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Existence of solution for a free boundary problem in a nonlinear piecewise homogeneous medium (1)

by

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ABSTRACT. – A free boundary problem arising from the bidimensional thermal modelling of aluminium electrolytic cells is studied. The medium is assumed piecewise homogeneous and nonlinear. A fixed domain method is proposed which leads to a weak formulation of the problem. Existence of weak solution is proved by regularizing the contact condition between the homogeneous subdomains and passing to the limit.

Key words: Free boundary, aluminium electrolysis, piecewise homogeneous media, weak solution, existence.

RÉSUMÉ. – Dans cet article on étudie un problème de frontière libre qui apparaît dans la modélisation thermoélectrique des cuves électrolytiques d'aluminium. Le domaine physique est supposé homogène par morceaux et nonlinéaire. On utilise une méthode de domaine fixe qui conduit à une formulation variationnelle du problème. L'existence de solution faible est démontré par régularisation de la condition de transmission entre les sousdomaines homogènes et passage à la limite.

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1. INTRODUCTION

In this paper, a free boundary problem motivated by the thermal modelling of an aluminium electrolytic cell is studied.

Aluminium is produced by reduction of alumina dissolved in an electrolytic bath based on molten cryolite (see [11]). This complex process, called Hall-Héroult, involves thermoelectrical and magnetohydrodynamical phenomena, electrochemical reactions, complex phase equilibria and so on (see [12]).

The Hall-Héroult process takes place in an electrolytic cell (see Fig. 1) which consists of a rectangular steel shell with an inner covering of insulating and refractory materials. Inside this, there is a linning of prebaked carbon cathode blocks with embedded steel current collector bars. Both the liquid metal and the electrolytic bath are upon these blocks. A frozen bath layer, the so-called ledge, protects the side wall of the cell from corrosive electrolyte. This ledge also reduces the heat loss from the cathode and works as a heat sink when extra power is supplied to the cell, thus playing a major role in the thermal behaviour of the cell.

The outline of this paper is as follows: in section 2, we recall the main features characterizing the thermoelectrical behaviour of an electrolytic cell. The unknowns are the temperature, the electric potential and the profile of the ledge which becomes a free boundary.

Theoretical analysis of this problem is extremely difficult due to the coupling between thermal and electric equations, the nonhomogeneity of the domain, the physical nonlinearities and the free boundary. In [4], a discretized thermoelectrical problem is introduced and an iterative algorithm is used to compute the solution for a test example and real industrial electrolytic cells.

As a first step, in [6] we study the free boundary problem in the ledge which is both piecewise homogeneous and nonconductor, and consequently only the thermal phenomenon is considered. Both, existence and uniqueness of solution are demonstrated assuming that the ledge is linear (i.e. the thermal conductivity coefficients depend on space variables but not on temperature).

In the present paper we also study the thermal submodel. The difference with respect to the case considered in [6] is that now thermal conductivity also depends on temperature. This fact leads to a nonlinear diffusion term which makes more difficult the mathematical analysis. Indeed, since thermal conductivity also depends on the space variable, to avoid this nonlinearity by using a global Kirchhoff transformation is not allowed. To overcome this difficulty we use domain decomposition methods by considering two homogeneous subdomains (which can be distinguished in the ledge) corresponding to the levels of bath and aluminium.

In section 3 we introduce a weak formulation in a fixed domain. Mathematically, this problem is a stationary one phase Stefan problem with source at the free boundary (see [17]).

Section 4 is devoted to proving an existence theorem for an auxiliary problem, depending on a parameter, which regularizes the contact condition between the homogeneous subdomains. After setting some a priori estimates in section 5, existence of a weak solution for the thermal problem is proved in section 6.

2. THE THERMOELECTRICAL PROBLEM

In this section we describe the thermoelectrical behaviour of the cathode of an aluminium electrolytic cell.

The voltage drop between the anode and the cathode causes an increasing of the temperature due to the Joule effect. Likewise, the potential distribution of the electrolytic cell depends on the temperature through the electrical conductivities of the materials. Therefore, from a mathematical point of view, the full problem couples both a thermal and an electrical problem, and it is similar to the so-called thermistor problem (see [13] and the references therein). However two additional difficulties appear in the present problem. Firstly, the domain of the model is not homogeneous and then physical parameters depend not only on temperature but on position x as well. Secondly, there is a free boundary: the profile of the ledge, called S in Fig. 1.

The boundary conditions for the electric problem are given by the knowledge of the current density through the cathodic bar. Moreover, the heat flux through the exterior boundaries due to the losses by convection an radiation leads to the boundary conditions for the thermal problem.

The ledge, being a nonconductor, is actually a fundamental part of the cell from the thermal point of view. In the recent years, several attempts have been made to determine the heat flux through the surface S (see [1], [18]). Since the temperature is almost uniform in the liquid phase due to the strong horizontal flow caused by the electromagnetic field, we assume that the temperature, T, is equal to the solidus temperature of the bath, called T_s, in S. The heat flux on S is given by

$$k(x,T)\frac{\partial T}{\partial n} = h(x_2)n_1(x), \tag{2.1}$$

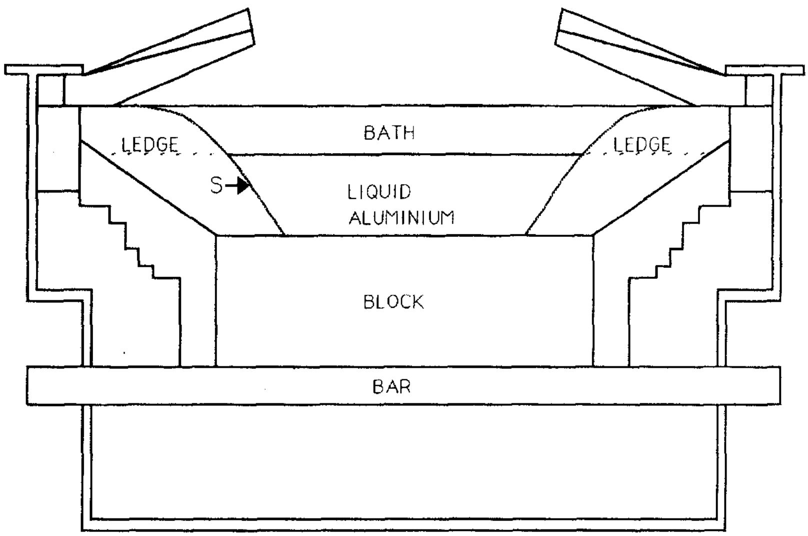

Fig. 1. – A section of the cell cathode.

where k is the thermal conductivity depending both on the space variable $x = (x_1, x_2)$ and on the temperature, n_1 represents the first component of the outward unit normal vector to the ledge at S at point x and $h(x_2)$ is a function to be given which only depends on x_2. In practice, h has to be identified from experimental measurements because it depends on factors as the electrolyte composition. In [3], a method to identify the function h from experimental measurements of the ledge profile is developed. Including n_1 is not only convenient from the mathematical point of view but it also makes sense from the physical one because the heat transfer depends on the slope of the free boundary: the greater the slope the greater the heat transfer.

In [5], this full thermoelectrical problem is discretized using pentahedral finite elements of six degrees of freedom and numerical results are given for real industrial situations.

The difficulties appearing on the theoretical treatment of this coupled problem, as the nonlinearities on the physical characteristics of the materials or the free boundary, lead us to consider a simplified problem taking place on the (unknown !) domain occupied by the ledge.

3. STATEMENT OF THE PROBLEM

As a approach to the theoretical study of the full thermoelectrical problem, we consider a simplified bidimensional submodel. The ledge is the domain where the problem is now posed; it is formed by two layers corresponding to the bath and the metal levels and, as a consequence, the thermal conductivity is different in these two levels (see Fig. 2). Since it is a nonconductor material, release of heat by Joule effect does not occur and then we can consider only the thermal part of the problem which becomes a one phase Stefan problem with source at the free boundary (see [17]). A similar problem is developed in [16] for the evolutionary and multiphase version but it does not cover the present situation.

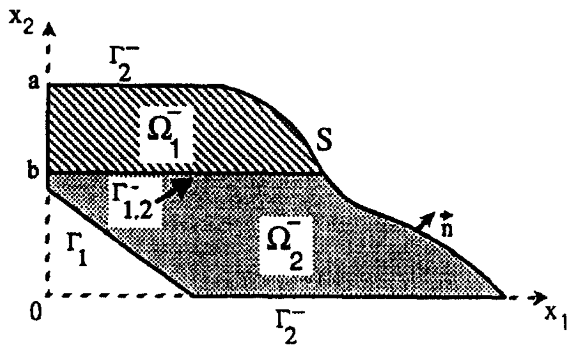

Fig. 2. – Ω^- domain.

Let Ω^- be the ledge. We assume that Ω^- can be written as

$$\Omega^- = \{(x_1, x_2) \in \mathbb{R}^2,\ 0 < x_2 < a,\ f_1(x_2) < x_1 < f_2(x_2)\}, \tag{3.1}$$

where a is a positive real number, $f_i,\ i = 1, 2$ are Lipschitz functions and

$$f_1(x_2) < f_2(x_2),\ \forall x_2 \in [0, a]. \tag{3.2}$$

Actually, f_2 is an unknown function corresponding to the free boundary.

The solidified bath and metal are denoted by Ω_1^- and Ω_2^-, respectively and we suppose that they are given by

$$\Omega_1^- = \Omega^- \cap \{(x_1, x_2) \in \mathbb{R}^2, b < x_2 < a\}, \tag{3.3}$$

$$\Omega_2^- = \Omega^- \cap \{(x_1, x_2) \in \mathbb{R}^2, 0 < x_2 < b\}, \tag{3.4}$$

where b is a positive real number such that $b < a$ (see Fig. 2).

Let $\Gamma_{1,2}^- = \partial\Omega_1^- \cap \partial\Omega_2^-$ and Γ_1 and S be the graphs of the functions f_1 and f_2, respectively. Finally,

$$\Gamma_2^- = \partial\Omega^- \cap ([x_2 = 0] \cup [x_2 = a]). \tag{3.5}$$

Notice that $\Gamma_1 \cup \Gamma_2^-$ is the part of the boundary of Ω^- different from the free boundary S. We denote

$$S_i = S \cap \partial\Omega_i^-, \ i = 1, 2. \tag{3.6}$$

Since Ω^- is piecewise homogeneous, the thermal conductivity can be written as follows:

$$k(x, T) = \begin{cases} k_1(T(x)) & \text{if } x \in \Omega_1^- \\ k_2(T(x)) & \text{if } x \in \Omega_2^-. \end{cases} \tag{3.7}$$

On the other hand, a Robin boundary condition is assumed on Γ_1 involving a convective coefficient α and the convective temperature of surroundings, T_c.

We shall assume all along the following assumptions on the data

(H1) For $i = 1, 2$, $k_i(s) : \mathbb{R} \to \mathbb{R}$ are continuous and there exist positive constants k_{min} and k_{max}, such that $k_{min} \leq k_i(s) \leq k_{max}$.

(H2) T_s is a positive constant.

(H3) $h \in L^\infty(0, a)$ is nonnegative where a is the height of the domain (see Fig. 2).

(H4) The function α only depends on the space variable and belongs to $L^\infty(\Gamma_1)$. Moreover, $\alpha(x) \geq \alpha_{min} > 0$ a.e. on Γ_1.

(H5) $T_c \in L^\infty(0, a)$, with $0 < T_{min} \leq T_c < T_s$ a.e. on $(0, a)$.

(H6) $h\, n_1 + \alpha(T_s - T_c) \geq 0$ a.e. on Γ_1, where n_1 denotes the first component of the outward unit normal vector to Γ_1. We assume that $-1 \leq n_1 < 0$.

