^N), \quad \forall u \in L^2(\mathbb{R}^N) \}.$$

THEOREM III.2. — *Under assumptions (8), (50), (51), (55), (52') and if $N \geq 3$, $c \equiv 0$; the conclusions of Theorem III.1 hold if we replace $H^1(\mathbb{R}^N)$ by H .*

REMARK III.1. — In the case of coefficients independent of x , the existence result is of course contained in the results of section II (at least if $N \geq 3$).

REMARK III.2. — It is possible to relax considerably assumption (50), but we will skip such extensions here. Notice that (50) holds if

$$f(x, t) = \sum_{i=1}^m \alpha_i(x) |t|^{p_i-1} t, \quad \text{where } m \geq 1, \quad \alpha_i \geq \alpha > 0 \quad \text{and } p_i > 1.$$

We will only describe the main new points in the proof of Theorems III.1-2: first of all, we remark that if (u_n) is a minimizing sequence of (53), then (u_n) is bounded in H^1 . Indeed we only need to observe that in view of (50) we have:

$$0 \leq F(x, t) \leq \frac{\theta}{1 + \theta} f(x, t) t \quad \text{on } \mathbb{R}^N \times \mathbb{R};$$

therefore if $J(u_n) = 0$ we have:

$$\begin{aligned} C &\geq \mathcal{E}(u_n) = \int_{\mathbb{R}^N} \frac{1}{2} f(x, u_n) u_n - F(x, u_n) dx \\ &\geq \left(\frac{1}{2} - \frac{\theta}{1 + \theta} \right) \int_{\mathbb{R}^N} f(x, u_n) u_n dx \\ &\geq \nu \|u_n\|_{H^1}^2. \end{aligned}$$

We next apply the concentration-compactness method with

$$\rho_n = |\nabla u_n|^2 + u_n^2$$

if vanishing takes place, we already know by Lemma I.1 that:

$$u_n \xrightarrow{n} 0 \text{ in } L^\alpha(\mathbb{R}^N), \quad \text{for } 2 < \alpha < \frac{2N}{N-2}.$$

Then, because of (50), (51), this implies: $\int_{\mathbb{R}^N} f(x, u_n) u_n + F(x, u_n) dx \xrightarrow{n} 0$; and since $J(u_n) = 0$, we finally obtain: $u_n \xrightarrow{n} 0$ in $H^1(\mathbb{R}^N)$. Next, remarking that in view of (50), (52) we have:

$$0 \leq f(x, t)t \leq \varepsilon |t|^2 + C_\varepsilon |t|^{\frac{2N}{N-2}}, \quad \forall t \in \mathbb{R}, \quad \forall x \in \mathbb{R}^N;$$

we deduce from the constraint $J(u_n) = 0$,

$$\begin{aligned} \nu \|u_n\|_{H^1}^2 &\leq \int_{\mathbb{R}^N} f(x, u_n) u_n dx \leq \varepsilon \|u_n\|_{L^2}^2 + C_\varepsilon \int_{\mathbb{R}^N} |u_n|^{\frac{2N}{N-2}} dx \\ &\leq \varepsilon \|u_n\|_{H^1}^2 + C_\varepsilon \|u_n\|_{H^1}^{2N/(N-2)}. \end{aligned}$$

And if $\|u_n\|_{H^1} \xrightarrow{n} 0$, this would imply for n large enough: $u_n \equiv 0$. The contradiction rules out vanishing.

To rule out dichotomy, we just need to observe that:

$$0 < I < I_{-\alpha} = \text{Inf} \left\{ \mathcal{E}(u) - \frac{1}{2} J(u)/u \in H^1, \quad J(u) = -\alpha \right\}, \quad \forall \alpha > 0.$$

Indeed if $u \in H^1$, $J(u) = -\alpha$, there exists $t = t(u) \in]0, 1[$ such that: $J(tu) = 0$. In addition one checks easily that for all $u \in H^1$ satisfying:

$$I_{-\alpha} \leq \mathcal{E}(u) \leq C, \quad J(u) = -\alpha$$

then $t = t(u) \leq k < 1$. If it were not the case there would exist (u_n) bounded in H^1 , $t_n \xrightarrow{n} 1$ such that: $J(t_n u_n) = 0$, $J(u_n) = -\alpha$. Therefore we would obtain:

$$\int_{\mathbb{R}^N} f(x, t_n u_n) u_n - f(x, u_n) u_n dx \xrightarrow{n} -\alpha < 0.$$

But clearly since (u_n) is bounded in H^1 :

$$\begin{aligned} \int_{\mathbb{R}^N} |f(x, t_n u_n) u_n - f(x, u_n) u_n| dx &\leq \varepsilon(\delta) + \int_{\delta \leq |u_n| \leq 1/\delta} |f(x, t_n u_n) - f(x, u_n)| |u_n| dx \\ &\leq \varepsilon(\delta) + \mu_n^\delta \text{meas} \{ |u_n| \geq \delta \} \\ &\leq \varepsilon(\delta) + \mu_n^\delta \frac{1}{\delta^2} |u_n|_{L^2}^2. \end{aligned}$$

with $\varepsilon(\delta) \rightarrow 0$ as $\delta \rightarrow 0$, $\mu_n^\delta \rightarrow 0$ for $\delta > 0$ fixed. And we conclude. It is then straightforward to complete the proof of Theorem III.1. ■

