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On the global Cauchy problem for some non linear Schrödinger equations

by

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ABSTRACT. — We study the Cauchy problem for a class of non linear Schrödinger equations in space time dimension $n + 1$. We look for solutions which are continuous functions of time with values in the Sobolev space $H^k(\mathbb{R}^n)$, $k > n/2$. Under suitable assumptions on the interactions, we prove in particular the existence of global solutions for $n \leq 7$. The global existence proof breaks down for $n \geq 8$.

RÉSUMÉ. — On étudie le problème de Cauchy pour une classe d'équations de Schrödinger non linéaires en dimension $n + 1$ d'espace temps. On cherche des solutions fonctions continues du temps à valeurs dans l'espace de Sobolev $H^k(\mathbb{R}^n)$, $k > n/2$. Sous des hypothèses convenables sur les interactions, on prouve en particulier l'existence de solutions globales pour $n \leq 7$. La démonstration d'existence globale ne s'applique pas pour $n \geq 8$.

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INTRODUCTION

Non linear evolution equations are attracting an increasing interest, both from the mathematical and from the physical point of view. Among those equations, the non linear Schrödinger (NLS) equation

$$i \frac{d\varphi}{dt} = - \frac{1}{2} \Delta\varphi + f_0(\varphi) , \tag{0.1}$$

where φ is a complex function defined in space time $\mathbb{R}^{n+1}$, Δ is the Laplace operator in $\mathbb{R}^n$, and f_0 a local non linear interaction, has been considered in the last few years by several authors ([*2*], [*5*]-[*11*], [*14*], [*16*]-[*18*]) as regards both the Cauchy problem and the theory of scattering. Here we are concerned with the Cauchy problem only. For suitably regular interactions, the local Cauchy problem can be conveniently treated by Segal's non linear semigroup theory [*15*] or a slight generalization thereof [*7*] [*10*]. For that purpose, one recasts the Cauchy problem for the equation (0.1) with initial data $\varphi(t_0, .) = \varphi_0(.)$ at initial time t_0 in the form of the integral equation

$$\varphi(t) = U(t-t_0)\varphi_0 - i \int_{t_0}^{t} d\tau U(t - \tau) f_0(\varphi(\tau)) , \tag{0.2}$$

where U is the one parameter group generated by the free equation,

$$U(t) = \exp\left(i \frac{t}{2} \Delta \right). \tag{0.3}$$

One then looks for solutions of (0.2) as continuous functions of time from an interval I containing t_0 to some Banach space X. Under suitable assumptions on f_0 and for suitable choices of X, one can prove the existence of such solutions for sufficiently small I by applying the contraction mapping theorem. The global Cauchy problem, namely the problem of extending the previous solutions to all times, can be handled for suitable interactions f_0 by the standard method of *a priori* estimates. For that purpose, one exploits the conservation laws of the L^2-norm and of the energy for the equation (0.1) to derive *a priori* estimates of the solutions in the Sobolev space H^1 (see definition (1.2) below). The second step fits more or less smoothly to the first one, depending on the choice of X and its relation with H^1. The optimal choice would be of course $X = H^1$, but that choice provides a satisfactory local theory only for $n = 1$. Failing to make that choice for $n \geq 2$, one is naturally led to follow either of two courses: one can choose for X a larger space $X \supset H^1$. The local Cauchy problem then becomes more complicated, as well as the proof of the conservation laws. However the global problem becomes simple, since *a priori* control of the solutions in H^1 immediately implies control in X, thereby leading to a global existence result with initial data in H^1, for arbitrary space dimension [7] [10]. Alternatively, one can choose for X a smaller space $X \subset H^1$, in the present case $X = H^k$ with $k > n/2$ (see definition (1.2) below). The local problem then becomes simple, as well as the proof of the conservation laws and of the smoothness properties of the solutions. However, the global problem now becomes harder, since *a priori* control in H^1 no longer implies *a priori* control in X, and the latter has to be obtained recursively from the integral equation (0.2). This last step has been performed so far for $n \leq 3$ only [2]. The present note is devoted to a study of that question in higher dimensions. The main result is that under suitable and natural assumptions on f_0, global existence of solutions of (0.1) or (0.2) holds for $n \leq 7$. However the present proof, based on standard Sobolev estimates, breaks down in dimension $n \geq 8$. When contrasted with the global existence results in H^1 for arbitrary n mentioned above, that fact suggests that smoothness properties may fail to hold in high dimensions for the global solutions thereby obtained.

The corresponding problem for the non linear Klein-Gordon equation has been treated by Pecher, Brenner and von Wahl [3] [4] [12] [13] with similar results: global existence in H^k, $k \geq n/2$, can be proved under suitable and reasonable assumptions on the interactions for $n \leqslant 9$. The proof however, is technically more complicated and requires the use of Besov spaces in dimensions $n > 6$.

This paper is organized as follows. In section 1, we recall briefly the theory of the local Cauchy problem, including smoothness properties and the relevant conservation laws, for the NLS equation in H^k, $k > n/2$. The proofs are standard and will be omitted. A detailed exposition will be found in [10]. In section 2, we derive the basic estimates needed for the global Cauchy problem, and state the main result in precise form.

1) THE LOCAL CAUCHY PROBLEM

In this section, we state without proof the basic results on the local Cauchy problem for the NLS equation (0.1) or (0.2). For any interval $I \subset R$ and any Banach space X, we denote by $\mathscr{C}(I, X)$ the space of continuous functions from I to X, and for any positive integer l, by $\mathscr{C}^l(I, X)$ the space of l times continuously differentiable functions from I to X. If I is compact, $\mathscr{C}(I, X)$ is a Banach space when equipped with the norm

$$| \varphi |_I = \operatorname*{Sup}_{t \in I} \| \varphi(t) \|_X . \tag{1.1}$$

We shall use the spaces $L^q \equiv L^q(\mathbb{R}^n)$, $1 \leq q \leq \infty$, with corresponding norms denoted by $\|\cdot\|_q$, and the Sobolev spaces $H^k \equiv H^k(\mathbb{R}^n)$, with k a non negative integer, defined by

$$H^k \equiv \left\{ \varphi : \| \varphi \|_{k,2}^2 \equiv \sum_{\alpha:|\alpha| \leqslant k} \| D^\alpha \varphi \|_2^2 < \infty \right\}, \tag{1.2}$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multiindex of space derivatives, and

$$|\alpha| = \alpha_1 + \ldots + \alpha_n .$$

It is well known that the free group $U(t)$ defined by (0.3) is unitary in H^k for all $k \geq 0$.