Physically, the assumption (H6) establishes an upper bound in the heat source at the free boundary. From the theoretical point of view, it is needed in order to prove that the solution of our problem is less or equal than T_s. In [5] and for a onedimensional version, solutions without this property are obtained if (H6) does not hold.

Throughout this paper we use standard notations for Sobolev spaces and norms. We also denote

$$H(div, \Omega) = \left\{ \mathbf{v} \in (L^2(\Omega))^2;\ div\, \mathbf{v} \in L^2(\Omega) \right\}, \tag{3.8}$$

$$H_{00}^{1/2}(\Gamma_0) = \{\mu \in L^2(\Gamma_0);\ \exists z \in H^1(\Omega) \text{ such that } z\,|_{\Gamma_0} = \mu \text{ and } z\,|_{(\partial\Omega \setminus \Gamma_0)} = 0\}, \tag{3.9}$$

where Γ_0 is an open set of the boundary of Ω.

Let us consider the following free boundary problem:

- **Problem (P)**

Find T_i in $H^1(\Omega_i^-)$ and S_i, $i = 1, 2$ such that

$$-\nabla.(k_i(T_i(x))\nabla T_i(x)) = 0 \text{ in } \Omega_i^-, \tag{3.10}$$

$$k_i(T_i(x))\frac{\partial T_i}{\partial n}(x) = 0 \text{ on } \Gamma_2^- \cap \partial\Omega_i^-, \tag{3.11}$$

$$T_i(x) = T_s \text{ on } S_i, \tag{3.12}$$

$$k_i(T_i(x))\frac{\partial T_i}{\partial n}(x) = h(x_2)n_1(x) \text{ on } S_i, \tag{3.13}$$

$$k_i(T_i(x))\frac{\partial T_i}{\partial n}(x) + \alpha(x_2)(T_i(x) - T_c(x_2)) = 0 \text{ on } \Gamma_1 \cap \partial\Omega_i^-, \tag{3.14}$$

for $i = 1, 2$. Moreover, we must impose the transmission conditions on $\Gamma_{1,2}^-$:

$$T_1(x) = T_2(x), \tag{3.15}$$

$$k_1(T_1(x))\frac{\partial T_1}{\partial n_{\Omega_1^-}}(x) + k_2(T_2(x))\frac{\partial T_2}{\partial n_{\Omega_2^-}}(x) = 0, \tag{3.16}$$

$n_{\Omega_i^-}$ being the outward unit normal vector to Ω_i^-, $i = 1, 2$. The conditions (3.15) and (3.16) express the requirement for the temperature and the heat flux not to have jumps on $\Gamma_{1,2}^-$.

Equality (3.10) holds in the distributional sense and then $k_i(T_i(x))\nabla T_i(x)$ belongs to $H(div, \Omega_i^-)$, $i = 1, 2$. The boundary condition (3.11) holds on $(H_{00}^{1/2}(\Gamma_2^- \cap \partial\Omega_i^-))'$, and analogously with (3.13), (3.14) and (3.16).

For theoretical and numerical purposes, it is interesting to embed the problem (P) into another one defined in a fixed domain. For this purpose, we consider the sets Ω_1^0 and Ω_2^0 called the fictitious domains (see Fig. 3).

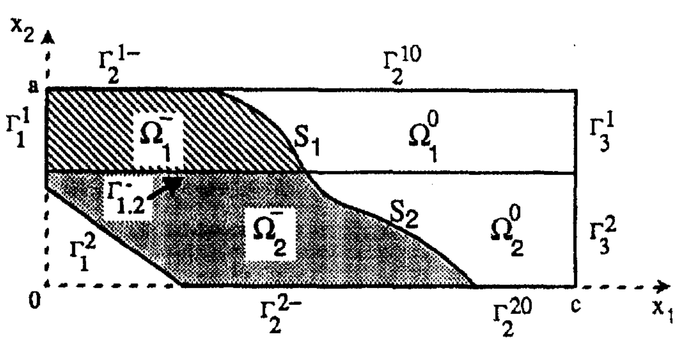

Fig. 3. – Ω domain.

We define Ω_i as the interior of the set $\Omega_i^- \cup S_i \cup \Omega_i^0$, and

$$\Gamma_{1,2} = \partial\Omega_1 \cap \partial\Omega_2, \tag{3.17}$$
$$\Gamma_1^i = \Gamma_1 \cap \partial\Omega_i, \tag{3.18}$$
$$\Gamma_2^{i-} = \Gamma_2^- \cap \partial\Omega_i^-, \tag{3.19}$$
$$\Gamma_3^i = \{(x_1, x_2) \in \partial\Omega_i;\ x_1 = c\},\ \Gamma_3 = \Gamma_3^1 \cup \Gamma_3^2 \tag{3.20}$$
$$\Gamma_2^{i0} = \partial\Omega_i \setminus (\Gamma_1^i \cup \Gamma_{1,2} \cup \Gamma_3^i \cup \Gamma_2^{i-}), \tag{3.21}$$
$$\Gamma_2^i = \Gamma_2^{i-} \cup \Gamma_2^{i0},\ \Gamma_2 = \Gamma_2^1 \cup \Gamma_2^2, \tag{3.22}$$

where the meaning of c is clear from Fig. 3. We define Ω as the interior of the set $\Omega_1 \cup \Omega_2 \cup \Gamma_{1,2}$ with boundary $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$. Moreover we set

$$\mathcal{H}^1(\Omega) = H^1(\Omega_1) \times H^1(\Omega_2), \tag{3.23}$$

with the standard product norm, i.e.

$$\| (u_1, u_2) \| = (\| u_1 \|_{1,2,\Omega_1}^2 + \| u_2 \|_{1,2,\Omega_2}^2)^{1/2}. \tag{3.24}$$

As $H^1(\Omega)$ is continuously imbedded into $\mathcal{H}^1(\Omega)$, hereafter a function $T \in H^1(\Omega)$ will be denoted by $(T_1, T_2) \in \mathcal{H}^1(\Omega)$ with $T_i = T\,|_{\Omega_i}$, $i = 1, 2$.

We consider the weak problem:

- **Problem (WP)**

Find $T = (T_1, T_2) \in H^1(\Omega)$, $q_i \in L^\infty(\Omega_i)$, $i = 1, 2$ such that

$$\sum_{i=1}^{2} \int_{\Omega_i} k_i(T_i)\nabla T_i.\nabla z_i dx + \sum_{i=1}^{2} \int_{\Omega_i} hq_i \frac{\partial z_i}{\partial x_1} dx + \sum_{i=1}^{2} \int_{\Gamma_1^i} \alpha(T_i - T_c) z_i d\Gamma$$
$$= \sum_{i=1}^{2} \int_{\Gamma_3^i} hz_i d\Gamma, \quad \forall (z_1, z_2) \in H^1(\Omega) \tag{3.25}$$

$$q_i \in H(T_i - T_s), \quad i = 1, 2, \tag{3.26}$$

where H denotes the multivalued Heaviside function given by

$$H(r) = \begin{cases} 0 & \text{if} \quad r < 0, \\ [0, 1] & \text{if} \quad r = 0, \\ 1 & \text{if} \quad r > 0. \end{cases} \tag{3.27}$$

For a solution of the problem (WP), let the sets Ω^-, S, Ω_i^-, Ω_i^0, Γ_2^- and $\Gamma_{1,2}^-$ be defined by

$$\Omega^- = \{x \in \Omega : T(x) < T_s\}, \tag{3.28}$$

$$S = \partial\Omega^- \cap \Omega, \tag{3.29}$$

$$\Omega_i^- = \Omega^- \cap \Omega_i, \tag{3.30}$$

$$\Omega_i^0 = \Omega_i \backslash (\Omega_i^- \cup S_i), \tag{3.31}$$

$$\Gamma_2^- = \Gamma_2 \cap \partial\Omega^-, \tag{3.32}$$

$$\Gamma_{1,2}^- = \partial\Omega_1^- \cap \partial\Omega_2^-, \tag{3.33}$$

S_i, Γ_2^{i-} and Γ_2^{i0} being defined in (3.6), (3.19) and (3.21) respectively.

Notice that q_i is equal to zero in Ω_i^- by (3.26).

Remark 3.1. – If the solution (T_1, T_2) of the problem (WP) is continuous and less or equal than T_s and S has twodimensional Lebesgue measure zero, then the set equality

$$\Gamma_{1,2} = \Gamma_{1,2}^0 \cup \Gamma_{1,2}^- \tag{3.34}$$

holds where $\Gamma_{1,2}^0 = \partial\Omega_1^0 \cap \partial\Omega_2^0$ (see [15] for further details).

PROPOSITION 3.1. – *If there exists a regular solution (T_1, T_2, q_1, q_2) of the problem (WP) such that $T_i(x) \le T_s$ a.e. in Ω_i and furthermore $\Gamma_1^i \subset \partial\Omega_i^-$, $\Gamma_3^i \subset \partial\Omega_i^0$ and Ω_i^- and Ω_i^0 are open sets with Lipschitz boundary, $(i = 1, 2)$, then (T_1, T_2, q_1, q_2, S) is a solution of the problem:*

$$-\nabla.(k_i(T_i(x))\nabla T_i(x)) = 0 \text{ in } \Omega_i^-, \tag{3.35}$$

$$T_i(x) < T_s, \quad q_i(x) = 0 \text{ in } \Omega_i^-, \tag{3.36}$$

$$-h(x_2)\frac{\partial q_i}{\partial x_1}(x) = 0 \text{ in } \Omega_i^0, \tag{3.37}$$

$$T_i(x) = T_s, \quad 0 \le q_i(x) \le 1 \text{ in } \Omega_i^0, \tag{3.38}$$

$$T_i(x) = T_s \text{ on } S_i, \tag{3.39}$$

$$k_i(T_i(x))\frac{\partial T_i}{\partial n}(x) = h(x_2)q_i(x)n_1(x) \text{ on } S_i, \tag{3.40}$$

$$k_i(T_i(x))\frac{\partial T_i}{\partial n}(x) = 0 \text{ on } \Gamma_2^i, \tag{3.41}$$

$$h(x_2)q_i(x) = h(x_2) \text{ on } \Gamma_3^i, \tag{3.42}$$

$$k_i(T_i(x))\frac{\partial T_i}{\partial n}(x) + \alpha(x_2)(T_i(x) - T_c(x_2)) = 0 \text{ on } \Gamma_1^i, \tag{3.43}$$

$$T_1(x) = T_2(x) \text{ on } \Gamma_{1,2}, \tag{3.44}$$

$$k_1(T_1(x))\frac{\partial T_1}{\partial n_{\Omega_1}}(x) + k_2(T_2(x))\frac{\partial T_2}{\partial n_{\Omega_2}}(x) = 0 \text{ on } \Gamma_{1,2}, \tag{3.45}$$

for $i = 1, 2$ *and* n_1 *as in equation (2.1).*

In (3.40) q_i represents the trace of q_i restricted to Ω_i^0 on S_i.

Proof. – If we choose $z \in \mathcal{D}(\Omega_i^-)$ (the usual space of functions of class $\mathcal{C}^\infty$ with compact support in Ω_i^-) and $z \in \mathcal{D}(\Omega_i^0)$, we classically have (3.35) and (3.37) in $\mathcal{D}'(\Omega_i^-)$ and in $\mathcal{D}'(\Omega_i^0)$, respectively.