We conclude this section by another minimization problem over a manifold given by a constraint necessarily satisfied by the solutions of the scalar fields equation:

$$-a_{ij} \partial_{ij} u = f(u), \quad \text{in } \mathbb{R}^N, \quad \nabla u \in L^2(\mathbb{R}^N), \quad u \in L^{2N/(N-2)}(\mathbb{R}^N), \quad F(u) \in L^1(\mathbb{R}^N)$$

where f satisfies (26), (27). We already saw that any such solution satisfies:

$$(23') \quad \frac{N-2}{2N} \int_{\mathbb{R}^N} a_{ij} \partial_i u \partial_j u dx = \int_{\mathbb{R}^N} F(u) dx.$$

Therefore the general idea given in the beginning of this section motivates the introduction of the following minimization problem:

$$(56) \quad I = \text{Min} \left\{ \int_{\mathbb{R}^N} a_{ij} \partial_i u \partial_j u dx / \nabla u \in L^2(\mathbb{R}^N), \quad u \in L^{2N/(N-2)}(\mathbb{R}^N), \right. \\ \left. u \neq 0, \quad F(u) \in L^1(\mathbb{R}^N), \quad \frac{N-2}{2N} \int_{\mathbb{R}^N} a_{ij} \partial_i u \partial_j u dx = \int_{\mathbb{R}^N} F(u) dx \right\}.$$

This problem when the minimizing functions u are restricted to be spherically symmetric (if $a_{ij} = \delta_{ij}$) was considered in M. Struwe [38]. We have the:

THEOREM III.3. — *Under assumptions (26), (27) and if $N \geq 3$, $F(\zeta) > 0$ for some ζ in \mathbb{R} , then any minimizing sequence (u_n) of (56) satisfies, up to a translation, the following properties: ∇u_n , u_n , $F(u_n)$ are relatively compact in L^2 , $L^{2N/(N-2)}$, L^1 respectively.*

We skip the proof since it is very similar to those we already made.

IV. MULTIPLE CONSTRAINTS AND SYSTEMS

We consider first the analogue of the heuristic principle given in Part 1 for problems with multiple constraints. In the following sections we pre-

sent some examples of such problems. With the same notations as in section I of Part 1, we consider:

$$I(\lambda_1, \dots, \lambda_m) = \text{Inf} \{ \mathcal{E}(u)/u \in H, J_i(u) = \lambda_i, 1 \leq i \leq m \}$$

where J_i are of the same form as J in Section I of Part 1, where $m \geq 2$ is given. We again introduce the « problem at infinity »:

$$I^\infty(\lambda_1, \dots, \lambda_m) = \text{Inf} \{ \mathcal{E}^\infty(u)/u \in H, J_i^\infty(u) = \lambda_i, 1 \leq i \leq m \}.$$

Then exactly as in Part 1, we see that we have in general:

$$I(\lambda_1, \dots, \lambda_m) \leq I(\alpha_1, \dots, \alpha_m) + I^\infty(\lambda_1 - \alpha_1, \dots, \lambda_m - \alpha_m) \quad \text{for all } \alpha_i \in [0, \lambda_i].$$

Then the following analogues of (S.1)-(S.2) are equivalent to the compactness of all minimizing sequences:

$$(S.1') \quad I(\lambda_1, \dots, \lambda_m) < I(\alpha_1, \dots, \alpha_m) + I^\infty(\lambda_1 - \alpha_1, \dots, \lambda_m - \alpha_m) \\ \text{for all } \alpha_i \in [0, \lambda_i] \quad \text{such that: } \sum_{i=1}^m \alpha_i < \sum_{i=1}^m \lambda_i$$

$$(S.2') \quad I^\infty(\lambda_1, \dots, \lambda_m) < I^\infty(\alpha_1, \dots, \alpha_m) + I^\infty(\lambda_1 - \alpha_1, \dots, \lambda_m - \alpha_m) \\ \text{for all } \alpha_i \in [0, \lambda_i] \quad \text{such that: } 0 < \sum_{i=1}^m \alpha_i < \sum_{i=1}^m \lambda_i.$$

Many problems of this kind arise when u is vector-valued and some constraints are imposed on each component of u (see section IV.1 below for an example).

IV.1. Examples.