We now state the main results. In everything that follows, f_0 is a $\mathscr{C}^l$ function from $\mathbb{C}$ to $\mathbb{C}$, for values of l that will be specified as needed. The first result concerns local existence and uniqueness.

PROPOSITION 1.1. — Let k be an integer, $k > n/2$, let $f_0 \in \mathscr{C}^{k+1}$ with $f_0(0) = 0$, and let $\varphi_0 \in H^k$. Then

(1) There exists $T > 0$, depending on φ_0 through $\| \varphi_0 \|_{k,2}$ only, such that for any $t_0 \in \mathbb{R}$, the equation (0.2) has a unique solution φ in $\mathscr{C}([t_0 - T, t_0 + T], H^k)$.

(2) For any interval I of R and any $t_0 \in I$, the equation (0.2) has at most one solution in $\mathscr{C}(I, H^k)$.

The next result concerns smoothness of the solutions.

PROPOSITION 1.2. — Let k and k' be integers, $k' \geq k > n/2$, let $f_0 \in \mathscr{C}^{k'}$ with $f_0(0) = 0$, and let $\varphi_0 \in H^{k'}$. Let I be an interval of $\mathbb{R}$, let $t_0 \in I$ and let $\varphi \in \mathscr{C}(I, H^k)$ be a solution of the equation (0.2). Then $\varphi \in \mathscr{C}^l(I, H^{k'-2l})$ for any integer l, $0 \leq l \leq k'/2$. If $k' \geq 2$, the differential equation (0.1) holds in $H^{k'-2}$.

We now turn to the conservation laws of the L^2-norm and of the energy. For that purpose, we assume in addition that the interaction f_0 satisfies the following assumption

ASSUMPTION 1.1. — For all $z \in \mathbb{C}$ and $\omega \in \mathbb{C}$ with $|\omega| = 1$, $f_0(\omega z) = \omega f_0(z)$. Furthermore $f_0(\rho)$ is real for all $\rho \in \mathbb{R}^+$.

It follows from the Assumption 1.1 that f_0 can be written as

$$f_0(z) = \frac{\partial V(z)}{\partial \bar{z}} \tag{1.3}$$

where V is the function from $\mathbb{C}$ to $\mathbb{R}$ defined by

$$V(z) = V(|z|) = 2 \int_0^{|z|} f_0(\rho) d\rho \qquad \text{for all} \quad z \in \mathbb{C} . \tag{1.4}$$

The energy is then defined for sufficiently regular φ as

$$E(\varphi) = \frac{1}{2} \| \nabla\varphi \|_2^2 + \int dx V(\varphi(x)) . \tag{1.5}$$

The relevant conservation laws for the solutions of (0.1) can then be stated as follows.

PROPOSITION 1.3. — Let k be an integer, $k > n/2$, let $f_0 \in \mathscr{C}^{k+1}$ with $f_0(0) = 0$, and let f_0 satisfy the assumption 1.1. Let I be an interval of $\mathbb{R}$, let $t_0 \in I$ and $\varphi_0 \in H^k$, Let φ be solution of the equation (0.2) in $\mathscr{C}(I, H^k)$. Then for all s and t in I, φ satisfies the equalities

$$\| \varphi(s) \|_2 = \| \varphi(t) \|_2 .$$
$$E(\varphi(s)) = E(\varphi(t)) .$$

We conclude this section with an estimate to the effect that the norm in H^1 is controlled by the L^2-norm and the energy, so that the conservation laws of Proposition 1.3 imply that the solutions of (0.2) are uniformly bounded in H^1. For that purpose, we need an additional assumption of lower boundedness of V, so that the kinetic and potential parts of the energy (1.5) cannot become separately infinite with opposite signs. We state that assumption in the form

$$V(\rho) \geqslant - C_3(\rho^2 + \rho^{p_3+1}) \tag{1.6}$$

with

$$p_3 < 1 + 4/n . \tag{1.7}$$

PROPOSITION 1.4. — Let $V(z)=V(|z|)$ satisfy (1.6), (1.7), let $\varphi \in H^1$ be such that $V(\varphi) \in L^1$ and define $E(\varphi)$ by (1.5).
Then

$$\| \nabla\varphi \|_2^2 \leqslant 2(1-\sigma)^{-1} \{ E(\varphi) + C_3 \| \varphi \|_2^2 \} + C_4 \| \varphi \|_2^{2+4\sigma/(n(1-\sigma))} \tag{1.8}$$

where σ is defined by $p_3 = 1 + 4\sigma/n$ (so that $0 \leq \sigma < 1$) and C_4 is a non negative constant (independent of φ).

The proof is elementary and can be found in [7, Lemma 3.2]. In Proposition 1.4 as well as in the subsequent global existence results that depend on it, one can also assume that $p_3 = 1 + 4/n$ provided $\| \varphi \|_2$ is sufficiently small, depending on V. One then obtains an estimate similar to, but slightly different from (1.8). This minor extension will not be mentioned further.

2) THE GLOBAL CAUCHY PROBLEM

In this section, we study the global Cauchy problem for the equation (0.1) in the space H^k, $k > n/2$, and we prove the existence of global solutions for

$n \leq 7$. In view of Propositions 1.1, 1.3 and 1.4, global existence will follow by standard arguments from the fact that for solutions in H^k, *a priori* control in H^1 implies *a priori* control in H^k, and this section is mainly devoted to a proof of that property. For completeness, we first state without proof the (elementary) global existence result for $n = 1$.