By definition of Ω_i^0 and Ω_i^-, we have $T_i = T_s$ in Ω_i^0 and $q_i = 0$ in Ω_i^-, respectively. Thus, by applying the Green formula, equation (3.25) implies

$$\sum_{i=1}^{2} \int_{\Gamma_1^i \cup \Gamma_2^i \cup S_i} k_i(T_i) \frac{\partial T_i}{\partial n} z_i d\Gamma + \int_{\Gamma_{1,2}} k_1(T_1) \frac{\partial T_1}{\partial n_{\Omega_1}} z_1 d\Gamma$$
$$+ \int_{\Gamma_{1,2}} k_2(T_2) \frac{\partial T_2}{\partial n_{\Omega_2}} z_2 d\Gamma + \sum_{i=1}^{2} \int_{S_i} h q_i (-n_1) z_i d\Gamma + \sum_{i=1}^{2} \int_{\Gamma_3^i} h q_i z_i d\Gamma$$
$$+ \sum_{i=1}^{2} \int_{\Gamma_1^i} \alpha (T_i - T_c) z_i d\Gamma = \sum_{i=1}^{2} \int_{\Gamma_3^i} h z_i d\Gamma, \quad \forall (z_1, z_2) \in H^1(\Omega), \quad (3.46)$$

taking into account that $\nabla T_i = 0$ in $\Omega \setminus \Omega^-$. From (3.46), we deduce (3.40), (3.41), (3.42), (3.43) and (3.45). Finally, since $(T_1, T_2) \in H^1(\Omega)$, (3.44) holds true. □

Notice that if Ω_i^0 is connected and $\Gamma_3^i \subset \partial\Omega_i^0$, (3.37) and (3.42) imply $hq_i = h$ in Ω_i^0, and therefore from (3.40) we deduce (3.13).

The problem (WP) is similar to those arising in the dam problem (see [9]), and in the lubrication with cavitation problem (see [2], [10]). The differences lie in both the coefficients of the partial differential operators and the boundary conditions. In [6], an easier problem is considered, in that thermal conductivity does not depend on temperature and hence the differential operator in equation (3.10) is linear. For this problem, existence and uniqueness of solution are proved. However, the technique developed in that paper can not be directly applied to the present problem.

In the following sections we are concerned with existence of solution of the problem (WP). The proof is laborious and is based on defining a regularized problem using maximal monotone operators techniques.

4. THE REGULARIZED PROBLEM

In this section we introduce the regularized problem, called problem (AP$_\lambda$). An existence result for this problem is given after both a Kirchhoff transformation and an approximation technique are used.

Let us consider the operator

$$\mathcal{A} : \mathcal{H}^1(\Omega) \times L^\infty(\Omega_1) \times L^\infty(\Omega_2) \to (\mathcal{H}^1(\Omega))', \tag{4.1}$$

defined by

$$< \mathcal{A}(T_1, T_2, q_1, q_2), (z_1, z_2) > = \sum_{i=1}^{2} \int_{\Omega_i} k_i(T_i) \nabla T_i . \nabla z_i dx$$
$$+ \sum_{i=1}^{2} \int_{\Omega_i} h q_i \frac{\partial z_i}{\partial x_1} dx + \sum_{i=1}^{2} \int_{\Gamma_1^i} \alpha T_i z_i d\Gamma, \tag{4.2}$$

and the element $\mathcal{F}$ of $(\mathcal{H}^1(\Omega))'$ given by

$$\mathcal{F}(z_1, z_2) = \sum_{i=1}^{2} \int_{\Gamma_1^i} \alpha T_c z_i d\Gamma + \sum_{i=1}^{2} \int_{\Gamma_3^i} h z_i d\Gamma. \tag{4.3}$$

We are able to prove the following

PROPOSITION 4.2. – *Let (T_1, T_2, q_1, q_2) be an element of $\mathcal{H}^1(\Omega) \times L^\infty(\Omega_1) \times L^\infty(\Omega_2)$ such that*

$$\mathcal{F} - \mathcal{A}(T_1, T_2, q_1, q_2) \in \gamma_{1,2}^* \partial I_{\{0\}} (\gamma_{1,2}(T_1, T_2)), \tag{4.4}$$
$$q_i \in H(T_i - T_s), \ i = 1, 2, \tag{4.5}$$

then (T_1, T_2, q_1, q_2) is a solution of problem (WP), where:

- *$\partial I_{\{0\}}$ is the subdifferential of $I_{\{0\}}$ which is the indicator function of the set $\{0\}$ in $H^{1/2}(\Gamma_{1,2})$.*
- *$\gamma_{1,2}$ is defined by*

$$\begin{aligned} \gamma_{1,2} : \mathcal{H}^1(\Omega) &\longrightarrow H^{1/2}(\Gamma_{1,2}) \\ (z_1, z_2) &\to z_1 \mid_{\Gamma_{1,2}} - z_2 \mid_{\Gamma_{1,2}} \end{aligned} \tag{4.6}$$

- *$\gamma_{1,2}^*$ is the adjoint operator of $\gamma_{1,2}$.*

Proof. – Since $I_{H^1(\Omega)} = I_{\{0\}} \circ \gamma_{1,2}$, by using the chain rule of subdifferential calculus we obtain

$$\mathcal{F} - \mathcal{A}(T_1, T_2, q_1, q_2) \in \partial I_{H^1(\Omega)}(T_1, T_2). \tag{4.7}$$

Therefore,

$$< \mathcal{A}(T_1, T_2, q_1, q_2), (z_1, z_2) - (T_1, T_2) > + I_{H^1(\Omega)}(z_1, z_2) - I_{H^1(\Omega)}(T_1, T_2) \\ \geq \mathcal{F}((z_1, z_2) - (T_1, T_2)), \quad \forall (z_1, z_2) \in \mathcal{H}^1(\Omega), \qquad (4.8)$$

and we deduce that (T_1, T_2) belongs to $H^1(\Omega)$. Taking into account that $H^1(\Omega)$ is a subspace of $\mathcal{H}^1(\Omega)$, we easily deduce

$$< \mathcal{A}(T_1, T_2, q_1, q_2), (z_1, z_2) >= \mathcal{F}((z_1, z_2)), \ \forall (z_1, z_2) \in H^1(\Omega), \quad (4.9)$$

and the proof is complete. □

Taking into account (4.4), we define an auxiliary problem by replacing the maximal monotone operator $\partial I_{\{0\}}$ by its Yosida approximation given by

$$(\partial I_{\{0\}})_\lambda(s) = \frac{s}{\lambda}. \qquad (4.10)$$

• **Problem (AP$_\lambda$)**

For a fixed $\lambda > 0$, find $(T_1^\lambda, T_2^\lambda, q_1^\lambda, q_2^\lambda)$ in $\mathcal{H}^1(\Omega) \times L^\infty(\Omega_1) \times L^\infty(\Omega_2)$ such that

$$\sum_{i=1}^{2} \int_{\Omega_i} k_i(T_i^\lambda) \nabla T_i^\lambda . \nabla z_i dx \\ + \sum_{i=1}^{2} \int_{\Omega_i} h q_i^\lambda \frac{\partial z_i}{\partial x_1} dx + \sum_{i=1}^{2} \int_{\Gamma_1^i} \alpha (T_i^\lambda - T_c) z_i d\Gamma \\ + \int_{\Gamma_{1,2}} \frac{(T_1^\lambda - T_2^\lambda)}{\lambda} (z_1 - z_2) d\Gamma \\ = \sum_{i=1}^{2} \int_{\Gamma_3^i} h z_i d\Gamma, \ \forall (z_1, z_2) \in \mathcal{H}^1(\Omega), \qquad (4.11)$$

$$q_i^\lambda \in H(T_i^\lambda - T_s), \quad i = 1, 2. \qquad (4.12)$$

Notice that this problem couples the subdomains Ω_1 and Ω_2 through the integral on the boundary $\Gamma_{1,2}$ in (4.11).

The following result establishes a lower bound for the solutions of the problem (AP$_\lambda$).

PROPOSITION 4.3. – *Under the assumptions (H1)-(H5), let* $(T_1^\lambda, T_2^\lambda, q_1^\lambda, q_2^\lambda)$ *be a solution of (AP$_\lambda$), then* $T_i^\lambda \geq T_{min}$ *a.e. in* Ω_i, $i = 1, 2$.

Proof. – Let us choose $z_i = (T_{min} - T_i^\lambda)^+$ as a test function in (4.11), we have

$$
\begin{aligned}
&-\sum_{i=1}^{2}\int_{\Omega_i} k_i(T_i^\lambda) \mid \nabla(T_{min} - T_i^\lambda)^+ \mid^2 dx \\
&+\sum_{i=1}^{2}\int_{\Gamma_1^i} \alpha(T_i^\lambda - T_c)(T_{min} - T_i^\lambda)^+ d\Gamma \\
&+\int_{\Gamma_{1,2}} \frac{(T_1^\lambda - T_2^\lambda)}{\lambda}((T_{min} - T_1^\lambda)^+ - (T_{min} - T_2^\lambda)^+)d\Gamma \\
&=\sum_{i=1}^{2}\int_{\Gamma_3^i} h(T_{min} - T_i^\lambda)^+ d\Gamma, \qquad (4.13)
\end{aligned}
$$

since $q_i^\lambda = 0$ if $T_i^\lambda \leq T_{min}$. We distinguish the sets

$$B = \Gamma_{1,2} \cap [T_{min} > T_1^\lambda] \cap [T_{min} > T_2^\lambda], \qquad (4.14)$$

$$C_1 = \Gamma_{1,2} \cap [T_{min} > T_1^\lambda] \cap [T_{min} \leq T_2^\lambda], \qquad (4.15)$$

$$C_2 = \Gamma_{1,2} \cap [T_{min} \leq T_1^\lambda] \cap [T_{min} > T_2^\lambda], \qquad (4.16)$$

where hereafter [.] denotes the set of points verifying the condition into brackets. The third integral of the left hand side in (4.13) verifies

$$
\begin{aligned}
&\int_{\Gamma_{1,2}} \frac{(T_1^\lambda - T_2^\lambda)}{\lambda}((T_{min} - T_1^\lambda)^+ - (T_{min} - T_2^\lambda)^+)d\Gamma \\
&= -\int_B \frac{(T_1^\lambda - T_2^\lambda)^2}{\lambda} d\Gamma \\
&+\sum_{i=1}^{2}\int_{C_i} (-1)^{i+1}\frac{(T_1^\lambda - T_2^\lambda)}{\lambda}(T_{min} - T_i^\lambda)^+ d\Gamma. \qquad (4.17)
\end{aligned}
$$

Since the right hand side of (4.17) is nonpositive, we deduce the same property for the left hand side of (4.13) while its right hand side is nonnegative. Therefore, $(T_{min} - T_i^\lambda)^+ = 0$ on Γ_3^i, $i = 1, 2$, and applying the Poincaré's inequality we obtain the result. □

4.1. An equivalent problem

We are now concerned with the existence of solutions of problem (AP_λ). Let us consider the Kirchhoff transformation given in each domain Ω_i by the function $\beta_i : [T_{min}, +\infty) \to \mathbb{R}$, defined by

$$\beta_i(t) = \int_0^t k_i(s)ds. \tag{4.18}$$

We define U_i^λ as the function

$$U_i^\lambda = \beta_i \circ T_i^\lambda, \tag{4.19}$$

and the constants

$$U_{i,min} = \beta_i(T_{min}), \tag{4.20}$$

$$U_{i,s} = \beta_i(T_s), \tag{4.21}$$

$i = 1, 2$. Notice that, in general, $U_{1,s}$ is not equal to $U_{2,s}$.

The proof of the following lemma is easy to obtain and is given in [15].

LEMMA 4.1. – *i) Both β_i and β_i^{-1} are increasing differentiable functions,* $i = 1, 2$.

ii) $\beta_i^{-1}(t)$ satisfies a Lipschitz condition with constant $\dfrac{1}{k_{min}}$.

iii) $(\beta_i^{-1}(t) - \beta_i^{-1}(\tilde{t}))(t - \tilde{t}) \geq \frac{1}{k_{max}}(t - \tilde{t})^2$.

