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Comparison theorems for some quasilinear degenerate elliptic operators and applications to symmetry and monotonicity results

by

Lucio DAMASCELLI

Dipartimento di Matematica, Università di Roma " Tor Vergata ",
Via della Ricerca Scientifica, 00133 Roma, Italy.

ABSTRACT. – We prove some weak and strong comparison theorems for solutions of differential inequalities involving a class of elliptic operators that includes the p-laplacian operator. We then use these theorems together with the method of moving planes and the sliding method to get symmetry and monotonicity properties of solutions to quasilinear elliptic equations in bounded domains.

RÉSUMÉ. – Nous prouvons quelques théorèmes de comparaison faible et fort pour solutions de certaines inéqualités différentielles liées à une classe d'opérateurs elliptiques qui comprend le p-laplacien. Ces théorèmes sont utilisés avec la méthode de « déplacement d'hyperplanes » et la méthode de « translation » pour obtenir des propriétés de symétrie et de monotonie des solutions d'équations elliptiques quasilinéaires dans des domaines bornés.

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1. INTRODUCTION AND STATEMENT OF THE RESULTS

In recents years several researches were devoted to the study of properties of solutions to elliptic equations involving the p-laplacian operator (see

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[1], [4], [7]-[11] and the references therein). The difficulties in extending properties of solutions of strictly elliptic equations to solutions of p-Laplace equations are mainly due to the degeneracy of the p-laplacian operator. In particular comparison principles widely used for strictly elliptic operators are not available when considering degenerate operators. In this paper we consider a class of second order quasilinear elliptic operators with a "growth of degree $p-1$", $1 < p < \infty$, which includes the p-laplacian operator and prove for them some comparison results. More precisely we consider the operator $-\operatorname{div} A(x, Du)$ in an open set $\Omega \subset \mathbb{R}^N$, $N \geq 2$, and we make the following assumptions on A:

$$A \in C^0(\overline{\Omega} \times \mathbb{R}^N; \mathbb{R}^N) \cap C^1(\overline{\Omega} \times \mathbb{R}^N \setminus \{0\}; \mathbb{R}^N) \tag{1-1}$$

$$A(x, 0) = 0 \qquad \forall\, x \in \Omega \tag{1-2}$$

$$\sum_{i,j=1}^{N} \left| \frac{\partial A_j}{\partial \eta_i}(x, \eta) \right| \leq \Gamma |\eta|^{p-2} \qquad \forall\, x \in \Omega,\ \eta \in \mathbb{R}^N \setminus \{0\} \tag{1-3}$$

$$\begin{gathered} \sum_{i,j=1}^{N} \frac{\partial A_j}{\partial \eta_i}(x, \eta) \xi_i \xi_j \geq \gamma |\eta|^{p-2} |\xi|^2 \\ \forall\, x \in \Omega,\ \eta \in \mathbb{R}^N \setminus \{0\},\ \xi \in \mathbb{R}^N \end{gathered} \tag{1-4}$$

with $1 < p < \infty$ and for suitable constants $\gamma, \Gamma \geq 0$.

In the case of the p-laplacian operator $A = A(\eta) = |\eta|^{p-2}\eta$.

In section 2 we prove different forms of weak and strong (maximum and) comparison principles. The proofs are based on simple estimates contained in Lemma 2.1 below that "explains" why maximum principles hold without special hypotheses about the degeneracies, while comparison principles are not in general available if $p \neq 2$ in their full generality (see the remark after Lemma 2.1).

We begin with forms of weak maximum and comparison principles that extend to general p a similar theorem proved in [3] for $p = 2$. If the constant Λ which appears in these theorems is zero then they are formulations of classical weak principles, while if $\Lambda > 0$ they are weak formulations of the " maximum principle in small domains " proposed in [2] for strong solutions of strictly elliptic differential inequalities.

In what follows Ω will be an open set in $\mathbb{R}^N$, $N \geq 2$ and A a function satisfying (1-1)-(1-4) for $p \in (1, \infty)$. Moreover all inequalities are meant to be satisfied in a weak sense.

THEOREM 1.1 (Weak Maximum Principle). – *Suppose* Ω *is bounded and* $u \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$, $1 < p < \infty$, *satisfies*

$$\text{(1-5)} \qquad -\operatorname{div} A(x, Du) + g(x, u) - \Lambda |u|^{p-2} u \ \leq 0 \quad [\ \geq 0\ ] \qquad \text{in } \Omega$$

where $\Lambda \geq 0$ *and* $g \in C(\overline{\Omega} \times \mathbb{R})$ *satisfies* $g(x, s) \geq 0$ *if* $s \geq 0$ $[g(x, s) \leq 0$ *if* $s \leq 0]$. *Let* $\Omega' \subseteq \Omega$ *be open and suppose* $u \leq 0$ $[\geq 0]$ *on* $\partial\Omega'$.