PROPOSITION 2.1. — Let $n = 1$ and let $f_0 \in \mathscr{C}^2$ with $f_0(0) = 0$. Let f_0 and V satisfy the assumption 1.1 and (1.6), (1.7). Let $t_0 \in \mathbb{R}$ and $\varphi_0 \in H^1$. Then the equation (0.2) has a unique solution in $\mathscr{C}(\mathbb{R}, H^1)$. That solution is uniformly bounded in H^1.

We now turn to the case of dimension $n \geq 2$. In addition to the spaces H^k defined by (1.2), it will be convenient to use the more general Sobolev spaces [*1*]

$$W^{k,q} = \left\{ \varphi : \| \varphi \|_{k,q}^q \equiv \sum_{\alpha:|\alpha| \leqslant k} \| D^\alpha \varphi \|_q^q < \infty \right\}, \tag{2.1}$$

with k a non negative integer, and $1 \leq q \leq \infty$. In particular, $H^k = W^{k,2}$. We shall use extensively the Sobolev inequalities in the general form

$$\| \varphi \|_p \leqslant C \| \varphi \|_r^{1-\sigma} \left\{ \sum_j \| D_j \varphi \|_q \right\}^\sigma \tag{2.2}$$

where $1 \leq p, q, r \leq \infty$, $r \leq p$, $0 \leq \sigma \leq 1$, $\sigma < 1$ if $p = \infty$, and

$$\frac{1}{p} = \frac{1-\sigma}{r} + \sigma\left(\frac{1}{q} - \frac{1}{n}\right). \tag{2.3}$$

These inequalities imply various embedding theorems between the Sobolev spaces. In particular $W^{k,q} \subset L^\infty$ if $kq > n$. The estimates of this section will make an essential use of the following well known smoothing property of the free group $U(t)$ (see for instance Lemma 1.2 in [*7*]): for any $t \neq 0$ and any pair of dual indices q and $\bar{q}$, $1 \leq \bar{q} \leq 2 \leq q \leq \infty$, $\frac{1}{q} + \frac{1}{\bar{q}} = 1$, $U(t)$ is bounded from $L^{\bar{q}}$ to L^q (more generally from $W^{k,\bar{q}}$ to $W^{k,q}$) with bound

$$\| U(t)\varphi \|_{k,q} \leqslant (2\pi |t|)^{-\delta(q)} \| \varphi \|_{k,\bar{q}} \tag{2.4}$$

for all $\varphi \in W^{k,\bar{q}}$, with

$$\delta(q) = n/2 - n/q. \tag{2.5}$$

We shall use that estimate in the range

$$1/2 - 1/n < 1/q \leqslant 1/2 \tag{2.6}$$

for which $0 \leq \delta(q) < 1$, so that the estimating factor in (2.4) is integrable near $t = 0$, and for which in addition $H^1 \subset L^q$.

In all this section, we shall denote by $f_0', f_0'', \dots, f_0^{(l)}$ the sets of first, second, ..., l-th order derivatives of f_0 with respect to z and $\bar{z}$, by $|f_0^{(l)}|$ the maximum of the moduli of those derivatives, and by $\nabla\varphi, \dots, \nabla^k\varphi$ the sets of first, ..., k-th order space derivatives of φ.

We now begin the proof of the *a priori* estimate in H^k of solutions of (0.1) or (0.2) that are *a priori* bounded in H^1. The first step is valid for all $n \geq 2$ and consists in estimating φ in $W^{1,q}$ or equivalently $\nabla\varphi$ in L^q for some (large) q in the range (2.6).

LEMMA 2.1. — Let $n \geq 2$ and $k > n/2$. Let $f_0 \in \mathscr{C}^k$ satisfy $f_0(0) = f_0'(0) = 0$ and

$$|f_0'(z)| \leqslant C|z|^{p_2-1} \qquad \text{for} \quad |z| \geqslant 1, \tag{2.8}$$

with

$$0 \leqslant p_2 - 1 < 4/(n-2). \tag{2.9}$$

Let $t_0 \in \mathbb{R}$ and $\varphi_0 \in H^k$. Let I be an interval of $\mathbb{R}$ containing t_0, let $\varphi \in \mathscr{C}(I, H^k)$ be a solution of the equation (0.2) and assume that φ is uniformly bounded in H^1, namely

$$\operatorname*{Sup}_{t\in I} \|\varphi(t)\|_{1,2} \leqslant M_{1,2} < \infty. \tag{2.10}$$

Let q in the range (2.6) satisfy

$$p_2 - 1 \leqslant 4 \quad \delta(q)/(n-2), \qquad (q > 2 \quad \text{if} \quad n = 2). \tag{2.11}$$

Then φ is estimated *a priori* in $W^{1,q}$. More precisely

$$\operatorname*{Sup}_{t\in I} \|\varphi(t)\|_{1,q} \leqslant M_{1,q} < \infty, \tag{2.12}$$

where $M_{1,q}$ depends only on I (through the length $|I|$ of I), on φ_0 (through $\|\varphi_0\|_{2,2}$) and on $M_{1,2}$.