We state the following

• **Problem** $(\widehat{AP}_\lambda)$

For a fixed $\lambda > 0$, find $(U_1^\lambda, U_2^\lambda, q_1^\lambda, q_2^\lambda)$ in $\mathcal{H}^1(\Omega) \times L^\infty(\Omega_1) \times L^\infty(\Omega_2)$ such that

$$\begin{aligned}
&\sum_{i=1}^2 \int_{\Omega_i} \nabla U_i^\lambda . \nabla z_i dx \\
&+ \sum_{i=1}^2 \int_{\Omega_i} h q_i^\lambda \frac{\partial z_i}{\partial x_1} dx + \sum_{i=1}^2 \int_{\Gamma_1^i} \alpha(\beta_i^{-1}(U_i^\lambda) - T_c) z_i d\Gamma \\
&+ \int_{\Gamma_{1,2}} \frac{(\beta_1^{-1}(U_1^\lambda) - \beta_2^{-1}(U_2^\lambda))}{\lambda}(z_1 - z_2) d\Gamma \\
&= \sum_{i=1}^2 \int_{\Gamma_3^i} h z_i d\Gamma, \; \forall (z_1, z_2) \in \mathcal{H}^1(\Omega),
\end{aligned} \tag{4.22}$$

$$q_i^\lambda \in H(U_i^\lambda - U_{i,s}), \quad i = 1, 2. \tag{4.23}$$

PROPOSITION 4.4. – *The problem (AP_λ) is equivalent to the problem $(\widehat{AP}_\lambda)$.*

Proof. – i) Due to the monotonicity of both β_i and β_i^{-1}, the following set identities hold

$$[T_i^\lambda < T_s] = [U_i^\lambda < U_{i,s}], \tag{4.24}$$
$$[T_i^\lambda > T_s] = [U_i^\lambda > U_{i,s}], \tag{4.25}$$
$$[T_i^\lambda = T_s] = [U_i^\lambda = U_{i,s}], \tag{4.26}$$

and, consequently, we deduce the equivalence

$$q_i^\lambda \in H(T_i^\lambda - T_s) \Leftrightarrow q_i^\lambda \in H(U_i^\lambda - U_{i,s}),\ i = 1, 2. \tag{4.27}$$

ii) Let $(T_1^\lambda, T_2^\lambda, q_1^\lambda, q_2^\lambda)$ be a solution of (AP_λ). Since β_i is a differentiable function, we have $U_i^\lambda \in H^1(\Omega_i)$ and $\nabla U_i^\lambda = k_i(T_i^\lambda)\nabla T_i^\lambda$, $i = 1, 2$ (see [14]). Hence, from (4.11), (4.12) and the equivalence (4.27), we deduce that $(U_1^\lambda, U_2^\lambda, q_1^\lambda, q_2^\lambda)$ is a solution of $(\widehat{\mathrm{AP}}_\lambda)$.

Conversely, given $(U_1^\lambda, U_2^\lambda, q_1^\lambda, q_2^\lambda)$ a solution of $(\widehat{\mathrm{AP}}_\lambda)$, we define $T_i^\lambda = \beta_i^{-1}(U_i^\lambda)$, $i = 1, 2$. Using that β_i^{-1} has derivative and lemma 4.1 ii), we obtain $T_i^\lambda \in H^1(\Omega_i)$ and $\nabla T_i^\lambda = (k_i(\beta_i^{-1}(U_i^\lambda)))^{-1}\nabla U_i^\lambda$, $i = 1, 2$. Therefore, from (4.22), (4.23) and the equivalence (4.27), we deduce that $(T_1^\lambda, T_2^\lambda, q_1^\lambda, q_2^\lambda)$ is a solution of (AP_λ). □

Remark 4.2. – Notice that, given $(T_1^\lambda, T_2^\lambda) \in H^1(\Omega)$, the new variable $(U_1^\lambda, U_2^\lambda)$ does not belong, in general, to $H^1(\Omega)$.

COROLLARY 4.1. – *Under the assumptions (H1)-(H5), let $(U_1^\lambda, U_2^\lambda, q_1^\lambda, q_2^\lambda)$ be a solution of problem $(\widehat{AP}_\lambda)$ then*

$$U_i^\lambda \geq U_{i,min} \text{ a.e. in } \Omega_i, \tag{4.28}$$

where $U_{i,min}$ is given by (4.20), $i = 1, 2$.

Proof. – As in the proof of the proposition 4.4, we deduce that $(\beta_1^{-1}(U_1^\lambda), \beta_2^{-1}(U_2^\lambda), q_1^\lambda, q_2^\lambda)$ is a solution of (AP_λ). Applying the proposition 4.3 and the monotonicity of β_i, (4.28) is deduced. □

4.2. A penalized problem

The proof of existence of solution of the problem $(\widehat{\mathrm{AP}}_\lambda)$ goes through the definition of the following regularized problem:

- **Problem $(\widehat{\mathbf{AP}}_{\lambda\epsilon})$**

For fixed $\lambda > 0$ and $\epsilon > 0$, find $(U^\lambda_{1\epsilon}, U^\lambda_{2\epsilon}) \in \mathcal{H}^1(\Omega)$ such that

$$\sum_{i=1}^{2}\int_{\Omega_i} \nabla U^\lambda_{i\epsilon}.\nabla z_i dx + \sum_{i=1}^{2}\int_{\Omega_i} hH_\epsilon(U^\lambda_{i\epsilon} - U_{i,s})\frac{\partial z_i}{\partial x_1}dx$$
$$+\sum_{i=1}^{2}\int_{\Gamma_1^i} \alpha(\beta_i^{-1}(U^\lambda_{i\epsilon}) - T_c)z_i d\Gamma$$
$$+\int_{\Gamma_{1,2}} \frac{(\beta_1^{-1}(U^\lambda_{1\epsilon}) - \beta_2^{-1}(U^\lambda_{2\epsilon}))}{\lambda}(z_1 - z_2)d\Gamma$$
$$=\sum_{i=1}^{2}\int_{\Gamma_3^i} hz_i d\Gamma, \quad \forall (z_1, z_2) \in \mathcal{H}^1(\Omega), \tag{4.29}$$

where $U_{i,s}$ is defined by (4.21), $i = 1, 2$, and

$$H_\epsilon(s) = \begin{cases} 0 & \text{if } s \leq 0 \\ \frac{s}{\epsilon} & \text{if } 0 \leq s \leq \epsilon \\ 1 & \text{if } s \geq \epsilon \end{cases} \tag{4.30}$$

is the Yosida regularization of the Heaviside multivalued function H.

Let $A_i : H^1(\Omega_i) \to H^1(\Omega_i)'$ be the nonlinear operator defined by

$$< A_i(v), z >= \int_{\Omega_i} \nabla v.\nabla z dx + \int_{\Gamma_1^i} \alpha\beta_i^{-1}(v) z d\Gamma + \int_{\Gamma_{1,2}} \frac{\beta_i^{-1}(v)}{\lambda} z d\Gamma, \tag{4.31}$$

and, for $w \in H^1(\Omega_i)$, let $F_i(w) \in H^1(\Omega_i)'$ be given by

$$< F_i(w), z > = -\int_{\Omega_i} hH_\epsilon(w - U_{i,s})\frac{\partial z}{\partial x_1}dx$$
$$+\int_{\Gamma_1^i} \alpha T_c z d\Gamma + \int_{\Gamma_{1,2}} \frac{g_i}{\lambda} z d\Gamma + \int_{\Gamma_3^i} hz d\Gamma. \tag{4.32}$$

The idea for proving the existence of problem $(\widehat{\text{AP}}_{\lambda\epsilon})$ is to apply the Shauder fixed point theorem to the operator

$$\begin{aligned} \mathcal{L} : L^2(\Gamma_{1,2}) \times L^2(\Gamma_{1,2}) &\to L^2(\Gamma_{1,2}) \times L^2(\Gamma_{1,2}) \\ (g_1, g_2) &\to \mathcal{L}(g_1, g_2) = (\beta_2^{-1}(V^\lambda_{2\epsilon}), \beta_1^{-1}(V^\lambda_{1\epsilon})), \end{aligned} \tag{4.33}$$

$V^\lambda_{i\epsilon}$ being the solution of the following

• **Problem** $(\widehat{\mathbf{AP}}^{i}_{\lambda\,\epsilon})$

For fixed $\lambda > 0$ and $\epsilon > 0$, and given $g_i \in L^2(\Gamma_{1,2})$, find $V^{\lambda}_{i\epsilon}$ in $H^1(\Omega_i)$ such that

$$< A_i(V^{\lambda}_{i\epsilon}), z_i >=< F_i(V^{\lambda}_{i\epsilon}), z_i >, \qquad \forall z_i \in H^1(\Omega_i), \tag{4.34}$$

for $i = 1, 2$.

Notice that, in order to define the operator $\mathcal{L}$, two uncoupled problems posed in Ω_1 and Ω_2 have to be solved, namely $(\widehat{\mathrm{AP}}^{i}_{\lambda\,\epsilon})$, $i = 1, 2$.

Remark 4.3. – If we set $g_i = \beta_j^{-1}(V^{\lambda}_{j\epsilon})$ for $j \neq i$ in (4.32), the equation (4.34) becomes (4.29) for the test function $z_i \in H^1(\Omega_i)$ and $z_j = 0 \in H^1(\Omega_j)$.

We prove first that $\mathcal{L}$ is well defined:

PROPOSITION 4.5. – *Under the assumptions (H1)-(H5), there exists a solution* $V^{\lambda}_{i\epsilon}$ *of problem* $(\widehat{AP}^{i}_{\lambda\,\epsilon})$, $i = 1, 2$.

Proof. – We consider L_i the mapping which associates to $w \in H^1(\Omega_i)$ the solution of the nonlinear problem:

$$< A_i(W^{\lambda}_{i\epsilon}), z_i >=< F_i(w), z_i >, \qquad \forall z_i \in H^1(\Omega_i). \tag{4.35}$$

Step 1. – A_i verifies the following properties:

• A_i is a continuous operator:

Applying the Cauchy-Schwarz inequality, the lemma 4.1, ii) and the continuity of the trace, we deduce

$$|< A_i(v_n) - A_i(v), z_i >| \leq K \parallel v_n - v \parallel_{1,2,\Omega_i} \parallel z_i \parallel_{1,2,\Omega_i}, \tag{4.36}$$

where K is a constant depending on $\parallel \alpha \parallel_{\infty,\Gamma_1^i}$, k_{min} and λ.

• A_i is a strongly monotone operator:

Using lemma 4.1, iii), we have

$$\begin{aligned} < A_i(v_1) - A_i(v_2), v_1 - v_2 > &\geq \int_{\Omega_i} | \nabla(v_1 - v_2) |^2 \, dx \\ &+ \frac{\alpha_{min}}{k_{max}} \int_{\Gamma_1^i} (v_1 - v_2)^2 d\Gamma + \frac{1}{\lambda k_{max}} \int_{\Gamma_{1,2}} (v_1 - v_2)^2 d\Gamma \\ &\geq C_i^{\star} \parallel v_1 - v_2 \parallel^2_{1,2,\Omega_i}, \end{aligned} \tag{4.37}$$

with $C_i^{\star}$ depending on α_{min}, k_{max} and λ.

• A_i is a coercive operator:

Choosing $v_2 = 0$ in (4.37) and taking into account that $A_i(0) = 0$, we deduce

$$< A_i(v_1), v_1 >\geq C_i^\star \parallel v_1 \parallel_{1,2,\Omega_i}^2, \tag{4.38}$$

and then

$$\lim_{\|v_1\|_{1,2,\Omega_i}\to\infty} \frac{< A_i(v_1), v_1 >}{\parallel v_1 \parallel_{1,2,\Omega_i}} = +\infty. \tag{4.39}$$

Since $F_i(w) \in H^1(\Omega_i)'$, by applying the Minty-Browder theorem (see [7]), we obtain the existence of a unique $W_{i\epsilon}^\lambda \in H^1(\Omega_i)$ such that $A_i(W_{i\epsilon}^\lambda) = F_i(w)$. Therefore, L_i is well defined.