Then there exists a constant $c > 0$, *depending on p and on* γ, Γ *in* (1-3), (1-4), *such that if* $\Lambda\left(\frac{|\Omega'|}{\omega_N}\right)^{\frac{p}{N}} < c$ *then* $u \leq 0$ $\quad [\geq 0]$ *in* Ω' *(where* $|\ |$ *stands for the Lebesgue measure and* ω_N *is the measure of the unit ball in* $\mathbb{R}^N$*). In particular if* $\Lambda = 0$ *then* Ω' *can be an arbitrary open subset of* Ω.

Let us put, if u,v are functions in $W^{1,\infty}(\Omega)$ and $A \subseteq \Omega$

$$M_A = M_A(u, v) = \sup_A(|Du| + |Dv|),$$

$$m_A = m_A(u, v) = \inf_A(|Du| + |Dv|)$$

THEOREM 1.2 (Weak Comparison Principle). – *Let* Ω *be bounded and* $u, v \in W^{1,\infty}(\Omega)$ *satisfy*

$$\text{(1-6)} \quad -\operatorname{div} A(x, Du) + g(x, u) - \Lambda u \leq -\operatorname{div} A(x, Dv) + g(x, v) - \Lambda v \ \ \text{in } \Omega$$

where $\Lambda \geq 0$ *and* $g \in C(\overline{\Omega} \times \mathbb{R})$ *is such that for each* $x \in \Omega$ $g(x, s)$ *is nondecreasing in s for* $|s| \leq \max\{\|u\|_{L^\infty}, \|v\|_{L^\infty}\}$. *Let* $\Omega' \subseteq \Omega$ *be open and suppose* $u \leq v$ *on* $\partial\Omega'$.

(a) *if* $\Lambda = 0$ *then* $u \leq v$ *in* Ω', $\forall\, p > 1$.

(b) *if* $p = 2$ *there exists* $\delta > 0$, *depending on* Λ *and* γ, Γ *in* (1-3), (1-4), *such that if* $|\Omega'| < \delta$ *then* $u \leq v$ *in* Ω'.

(c) *if* $1 < p < 2$ *there exist* δ, $M > 0$, *depending on p*, Λ, γ, Γ, $|\Omega|$ *and* M_Ω, *such that the following holds: if* $\Omega' = A_1 \cup A_2$ *with* $|A_1 \cap A_2| = 0$, $|A_1| < \delta$ *and* $M_{A_2} < M$ *then* $u \leq v$ *in* Ω'.

(d) *if* $p > 2$ *and* $m_\Omega > 0$, *there exist* δ, $m > 0$, *depending on p*, Λ, γ, Γ, $|\Omega|$ *and* m_Ω, *such that the following holds: if* $\Omega' = A_1 \cup A_2$ *with* $|A_1 \cap A_2| = 0$, $|A_1| < \delta$ *and* $m_{A_2} > m$ *then* $u \leq v$ *in* Ω'.

Remark 1.1. – As we shall see from the proof if $p \geq 2$ it is enough to suppose $u, v \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$. If $p > 2$ and $\Lambda > 0$ to use the theorem we need to know that $|Du| + |Dv|$ is bounded from below by a positive constant, and this is a serious restriction in applications. On the contrary if $1 < p \leq 2$ we do not have to worry about the degeneracies (provided

$u, v \in W^{1,\infty}(\Omega)$ if $p < 2$) and this makes the theorem useful, as we shall see in section 3. $\square$

Remark 1.2. – The typical way we use Theorem 1.2 is the following. Suppose that Ω is a bounded domain, $f \in C(\overline{\Omega} \times \mathbb{R})$ and $u, v \in W^{1,\infty}(\Omega)$ are respectively a weak subsolution and a weak supersolution of the equation

$$(1\text{-}7) \qquad -\text{div}\, A(x, Dz) = f(x, z) \quad \text{in } \Omega$$

Then u and v satisfy (1-6) with $g(x, s) = \Lambda s - f(x, s)$, $\forall\ \Lambda \geq 0$.

(i) Let $f(x, s)$ be nonincreasing in s for x fixed and $|s| \leq \max\{\|u\|_{L^\infty}, \|v\|_{L^\infty}\}$. If $u \leq v$ on $\partial\Omega'$ for an open subset Ω' of Ω, then $u \leq v$ in Ω' by Theorem 1.2 (a). This particular case of Theorem 1.2 (a) has been proved in [10] by Tolksdorf.

In particular if u and v are both solutions of equation 1-7 and have the same boundary data on $\partial\Omega$ then they must coincide.

(ii) Suppose next that $f(x, s)$ is not nonincreasing, but there exists a $\Lambda > 0$ such that $g(x, s) = f(x, s) - \Lambda s$ is nonincreasing in s for $|s| \leq \max\{\|u\|_{L^\infty}, \|v\|_{L^\infty}\}$ (e.g. $f(x, s)$ is (semi)locally Lipschitz continuous in s uniformly in x).

If $1 < p \leq 2$ by Theorem 1.2 (b) and (c) (with $A_2 = \emptyset$) there exists $\delta > 0$ such that for any open set $\Omega' \subseteq \Omega$ with $|\Omega'| < \delta$ the inequality $u \leq v$ on $\partial\Omega'$ implies that $u \leq v$ in Ω'.