We will first begin with the easy example of Hartree systems: then, see for example E. H. Lieb and B. Simon [23], we consider

$$I = \text{Inf} \left\{ \frac{1}{2} \int_{\mathbb{R}^3} \sum_{i=1}^m |\nabla u_i|^2 - V(x)u_i^2 dx \right. \\ \left. + \frac{1}{2} \sum_{i < j} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{u_i^2(x)u_j^2(y)}{|x - y|} dx dy / u_i \in H^1(\mathbb{R}^3), \quad |u_i|_{L^2}^2 = 1 \right\};$$

where $m \geq 2$, $V(x) = \sum_{i=1}^M \frac{z_i}{|x - x_i|}$ and $z_i > 0, x_i \in \mathbb{R}^3$. We denote by $Z = \sum_i z_i$, and by $I(\lambda_1, \dots, \lambda_m)$ the same problem with the constraints

$|u_i|_{L^2}^2 = \lambda_i$. Clearly $I^\infty(\mu_1, \dots, \mu_m) = 0$, for all $\mu_i \geq 0$ and I is a non-increasing function of each variable λ_i . Therefore (S.1') is equivalent to:

$$I < I(1, \dots, \alpha_i, \dots, 1) \quad \text{for all } \alpha_i \in [0, 1[, \quad \text{for all } 1 \leq i \leq m.$$

And this will be the case if

$$\text{Inf} \left\{ \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 - V(x)u^2 dx + \sum_{k=1}^{m-1} \frac{1}{2} \int_{\mathbb{R}^3} u^2 \left(w_k^2 * \frac{1}{|x|} \right) dx / u \in H^1(\mathbb{R}^3), |u|_{L^2} = 1 \right\} \leq -\alpha < 0$$

for some α independent of w_1, \dots, w_{m-1} satisfying: $w_i \in H^1, |w_i|_{L^2} = 1$.

Choosing radial test functions u , it is easily seen that this condition holds if $Z > (m - 1)$: indeed we have for such u

$$\begin{aligned} & \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 + V(x)u^2 dx + \sum_{k=1}^{m-1} \frac{1}{2} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{u^2(x)w_k^2(y)}{\max(|x|, |y|)} dx dy \\ & \leq \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 - V(x)u^2 + \frac{(m-1)}{|x|} u^2 dx. \end{aligned}$$

We see that we recover the main result of [23] (observe that the above argument is in fact the same as the final argument in the proof of [23]).

We now explain how one adapts the concentration-compactness method to such problems: we consider now a problem slightly more difficult than the preceding one (since in the following example \mathcal{E} is no more weakly lower semicontinuous). We study the « system » analogue of the Choquard-Pekar problem:

$$\begin{aligned} I = \text{Inf} & \left\{ \sum_{i=1}^m \int_{\mathbb{R}^N} \frac{1}{2} |\nabla u_i|^2 + \frac{1}{2} V_i(x)u_i^2 dx \right. \\ & \left. - \sum_{i \neq j} \frac{1}{4} \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{u_i^2(x)u_j^2(y)}{|x-y|} dx dy / u_i \in H^1(\mathbb{R}^N), |u_i|_{L^2}^2 = 1 \text{ for all } 1 \leq i \leq m \right\} \end{aligned}$$

where $m \geq 2$ (the case $m = 1$ was treated in Part 1 [24]) and where $V_i \in L^p(\mathbb{R}^N) + L^q(\mathbb{R}^N)$ with $\frac{N}{2} \leq p, q < \infty$, for all $1 \leq i \leq m$.

Then the heuristic principle briefly sketched in the introduction of this section states that *all minimizing sequences $u^n = (u_1^n, \dots, u_m^n)$ are relatively compact in $H^1(\mathbb{R}^N)$ if and only if:*

$$I < I^\infty(\alpha_1, \dots, \alpha_m) + I(1 - \alpha_1, \dots, 1 - \alpha_m), \quad \forall \alpha_i \in [0, 1], \quad 0 < \sum_{i=1}^m \alpha_i \leq m;$$

where $I(\lambda_1, \dots, \lambda_m)$ denotes the value of the infimum of the same problem but with constraints: $\|u_i\|_{L^2}^2 = d_i \geq 0$.

Indeed we use the concentration-compactness method with $\rho_n = \sum_{i=1}^m (u_i^n)^2$: if $Q_n(t) \xrightarrow{n} Q(t)$ on \mathbb{R}_+ and $l = \lim_{t \rightarrow +\infty} Q(t)$, then $0 \leq l \leq m$. If $l = 0$, we argue as usual; while if $l \in]0, m[$, we decompose ρ_n and thus all the $(u_i^n)_i$ simultaneously in two sequences $v^n = (v_1^n, \dots, v_m^n)$ and $w^n = (w_1^n, \dots, w_m^n)$ such that: $\|u^n - (v^n + w^n)\|_{L^2} \leq \varepsilon$,

$$\mathcal{E}(u^n) \geq \mathcal{E}(v^n) + \mathcal{E}^\infty(w^n) - \delta(\varepsilon) \quad \text{with} \quad \delta(\varepsilon) \xrightarrow{\varepsilon \rightarrow 0} 0$$

$$\text{dist}(\text{Supp } |v^n|, \text{Supp } |w^n|) \xrightarrow{n} +\infty, \quad \left| \sum_{i=1}^m \|v_i^n\|_{L^2}^2 - l \right| \leq \varepsilon.$$

Therefore without loss of generality we may assume that $\|v_i^n\|_{L^2}^2 \xrightarrow{n} \alpha_i$ where: $0 \leq \alpha_i \leq 1$, $\left| \sum_{i=1}^m \alpha_i - l \right| \leq \varepsilon$. It is then easy to rule out this case using the strict subadditivity inequalities. And we conclude easily by arguments very similar to those made before.