Step 2. – L_i is compact.

Indeed, it is enough to prove the complete continuity of L_i. For this purpose, let $\{w_n\}$ be a sequence in $H^1(\Omega_i)$ which converges weakly to $w \in H^1(\Omega_i)$.

Let $\{W_{i\epsilon}^{\lambda n}\}$ be the sequence defined by

$$W_{i\epsilon}^{\lambda n} = L_i(w_n), \quad \forall n \in \mathbb{N}. \tag{4.40}$$

Then, $W_{i\epsilon}^{\lambda n}$ y $W_{i\epsilon}^\lambda$ are the unique solutions of

$$< A_i(W_{i\epsilon}^{\lambda n}), z_i >=< F_i(w_n), z_i >, \forall z_i \in H^1(\Omega_i), \tag{4.41}$$

and

$$< A_i(W_{i\epsilon}^{\lambda}), z_i >=< F_i(w), z_i >, \quad \forall z_i \in H^1(\Omega_i), \tag{4.42}$$

respectively. By substracting (4.42) from (4.41), taking $z_i = W_{i\epsilon}^{\lambda n} - W_{i\epsilon}^\lambda$ as a test function, and applying the definition of A_i (see (4.31)), we deduce

$$\begin{aligned} &< A_i(W_{i\epsilon}^{\lambda n}) - A_i(W_{i\epsilon}^{\lambda}), W_{i\epsilon}^{\lambda n} - W_{i\epsilon}^{\lambda} > \\ &= -\int_{\Omega_i} h(H_\epsilon(w_n - U_{i,s}) - H_\epsilon(w - U_{i,s}))\frac{\partial(W_{i\epsilon}^{\lambda n} - W_{i\epsilon}^{\lambda})}{\partial x_1}dx. \end{aligned} \tag{4.43}$$

From (4.37), the Cauchy-Schwarz inequality and the Lipschitz continuity of H_ϵ, it follows that

$$C_i^\star \parallel W_{i\epsilon}^{\lambda n} - W_{i\epsilon}^{\lambda} \parallel_{1,2,\Omega_i}\leq \frac{\parallel h \parallel_{\infty,\Omega_i}}{\epsilon} \parallel w_n - w \parallel_{0,2,\Omega_i}. \tag{4.44}$$

Since $H^1(\Omega_i)$ is compactly imbedded in $L^2(\Omega_i)$, the complete continuity of L_i is now clear.

Step 3. – L_i maps $H^1(\Omega_i)$ in a ball.

Indeed, by taking $z_i = W_{i\epsilon}^\lambda$ as a test function in (4.35) and applying the definition of A_i we have

$$< A_i(W_{i\epsilon}^\lambda), W_{i\epsilon}^\lambda > = < F_i(w), W_{i\epsilon}^\lambda > . \tag{4.45}$$

Using (4.38), the Cauchy-Schwarz inequality, the fact that $| H_\epsilon(s) | \leq 1$ and the continuity of the trace we deduce

$$\begin{aligned} C_i^\star \| W_{i\epsilon}^\lambda \|_{1,2,\Omega_i}^2 \leq & \| h \|_{0,2,\Omega_i} \| \frac{\partial W_{i\epsilon}^\lambda}{\partial x_1} \|_{0,2,\Omega_i} \\ & + \| \alpha \|_{\infty,\Gamma_1^i} \| T_c \|_{0,2,\Gamma_1^i} \| W_{i\epsilon}^\lambda \|_{0,2,\Gamma_1^i} \\ & + \frac{1}{\lambda} \| g_i \|_{0,2,\Gamma_{1,2}} \| W_{i\epsilon}^\lambda \|_{0,2,\Gamma_{1,2}} + \| h \|_{0,2,\Gamma_3^i} \| W_{i\epsilon}^\lambda \|_{0,2,\Gamma_3^i} \\ & \leq \widehat{C} \| W_{i\epsilon}^\lambda \|_{1,2,\Omega_i}, \end{aligned} \tag{4.46}$$

therefore $W_{i\epsilon}^\lambda$ belongs to the ball of $H^1(\Omega_i)$ with center 0 and radius $R = \dfrac{\widehat{C}}{C^\star}$.

Finally, the existence of a function $V_{i\epsilon}^\lambda$ satisfying (4.34) results from the Schauder fixed point theorem. □

The proof of the following result is similar to that obtained in [8], [9]. The difference comes from the boundary integrals of (4.34).

PROPOSITION 4.6. – *Under the assumptions (H1)-(H5), the solution $V_{i\epsilon}^\lambda$ of the problem $(\widehat{AP}^i_{\lambda\epsilon})$ is unique, $i = 1, 2$.*

Proof. – Let $V_{i\epsilon}^{\lambda 1}$ and $V_{i\epsilon}^{\lambda 2}$ be two solutions of (4.34) and $Q_{i\epsilon}^\lambda = V_{i\epsilon}^{\lambda 1} - V_{i\epsilon}^{\lambda 2}$. We consider the function

$$p_\delta(x) = \begin{cases} (1 - \frac{\delta}{x})^+ & \text{si} \quad x > 0 \\ 0 & \text{si} \quad x \leq 0, \end{cases} \tag{4.47}$$

for a fixed $\delta > 0$. Since p_δ is a Lipschitz function, $p_\delta(Q_{i\epsilon}^\lambda)$ belongs to $H^1(\Omega_i)$ (see [14]). From the equalities satisfied by $V_{i\epsilon}^{\lambda 1}$ and $V_{i\epsilon}^{\lambda 2}$ and taking $z_i = p_\delta(Q_{i\epsilon}^\lambda)$ as a test function, we obtain

$$\begin{aligned} & \delta \int_{\Omega_i} \frac{| \nabla (Q_{i\epsilon}^\lambda - \delta)^+ |^2}{(Q_{i\epsilon}^\lambda)^2} dx \\ & + \delta \int_{\Omega_i} h \{ H_\epsilon(V_{i\epsilon}^{\lambda 1} - U_{i,s}) - H_\epsilon(V_{i\epsilon}^{\lambda 2} - U_{i,s}) \} \frac{\frac{\partial (Q_{i\epsilon}^\lambda - \delta)^+}{\partial x_1}}{(Q_{i\epsilon}^\lambda)^2} dx \end{aligned}$$

$$+\int_{\Gamma_1^i} \alpha(\beta_i^{-1}(V_{i\epsilon}^{\lambda 1}) - \beta_i^{-1}(V_{i\epsilon}^{\lambda 2}))\frac{(Q_{i\epsilon}^{\lambda}-\delta)^+}{Q_{i\epsilon}^{\lambda}}d\Gamma$$
$$+\int_{\Gamma_{1,2}} \frac{(\beta_i^{-1}(V_{i\epsilon}^{\lambda 1}) - \beta_i^{-1}(V_{i\epsilon}^{\lambda 2}))}{\lambda}\frac{(Q_{i\epsilon}^{\lambda}-\delta)^+}{Q_{i\epsilon}^{\lambda}}d\Gamma = 0. \tag{4.48}$$

Taking into account the monotonicity of β_i^{-1}, the lemma 4.1, iii) and the fact that h is a nonnegative function, we deduce

$$\delta\int_{\Omega_i} \frac{|\nabla(Q_{i\epsilon}^{\lambda}-\delta)^+|^2}{(Q_{i\epsilon}^{\lambda})^2}dx + \frac{\alpha_{min}}{k_{max}}\int_{\Gamma_1^i}\frac{|(Q_{i\epsilon}^{\lambda}-\delta)^+|^2}{Q_{i\epsilon}^{\lambda}}d\Gamma$$
$$+\frac{1}{\lambda k_{max}}\int_{\Gamma_{1,2}}\frac{|(Q_{i\epsilon}^{\lambda}-\delta)^+|^2}{Q_{i\epsilon}^{\lambda}}d\Gamma$$
$$\leq \delta\int_{\Omega_i} h\,|\,H_\epsilon(V_{i\epsilon}^{\lambda 1}-U_{i,s}) - H_\epsilon(V_{i\epsilon}^{\lambda 2}-U_{i,s})\,|\left|\frac{\frac{\partial(Q_{i\epsilon}^{\lambda}-\delta)^+}{\partial x_1}}{(Q_{i\epsilon}^{\lambda})^2}\right|dx. \tag{4.49}$$

Since H_ϵ is a Lipschitz function, by using the Cauchy-Schwarz inequality we have

$$\delta\int_{\Omega_i} \frac{|\nabla(Q_{i\epsilon}^{\lambda}-\delta)^+|^2}{(Q_{i\epsilon}^{\lambda})^2}dx + \frac{\alpha_{min}}{k_{max}}\int_{\Gamma_1^i}\frac{|(Q_{i\epsilon}^{\lambda}-\delta)^+|^2}{Q_{i\epsilon}^{\lambda}}d\Gamma$$
$$+\frac{1}{\lambda k_{max}}\int_{\Gamma_{1,2}}\frac{|(Q_{i\epsilon}^{\lambda}-\delta)^+|^2}{Q_{i\epsilon}^{\lambda}}d\Gamma$$
$$\leq \frac{\delta\,\|h\|_{0,2,\Omega_i}}{\epsilon}\left\|\frac{|\nabla(Q_{i\epsilon}^{\lambda}-\delta)^+|}{Q_{i\epsilon}^{\lambda}}\right\|_{0,2,\Omega_i}. \tag{4.50}$$

This leads to

$$\left\|\frac{|\nabla(Q_{i\epsilon}^{\lambda}-\delta)^+|}{Q_{i\epsilon}^{\lambda}}\right\|_{0,2,\Omega_i} \leq \frac{\|h\|_{0,2,\Omega_i}}{\epsilon} = C_\epsilon, \tag{4.51}$$

with C_ϵ independent of δ. Thus, dividing (4.50) by δ and using the latter expression, we have

$$\int_{\Omega_i} \frac{|\nabla(Q_{i\epsilon}^{\lambda}-\delta)^+|^2}{(Q_{i\epsilon}^{\lambda})^2}dx + \frac{\alpha_{min}}{\delta k_{max}}\int_{\Gamma_1^i}\frac{|(Q_{i\epsilon}^{\lambda}-\delta)^+|^2}{Q_{i\epsilon}^{\lambda}}d\Gamma$$
$$+\frac{1}{\delta\lambda k_{max}}\int_{\Gamma_{1,2}}\frac{|(Q_{i\epsilon}^{\lambda}-\delta)^+|^2}{Q_{i\epsilon}^{\lambda}}d\Gamma \leq C_\epsilon^2. \tag{4.52}$$

By passing to the limit when δ goes to zero , we obtain that $(Q_{i\epsilon}^{\lambda}-\delta)^{+}=0$ a.e. on Γ_1^i and $\Gamma_{1,2}$. On the other hand, after an easy computation we deduce

$$\nabla \ln\left(1+\frac{(Q_{i\epsilon}^{\lambda}-\delta)^{+}}{\delta}\right)=\frac{\nabla(Q_{i\epsilon}^{\lambda}-\delta)^{+}}{Q_{i\epsilon}^{\lambda}}, \tag{4.53}$$

and from (4.52) we obtain

$$\int_{\Omega_i}\left|\nabla ln\left(1+\frac{(Q_{i\epsilon}^{\lambda}-\delta)^{+}}{\delta}\right)\right|^2 dx\leq C_\epsilon^2. \tag{4.54}$$

Now by using the Poincaré's inequality, and letting $\delta\to 0$, we obtain that $Q_{i\epsilon}^{\lambda}\leq 0$ a.e. in Ω_i. Interchanging the roles of $V_{i\epsilon}^{\lambda 1}$ and $V_{i\epsilon}^{\lambda 2}$, we deduce $Q_{i\epsilon}^{\lambda}=0$ a.e. in Ω_i, $i=1,2$ which completes the proof. □