Proof. — Since $H^1 \subset L^q$, it is sufficient to estimate $\nabla\varphi$ in L^q. By the Sobolev inequalities and (2.4), we obtain from the equation (0.2)

$$\|\nabla\varphi(t)\|_q \leqslant C\|\varphi_0\|_{2,2} + C\int_{t_0}^{t} d\tau\, |t-\tau|^{-\delta(q)} \|f_0'(\varphi(\tau))\nabla\varphi(\tau)\|_{\bar{q}}. \tag{2.13}$$

Now the assumptions on f_0 imply that one can decompose f_0' as

$$f_0'(z) = g_1(z) + g_2(z) \tag{2.14}$$

where $g_1, g_2 \in \mathscr{C}^{k-1}$ and

$$|g_1(z)| \leqslant C \operatorname{Min}(1, |z|), \tag{2.15}$$

$$\begin{cases} g_2(z) = 0 & \text{for} \quad z \leq 1, \\ |g_2(z)| \leqslant C|z|^{p_2-1} & \text{for} \quad z \geq 1. \end{cases} \tag{2.16}$$

We can therefore estimate the $L^{\bar{q}}$ norm in the right hand side of (2.13) by

$$\|\cdot\|_{\bar{q}} \leqslant C\{\|g_1(\varphi)\|_l \|\nabla\varphi\|_2 + \|g_2(\varphi)\|_{l/2} \|\nabla\varphi\|_q\} \tag{2.17}$$

with $1/l = 1/2 - 1/q$. Now $g_1(\varphi) \in L^2 \cap L^\infty$ with $\| g_1(\varphi) \|_2 \leq C \| \varphi \|_2$ and $\| g_1(\varphi) \|_\infty \leq C$, so that $\| g_1(\varphi) \|_l$ is estimated in terms of $\| \varphi \|_2$. On the other hand, by the Sobolev inequalities, $\| g_2(\varphi) \|_{l/2}$ is estimated by $\| \varphi \|_{1,2}$ provided $(p_2 - 1)l \leqslant 4n/(n-2)$, $l < \infty$ if $n = 2$, which is equivalent to (2.11).

The result now follows from (2.13), (2.17) and the preceding remarks by Gronwall's inequality. Q. E. D.

REMARK 2.1. — In Lemma 2.1, the assumptions that $\varphi \in \mathscr{C}(I, H^k)$ and $f_0 \in \mathscr{C}^k$ are unnecessarily strong. It would be sufficient that $\varphi \in \mathscr{C}(I, X')$ for some $X' \subset H^1 \cap W^{1,q}$ such that the equation (0.2) makes sense in $\mathscr{C}(I, X')$, and that f_0 satisfy the minimal order of differentiability needed for that purpose. In particular, one could take $X' = H^1 \cap W^{1,q}$, $\varphi_0 \in H^2$ and $f_0 \in \mathscr{C}^1$. The estimates in the proof of Lemma 2.1, supplemented with similar but simpler ones for the norm in L^q and with elementary continuity arguments, actually show that the right hand side of (0·2) is well defined from $\mathscr{C}(I, H^1 \cap W^{1,q})$ to $\mathscr{C}(I, W^{1,q})$. For simplicity, we have chosen to keep $X' = H^k$, since we know already from the local theory that the equation (0.2) makes sense in H^k. In the same spirit, if one assumes only $f_0 \in \mathscr{C}^1$, then the assumption $f_0'(0) = 0$ has to be replaced by

$$| f_0'(z) | \leqslant C | z |^{p_1 - 1} \qquad \text{for} \qquad | z | \leqslant 1 \tag{2.18}$$

with

$$p_1 \geqslant 1 + \frac{2}{n} \delta(q). \tag{2.19}$$

Of course for $n \geq 2$, q in the range (2.6) and $q > 2$, the conditions (2.18), (2.19) are equivalent to $f_0'(0) = 0$ as soon as $f_0 \in \mathscr{C}^2$.

In space dimensions 2 and 3, Lemma 2.1 implies the existence of global solutions in H^2, as we now show (see also [*2*]).

PROPOSITION 2.2. — Let $n = 2$ or 3, and let $f_0 \in \mathscr{C}^3$ with $f_0(0) = f_0'(0) = 0$, satisfying (2.8) and (2.9). Let f_0 and V satisfy the assumption 1.1 and (1.6), (1.7). Let $t_0 \in \mathbb{R}$ and $\varphi_0 \in H^2$. Then the equation (0.2) has a unique solution in $\mathscr{C}(\mathbb{R}, H^2)$. That solution is uniformly bounded in H^1.

Proof. — The result follows by standard arguments from Proposition 1.1 provided we can show that any solution in $\mathscr{C}(I, H^2)$ satisfies an *a priori* estimate in H^2. By Propositions 1.3 and 1.4, such a solution is uniformly bounded in H^1. It is therefore sufficient to estimate $\nabla^2 \varphi$ in L^2. For that purpose, we choose q in the range (2.6) sufficently large for (2.11) to hold and satisfying in addition $q \geq 4$. By Lemma 2.1, φ is estimated in $H^1 \cap W^{1,q}$ for that value of q. In particular $\nabla \varphi$ is estimated in L^4, and by the Sobolev inequalities, φ is estimated in L^∞ since $q \geq 4 > n$. From the equation (0.2). we then obtain for all $t \in I$

$$\| \nabla^2 \varphi(t) \|_2 \leqslant \| \nabla^2 \varphi_0 \|_2 + \int_{t_0}^{t} d\tau \| \nabla^2 f_0(\varphi(\tau)) \|_2 . \qquad (2.20)$$

The norm in the integral is estimated by

$$\| \cdot \|_2 \leqslant C \{ \| f_0'(\varphi) \|_\infty \; \| \nabla^2 \varphi \|_2 + \| f_0''(\varphi) \|_\infty \; \| \nabla \varphi \|_4^2 \} . \qquad (2.21)$$

By Lemma 2.1 and the remarks above, all the norms in the right hand side of (2.21) except $\| \nabla^2 \varphi \|_2$ have already been estimated. The required estimate of $\| \nabla^2 \varphi \|_2$ therefore follows from (2.20) and (2.21) by Gronwall's inequality. Q. E. D.

REMARK 2.2. — We note for future reference that, if $f_0 \in \mathscr{C}^2$ and regardless of the dimension n, the estimates (2.20) and (2.21) provide an *a priori* estimate of $\nabla^2 \varphi$ in L^2 provided one knows in advance that φ is estimated in L^∞ and $\nabla \varphi$ in L^4.