The above argument shows clearly that all the results we had with only one constraint may be easily transposed to systems, where we replace (S.1)-(S.2) by (S.1')-(S.2').

IV.2. The case of L^∞ constraints.

It is then clear enough that the idea of the concentration-compactness method rests on the splitting of integral functionals. Therefore it is not clear how the method should be applied on everywhere defined constraints like for example L^∞ constraints. We explain how the method has to be adapted on a problem arising in astrophysics (see Auchmuty [3], P. L. Lions [30]) involving two constraints one of which being a L^∞ type constraint:

$$(57) \quad I_\lambda = \text{Inf} \left\{ \iint_{\mathbb{R}^N \times \mathbb{R}^N} K(x-y)\rho(x)\rho(y)dx dy / 0 \leq \rho \leq 1 \text{ on } \mathbb{R}^N, \int_{\mathbb{R}^N} \rho dx = \lambda \right\}$$

where $\lambda > 0$ and $K \in M^p + M^q$ with $1 \leq p < q < \infty$ (if $p = 1$, we set $M^1 = L^1$).

THEOREM IV.1. — *The condition*

$$(S.2) \quad I_\lambda < I_\alpha + I_{\lambda-\alpha}, \quad \forall \alpha \in]0, \lambda[$$

is necessary and sufficient for the relative compactness in L^1 up to a translation of all minimizing sequences. If (S.2) holds, then all minimizing sequences are relatively compact in $L^p(\mathbb{R}^N)$ ($\forall p < \infty$) up to a translation and there exists a minimum.

REMARK IV.1. — If $K(x) = -\frac{1}{|x|^\mu}$ with $0 < \mu < M$, then by an easy scaling argument we obtain: $I_\lambda = \lambda^{2-\mu/N} I_1$ with $I_1 < 0$ and thus (S.2) holds.

The proof of the above result is exactly the same as those made before (see in particular section II of Part 1), remarking that the constraint $0 \leq \rho \leq 1$ is conserved when we split the minimizing sequence ρ_n into two parts ρ_n^1 , ρ_n^2 as we did in the proof of Lemma I.1 in Part 1 [24].

REMARK IV.2. — If we replace the constraint $0 \leq \rho \leq 1$ by: $0 \leq \bar{\rho} \leq \rho(x)$ where (for example) $\bar{\rho} \in C_b(\mathbb{R}^N)$, $\bar{\rho}(x) \rightarrow \rho^\infty$ as $|x| \rightarrow \infty$; then the same result holds replacing (S.2) by:

$$(S.1) \quad I_\lambda < I_\alpha + I_{\lambda-\alpha}^\infty, \quad \forall \alpha \in [0, \lambda[$$

where I_μ^∞ is the same problem but with the constraint: $0 \leq \rho \leq \rho^\infty$.

V. VARIANTS AND EXTENSIONS OF THE CONCENTRATION-COMPACTNESS PRINCIPLE

V.1. Unbounded domains.

We first want to explain that if we consider the analogues of the preceding problems in unbounded regions different from \mathbb{R}^N then, for general unbounded domains Ω , condition (S.1)—where the problem at infinity is defined as before—is necessary and sufficient for the compactness of all minimizing sequences. More precisely (S.1) is always sufficient while it is necessary if, for example, Ω satisfies:

$$(58) \quad \forall R < \infty, \quad \exists x_n \in \mathbb{R}^N, \quad |x_n| \xrightarrow{n} +\infty, \quad x_n + B_R \subset \Omega.$$

Let us consider one example corresponding to the determination of standing waves for nonlinear Schrödinger equations in an unbounded domain Ω of \mathbb{R}^N :

$$(59) \quad I_\lambda = \text{Inf} \left\{ \int_\Omega \frac{1}{2} |\nabla u|^2 + \frac{1}{2} V(x)u^2 - \frac{1}{p} |u|^p dx / u \in H_0^1(\Omega), \quad |u|_{L^2}^2 = \lambda \right\}$$

where $V \in L^\alpha(\Omega) + L^\beta(\Omega)$ with $\frac{N}{2} \leq \alpha, \beta < \infty$ and $2 < p < 2 + \frac{4}{N}$, $\lambda > 0$.

We introduce the problem at infinity:

$$(60) \quad I_\lambda^\infty = \text{Inf} \left\{ \int_{\mathbb{R}^N} \frac{1}{2} |\nabla u|^2 - \frac{1}{p} |u|^p dx / u \in H^1(\mathbb{R}^N), \quad |u|_{L^2}^2 = \lambda \right\}.$$

THEOREM V. 1. — Under assumption (58), the condition:

$$I_\lambda < I_\lambda^\infty$$

is necessary and sufficient for the relative compactness in $H_0^1(\Omega)$ of all minimizing sequences of (59).