We are now able to prove the following

PROPOSITION 4.7. – *Let $g_i\in L^2(\Gamma_{1,2})$ be such that $g_i\leq T_s+\dfrac{\epsilon}{k_{min}}$ a.e. on $\Gamma_{1,2}$.*

Under the assumptions (H1)-(H6), the solution $V_{i\epsilon}^{\lambda}$ of $(\widehat{AP}^{i}_{\lambda\epsilon})$ verifies

$$\beta_i^{-1}(V_{i\epsilon}^{\lambda})\ \leq T_s+\frac{\epsilon}{k_{min}} \text{ a.e. in } \Omega_i,\ i=1,2. \tag{4.55}$$

Proof. – Let $P_{i\epsilon}^{\lambda}=\beta_i^{-1}(V_{i\epsilon}^{\lambda})$. Taking $z_i=\left(P_{i\epsilon}^{\lambda}-T_s-\dfrac{\epsilon}{k_{min}}\right)^{+}$ as a test function in (4.34), we obtain

$$\begin{aligned}&\int_{\Omega_i}\nabla V_{i\epsilon}^{\lambda}.\nabla\left(P_{i\epsilon}^{\lambda}-T_s-\frac{\epsilon}{k_{min}}\right)^{+}dx+\int_{\Omega_i}h\frac{\partial\left(P_{i\epsilon}^{\lambda}-T_s-\frac{\epsilon}{k_{min}}\right)^{+}}{\partial x_1}dx\\&+\int_{\Gamma_1^i}\alpha(P_{i\epsilon}^{\lambda}-T_c)\left(P_{i\epsilon}^{\lambda}-T_s-\frac{\epsilon}{k_{min}}\right)^{+}d\Gamma\\&+\int_{\Gamma_{1,2}}\frac{(P_{i\epsilon}^{\lambda}-g_i)}{\lambda}\left(P_{i\epsilon}^{\lambda}-T_s-\frac{\epsilon}{k_{min}}\right)^{+}d\Gamma\\&=\int_{\Gamma_3^i}h\left(P_{i\epsilon}^{\lambda}-T_s-\frac{\epsilon}{k_{min}}\right)^{+}d\Gamma.\end{aligned} \tag{4.56}$$

Notice that $H_\epsilon(V_{i\epsilon}^{\lambda}-U_{i,s})=1$ a.e. in $\mathcal{N}=[P_{i\epsilon}^{\lambda}>T_s+\frac{\epsilon}{k_{min}}]$. Indeed, if $(x_1,x_2)\in\mathcal{N}$ then $V_{i\epsilon}^{\lambda}(x_1,x_2)>\beta_i(T_s+\frac{\epsilon}{k_{min}})$, and taking into account (H1), it follows that

$$V_{i\epsilon}^{\lambda}(x_1,x_2)>U_{i,s}+\int_{T_s}^{T_s+\frac{\epsilon}{k_{min}}}k_i(s)ds\geq U_{i,s}+\epsilon. \tag{4.57}$$

Applying the Green formula in (4.56) we obtain:

$$\int_{\Omega_i} k_i(P_{i\epsilon}^\lambda) \mid \nabla\left(P_{i\epsilon}^\lambda - T_s - \frac{\epsilon}{k_{min}}\right)^+ \mid^2 dx$$
$$+ \int_{\Gamma_1^i} (\alpha(P_{i\epsilon}^\lambda - T_c) + hn_1)\left(P_{i\epsilon}^\lambda - T_s - \frac{\epsilon}{k_{min}}\right)^+ d\Gamma$$
$$+ \int_{\Gamma_{1,2}} \frac{(P_{i\epsilon}^\lambda - g_i)}{\lambda}\left(P_{i\epsilon}^\lambda - T_s - \frac{\epsilon}{k_{min}}\right)^+ d\Gamma = 0, \tag{4.58}$$

since h only depends on x_2.

On the other hand, (H6) leads to

$$\alpha(P_{i\epsilon}^\lambda - T_c) + hn_1 > 0 \text{ a.e. on } \Gamma_1^i,\ i = 1,2. \tag{4.59}$$

Therefore, we deduce that all of the terms in the left hand side of (4.58) are nonnegative, and then all of them must be equal to zero. Thus $P_{i\epsilon}^\lambda \leq T_s + \frac{\epsilon}{k_{min}}$ a.e. on Γ_1^i, and applying the Poincaré's inequality, the result follows. □

COROLLARY 4.2. – *The assumptions being those of the proposition 4.7, the inequality* $V_{i\epsilon}^\lambda \leq U_{i,s} + \frac{k_{max}\epsilon}{k_{min}}$ *holds a.e. in* Ω_i, $i = 1, 2$.

Proof. – By using the proposition 4.7 and the monotonicity of β_i we get

$$V_{i\epsilon}^\lambda \leq \beta_i(T_s + \frac{\epsilon}{k_{min}}) \text{ a.e. in } \Omega_i. \tag{4.60}$$

Finally, from the definition of β_i and (H1) we obtain

$$V_{i\epsilon}^\lambda \leq \int_0^{T_s} k_i(s)ds + \int_{T_s}^{T_s+\frac{\epsilon}{k_{min}}} k_i(s)ds \leq U_{i,s} + \frac{k_{max}\epsilon}{k_{min}},\ i = 1, 2. \quad \square \tag{4.61}$$

PROPOSITION 4.8. – *Under the assumptions (H1)-(H6), there exists a solution* $(U_{1\epsilon}^\lambda, U_{2\epsilon}^\lambda)$ *of the coupled regularized problem* $(\widehat{AP}_{\lambda\epsilon})$*, defined by (4.29), such that* $U_{i\epsilon}^\lambda \leq U_{i,s} + \frac{k_{max}}{k_{min}}\epsilon$ *a.e. in* Ω_i, $i = 1, 2$.

Proof. – Let us consider the space $L^2(\Gamma_{1,2}) \times L^2(\Gamma_{1,2})$ with the norm

$$\| (f, g) \|_{L^2(\Gamma_{1,2})\times L^2(\Gamma_{1,2})} = \left(\| f \|_{0,2,\Gamma_{1,2}}^2 + \| g \|_{0,2,\Gamma_{1,2}}^2\right)^{\frac{1}{2}}, \tag{4.62}$$

and $\mathcal{L}$ the operator introduced in (4.33), where $V_{i\epsilon}^{\lambda}$ is the solution of problem $(\widehat{\text{AP}}^{i}_{\lambda\,\epsilon})$ corresponding to g_i, $i = 1, 2$. The propositions (4.5) and (4.6) imply that $\mathcal{L}$ is well defined.

Let $\{(g_1^n, g_2^n)\}$ be a bounded sequence in $L^2(\Gamma_{1,2}) \times L^2(\Gamma_{1,2})$. Then, $\{g_i^n\}$ is a bounded sequence in $L^2(\Gamma_{1,2})$, $i = 1, 2$.

We denote by $V_{i\epsilon}^{\lambda n}$ the solution of $(\widehat{\text{AP}}^{i}_{\lambda\,\epsilon})$ corresponding to g_i^n, $i = 1, 2$. If we set $z_i = V_{i\epsilon}^{\lambda n}$ as a test function in (4.34), we have

$$< A_i(V_{i\epsilon}^{\lambda n}) \,,\, V_{i\epsilon}^{\lambda n} > = < F_i(V_{i\epsilon}^{\lambda n}), V_{i\epsilon}^{\lambda n} > . \tag{4.63}$$

Applying (4.38), the Cauchy-Schwarz inequality and $| H_\epsilon(s) | \leq 1$, it follows that

$$\begin{aligned} C_i^\star \| V_{i\epsilon}^{\lambda n} \|_{1,2,\Omega_i}^2 &\leq \| h \|_{0,2,\Gamma_3^i} \| V_{i\epsilon}^{\lambda n} \|_{0,2,\Gamma_3^i} \\ &+ \| h \|_{0,2,\Omega_i} \| \frac{\partial V_{i\epsilon}^{\lambda n}}{\partial x_1} \|_{0,2,\Omega_i} + \| \alpha \|_{\infty,\Gamma_1^i} \| T_c \|_{0,2,\Gamma_1^i} \| V_{i\epsilon}^{\lambda n} \|_{0,2,\Gamma_1^i} \\ &+ \frac{1}{\lambda} \| g_i^n \|_{0,2,\Gamma_{1,2}} \| V_{i\epsilon}^{\lambda n} \|_{0,2,\Gamma_{1,2}} \leq \widehat{C} \| V_{i\epsilon}^{\lambda n} \|_{1,2,\Omega_i}, \end{aligned} \tag{4.64}$$

hence,

$$\| V_{i\epsilon}^{\lambda n} \|_{1,2,\Omega_i} \leq \tilde{C}, \tag{4.65}$$

where $\tilde{C}$ is a constant which depends on λ but not on ϵ. It follows that $\{V_{i\epsilon}^{\lambda n}\}$ is bounded in $H^1(\Omega_i)$, and then it has a subsequence $\{V_{i\epsilon}^{\lambda n_k}\}$ weakly convergent to an element $V_{i\epsilon}^{\lambda}$ in $H^1(\Omega_i)$, $i = 1, 2$. Consequently, $\{V_{i\epsilon}^{\lambda n_k}\}$ converges strongly to $V_{i\epsilon}^{\lambda}$ in $L^2(\Gamma_{1,2})$. Hence β_i^{-1} being Lipschitz continuous it follows that $\{\beta_i^{-1}(V_{i\epsilon}^{\lambda n_k})\}$ converges strongly to $\beta_i^{-1}(V_{i\epsilon}^{\lambda})$ in $L^2(\Gamma_{1,2})$, $i = 1, 2$. Thus $\mathcal{L}$ is compact.

We define

$$\mathcal{M} = \Big\{ (g_1, g_2) \in L^2(\Gamma_{1,2}) \times L^2(\Gamma_{1,2}) : \\ 0 \leq g_i \leq T_s + \frac{\epsilon}{k_{min}} \text{ a.e. in } \Gamma_{1,2},\ i = 1, 2 \Big\} \tag{4.66}$$

It is clear from proposition 4.7 that

$$\mathcal{L}(\mathcal{M}) \subset \mathcal{M}. \tag{4.67}$$

$\mathcal{M}$ being a closed bounded convex set, the existence of a fixed point of $\mathcal{L}$, denoted by $(U_{1\epsilon}^{\lambda}, U_{2\epsilon}^{\lambda})$, results from the Schauder fixed point theorem.

Finally, from corollary 4.2 we obtain

$$U_{i\epsilon}^{\lambda} \leq U_{i,s} + \frac{k_{max}}{k_{min}}\epsilon \text{ a.e. in } \Omega_i, \; i = 1, 2. \quad \square \tag{4.68}$$

In the next proposition we pass to the limit in ϵ.

PROPOSITION 4.9. – *Under the assumptions (H1)-(H6), there exists a solution* $(U_1^{\lambda}, U_2^{\lambda}, q_1^{\lambda}, q_2^{\lambda})$ *of the problem* $(\widehat{AP}_{\lambda})$*, defined by (4.22) and (4.23), such that* $U_i^{\lambda} \leq U_{i,s}$ *a.e. in* Ω_i, $i = 1, 2$.

Proof. – For a fixed $\epsilon > 0$, let $(U_{1\epsilon}^{\lambda}, U_{2\epsilon}^{\lambda})$ be a solution of $(\widehat{\mathrm{AP}}_{\lambda\epsilon})$.