This is a weak formulation and an extension to the case $1 < p \leq 2$ of the "maximum principle in small domains" of [2]. If $p > 2$ we get analogous results under nondegeneracy hypotheses.

(iii) In the case $1 < p < 2$ Theorem (1.2) (c) implies a quite interesting and singular result. In fact suppose again that $f(x, s) - \Lambda s$ is nonincreasing in s for s in the range of u and v and that $1 < p < 2$. Then by Theorem 1.2 (c) (with $A_1 = \emptyset$) there exists $M > 0$ such that for any open set $\Omega' \subseteq \Omega$ the inequality $u \leq v$ on $\partial\Omega'$ implies the inequality $u \leq v$ in Ω' provided $M_{\Omega'} = \sup_{\Omega'}(|Du| + |Dv|) < M$.

Note that this statement is a comparison principle which holds without any assumption on the size of Ω' but rather on the smallness of $|Du|$ and $|Dv|$. This is, in general, not true even when $p = 2$.

Furthermore, as we shall see in section 3, we can use the theorem more generally when we can decompose the domain in two subdomains, one having small measure while on the other the functions involved has small gradients. $\square$

Next we deal with a form of the strong comparison principle. The strong maximum principle is well known for the kind of operators we are talking about and can be obtained via Hopf Lemma (see [10] and [13] for particular cases) or as a consequence of a Harnack type inequality (see section 2). We shall follow the second approach to derive a strong comparison theorem. First we prove the following Harnack type comparison inequality.

THEOREM 1.3 (Harnack type comparison inequality). – *Suppose u, v satisfy*

$$(1\text{-}8)\quad -\operatorname{div} A(x, Du) + \Lambda u \leq -\operatorname{div} A(x, Dv) + \Lambda v, \qquad u \leq v \qquad in\ \Omega$$

where $\Lambda \in \mathbb{R}$ and $u, v \in W^{1,\infty}_{loc}(\Omega)$ if $p \neq 2$; $u, v \in W^{1,2}_{loc}(\Omega) \cap L^{\infty}_{loc}(\Omega)$ if $p = 2$. Suppose $\overline{B(x_0, 5\delta)} \subseteq \Omega$ and, if $p \neq 2$, $\inf_{B(x_0,5\delta)} (|Du| + |Dv|) > 0$. Then for any positive $s < \frac{N}{N-2}$ we have

$$(1\text{-}9)\qquad \|v - u\|_{L^s(B(x_0,2\delta))} \leq c\,\delta^{\frac{N}{s}} \inf_{B(x_0,\delta)} (v - u)$$

where c is a constant depending on N, p, s, Λ, δ, the constants γ, Γ in (1-3), (1-4), *and if $p \neq 2$ also on m and M, where $m = \inf_{B(x_0,5\delta)} (|Du| + |Dv|)$, $M = \sup_{B(x_0,5\delta)} (|Du| + |Dv|)$.*

Theorem 1.3 implies the following strong comparison principle.

THEOREM 1.4 (Strong Comparison Principle). – *Let $u, v \in C^1(\Omega)$ satisfy* (1-8) *and define $Z = \{x \in \Omega : |Du(x)| + |Dv(x)| = 0\}$ if $p \neq 2$, $Z = \emptyset$ if $p = 2$.*

If $x_0 \in \Omega \setminus Z$ and $u(x_0) = v(x_0)$ then $u = v$ in the connected component of $\Omega \setminus Z$ containing x_0.

Remark 1.3. – By the previous theorem if u, v satisfy (1-8) in a domain Ω and $|Du| + |Dv| > 0$ in Ω then $u > v$ in Ω unless u and v coincide in Ω. In [10] Tolskdorf proved (via Hopf Lemma) a strong comparison principle for solutions of a suitable quasilinear equation, under the hypothesis that one of the two functions is of class C^2 with its gradient away from zero. Since the solutions of problems involving the operator A are usually (for $p \neq 2$) in the the class $C^{1,\alpha}$ (see [4] and [11]), Theorem 1.4 is more natural and allows the functions to have vanishing gradients, although not simultaneously if $p \neq 2$. Moreover u and v need not to solve a particular equation. □

If in (1-8) $\Lambda = 0$ we can get further results, as the following corollaries show. The first one is a corollary to Theorem 1.2 (a) and, in the case when the set S defined below is compact, it has been proved in [8] by another method. The second one is a corollary to Theorem 1.4 (and Corollary 1.1).

COROLLARY 1.1. – *Suppose* $u, v \in C^1(\Omega)$ *satisfy*

$$-\mathrm{div}\, A(x, Du) \leq -\mathrm{div}\, A(x, Dv), \quad u \leq v \quad in\ \Omega \tag{1-10}$$

Let us define $S = \{x \in \Omega : u(x) = v(x)\}$. *If* S *is either discrete or compact in* Ω *then it is empty.*

COROLLARY 1.2. – *Let* $u, v \in C^1(\Omega)$ *satisfy* (1-10). *Let us define* $Z = \{x \in \Omega : Du(x) = Dv(x) = 0\}$ *and suppose that either*

(a) Ω *is connected and* Z *is discrete*

or

(b) Z *is compact and* $\Omega \setminus Z$ *is connected.*

Then $u < v$ *unless* $u \equiv v$.