The above result is just a consequence of the fact that (S.1) is in fact equivalent to the above strict inequality since we have easily by homogeneity: $I_{\theta\lambda} < \theta I_\lambda < 0, \forall \theta > 1, \forall \lambda > 0$. Then the fact that (S.1) is equivalent to the compactness of all minimizing sequences follows from the concentration-compactness method considering here (for example):

$$\rho_n = 1_\Omega(x)u_n^2.$$

It is now clear that we can adapt all the results of the preceding sections to the cases when the problems are given now in general unbounded domains Ω instead of \mathbb{R}^N .

We next want to consider problems which are specific to unbounded domains (distinct from \mathbb{R}^N) namely problems associated with (for example) the equation:

$$(61) \quad -\Delta u = 0 \quad \text{in } \Omega, \quad \frac{\partial u}{\partial n} = f(u) \quad \text{on } \partial\Omega, \quad u \neq 0$$

in addition one wants u to « vanish at infinity ». Here $f \in C(\mathbb{R}), f(0) = 0$ and n is the unit outward normal to $\partial\Omega$. To simplify the presentation we will only consider the case when $\Omega = \{x = (x_1, \dots, x_N), x_1 > 0\} - \Omega$ is an half-space, but it will clear from the arguments given below that we could treat as well much more general equations and domains. We will introduce two different minimization problems: the first one is given by

$$(62) \quad I_\lambda = \text{Inf} \left\{ \int_\Omega |\nabla u|^2 dx / u \in L^2(\Omega), u \in L^{2N/(N-2)}(\Omega), F(u) \in L^1(\partial\Omega), \int_{\partial\Omega} F(u) dx' = \lambda \right\}.$$

Recall that, in view of the trace theorems, if $\nabla u \in L^2(\Omega), u \in L^{2N/(N-2)}(\Omega)$ then one may define by density the trace of u on $\partial\Omega$ and $u \in L^q(\partial\Omega)$ where $q = 2(N-1)/(N-2)$. To simplify we will assume $N \geq 3$. We will assume that $f, F = \int_0^t f ds$ satisfy:

$$(63) \quad \lim_{t \rightarrow 0} \{f(t)t\}^+ |t|^{-q} = 0, \quad \lim_{|t| \rightarrow \infty} F(t) |t|^{-q} = 0.$$

The second minimization problem is given by:

$$(64) \quad I_\lambda = \text{Inf} \left\{ \int_\Omega |\nabla u|^2 + c(x)u^2 dx / u \in H^1(\Omega), \int_{\partial\Omega} |u|^p dx' = \lambda \right\}$$

where $2 < p < q$ and

$$(65) \quad c(x) \in C(\overline{\Omega}), \quad c(x) \rightarrow c^\infty > 0 \quad \text{as } |x| \rightarrow \infty, \quad c(x) > 0 \quad \text{on } \overline{\Omega}.$$

We then introduce the problem at infinity:

$$(64') \quad I_\lambda^\infty = \text{Inf} \left\{ \int_\Omega |\nabla u|^2 + c^\infty u^2 dx / u \in H^1(\Omega), \int_{\partial\Omega} |u|^p dx' = \lambda \right\}.$$

THEOREM V.2. — *We assume $N \geq 3$, (63) and $F(\zeta) > 0$ for some $\zeta \in \mathbb{R}$; then all minimizing sequences of (62) satisfy up to a translation of the form $y_n = (0, y_n^2, \dots, y_n^m)$:*

$\forall u_n, u_n, F(u_n)$ are relatively compact in $L^2(\Omega), L^{2N/(N-2)}(\Omega), L^1(\partial\Omega)$ respectively.

In particular there exists a minimum.

REMARK V.1. — Obviously any minimum of (62) satisfies:

$$-\Delta u = 0 \quad \text{in } \Omega, \quad \frac{\partial u}{\partial n} = \theta f(u) \quad \text{on } \partial\Omega,$$

where $\theta > 0$. Then choosing $\tilde{u}(\cdot) = u\left(\frac{\cdot}{\theta}\right)$, we obtain a solution of (61) —and it is easily proved that the minima of (62) lead by this argument to ground-state solutions of (61). ■

THEOREM V.3. — *We assume $2 < p < q$ ($q = +\infty$ if $N \leq 2$) and (65). Then all minimizing sequences of (64') are relatively compact in $H^1(\Omega)$ up to a translation of the form $y_n = (0, y_n^2, \dots, y_n^m)$.*

Then condition: $I_\lambda < I_\lambda^\infty$, is necessary and sufficient for the relative compactness of all minimizing sequences in $H^1(\Omega)$.

First of all, let us explain these results by remarking that, in the case of Theorem V.2, we have: $I = \lambda^{\frac{N-2}{N-1}} I_1 > 0$ (scaling) and thus (S.2) holds. While in Theorem V.3, the result follows from the fact that (S.1) reduces to $I_\lambda < I_\lambda^\infty$ since we have:

$$I_\lambda = \lambda^{2/p} I_1 > 0.$$

These results are still proved by the concentration-compactness method that is applied to sequences of bounded nonnegative measures instead

of L^1 functions: indeed in the first case, we introduce for any minimizing sequence (u_n) , the measure μ_n defined by:

$$\forall \varphi \in C_b(\mathbb{R}^N), \quad \int \varphi d\mu_n = \int_{\partial\Omega} \varphi F_2(u_n) dx'$$

where F_2 is built in a similar way as in the proof of Theorem II.3.