As in the proof of the proposition 4.8, we deduce that $\{(U_{1\epsilon}^{\lambda}, U_{2\epsilon}^{\lambda})\}$ is bounded in $\mathcal{H}^1(\Omega)$ independently of ϵ and so we can extract a subsequence of ϵ still denoted by ϵ such that

$$\{U_{i\epsilon}^{\lambda}\} \rightharpoonup U_i^{\lambda} \text{ in } H^1(\Omega_i) \text{ weakly}, \tag{4.69}$$

$$\{U_{i\epsilon}^{\lambda}\} \to U_i^{\lambda} \text{ in } L^2(\Omega_i) \text{ strongly}, \tag{4.70}$$

$$\{U_{i\epsilon}^{\lambda}\} \to U_i^{\lambda} \text{ a.e. in } \Omega_i. \tag{4.71}$$

Moreover, $\{U_{i\epsilon}^{\lambda}\}$ converges strongly to U_i^{λ} in $L^2(\Gamma_{1,2})$, $L^2(\Gamma_1^i)$ and $L^2(\Gamma_3^i)$, $i = 1, 2$. From (4.68) and (4.71) it follows that

$$U_i^{\lambda}(x) \leq U_{i,s} \text{ a.e. in } \Omega_i, \; i = 1, 2. \tag{4.72}$$

Since $\{H_{\epsilon}(U_{1\epsilon}^{\lambda} - U_{1,s}), H_{\epsilon}(U_{2\epsilon}^{\lambda} - U_{2,s})\}$ is bounded in $L^2(\Omega_1) \times L^2(\Omega_2)$, there exists $(q_1^{\lambda}, q_2^{\lambda})$ in $L^2(\Omega_1) \times L^2(\Omega_2)$ such that

$$\{H_{\epsilon}(U_{i\epsilon}^{\lambda} - U_{i,s})\} \rightharpoonup q_i^{\lambda} \text{ in } L^2(\Omega_i), \; i = 1, 2. \tag{4.73}$$

On the other hand, notice that q_i^{λ} belongs to the closed convex set $\mathcal{N}_i$ defined by

$$\mathcal{N}_i = \{f \in L^{\infty}(\Omega_i) : \; 0 \leq f \leq 1 \text{ a.e. in } \Omega_i\} \tag{4.74}$$

since this set is weakly closed.

Finally, in the set $[U_i^{\lambda} < U_{i,s}]$ we have

$$H_{\epsilon}(U_{i\epsilon}^{\lambda} - U_{i,s}) \to 0 \text{ a.e.}, \tag{4.75}$$

and applying the Lebesgue theorem we get

$$H_{\epsilon}(U_{i\epsilon}^{\lambda} - U_{i,s}) \to 0 \text{ in } L^2([U_i^{\lambda} < U_{i,s}]). \tag{4.76}$$

From (4.73), we deduce

$$H_{\epsilon}(U_{i\epsilon}^{\lambda} - U_{i,s}) \rightharpoonup q_i^{\lambda} \text{ in } L^2([U_i^{\lambda} < U_{i,s}]), \tag{4.77}$$

and by the uniqueness of the limit

$$q_i^{\lambda} = 0 \text{ a.e. in } [U_i^{\lambda} < U_{i,s}], \; i = 1, 2, \tag{4.78}$$

which completes the proof. $\square$

5. A PRIORI ESTIMATES

This section is devoted to obtaining some estimates for the solution of the auxiliary problem $(\widehat{\mathrm{AP}}_\lambda)$.

PROPOSITION 5.10. – *Under the assumptions (H1)-(H6), a solution* $(U_1^\lambda, U_2^\lambda, q_1^\lambda, q_2^\lambda)$ *of the problem* $(\widehat{AP}_\lambda)$*, defined by (4.22) and (4.23), satisfies*

$$U_i^\lambda \leq U_{i,s} \text{ a.e. in } \Omega_i,\ i = 1, 2. \tag{5.1}$$

Proof. – Taking $(z_1, z_2) = ((\beta_1^{-1}(U_1^\lambda) - T_s)^+, (\beta_2^{-1}(U_2^\lambda) - T_s)^+)$ as a test function in (4.22) and using $\nabla U_i^\lambda = k_i(\beta_i^{-1}(U_i^\lambda))\nabla\beta_i^{-1}(U_i^\lambda)$, we obtain

$$\begin{aligned}
&\sum_{i=1}^{2} \int_{\Omega_i} k_i(\beta_i^{-1}(U_i^\lambda)) \mid \nabla(\beta_i^{-1}(U_i^\lambda) - T_s)^+ \mid^2 dx \\
&+\sum_{i=1}^{2} \int_{\Omega_i} h q_i^\lambda \frac{\partial(\beta_i^{-1}(U_i^\lambda) - T_s)^+}{\partial x_1} dx \\
&+\sum_{i=1}^{2} \int_{\Gamma_1^i} \alpha(\beta_i^{-1}(U_i^\lambda) - T_c)(\beta_i^{-1}(U_i^\lambda) - T_s)^+ d\Gamma \\
&+\int_{\Gamma_{1,2}} \frac{(\beta_1^{-1}(U_1^\lambda) - \beta_2^{-1}(U_2^\lambda))}{\lambda}((\beta_1^{-1}(U_1^\lambda) - T_s)^+ - (\beta_2^{-1}(U_2^\lambda) - T_s)^+)d\Gamma \\
&= \sum_{i=1}^{2} \int_{\Gamma_3^i} h(\beta_i^{-1}(U_i^\lambda) - T_s)^+ d\Gamma.
\end{aligned} \tag{5.2}$$

Notice that $q_i^\lambda = 1$ a.e. in $[U_i^\lambda > U_{i,s}]$. By applying the Green formula and taking into account that h only depends on x_2 it follows that

$$\begin{aligned}
&\sum_{i=1}^{2} \int_{\Omega_i} k_i(\beta_i^{-1}(U_i^\lambda)) \mid \nabla(\beta_i^{-1}(U_i^\lambda) - T_s)^+ \mid^2 dx \\
&+\sum_{i=1}^{2} \int_{\Gamma_1^i} \big(\alpha(\beta_i^{-1}(U_i^\lambda) - T_c) + h n_1\big)(\beta_i^{-1}(U_i^\lambda) - T_s)^+ d\Gamma \\
&+\int_{\Gamma_{1,2}} \frac{(\beta_1^{-1}(U_1^\lambda) - \beta_2^{-1}(U_2^\lambda))}{\lambda}((\beta_1^{-1}(U_1^\lambda) - T_s)^+ - (\beta_2^{-1}(U_2^\lambda) - T_s)^+)d\Gamma = 0.
\end{aligned} \tag{5.3}$$

From (H6) we deduce that $\alpha(\beta_i^{-1}(U_i^\lambda) - T_c) + hn_1 > 0$ a.e. on Γ_1^i. On the other hand, as in the proof of the proposition 4.3, it follows that the third term of the expression (5.3) is nonnegative. Consequently all of the terms of (5.3) are nonnegative and then they are equal to zero. Thus $(\beta_i^{-1}(U_i^\lambda) - T_s)^+ = 0$ a.e. in Γ_1^i, $i = 1, 2$. By the Poincaré's inequality and the monotonicity of β_i^{-1}, $i = 1, 2$, we deduce (5.1). □

COROLLARY 5.3. – *Under the assumptions (H1)-(H6), a solution* $(T_1^\lambda, T_2^\lambda, q_1^\lambda, q_2^\lambda)$ *of the problem* (AP_λ) *defined by (4.11) and (4.12) verifies*

$$T_i^\lambda \le T_s \text{ a.e. in } \Omega_i,\ i = 1, 2. \tag{5.4}$$

Proof. – As in the proof of proposition 4.4, we obtain that $(U_1^\lambda, U_2^\lambda, q_1^\lambda, q_2^\lambda)$ is a solution of the problem $(\widehat{AP}_\lambda)$. By applying the proposition 5.10 it follows that $U_i^\lambda \le U_{i,s}$ a.e. in Ω_i and then using the monotonicity of β_i^{-1}, $i = 1, 2$, we have (5.4). □

PROPOSITION 5.11. – *Under the assumptions (H1)-(H6), a solution* $(U_1^\lambda, U_2^\lambda, q_1^\lambda, q_2^\lambda)$ *of the problem* $(\widehat{AP}_\lambda)$ *verifies*

$$\| \beta_1^{-1}(U_1^\lambda) - \beta_2^{-1}(U_2^\lambda) \|_{0,2,\Gamma_{1,2}} \le K\lambda^{\frac{1}{2}},\ \lambda > 0, \tag{5.5}$$

with K a constant which does not depend on λ.

Proof. – Taking $(z_1, z_2) = (\beta_1^{-1}(U_1^\lambda), \beta_2^{-1}(U_2^\lambda))$ as a test function in (4.22), we obtain

$$\sum_{i=1}^{2} \int_{\Omega_i} \frac{1}{k_i(\beta_i^{-1}(U_i^\lambda))} \mid \nabla U_i^\lambda \mid^2 dx + \sum_{i=1}^{2} \int_{\Omega_i} \frac{h q_i^\lambda}{k_i(\beta_i^{-1}(U_i^\lambda))} \frac{\partial U_i^\lambda}{\partial x_1} dx$$
$$+ \sum_{i=1}^{2} \int_{\Gamma_1^i} \alpha(\beta_i^{-1}(U_i^\lambda) - T_c)\beta_i^{-1}(U_i^\lambda) d\Gamma + \int_{\Gamma_{1,2}} \frac{(\beta_1^{-1}(U_1^\lambda) - \beta_2^{-1}(U_2^\lambda))^2}{\lambda} d\Gamma$$
$$= \sum_{i=1}^{2} \int_{\Gamma_3^i} h\beta_i^{-1}(U_i^\lambda) d\Gamma. \tag{5.6}$$

From proposition 5.10, we have $U_i^\lambda \le U_{i,s}$ a.e. in Ω_i. Using both $q_i^\lambda = 0$ a.e. in $[U_i^\lambda < U_{i,s}]$ and $\frac{\partial U_i^\lambda}{\partial x_1} = 0$ a.e. in $[U_i^\lambda = U_{i,s}]$, the second term of (5.6) vanishes and we have

$$\sum_{i=1}^{2} \int_{\Omega_i} \frac{1}{k_i(\beta_i^{-1}(U_i^\lambda))} \mid \nabla U_i^\lambda \mid^2 dx + \sum_{i=1}^{2} \int_{\Gamma_1^i} \alpha(\beta_i^{-1}(U_i^\lambda))^2 d\Gamma$$
$$+ \int_{\Gamma_{1,2}} \frac{(\beta_1^{-1}(U_1^\lambda) - \beta_2^{-1}(U_2^\lambda))^2}{\lambda} d\Gamma$$
$$= \sum_{i=1}^{2} \int_{\Gamma_3^i} h\beta_i^{-1}(U_i^\lambda) d\Gamma + \sum_{i=1}^{2} \int_{\Gamma_1^i} \alpha T_c \beta_i^{-1}(U_i^\lambda) d\Gamma. \tag{5.7}$$

Since all of the terms on the left hand side of (5.7) are nonnegative, we obtain

$$\int_{\Gamma_{1,2}} \frac{(\beta_1^{-1}(U_1^\lambda) - \beta_2^{-1}(U_2^\lambda))^2}{\lambda} d\Gamma \leq \sum_{i=1}^{2} \left(\int_{\Gamma_3^i} h\beta_i^{-1}(U_i^\lambda) d\Gamma + \int_{\Gamma_1^i} \alpha T_c \beta_i^{-1}(U_i^\lambda) d\Gamma \right). \tag{5.8}$$

By the Cauchy-Schwarz inequality, it follows that

$$\int_{\Gamma_{1,2}} \frac{(\beta_1^{-1}(U_1^\lambda) - \beta_2^{-1}(U_2^\lambda))^2}{\lambda} d\Gamma \leq \sum_{i=1}^{2} \| h \|_{\infty,\Gamma_3^i} \| \beta_i^{-1}(U_i^\lambda) \|_{0,2,\Gamma_3^i}\, meas(\Gamma_3^i)^{\frac{1}{2}} + \sum_{i=1}^{2} \| \alpha \|_{\infty,\Gamma_1^i} \| T_c \|_{0,2,\Gamma_1^i} \| \beta_i^{-1}(U_i^\lambda) \|_{0,2,\Gamma_1^i}, \tag{5.9}$$

where here and in the sequel $meas(\,.\,)$ stands for the Lebesgue measure of the set in parenthesis.