Remark 1.4. – Let $u, v \in C^1(\Omega) \cap L^\infty(\Omega)$ be respectively a weak subsolution and a weak supersolution of equation (1-7) with $u \leq v$ in Ω. Suppose that there exists a $\Lambda \geq 0$ such that $f(x, s) + \Lambda s$ is nondecreasing in s for s in the range of u and v (e.g. $f(x, s)$ is (semi)locally Lipschitz continuous in s uniformly in x). Then u and v satisfy (1-8) and Theorem 1.4 applies. In particular if $f(x, .)$ is nondecreasing we have (1-8) with $\Lambda = 0$ and we can use Corollary 1.1 or Corollary 1.2. □

In section 3 we apply the previous comparison theorems to the study of symmetry and monotonicity properties of solutions to quasilinear elliptic equations. For simplicity we consider here the case of the p-laplacian operator that we denote by Δ_p, so that $-\Delta_p u$ stands for $-\mathrm{div}\,(|Du|^{p-2}Du)$, but the same method applies to any operator that satisfy conditions (1-1)-(1-4) as well as natural symmetry conditions (see [3] for the case $p = 2$). Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geq 2$, which is convex and symmetric in the x_1-direction and consider the problem

$$\begin{cases} -\Delta_p u = f(u) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1-11}$$

In their famous paper ([5]) Gidas, Ni and Nirenberg used the method of moving planes to prove (among other results) that if $p = 2$ every classical solution to (1-11) is symmetric with respect to the hyperplane $T_0 = \{x = (x_1, x') \in \mathbb{R}^N : x_1 = 0\}$ and strictly increasing in x_1 for $x_1 < 0$, provided Ω is smooth and f is locally Lipschitz continuous. As a corollary if Ω is a ball then u is radial and radially decreasing. Since then many papers extended the results and the method in several directions. In particular Berestycki and Nirenberg in [2] improved the method by using a form of the maximum principle in domains with small measure. If $p \neq 2$ the problem is much more difficult since the operator is degenerate and partial results were obtained under special hypotheses on the solutions and/or on the nonlinearity. In [9] it is proved, using symmetrization methods, that if Ω is a ball, $p = N$ (the dimension of the space) and f is continuous with $f(s) > 0$ if $s > 0$, then u is radial and radially decreasing. In [7] symmetry results are obtained for solutions that in suitable spaces are isolated and have a nonzero index. In [1] symmetry in a ball is obtained under the crucial hypothesis that the gradient of the solution vanishes only at the center of the ball (which is then the only point of maximum).

Here, using the method of moving planes as in [2] and the above comparison results, we obtain the symmetry of the positive solutions when $1 < p < 2$ under quite general hypotheses on the set of the points where the gradient of the solution vanishes. In the general case we slightly generalize the result of [1] with a simpler proof. To state more precisely the symmetry results we need some notations.

Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geq 2$, convex and symmetric in the x_1-direction (*i.e.* for each $x' \in \mathbb{R}^{N-1}$ the set $\{x_1 \in \mathbb{R} : (x_1, x') \in \Omega\}$ is either empty or an open interval symmetric with respect to 0). For such a domain we set $-a = \inf_{x\in\Omega} x_1$ and for $-a < \lambda < a$ we define

$$T_\lambda = \{x \in \mathbb{R}^N : x_1 = \lambda\}, \quad \Omega_\lambda = \{x \in \Omega : x_1 < \lambda\},$$
$$\Omega^\lambda = \{x \in \Omega : x_1 > \lambda\}$$

If $x = (x_1, x')$ let $x_\lambda = (2\lambda - x_1, x')$ be the point corresponding to x in the reflection through T_λ and if u is a real function in Ω let us put $u_\lambda(x) = u(x_\lambda)$ whenever $x, x_\lambda \in \Omega$. Finally if $u \in C^1(\overline{\Omega})$ we put

$$Z = \{x \in \Omega : Du(x) = 0\}$$

and

$$Z_\lambda = \{x \in \Omega_\lambda : Du(x) = Du_\lambda(x) = 0\} \qquad \text{for } -a < \lambda \leq 0$$
$$Z^\lambda = \{x \in \Omega^\lambda : Du(x) = Du_\lambda(x) = 0\} \qquad \text{for } 0 \leq \lambda < a$$

THEOREM 1.5. – *Let* $1 < p < 2$ *and* $u \in C^1(\overline{\Omega})$ *a weak solution of* (1-11) *with* f *locally Lipschitz continuous. Suppose that the following condition holds:*

– *if* $\lambda < 0$ *and* C_λ *is a connected component of* Ω_λ *then* $C_\lambda \setminus Z_\lambda$ *is connected, with the analogous condition satisfied by* $C^\lambda \setminus Z^\lambda$ *for* $\lambda > 0$.