In the second case, μ_n is defined in a similar way replacing $F_2(u_n)$ by $|u_n|^p$. Then the concentration-compactness method is applied to the measures:

$$P_n = \{ |\nabla u_n|^2 + |u_n|^{2N/(N-2)} \} 1_\Omega + \mu_n$$

or

$$P_n = \{ |\nabla u_n|^2 + u_n^2 \} 1_\Omega + \mu_n.$$

V.2. Partial concentration-compactness.

In this section, we want to explain on a few examples—motivated by Mathematical Physics—that the concentration-compactness principle and method can be adapted to problems where the unboundedness (or translation invariance) of the domain « takes place only in some directions ». For example, let us consider the following problem motivated by fluid mechanics (see Amick and Toland [2], J. Bona, D. K. Bose and R. E. L. Turner [13], M. J. Esteban [20]):

$$(66) \quad I_\lambda = \text{Inf} \left\{ \int_\Omega a_{ij}(x) \partial_i u \partial_j u + c(x) u^2 dx / u \in H_0^1(\Omega), \int_\Omega K(x) |u|^p dx = \lambda \right\}$$

where $\Omega = \mathcal{O} \times \mathbb{R}^m$, and \mathcal{O} is bounded in \mathbb{R}^n , $m \geq 1$; where $2 < p < \frac{2N}{N-2}$ if $N \geq 3$, $2 < p < \infty$ if $N=2$ and $N=n+m$; and where

$$(67) \quad \begin{cases} \varphi(y, z) \rightarrow \varphi^\infty(y) \text{ as } |z| \rightarrow \infty, \text{ uniformly in } y \in \bar{\mathcal{O}}, \text{ for } \varphi = a_{ij}, c, K; \\ a_{ij} = a_{ji}, c, K \in C_b(\bar{\Omega}); \end{cases}$$

$$(68) \quad \exists v > 0, \quad \forall u \in H_0^1(\Omega), \quad \int_\Omega a_{ij}(x) \partial_i u \partial_j u + c(x) u^2 dx \geq v |\nabla u|_{L^2}^2.$$

THEOREM V.4. — *We assume (67), (68). All minimizing sequences of (66) are relatively compact in $H_0^1(\Omega)$ if and only if $I_\lambda < I_\lambda^\infty$, where*

$$(66') \quad I_\lambda^\infty = \text{Inf} \left\{ \int_\Omega a_{ij}^\infty(y) \partial_i u \partial_j u + c^\infty(y) u^2 dx / u \in H_0^1(\Omega), \int_\Omega K^\infty(y) |u|^p dx = \lambda \right\}.$$

In the case when $a_{ij} \equiv a_{ij}^\infty$, $c \equiv c^\infty$, $K \equiv K^\infty$, all minimizing sequences of (66) (= (66')) are relatively compact in $H_0^1(\Omega)$ up to a translation of the form $x_n = (0, z_n)$ with $z_n \in \mathbb{R}^m$.

REMARK V.2. — It will quite clear that we could treat as well much more general problems, that we can replace the Dirichlet boundary conditions by Neuman nconditions (or even nonlinear ones as in section V.1). Let us in particular consider the following variant (adapted from section III.2):

$$(69) \quad I = \text{Inf} \left\{ \int_{\Omega} \frac{1}{2} a_{ij}(x) \partial_i u \partial_j u + \frac{1}{2} c(x) u^2 - F(x, u) dx / u \in H_0^1(\Omega), \quad u \neq 0, \right. \\ \left. \int_{\Omega} a_{ij}(x) \partial_i u \partial_j u + c(x) u^2 - f(x, u) u dx = 0 \right\}$$

where $F(x, t) = \int_0^t f(x, s) ds, f \in C(\bar{\Omega} \times \mathbb{R})$ satisfies:

$$(50') \quad \left\{ \begin{array}{l} f \in C^{0,1}(\bar{\Omega} \times \mathbb{R}), \quad \frac{\partial f}{\partial t} \in \text{BUC}(\bar{\Omega} \times [-R, +R]) \quad \text{for all } R < \infty \\ \exists \theta \in]0, 1[, \quad 0 \leq \frac{f(x, t)}{t} \leq \theta \frac{\partial f}{\partial t}(x, t) \quad \text{on } \bar{\Omega} \times \mathbb{R}; \end{array} \right.$$

$$(51') \quad \lim_{|t| \rightarrow \infty} f(x, t) |t|^{-\frac{N+2}{N-2}} = 0, \quad \lim_{|t| \rightarrow \infty} |f(x, t)| > 0, \quad \text{uniformly in } x \in \bar{\Omega},$$

(if $N = 2, \frac{N+2}{N-2}$ may be replaced by any finite power);

$$(52') \quad f(y, z, t) \rightarrow f^\infty(y, t) \quad \text{as } |z| \rightarrow \infty, \quad \text{uniformly for } y \in \bar{\mathcal{O}}, \\ t \text{ bounded}$$

$$\exists \delta > 0, \quad |f(y, z, t) - f^\infty(y, t)| \leq \varepsilon(R) |t|^2 \quad \text{for } |z| \geq R, \quad |t| \leq \delta$$

with $\varepsilon(R) \rightarrow 0$ as $R \rightarrow \infty$.