Let K be the constant given by

$$K^2 = T_s \left(\sum_{i=1}^{2} \| h \|_{\infty,\Gamma_3^i}\, meas(\Gamma_3^i) + \sum_{i=1}^{2} \| \alpha \|_{\infty,\Gamma_1^i} \| T_c \|_{0,2,\Gamma_1^i}\, meas(\Gamma_1^i)^{\frac{1}{2}} \right). \tag{5.10}$$

Notice that K is independent of λ. Thus, from (5.9) and taking into account that $\beta_i^{-1}(U_i^\lambda) \leq T_s$ a.e. on Γ_3^i and Γ_1^i, $i = 1, 2$, we obtain (5.5). □

Remark 5.4. – By definition of β_i and using (H1), it is easy to deduce the following inequality

$$U_i^\lambda \leq \beta_i^{-1}(U_i^\lambda)\, k_{max}. \tag{5.11}$$

Thus, from (5.7), we obtain

$$\frac{1}{k_{max}} \left(\sum_{i=1}^{2} \int_{\Omega_i} | \nabla U_i^\lambda |^2 \, dx \right) + \frac{\alpha_{min}}{k_{max}^2} \left(\sum_{i=1}^{2} \int_{\Gamma_1^i} (U_i^\lambda)^2 d\Gamma \right) \leq K^2. \tag{5.12}$$

Consequently, $(U_1^\lambda, U_2^\lambda)$ is bounded in $\mathcal{H}^1(\Omega)$ by a constant which is independent of λ.

6. EXISTENCE OF A SOLUTION

We are now able to prove the main existence result

PROPOSITION 6.12. – *Under the assumptions (H1)-(H6), there exists a solution* (T_1, T_2, q_1, q_2) *of the problem (WP), defined by (3.25) and (3.26).*

Proof. – For a fixed $\lambda > 0$, we consider $(U_1^\lambda, U_2^\lambda, q_1^\lambda, q_2^\lambda)$ the solution of problem $(\widehat{\mathrm{AP}}_\lambda)$.

From the remark 5.4, $\{(U_1^\lambda, U_2^\lambda)\}$ is bounded in $\mathcal{H}^1(\Omega)$ independently of λ. Thus, we can extract a subsequence still denoted by λ such that

$$\{U_i^\lambda\} \rightharpoonup U_i \text{ in } H^1(\Omega_i) \text{ weakly,} \tag{6.1}$$

$$\{U_i^\lambda\} \to U_i \text{ in } L^2(\Omega_i) \text{ strongly ,} \tag{6.2}$$

$$\{U_i^\lambda\} \to U_i \text{ a.e. in } \Omega_i, \tag{6.3}$$

$$\{U_i^\lambda\} \to U_i \text{ in } L^2(\Gamma_{1,2}). \tag{6.4}$$

Furthermore, $\{U_i^\lambda\}$ converges to U_i strongly in $L^2(\Gamma_1^i)$ and $L^2(\Gamma_3^i)$, $i = 1, 2$. Then, from (6.3), the proposition 5.10 and the corollary 4.1, we obtain

$$U_{i,min} \leq U_i \leq U_{i,s} \text{ a.e. in } \Omega_i, \ i = 1, 2. \tag{6.5}$$

Since $\{(q_1^\lambda, q_2^\lambda)\}$ is bounded in $L^2(\Omega_1) \times L^2(\Omega_2)$, there exists (q_1, q_2) in $L^2(\Omega_1) \times L^2(\Omega_2)$ such that

$$\{q_i^\lambda\} \rightharpoonup q_i \text{ in } L^2(\Omega_i), \ i = 1, 2, \tag{6.6}$$

and q_i belongs to the weakly closed set $\mathcal{N}_i$ defined by (4.74). As in the proof of proposition 4.9, q_i vanishes a.e. in $[U_i < U_{i,s}]$, $i = 1, 2$. Thus, it follows that

$$q_i \in H(U_i - U_{i,s}), \ i = 1, 2. \tag{6.7}$$

On the other hand, if we take (z_1, z_2) in $H^1(\Omega)$ as a test function in (4.22), we get

$$\sum_{i=1}^{2} \int_{\Omega_i} \nabla U_i^\lambda . \nabla z_i dx + \sum_{i=1}^{2} \int_{\Omega_i} h q_i^\lambda \frac{\partial z_i}{\partial x_1} dx + \sum_{i=1}^{2} \int_{\Gamma_1^i} \alpha(\beta_i^{-1}(U_i^\lambda) - T_c) z_i d\Gamma = \sum_{i=1}^{2} \int_{\Gamma_3^i} h z_i d\Gamma, \quad \forall (z_1, z_2) \in H^1(\Omega), \tag{6.8}$$

and letting $\lambda \to 0$, we deduce

$$\sum_{i=1}^{2} \int_{\Omega_i} \nabla U_i . \nabla z_i dx + \sum_{i=1}^{2} \int_{\Omega_i} h q_i \frac{\partial z_i}{\partial x_1} dx + \sum_{i=1}^{2} \int_{\Gamma_1^i} \alpha(\beta_i^{-1}(U_i) - T_c) z_i d\Gamma = \sum_{i=1}^{2} \int_{\Gamma_3^i} h z_i d\Gamma, \quad \forall (z_1, z_2) \in H^1(\Omega). \tag{6.9}$$

For $i = 1, 2$, we set $T_i = \beta_i^{-1}(U_i)$. By taking into account the equivalence (4.27), it follows that

$$\sum_{i=1}^{2} \int_{\Omega_i} k_i(T_i) \nabla T_i . \nabla z_i dx + \sum_{i=1}^{2} \int_{\Omega_i} h q_i \frac{\partial z_i}{\partial x_1} dx + \sum_{i=1}^{2} \int_{\Gamma_1^i} \alpha(T_i - T_c) z_i d\Gamma = \sum_{i=1}^{2} \int_{\Gamma_3^i} h z_i d\Gamma, \quad \forall (z_1, z_2) \in H^1(\Omega), \tag{6.10}$$

$$q_i \in H(T_i - T_s), \quad i = 1, 2. \tag{6.11}$$

Thus, both $(T_1, T_2) \in \mathcal{H}^1(\Omega)$ and $(q_1, q_2) \in L^\infty(\Omega_1) \times L^\infty(\Omega_2)$ verify (3.25) and (3.26).

Finally, (5.5) leads to

$$\{\beta_1^{-1}(U_1^\lambda) - \beta_2^{-1}(U_2^\lambda)\} \to 0 \text{ in } L^2(\Gamma_{1,2}). \tag{6.12}$$

On the other hand, taking into account that β_i^{-1} is a Lipschitz function for $i = 1, 2$, (6.4) leads to

$$\{\beta_1^{-1}(U_1^\lambda) - \beta_2^{-1}(U_2^\lambda)\} \to T_1 - T_2 \text{ in } L^2(\Gamma_{1,2}). \tag{6.13}$$

Thus

$$T_1 = T_2 \text{ in } L^2(\Gamma_{1,2}), \tag{6.14}$$

and then $(T_1, T_2) \in H^1(\Omega)$ which finishes the proof. □

From the proof of the proposition 6.12, we deduce that $T_i \le T_s$ a.e. in Ω_i, $i = 1, 2$. Actually this property holds for every solution of (WP):

PROPOSITION 6.13. – *Under the assumptions (H1)-(H6), a solution of the problem (WP) satisfies*

$$T_i \le T_s \textit{ a.e. in } \Omega_i, \ i = 1, 2. \tag{6.15}$$

Proof. – Let us choose $z_i = (T_i - T_s)^+$ as a test function in (3.25), we have

$$\sum_{i=1}^{2} \int_{\Omega_i} k(T_i) \mid \nabla (T_i - T_s)^+ \mid^2 dx + \sum_{i=1}^{2} \int_{\Omega_i} h \frac{\partial (T_i - T_s)^+}{\partial x_1} dx$$
$$+ \sum_{i=1}^{2} \int_{\Gamma_1^i} \alpha (T_i - T_c)(T_i - T_s)^+ d\Gamma = \sum_{i=1}^{2} \int_{\Gamma_3^i} h (T_i - T_s)^+ d\Gamma, \quad (6.16)$$

since $q_i = 1$ a.e. in $[T_i > T_s]$, $i = 1, 2$. By applying the Green formula and taking into account that h only depends on x_2 it follows that

$$\sum_{i=1}^{2} \int_{\Omega_i} k(T_i) \mid \nabla (T_i - T_s)^+ \mid^2 dx$$
$$+ \sum_{i=1}^{2} \int_{\Gamma_1^i} (h n_1 + \alpha (T_i - T_c))(T_i - T_s)^+ d\Gamma = 0. \quad (6.17)$$

From (H6) we deduce that $hn_1 + \alpha(T_i - T_c) > hn_1 + \alpha(T_s - T_c) \geq 0$ a.e. on Γ_1^i, $i = 1, 2$. Consequently, all of the terms of (6.17) are nonnegative, and then they are equal to zero. Thus $(T_i - T_s)^+ = 0$ a.e. on Γ_1^i, $i = 1, 2$, and by the Poincaré's inequality we deduce (6.15). □

Remark 6.5. – A relationship between the parameters h, k_i, T_c, T_s, n_1 and c ensuring that S is indeed enclosed in Ω is an open problem. However, the following properties are easy to verify:

1) If $hn_1 + \alpha(T_s - T_c) = 0$ a.e. on Γ_1^i, $i = 1, 2$, then $T_i = T_s$, $q_i = 1$, $i = 1, 2$ is a solution of (WP), and, consequently, the liquid phase fills up the whole domain.

2) If $hn_1 + \alpha(T_s - T_c) < 0$ a.e. on Γ_1^i, $i = 1, 2$, then there exists a subset of Ω where the temperature is greater than T_s. Therefore, the assumption (H6) is necessary in order to obtain solutions of the initial problem (P).

3) The following "onedimensional" problem gives us an insight into the shape of the solution of the problem (WP):

Let us consider the problem (WP) taking place in the domain $\Omega = [0, 1] \times [0, 1]$. Then $n_1 = -1$ on Γ_1^i, $i = 1, 2$. We choose h, α and T_c as three constants and $k_i(T) = 1$, $i = 1, 2$. The solution is as follows:

i) For $h \leq \dfrac{\alpha(T_s - T_c)}{\alpha + 1}$,

$$T(x_1, x_2) = h x_1 + \frac{\alpha T_c + h}{\alpha},$$
$$q(x_1, x_2) = 0,$$

thus $T < T_s$ and no free boundary exists.

ii) For $\dfrac{\alpha(T_s - T_c)}{\alpha + 1} < h \leq \alpha(T_s - T_c)$,

$$T(x_1, x_2) = \begin{cases} hx_1 + \frac{\alpha T_c + h}{\alpha} & \text{if} \quad x_1 \leq \gamma \\ T_s & \text{if} \quad x_1 > \gamma \end{cases}$$

$$q(x_1, x_2) = \begin{cases} 0 & \text{if} \quad x_1 < \gamma \\ 1 & \text{if} \quad x_1 > \gamma \end{cases}$$

where γ is given by

$$\gamma = \frac{\alpha(T_s - T_c) - h}{\alpha h}.$$

The free boundary is given by $S = \{(\gamma, x_2),\ 0 \leq x_2 \leq 1\}$.

iii) For $h > \alpha(T_s - T_c)$,

$$T(x_1, x_2) = \frac{\alpha T_c + h}{\alpha},$$
$$q(x_1, x_2) = 1,$$

and then condition $T \leq T_s$ does not hold. Thus the liquid phase fills up the whole domain.

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