Then u *is symmetric with respect to the hyperplane* $T_0 = \{x \in \mathbb{R}^N : x_1 = 0\}$ *(i.e.* $u(x_1, x') = u(-x_1, x')$ *if* $(x_1, x') \in \Omega$*) and* $u(x_1, x')$ *is nondecreasing in* x_1 *for* $x_1 < 0$ *(and* $(x_1, x') \in \Omega$*).*

The condition in the above theorem is in particular satisfied if the set Z is discrete. In this case the solution is strictly monotone:

COROLLARY 1.3. – *Suppose that Z is discrete (and $1 < p < 2$). Then $u(x_1, x')$ is strictly increasing in x_1 for $x_1 < 0$ and if $\Omega = B(0, R)$ then u is radial and radially strictly decreasing.*

THEOREM 1.6. – *Let $u \in C^1(\overline{\Omega})$ be a weak solution of problem* (1-11), *where $p > 2$ and f is locally Lipschitz continuous. Suppose that the set where the gradient of u vanishes is contained in the hyperplane $T_0 = \{x \in \mathbb{R}^N : x_1 = 0\}$. Then u is symmetric with respect to T_0 and $u(x_1, x')$ is strictly increasing in x_1 for $x_1 < 0$.*

COROLLARY 1.4 [1]. – *Let Ω be a ball $B(0, R)$ in $\mathbb{R}^N$, $N \geq 2$ and suppose f is locally Lipschitz and $u \in C^1(\overline{\Omega})$ is a weak solution of* (1-11) *whose gradient vanishes only at the origin. Then u is radial and radially strictly decreasing.*

Next we apply the previous comparison principles together with the "sliding method" as in [2] to get the monotonicity of solutions to suitable quasilinear elliptic equations. We illustrate the method with a simple problem which is a generalization to the p-laplacian operator of an analogous problem studied in [2]. It shows that in some case the sliding method yields better results than the moving planes method for degenerate equations. This happens because we have a strict inequality between the functions involved on the boundary of suitable open sets and we can use Corollary 1.1. Let us begin with some notations. Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geq 2$, convex in the x_1-direction and consider the problem

$$\text{(1-12)} \qquad \begin{cases} -\Delta_p u = f(u) & \text{in } \Omega \\ u = \phi & \text{on } \partial\Omega \end{cases}$$

with f continuous and $\phi \in C^1(\partial\Omega)$ satisfying the following condition: if $x' = (x'_1, y)$, $x'' = (x''_1, y) \in \partial\Omega$ and $x'_1 < x''_1$ then

$$\text{(1-13)} \qquad \phi(x') < \phi(x'')$$

We consider solutions of (1-12) satisfying the following condition: if $x', x'' \in \partial\Omega$ are as before and $x = (x_1, y) \in \Omega$ with $x'_1 < x_1 < x''_1$ then

$$\text{(1-14)} \qquad \phi(x') < u(x) < \phi(x'')$$

If $\tau > 0$ let us put $\Omega_\tau = \Omega - \tau e_1$ (where $e_1 = (1, 0, \ldots 0)$) and $u_\tau(x) = u(x + \tau e_1)$ for $x \in \Omega_\tau$. Then we define $D_\tau = \Omega \cap \Omega_\tau$,

$\tau_1 = \sup\{\tau > 0 : D_\tau \neq \emptyset\}$ and $Z_\tau = \{x \in D_\tau : Du(x) = Du_\tau(x) = 0\}$ for $0 < \tau < \tau_1$.

THEOREM 1.7. – *Let $u \in C^1(\overline{\Omega})$ be a weak solution of* (1-12), (1-14) *with f locally Lipschitz continuous and $1 < p < 2$. Suppose that for each $\tau \in (0, \tau_1)$ and each connected component C_τ of D_τ the set $C_\tau \setminus Z_\tau$ is connected. Then u is nondecreasing in the x_1-direction (i.e. $u(x_1, y) \leq u(x_2, y)$ if $x_1 < x_2$) and if the set $Z = \{x \in \Omega : Du(x) = 0\}$ is discrete then u is strictly increasing in the x_1-direction.*

THEOREM 1.8. – *Suppose that f is continuous and nondecreasing and $u \in C^1(\overline{\Omega})$ is a weak solution of* (1-12), (1-14) *with $1 < p < \infty$. Then u is strictly increasing in the x_1-direction and is the only solution to the problem* (1-12) *that satisfy* (1-14).

Remark 1.5. – Note that no hypotheses on p, Z or Z_τ are required in Theorem 1.8 by assuming f nondecreasing and only continuous. □

2. PROOF OF COMPARISON THEOREMS

In this section we prove the comparison theorems stated in section 1.

Throughout this section Ω will be an open set in $\mathbb{R}^N$, $N \geq 2$, and A a function that satisfy (1-1)–(1-4) for a p with $1 < p < \infty$. We begin with a simple lemma that provides the estimates necessary for the sequel.