We denote by I^∞ the Infimum corresponding to $a_{ij}^\infty, c^\infty, f^\infty$.

THEOREM V.5. — We assume (67), (68), (50'), (51'), (52'). Then any minimizing sequence of (69) is relatively compact in $H_0^1(\Omega)$ if and only if $I < I^\infty$. If this condition holds, there exists a minimum u of (69) and this minimum satisfies:

$$-\frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial u}{\partial x_j} \right) + c(x) u = f(x, u) \quad \text{in } \Omega, \quad u \in H_0^1(\Omega), \quad u \neq 0.$$

If $a_{ij} \equiv a_{ij}^\infty, c \equiv c^\infty, f \equiv f^\infty$, any minimizing sequence of (69) is relatively compact in $H_0^1(\Omega)$ up to a translation of the form $x_n = (0, z_n)$ with $x_n \in \mathbb{R}^m$; and the above conclusion holds.

REMARK V.3. — The condition $I < I^\infty$ holds if for example: $(a_{ij}) \leq (a_{ij}^\infty), c \leq c^\infty$ and $F \geq F^\infty$, with some strict inequality at some point of $\bar{\Omega}$.

The proof of these results is easily adapted from the concentration-compactness method and the arguments of section I of Part I [24]: indeed consider, for example, $\rho_n = |\nabla u_n|^2 + u_n^2$ and introduce the (partial) concentration function Q_n on \mathbb{R}_+ :

$$Q_n(t) = \text{Sup}_{z \in \mathbb{R}^m} \int_{\emptyset x(z + B_t)} |\nabla u_n|^2 + u_n^2 dx.$$

Then it is easy to adapt the proof of Lemma I.1 of Part 1 [24] and we may mimick the arguments we did before.

By the same type of arguments, one may treat the problem of *global vortex rings* (in two and three dimensions): we may then recover the general existence results obtained by H. Berestycki and P. L. Lions [11], extending those of Fraenkel and Berger [21]. In addition we obtain the compactness of all minimizing sequences up to a translation of the form $x_n = (0, z_n)$ with $x_n \in \mathbb{R}$.

We can perform a similar treatment of the *rotating stars* problem—thus recovering results of P. L. Lions [30]—i. e.:

$$I_\lambda = \text{Inf} \left\{ \int_{\mathbb{R}^3} j(\rho) + V(r)\rho dx - \frac{1}{2} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{\rho(x)\rho(y)}{|x-y|} dx dy / 0 \leq \rho, \rho \in L^1(\mathbb{R}^3), \right. \\ \left. \rho(x, y, z) = \rho((x^2 + y^2)^{1/2}, z), \int_{\mathbb{R}^3} \rho dx = \lambda \right\}$$

where $V \geq 0$, $r = (x^2 + y^2)^{1/2}$, $\lambda > 0$, $j \in C_+(\mathbb{R}_+)$ satisfies:

$$j(0) = j'(0) = 0, j \text{ is strictly convex, } \lim_{t \rightarrow +\infty} j(t)t^{-4/3} = +\infty$$

Then we can prove that any minimizing sequence of the above problem is relatively compact in $L^1(\mathbb{R}^3)$ up to a translation in z if and only if $I_\lambda < 0$. We just need to apply the above arguments with:

$$Q_n(t) = \text{Sup}_{z \in \mathbb{R}} \int_{(x^2 + y^2 + (\eta - z)^2 \leq t^2)} \rho_n(x, y, \eta) dx dy d\eta.$$

V.3. Applications to best constants.

The concentration-compactness method may be applied to show that best constants in various functional inequalities are attained: we will give here a few examples (the list of applications is by no means complete); the main restriction being that we need some form of local compactness which excludes the interesting limit cases (for the treatment of such cases, see P. L. Lions [31] [32] [33]).

Example 1. — If $D^m u \in L^p$, $u \in L^q$ with $m \geq 1$, $1 \leq p < \infty$, $q \neq \frac{Np}{N-mp}$ if $p < \frac{N}{m}$, $1 \leq q < \infty$, and if we denote by I the open interval with end-points q and $\frac{Np}{N-mp}$ (if $p < \frac{N}{m}$, $+\infty$ if $\frac{N}{m} \leq p$), then:

$$|u|_{L^r(\mathbb{R}^N)} \leq C |u|_{L^q(\mathbb{R}^N)}^\theta |D^m u|_{L^p(\mathbb{R}^N)}^{1-\theta}$$

where
$$\theta = \frac{q}{\alpha} \frac{Np + mp\alpha - N\alpha}{Np + mpq - Nq}, \quad \alpha \in I.$$