REMARK 2.3. — If one were willing to estimate $\| \nabla \varphi \|_4^2$ by the Sobolev inequalities in terms of $\| \nabla \varphi \|_q$ and at most linearly in terms of $\| \nabla^2 \varphi \|_2$, one could replace the condition $q \geqslant 4$ by the weaker one $q > n$. This remark has no incidence on the global existence problem in H^2.

We assume from now on that $n \geqslant 4$. This implies that $q < n$ and $q < 4$ for all q in the range (2.6). In particular we can no longer estimate directly φ in H^2 by the arguments given in the proof of Proposition 2.2. Instead of that, we first estimate φ in $W^{2,q}$ for suitably large q in the range (2.6). It is at this point that the proof breaks down for $n \geq 8$.

LEMMA 2.2. — Let $4 \leqslant n < 8$ and $k > n/2$. Let $f_0 \in \mathscr{C}^k$ satisfy $f_0(0) = f_0'(0) = 0$, the conditions (2.8), (2.9), and in addition

$$| f_0''(z) | \leqslant C(1 + | z |^{p_4 - 2}) \qquad (2.22)$$

with

$$1 \leqslant p_4 - 1 < 4/(n-4) . \qquad (2.23)$$

Let $t_0 \in \mathbb{R}$ and $\varphi_0 \in H^k$. Let I be an interval of $\mathbb{R}$ containing t_0, let $\varphi \in \mathscr{C}(I, H^k)$ be a solution of the equation (0.2) satisfying (2.10). Let q in the range (2.6) satisfy (2.11) and in addition

$$p_4 - 1 \leqslant 4 \qquad \delta(q) \qquad (n - 2 - 2\delta(q))^{-1} \qquad (2.24)$$

(and therefore, by comparison with (2.23),

$$\delta(q) \geqslant \frac{1}{3}\left(\frac{n}{2} - 1\right) \quad (2.24'))$$

Then φ is estimated *a priori* in $W^{2,q}$, more precisely

$$\operatorname*{Sup}_{t\in I} \| \varphi(t) \|_{2,q} \leqslant M_{2,q} < \infty \tag{2.25}$$

where $M_{2,q}$ depends only on I (through its length $|I|$), on φ_0 (through $\| \varphi_0 \|_{3,2}$) and on $M_{1,2}$.

Proof. — By Lemma 2.1, we know already that φ is estimated in $W^{1,q}$, and it is therefore sufficient to estimate $\nabla^2\varphi$ in L^q. By the Sobolev inequalities and (2.4), we obtain from the equation (0.2)

$$\| \nabla^2\varphi(t) \|_q \leqslant C \| \varphi_0 \|_{3,2} + C \int_{t_0}^{t} d\tau \, | t - \tau |^{-\delta(q)} \times \{ \| f_0'(\varphi(\tau))\nabla^2\varphi(\tau) \|_{\bar{q}} + \| f_0''(\varphi(\tau))\nabla\varphi(\tau)^2 \|_{\bar{q}} \} . \tag{2.26}$$

We estimate the first norm in the integrand by

$$\| f_0'(\varphi)\nabla^2\varphi \|_{\bar{q}} \leqslant \| f_0'(\varphi) \|_{l/2} \| \nabla^2\varphi \|_q \tag{2.27}$$

where again $1/l = 1/2 - 1/q$, so that for $n \geqslant 4$ and q in the range (2.6), one has $q < 4 \leqslant n < l$. By the same argument as in the proof of Lemma 2.1, and with the simplification that now $l/2 \geqslant 2$, one sees easily that $\| f_0'(\varphi) \|_{l/2}$ is estimated in terms of $M_{1,2}$ alone under the assumptions (2.8), (2.9), (2.11).

We next estimate the second norm in the integrand of (2.26) by

$$\| f_0''(\varphi)(\nabla\varphi)^2 \|_{\bar{q}} \leqslant C \{ \| \nabla\varphi \|_{2\bar{q}}^2 + \| \, |\varphi|^{p_4 - 2} \|_m \| \nabla\varphi \|_{2s}^2 \} , \tag{2.28}$$

where we have used (2.22), and with $1/\bar{q} = 1/m + 1/s$.

The first term in the right hand side of (2.28) is the special case of the second one with $p_4 = 2$ and $m = \infty$, and we concentrate on the latter. We estimate $\nabla\varphi$ in L^{2s} by interpolation between L^2 and L^q if $2s \leqslant q$ (one always has $2s \geq 2\bar{q} \geq 2$) and by the Sobolev inequality if $2s > q$. We obtain in the latter case

$$\| \nabla\varphi \|_{2s}^2 \leqslant \| \nabla\varphi \|_q^{2-\sigma} \| \nabla^2\varphi \|_q^\sigma \tag{2.29}$$

with

$$\frac{1}{\bar{q}} - \frac{1}{m} = \frac{2}{q} - \frac{\sigma}{n}, \tag{2.30}$$

provided $0 \leq \sigma \leq 2$. We need not worry about the possibility of σ becoming negative, since that case corresponds to the harmless situation where $2s \leq q$. We substitute (2.29) into (2.28) and estimate the second term in the right hand side in terms of the norm of φ in $H^1 \cap W^{1,q}$, and at most linearly in terms of the norm of $\nabla^2\varphi$ in L^q by imposing that $\sigma \leq 1$ and that the norm of φ in $L^{(p_4-2)m}$ be controlled through the Sobolev inequalities by the norm of φ in $H^1 \cap W^{1,q}$. We end up with the condition

$$0 \leqslant (p_4 - 2)\left(\frac{1}{q} - \frac{1}{n}\right) \leqslant \frac{1}{m} \leqslant -\frac{1}{2} + \frac{1}{n}(1 + 3\delta(q)) , \tag{2.31}$$

where the last inequality is simply $\sigma \leq 1$ rewritten by using (2.30) and the definition of $\delta(q)$. Upon elimination of m, or equivalently omission of the middle member, (2.31) becomes identical with (2.24). The special case $p_4 = 2$, $m = \infty$ in (2.31), corresponding to the first term in the right hand side of (2.28), reduces to (2.24′), an immediate consequence of (2.24) and the condition $p_4 \geq 2$. Furthermore, (2.24) together with $p_4 \geq 2$ can be satisfied for any p_4 satisfying (2.23) by taking $\delta(q)$ sufficiently close to 1, namely q sufficiently large. The condition (2.23) implies $n < 8$.
Finally, under the conditions (2.22), (2.23), (2.24), by substituting (2.29) into (2.28) and then into (2.26) and using in addition (2.27), one obtains a sublinear integral inequality for $\| \nabla^2 \varphi \|_q$ from which the required estimate follows by Gronwall's inequality. Q. E. D.