LEMMA 2.1. – *There exist constants c_1, c_2, depending on p and on the constants γ, Γ in* (1-3), (1-4), *such that $\forall\, \eta, \eta' \in \mathbb{R}^N$ with $|\eta| + |\eta'| > 0$, $\forall\, x \in \Omega$:*

$$|A(x,\eta) - A(x,\eta')| \leq c_1(|\eta| + |\eta'|)^{p-2}|\eta - \eta'| \tag{2-1}$$

$$[A(x,\eta) - A(x,\eta')] \cdot [\eta - \eta'] \geq c_2(|\eta| + |\eta'|)^{p-2}|\eta - \eta'|^2 \tag{2-2}$$

where the dot stands for the scalar product in $\mathbb{R}^N$. In particular, since (1-2) *holds, we have for any $x \in \Omega$, $\eta \in \mathbb{R}^N$:*

$$|A(x,\eta)| \leq c_1|\eta|^{p-1} \tag{2-3}$$

$$A(x,\eta) \cdot \eta \geq c_2|\eta|^p \tag{2-4}$$

Moreover for each $x \in \Omega$, η, $\eta' \in \mathbb{R}^N$ we have:

$$|A(x,\eta) - A(x,\eta')| \leq c_1|\eta - \eta'|^{p-1} \quad \text{if } 1 < p \leq 2 \tag{2-5}$$

$$[A(x,\eta) - A(x,\eta')] \cdot [\eta - \eta'] \geq c_2|\eta - \eta'|^p \quad \text{if } p \geq 2 \tag{2-6}$$

Proof. – Since (2-1) and (2-2) are symmetric in η, η' we can suppose $|\eta'| \geq |\eta|$, $|\eta'| > 0$. From (1-1),(1-2) we get for $j = 1 \dots N$:

$$A_j(x,\eta) - A_j(x,\eta') = \int_0^1 \sum_{i=1}^N \frac{\partial A_j}{\partial \eta_i}(x, \eta' + t(\eta - \eta'))(\eta_i - \eta_i')\, dt$$

Using (1-3), (1-4) we have that

$$\text{(2-7)} \qquad |A(x,\eta) - A(x,\eta')| \leq \Gamma |\eta - \eta'| \int_0^1 |\eta' + t(\eta - \eta')|^{p-2}\, dt$$

$$\text{(2-8)} \quad [A(x,\eta) - A(x,\eta')] \cdot [\eta - \eta'] \geq \gamma |\eta - \eta'|^2 \int_0^1 |\eta' + t(\eta - \eta')|^{p-2}\, dt$$

Since $|\eta' + t(\eta - \eta')| = |(1-t)\eta' + t\eta| \leq |\eta| + |\eta'| \quad \forall t \in [0,1]$ if $p \geq 2$ (2-7) yields (2-1), while if $1 < p \leq 2$ (2-8) yields (2-2).

To get (2-1) for $1 < p < 2$ we have to prove that

$$\text{(2-9)} \qquad \int_0^1 |\eta' + t(\eta - \eta')|^{p-2}\, dt \leq c(|\eta| + |\eta'|)^{p-2} \qquad (1 < p < 2)$$

Analogously to get (2-2) for $p > 2$ we have to prove that

$$\text{(2-10)} \qquad \int_0^1 |\eta' + t(\eta - \eta')|^{p-2}\, dt \geq c(|\eta| + |\eta'|)^{p-2} \qquad (p > 2)$$

To this end observe that if $|\eta - \eta'| \leq \frac{|\eta'|}{2}$ then (since $|\eta'| \geq |\eta|$)

$$|\eta' + t(\eta - \eta')| \geq |\eta'| - |\eta - \eta'| \geq \frac{|\eta'|}{2} \geq \frac{|\eta'| + |\eta|}{4}$$

so that (2-9) and (2-10) hold with $c = (\frac{1}{4})^{p-2}$.

If instead $|\eta - \eta'| > \frac{|\eta'|}{2} > 0$ and we put $t_0 = \frac{|\eta'|}{|\eta - \eta'|} \in (0,2)$ then

$$\begin{aligned} |\eta' + t(\eta - \eta')| &\geq \big||\eta'| - t|\eta - \eta'|\big| \\ &= |t_0 - t||\eta - \eta'| \geq |t_0 - t| \frac{|\eta'|}{2} \geq |t_0 - t| \frac{|\eta| + |\eta'|}{4} \end{aligned}$$

If $1 < p < 2$, for any $t_0 \in (0,2)$ we have that $\int_0^1 |t_0 - t|^{p-2}\, dt \leq 2\int_0^1 z^{p-2}\, dz = \frac{2}{p-1}$ so that (2-9) holds with $c = (\frac{1}{4})^{p-2} \frac{2}{p-1}$. If $p > 2$, for any $t_0 \in (0,2)$ we have that $\int_0^1 |t_0 - t|^{p-2}\, dt \geq \int_0^{\frac{1}{2}} z^{p-2}\, dz = \frac{1}{p-1}(\frac{1}{2})^{p-1}$ so that (2-10) holds with $c = (\frac{1}{4})^{p-2} \frac{1}{p-1} (\frac{1}{2})^{p-1}$.