Example 2. — If $Du \in L^p$, $u \in L^q$ with $1 \leq p < \infty$, $q \neq \frac{Np}{N-p}$ if $p < N$, $1 \leq q < \infty$ and if α, β, μ are such that: $0 < \mu < N$, $0 < \alpha < \infty$, $0 < \beta < \infty$ and

$$\frac{\alpha}{r} + \frac{\mu}{N} = 1 + \frac{1}{\beta}, \quad \text{for some } r \in I;$$

then

$$\left| |u|^\alpha * \frac{1}{|x|^\mu} \right|_{L^\beta(\mathbb{R}^N)}^{1/\alpha} \leq C |Du|_{L^p(\mathbb{R}^N)}^\theta |u|_{L^q(\mathbb{R}^N)}^{1-\theta}$$

where θ satisfies:
$$\frac{N-\mu\beta}{\alpha\beta} = \theta \frac{N-p}{p} + (1-\theta) \frac{N}{q}.$$

Example 3. — If $Du \in L^p$, $u \in L^q$ with $1 \leq p < \infty$, $q \neq \frac{Np}{N-p}$ if $p < N$, $1 \leq q < \infty$ and if α, μ are such that: $0 < \mu < N$, $0 < \alpha < \infty$, $\frac{2\alpha}{r} + \frac{\mu}{N} = 2$, for some $r \in I$; then

$$\left\{ \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u|^\alpha(x) |u|^\alpha(y)}{|x-y|^\mu} dx dy \right\}^{1/2\alpha} \leq C |Du|_{L^p(\mathbb{R}^N)}^\theta |u|_{L^q(\mathbb{R}^N)}^{1-\theta}$$

where θ satisfies:
$$\frac{2N-\mu}{2\alpha} = \theta \frac{N-p}{p} + (1-\theta) \frac{N}{q}.$$

Example 4. — If $u \in W^{1,p}(\Omega)$ with $\Omega = \{x = (x_1, \dots, x_N) / x_1 > 0\}$, $1 \leq p < \infty$, and if we denote by $I = \left] p, \frac{(N-1)p}{(N-p)} \right[$ ($+\infty$ if $N \leq p$); then:

$$|u|_{L^r(\Omega)} \leq C |\nabla u|_{L^p(\Omega)}^\theta |u|_{L^p(\Omega)}^{1-\theta}$$

for $\alpha \in I$, where $\theta = \frac{N}{p} - \frac{N-1}{\alpha}$.

Example 5. — If $u \in L^p \cap L^q$ with $1 \leq p < q \leq +\infty$ and if α, β, μ satisfy:

$0 < \mu < N$, $0 < \alpha < \infty$, $0 < \beta < \infty$, $\frac{\alpha}{r} + \frac{\mu}{N} = 1 + \frac{1}{\beta}$ with $r \in]p, q[$; then:

$$\left| |u|^\alpha * \frac{1}{|x|^\mu} \right|_{L^\beta}^{1/\alpha} \leq C |u|_{L^p}^\theta |u|_{L^q}^{1-\theta}$$

where θ satisfies: $\frac{N-\mu\beta}{\alpha\beta} = \theta \frac{N}{p} + (1-\theta) \frac{N}{q}$.

Example 6. — If $u \in L^p \cap L^q$ with $1 \leq p < q \leq +\infty$ and if α, μ are such that: $0 < \alpha < \infty$, $0 < \mu < N$, $\frac{2\alpha}{r} + \frac{\mu}{N} = 2$ with $r \in]p, q[$; then:

$$\left\{ \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u|^\alpha(x) |u|^\alpha(y)}{|x-y|^\mu} dx dy \right\}^{1/2} \leq C |u|_{L^p}^\theta |u|_{L^q}^{1-\theta}$$

where $\frac{2N-\mu}{2\alpha} = \theta \frac{N}{p} + (1-\theta) \frac{N}{q}$.

In all these examples, we can prove not only that the best possible constant C is attained but that all minimizing sequences are relatively compact up to a translation and a change of scale. Indeed it is obvious that all these inequalities are scale invariant and thus we may without loss of generality look at the associated minimization problem with two constraints like in Example 1:

$$I(\lambda, \mu) = \text{Inf} \left\{ - \int_{\mathbb{R}^N} |u|^\alpha dx / |D^m u| \in L^p, u \in L^q, |D^m u|_{L^p}^p = \lambda, |u|_{L^q}^q = \mu \right\};$$

for some λ, μ . Now using the homogeneity and again scaling arguments we find:

$$I(\lambda, \mu) = \lambda^\gamma \mu^\delta I(1, 1) < 0$$

with $\gamma = (Nq - N\alpha)(Nq - mpq - Np)^{-1}$, $\delta = (N\alpha - mp\alpha - Np)(Nq - mpq - Np)^{-1}$. Remarking that $\gamma + \delta = (Nq - mp\alpha - Np)(Nq - mpq - Np)^{-1} > 1$, we deduce: $I(\lambda, \mu) < I(\alpha, \beta) + I(\lambda - \alpha, \mu - \beta)$, $\forall \alpha \in]0, \lambda[, \forall \beta \in]0, \mu[$. And in view of the arguments of section IV.1, we may conclude and our claim is proved.

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