REMARK 2.4. — A remark similar to Remark 2.1 applies to the regularity assumptions on φ and f_0 in Lemma 2.2. What is actually needed is that $\varphi \in \mathscr{C}(\mathrm{I}, \mathrm{X}')$ for some $\mathrm{X}' \subset \mathrm{H}^1 \cap \mathrm{W}^{2,q}$ such that the equation (0.2) makes sense in $\mathscr{C}(\mathrm{I}, \mathrm{X}')$, and that f_0 satisfy the minimal order of differentiability needed for that purpose. In particular, one could take $\mathrm{X}' = \mathrm{H}^1 \cap \mathrm{W}^{2,q}$, $\varphi_0 \in \mathrm{H}^3$ and $f_0 \in \mathscr{C}^2$. We have taken again $\mathrm{X}' = \mathrm{H}^k$ for simplicity.

We are now in a position to prove the existence of global solutions in H^3 for space dimensions $n = 4$ and 5.

PROPOSITION 2.3. — Let $n = 4$ or 5, and let $f_0 \in \mathscr{C}^4$ with $f_0(0) = f_0'(0) = 0$, satisfying (2.8), (2.9) and (2.22), (2.23). Let f_0 and V satisfy the assumption 1.1 and (1.6), (1.7). Let $t_0 \in \mathbb{R}$ and $\varphi_0 \in \mathrm{H}^3$. Then the equation (0.2) has a unique solution in $\mathscr{C}(\mathbb{R}, \mathrm{H}^3)$. That solution is uniformly bounded in H^1.

Proof. — The result follows again by standard arguments from Proposition 1.1 provided we can show that any solution in $\mathscr{C}(\mathrm{I}, \mathrm{H}^3)$ satisfies an *a priori* estimate in H^3. By Proposition 1.3 and 1.4, such a solution is uniformly bounded in H^1. It is therefore sufficient to estimate $\nabla^2 \varphi$ and $\nabla^3 \varphi$ in L^2. For that purpose, we choose q in the range (2.6) sufficiently large for (2.11) and (2.24) to hold, and satisfying in addition $q \geq 3$, a condition compatible with (2.6) for $n < 6$. By Lemma 2.2, φ is estimated in $\mathrm{H}^1 \cap \mathrm{W}^{2,q}$ for that value of q. By the Sobolev inequalities this implies that φ is estimated in L^∞ (since $q \geq 3 > n/2$) and that $\nabla \varphi$ is estimated in L^6 (since $1/q \leq 1/3 < 1/6 + 1/n$) and *a fortiori* in L^4. By Remark 2.2, this in turn implies that φ is estimated in H^2, and it remains only to show that $\nabla^3 \varphi$ is estimated in L^2. For that purpose, we infer from the equation (0.2) that for all $t \in \mathrm{I}$

$$\| \nabla^3 \varphi(t) \|_2 \leqslant \| \nabla^3 \varphi_0 \|_2 + \mathrm{C} \int_{t_0}^{t} d\tau \, \{ \| f_0'(\varphi(\tau)) \nabla^3 \varphi(\tau) \|_2 + \| f_0''(\varphi(\tau)) \nabla^2 \varphi(\tau) \nabla \varphi(\tau) \|_2 + \| f_0'''(\varphi(\tau)) \nabla \varphi(\tau)^3 \|_2 \} . \quad (2.32)$$

We estimate the integrand in (2.32) as

$$\| f_0'(\varphi) \|_\infty \| \nabla^3 \varphi \|_2 + \| f_0''(\varphi) \|_\infty \| \nabla^2 \varphi \|_3 \, \| \nabla \varphi \|_6 + \| f_0'''(\varphi) \|_\infty \| \nabla \varphi \|_6^3 . \tag{2.33}$$

Since we have already estimated φ in L^∞, $\nabla\varphi$ in L^6 and $\nabla^2\varphi$ in L^3, the last two terms in (2.33) are also estimated, while the first term is linear in the yet uncontrolled norm $\| \nabla^3 \varphi \|_2$. The required estimate on this last norm follows therefore from (2.32), (2.33) and Gronwall's inequality. Q. E. D.

REMARK 2.5. — We note for future reference that, if $f_0 \in \mathscr{C}^3$ and regardless of the dimension n, the estimates (2.32), (2.33) provide an *a priori* estimate of $\nabla^3\varphi$ in L^2 provided one knows in advance that φ is estimated in L^∞, $\nabla\varphi$ in L^6 and $\nabla^3\varphi$ in L^3.

REMARK 2.6. — If one were willing to estimate $\| \nabla\varphi \|_6^2$ and $\| \nabla^2\varphi\nabla\varphi \|_2$ by the Sobolev inequalities in terms of $\| \nabla^2\varphi \|_q$ and at most linearly in terms of $\| \nabla^3\varphi \|_2$, one could replace the condition $q \geq 3$ by the weaker one $q > n/2$.

We finally turn to the case of dimensions 6 and 7. This implies that $q < 3$ for all q in the range (2.6). Following the same method as previously, we postpone the estimate of the L^2-norms of $\nabla^2\varphi$, $\nabla^3\varphi$ and estimate first φ in $W^{3,q}$ for suitable q in the range (2.6).