Finally (2-5) and (2-6) are immediate consequences of (2-1) and (2-2) because $|\eta - \eta'| \le |\eta| + |\eta'| \quad \forall \eta, \eta' \in \mathbb{R}^N$. $\square$

Remark 2.1. – In our applications η, η' will be gradients of $C^1(\overline{\Omega})$ functions, so that they will be bounded but possibly approaching zero. If $1 < p < 2$ then (2-1) blows up when $|\eta| + |\eta'|$ approaches zero and the natural estimates are (2-5) and (2-2). Unfortunately (2-5) and (2-2) are not symmetric, in the sense that the former is an estimate "of order p", while the latter is an "order 2" estimate. Analogously if $p > 2$ the natural estimates are (2-1) ("of order 2") and (2-6) ("of order p") which are asymmetric. This is the reason why we are forced to use (2-1) and (2-2), both of the same "order 2", when studying comparison principles. If $p \neq 2$ this causes problems when the gradients of the functions involved are close to zero and requires special hypotheses on the sets where their gradients vanish (of course no problem arises when $p = 2$). Note however that (when $\eta' = 0$) (2-3) and (2-4) are both of the same "order p" for each $p > 1$ and this explains why maximum principles hold without restrictions for any $p > 1$, while comparison principles are, in general, not available when $p \neq 2$. $\square$

If $u, v \in W^{1,p}_{loc}(\Omega) \cap L^\infty_{loc}(\Omega)$ and $\beta \in C^0(\overline{\Omega} \times \mathbb{R})$ we say that (in a weak sense)

$$(2\text{-}11) \quad -\operatorname{div} A(x, Du) + \beta(x, u) \le \begin{cases} -\operatorname{div} A(x, Dv) + \beta(x, v) \\ 0 \end{cases} \quad \text{in } \Omega$$

if for each nonnegative $\varphi \in C_c^\infty(\Omega)$ we have

(2-12)

$$\int_\Omega [A(x, Du) \cdot D\varphi + \beta(x, u)\varphi]\, dx \le \begin{cases} \int_\Omega [A(x, Dv) \cdot D\varphi + \beta(x, v)\varphi]\, dx \\ 0 \end{cases}$$

If Ω is bounded and $u, v \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ since β is continuous and (2-3) holds, by a density argument (2-12) holds for any nonnegative $\varphi \in W_0^{1,p}(\Omega)$.

Similarly by $u \le v$ on $\partial\Omega$ (in the weak sense) we mean $(u - v)^+ \in W_0^{1,p}(\Omega)$. Of course if u and v are continuous in $\overline{\Omega}$ and satisfy $u \le v$ pointwisely on $\partial\Omega$ then they satisfy the inequality also weakly.

In the sequel we shall use the following

LEMMA 2.2 (Poincaré's inequality). – *Let Ω be a bounded open set and suppose $\Omega = A \cup B$, with A, B measurable subset of Ω. If $u \in W_0^{1,p}(\Omega)$, $1 < p < \infty$, then*

$$(2\text{-}13) \quad \|u\|_{L^p(\Omega)} \le \omega_N^{-\frac{1}{N}} |\Omega|^{\frac{1}{Np}} [|A|^{\frac{1}{Np'}} \|Du\|_{L^p(A)} + |B|^{\frac{1}{Np'}} \|Du\|_{L^p(B)}]$$

where $p' = \frac{p}{p-1}$.

Proof. – We slightly modify the proof in [6], where the lemma is proved for $A = \Omega$, $B = \emptyset$, using potential estimates. Define $h(x,y) = |x-y|^{1-N}$ and suppose C is a measurable subset of Ω. If $R > 0$ is such that $|C| = |B(x,R)|$ observe that

$$\begin{aligned}(2\text{-}14)\qquad \int_C h\,dy &= \int_{C\cap B(x,R)} h\,dy + \int_{C\setminus B(x,R)} h\,dy \\ &\le \int_{C\cap B(x,r)} h\,dy + \int_{B(x,R)\setminus C} h\,dy \\ &= \int_{B(x,R)} h\,dy = N\omega_N R = N\omega_N\left(\frac{|C|}{\omega_N}\right)^{\frac{1}{N}}\end{aligned}$$

If $f \in L^p(C)$ by Fubini's Theorem for almost every $x \in \Omega$ $f(y)\,(h(x,y))^{\frac{1}{p}} \in L^p(C)$. Let us define $V_C\, f(x) = \int_C f(y)\, h(x,y)\, dy$. Then we have by (2-14) and Hölder's inequality

$$\begin{aligned}|V_C\, f(x)| &\le \int_C |f|\, h^{\frac{1}{p}}\, h^{\frac{1}{p'}}\, dy \\ &\le \left(\int_C |f(y)|^p\, h(x,y)\, dy\right)^{\frac{1}{p}} \left(\int_C h(x,y)\, dy)^{\frac{1}{p'}}\right) \\ &\le \left[N\omega_N\left(\frac{|C|}{\omega_N}\right)^{\frac{1}{N}}\right]^{\frac{1}{p'}} \left(\int_C |f(y)|^p\, h(x,y)\, dy\right)^{\frac{1}{p}}\end{aligned}$$