LEMMA 2.3. — Let $n=6$ or 7 and $k > n/2$. Let $f_0 \in \mathscr{C}^k$ satisfy $f_0(0) = f_0'(0) = 0$, the conditions (2.8), (2.9), (2.22), (2.23) and in addition

$$| f_0'''(z) | \leqslant C(1 + | z |^{p_5 - 3}) \tag{2.34}$$

with

$$2 \leqslant p_5 - 1 < 4/(n-6). \tag{2.35}$$

Let $t_0 \in \mathbb{R}$ and $\varphi_0 \in H^k$. Let I be an interval of $\mathbb{R}$ containing t_0, let $\varphi \in \mathscr{C}(I, H^k)$ be a solution of the equation (0.2) satisfying (2.10). Let q in the range (2.6) satisfy (2.11), (2.24) and in addition

$$p_5 - 1 \leqslant 4 \quad \delta(q) \quad (n - 4 - 2\delta(q))^{-1} . \tag{2.36}$$

Then φ is estimated in $W^{3,q}$, more precisely

$$\sup_{t \in I} \| \varphi(t) \|_{3,q} \leqslant M_{3,q} < \infty , \tag{2.37}$$

where $M_{3,q}$ depends only on I (through its length $| I |$), on φ_0 (through $\| \varphi_0 \|_{4,2}$) and on $M_{1,2}$.

Proof. — By Lemma 2.2, we know already that φ is estimated in $W^{2,q}$

and it is therefore sufficient to estimate $\nabla^3\varphi$ in L^q. By the Sobolev inequalities and (2.4), we obtain from the equation (0.2)

$$\|\nabla^3\varphi(t)\|_q \leqslant C\|\varphi_0\|_{4,2} + C\int_{t_0}^{t} d\tau\,|t-\tau|^{-\delta(q)}\{\|f_0'(\varphi(\tau))\nabla^3\varphi(\tau)\|_{\bar q} + \|f_0''(\varphi(\tau))\nabla^2\varphi(\tau)\nabla\varphi(\tau)\|_{\bar q} + \|f_0'''(\varphi(\tau))\nabla\varphi(\tau)^3\|_{\bar q}\}. \quad (2.38)$$

We estimate the first norm in the integrand by

$$\|f_0'(\varphi)\nabla^3\varphi\|_{\bar q} \leqslant \|f_0'(\varphi)\|_{l/2}\|\nabla^3\varphi\|_q \quad (2.39)$$

where again $1/l = 1/2 - 1/q$. As in the proof of Lemma 2.2, $\|f_0'(\varphi)\|_{l/2}$ is estimated in terms of $M_{1,2}$ alone.
We estimate the second norm in the integrand of (2.38) by

$$\|f_0''(\varphi)\nabla^2\varphi\nabla\varphi\|_{\bar q} \leqslant C\|\nabla^2\varphi\|_q\{\|\nabla\varphi\|_{l/2} + \|\varphi^{p_4-2}\|_m\|\nabla\varphi\|_r\} \quad (2.40)$$

where we have used (2.22) and with $2/l = 1/m + 1/r$. We concentrate on the last term in the bracket, of which the first term is the special case $p_4=2$, $m=\infty$. Since $l > n \geq 6$, we have $r > 3 > q$. We estimate $\nabla\varphi$ in L^r by the Sobolev inequalities in terms of $\|\nabla\varphi\|_q$ and $\|\nabla^2\varphi\|_q$, and by the same calculation as in the proof of Lemma 2.2, with one factor $\|\nabla\varphi\|_q$ replaced by $\|\nabla^2\varphi\|_q$, we find that under the condition (2.24), the right hand side of (2.40) is estimated in terms of the norm of φ in $H^1 \cap W^{2,q}$.

We estimate finally the third norm in the integrand of (2.38) by

$$\|f_0'''(\varphi)(\nabla\varphi)^3\|_{\bar q} \leqslant C\{\|\nabla\varphi\|_{3\bar q}^3 + \|\varphi^{p_5-3}\|_{m'}\|\nabla\varphi\|_{3s'}^3\}, \quad (2.41)$$

where we have used (2.34), and with $1/\bar q = 1/m' + 1/s'$. We concentrate on the last term in the bracket, of which the first one is the special case $p_5 = 3$, $m' = \infty$. Note that $3s' \geqslant 3\bar q > 9/2 > q$. We estimate $\nabla\varphi$ in $L^{3s'}$ by the Sobolev inequalities in terms of $\|\nabla\varphi\|_q$ and $\|\nabla^2\varphi\|_q$ for low values of s', and in terms of $\|\nabla^2\varphi\|_q$ and $\|\nabla^3\varphi\|_q$ for high values of s'. We obtain in the latter case

$$\|\nabla\varphi\|_{3s'}^3 \leqslant \|\nabla^2\varphi\|_q^{3-\sigma'}\|\nabla^3\varphi\|_q^{\sigma'} \quad (2.42)$$

with

$$\frac{1}{\bar q} - \frac{1}{m'} = \frac{3}{q} - \frac{3}{n} - \frac{\sigma'}{n} \quad (2.43)$$

provided $0 \leq \sigma' \leq 3$. We disregard the lower inequality which is connected with the harmless low values of s', and replace the higher one by the stronger sublinearity condition $\sigma' \leq 1$ as in the proof of Lemma 2.2. We impose in addition that the norm of φ in $L^{(p_5-3)m'}$ be controlled through the Sobolev inequalities by the norm of φ in $H^1 \cap W^{2,q}$ and obtain finally that the right hand side of (2.41) is estimated in terms of the norm of φ in $H^1 \cap W^{2,q}$ and, at most linearly, in terms of the norm of $\nabla^3\varphi$ in L^q provided

$$0 \leqslant (p_5 - 3)\left(\frac{1}{q} - \frac{2}{n}\right) \leqslant \frac{1}{m'} \leqslant -1 + \frac{4}{n}(1 + \delta(q)), \quad (2.44)$$

where the last inequality is simply $\sigma' \leq 1$ rewritten by using (2.43) and the definition of $\delta(q)$. Upon elimination of m', (2.44) becomes identical with (2.36). The special case $p_5 = 3$, $m' = \infty$ in (2.44) reduces to the condition $1 + \delta(q) \geq n/4$, an immediate consequence of (2.36) and of the condition $p_5 \geq 3$. The condition (2.36) together with $p_5 \geq 3$ can be satisfied for any p_5 satisfying (2.35) by taking $\delta(q)$ sufficiently close to 1.

Finally, under the conditions stated in the Lemma, one obtains a sublinear integral inequality for $\| \nabla^3 \varphi \|_q$, from which the required estimate follows by Gronwall's inequality. Q. E. D.

REMARK 2.7. — A remark similar to Remarks 2.1 and 2.4 applies to the regularity assumptions on φ and f_0 in Lemma 2.3. What is actually needed is that $\varphi \in \mathscr{C}(\mathrm{I}, \mathrm{X}')$ for some $\mathrm{X}' \subset \mathrm{H}^1 \cap \mathrm{W}^{3,q}$ such that the equation (0.2) makes sense in $\mathscr{C}(\mathrm{I}, \mathrm{X}')$, and that f_0 satisfy the minimal order of differentiability needed for that purpose. In particular one could take $\mathrm{X}' = \mathrm{H}^1 \cap \mathrm{W}^{3,q}$, $\varphi_0 \in \mathrm{H}^4$ and $f_0 \in \mathscr{C}^3$. We have taken again $\mathrm{X}' = \mathrm{H}^k$ for simplicity.

We are now in a position to prove the existence of global solutions in H^4 for space dimensions $n = 6$ and 7.

PROPOSITION 2.4. — Let $n=6$ or 7 and let $f_0 \in \mathscr{C}^5$ with $f_0(0) = f_0'(0) = 0$, satisfying (2.8), (2.9), (2.22), (2.23) and (2.34), (2.35). Let f_0 and V satisfy the assumption 1.1 and (1.6), (1.7). Let $t_0 \in \mathbb{R}$ and $\varphi_0 \in \mathrm{H}^4$. Then the equation (0.2) has a unique solution in $\mathscr{C}(\mathbb{R}, \mathrm{H}^4)$. That solution is uniformly bounded in H^1.

Proof. — In the same way as in the proofs of Propositions 2.2 and 2.3, it is sufficient to prove that for any solution φ in $\mathscr{C}(\mathrm{I}, \mathrm{H}^4)$, $\nabla^2\varphi$, $\nabla^3\varphi$ and $\nabla^4\varphi$ are estimated *a priori* in L^2. For that purpose, we choose q in the range (2.6) sufficiently large for (2.11), (2.24) and (2.36) to hold, and satisfying in addition $q \geq 8/3$, a condition compatible with (2.6) for $n < 8$. By Lemma 2.3, φ is estimated in $\mathrm{H}^1 \cap \mathrm{W}^{3,q}$ for that value of q. By the Sobolev inequalities, this implies that φ is estimated in L^∞ (since $q \geqslant 8/3 > n/3$), that $\nabla\varphi$ is estimated in L^8 (since $1/q \leq 3/8 < 1/8 + 2/n$), and *a fortiori* in L^4, so that $\nabla^2\varphi$ is estimated in L^2, by Remark 2.2. One can then estimate $\nabla^2\varphi$ in L^4 and *a fortiori* in L^3 by the Sobolev inequalities, since $1/q \leq 3/8 < 1/4 + 1/n$, so that $\nabla^3\varphi$ is estimated in L^2, by Remark 2.5. It remains only to show that $\nabla^4\varphi$ is estimated in L^2. For that purpose, we infer from the equation (0.2) that for all $t \in \mathrm{I}$

$$\begin{aligned} \| \nabla^4\varphi(t) \|_2 \leqslant \| \nabla^4\varphi_0 \|_2 + \mathrm{C} \int_{t_0}^{t} d\tau \, \{ \| f_0'(\varphi(\tau))\nabla^4\varphi(\tau) \|_2 \\ + \| f_0''(\varphi(\tau))\nabla^2\varphi(\tau)^2 \|_2 + \| f_0''(\varphi(\tau))\nabla^3\varphi(\tau)\nabla\varphi(\tau) \|_2 \\ + \| f_0'''(\varphi(\tau))\nabla^2\varphi(\tau)\nabla\varphi(\tau)^2 \|_2 + \| f_0^{(4)}(\varphi(\tau))\nabla\varphi(\tau)^4 \|_2 \} . \quad (2.45) \end{aligned}$$

We estimate the integrand in (2.45) as

$$\| f_0'(\varphi) \|_\infty \| \nabla^4\varphi \|_2 + \| f_0''(\varphi) \|_\infty \{ \| \nabla^2\varphi \|_4^2 + \| \nabla^3\varphi \|_{8/3} \| \nabla\varphi \|_8 \} + \| f_0'''(\varphi) \|_\infty \| \nabla^2\varphi \|_4 \| \nabla\varphi \|_8^2 + \| f_0^{(4)}(\varphi) \|_\infty \| \nabla\varphi \|_8^4 . \quad (2.46)$$

Since we have already estimated φ in L^∞, $\nabla\varphi$ in L^8, $\nabla^2\varphi$ in L^4 and $\nabla^3\varphi$ in $L^{8/3}$, all terms but the first one in (2.46) are also estimated, while the first one is linear in the yet uncontrolled norm $\| \nabla^4\varphi \|_2$. The required estimate of that norm follows therefore from (2.45), (2.46) and Gronwall's inequality. Q. E. D.

REMARK 2.8. — If one were willing to estimate all the norms in the integrand of (2.45) in terms of the norms of φ in $H^1 \cap W^{3,q}$ and at most linearly in terms of $\| \nabla^4\varphi \|_2$, one could replace the condition $q \geq 8/3$ by the weaker one $q > n/3$.

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