Taking the p power and integrating in x over Ω we obtain, using again Fubini's Theorem and (2-14) with $C = \Omega$ and the role of x and y interchanged,

$$(2\text{-}15)\qquad \|V_C\, f\|_{L^p(\Omega)} \le N\omega_N\left(\frac{|C|}{\omega_N}\right)^{\frac{1}{Np'}}\left(\frac{|\Omega|}{\omega_N}\right)^{\frac{1}{Np}}\|f\|_{L^p(C)}$$

Now if $u \in C_c^\infty(\Omega)$ then we have the representation (see Lemma 7.14 in [6])

$$u(x) = \frac{1}{N\omega_N}\int_\Omega |x-y|^{-N} Du(y)\cdot(x-y)\, dy$$

so that if $\Omega = A \cup B$ we have that $|u(x)| \le \frac{1}{N\omega_N}\,[\,V_A\,|Du|(x) + V_B\,|Du|(x)]$. From (2-15) we obtain (2-13) for $u \in C_c^\infty(\Omega)$ and the general case follows by density. □

Proof of Theorem 1.1. – Let us prove the assertion when $u \leq 0$ on $\partial\Omega'$, the other case being perfectly analogous (with u^+ substituted by u^-). By hypothesis $u^+ \in W_0^{1,p}(\Omega')$ and can be used as a test function in (2-12) yielding

$$\int_{[u\geq 0]} A(x, Du) \cdot Du \, dx + \int_{[u\geq 0]} g(x,u)u \, dx - \Lambda \int_{[u\geq 0]} |u|^p \, dx \leq 0$$

where $[u \geq 0] = \{x \in \Omega' : u(x) \geq 0\}$. Since $g(x,u)u \geq 0$ and (2-4) holds we get

$$c_2 \int_{\Omega'} |Du^+|^p \, dx = c_2 \int_{[u\geq 0]} |Du|^p \, dx \leq \Lambda \int_{[u\geq 0]} |u|^p \, dx = \Lambda \int_{\Omega'} |u^+|^p \, dx$$

where c_2 is the constant in (2-4), and from (2-13) (with $B = \emptyset$) we infer that

$$c_2 \int_{\Omega'} |Du^+|^p \, dx \leq \Lambda \left(\frac{|\Omega'|}{\omega_N}\right)^{\frac{p}{N}} \int_{\Omega'} |Du^+|^p \, dx$$

So if $c_2 > \Lambda\left(\frac{|\Omega'|}{\omega_N}\right)^{\frac{p}{N}}$ it must be $0 = \int_{\Omega'} |Du^+|^p \, dx = \|u^+\|^p_{W_0^{1,p}(\Omega')}$ and $u^+ = 0$ in Ω'. $\square$

Proof of Theorem 1.2. – It is analogous to the previous proof with estimate (2-4) substituted by (2-2) and (2-6). Using $(u - v)^+ \in W_0^{1,p}(\Omega')$ as a test function we get

$$\begin{aligned}\int_{[u\geq v]} [A(x, Du) - A(x, Dv)] \cdot (Du - Dv) \, dx& \\ + \int_{[u\geq v]} [g(x,u) - g(x,v)](u - v) \, dx& \\ - \Lambda \int_{[u\geq v]} (u - v)^2 \, dx \leq 0&\end{aligned}$$

Since $g(x,u) \geq g(x,v)$ if $u \geq v$ we get

$$\int_{[u\geq v]} [A(x, Du) - A(x, Dv)] \cdot (Du - Dv) \, dx \leq \Lambda \int_{[u\geq v]} (u - v)^2$$

If $p > 2$ and $\Lambda = 0$ from (2-6) we get $c_2 \int_{\Omega'} |D(u - v)^+|^p \, dx \leq 0$ so that $(u - v)^+ = 0$ in Ω' and we have (a) in the case of $p > 2$.

In all other cases we use the estimate (2-2): if $p = 2$ we get, using (2-13) (with $B = \emptyset$) as in the previous theorem

$$c_2 \int_{\Omega'} |D(u-v)^+|^2 \, dx \le \Lambda \left(\frac{|\Omega'|}{\omega_N} \right)^{\frac{2}{N}} \int_{\Omega'} |D(u-v)^+|^2 \, dx$$

where c_2 is the constant in (2-2). So if $\Lambda \left(\frac{|\Omega'|}{\omega_N} \right)^{\frac{2}{N}} < c_2$ we get $\|(u-v)^+\|_{W_0^{1,2}(\Omega')} = 0$ so that $(u-v)^+ = 0$ in Ω' and we have (a) and (b) for $p = 2$.

If $1 < p < 2$ and $\Omega' = A_1 \cup A_2$ with $|A_1 \cap A_2| = 0$ we have, using (2-13) for $p = 2$,

$$\begin{aligned} c_2 \, M_\Omega^{p-2} & \int_{A_1 \cap [u \ge v]} |D(u-v)|^2 \, dx + c_2 \, M_{A_2}^{p-2} \int_{A_2 \cap [u \ge v]} |D(u-v)|^2 \, dx \\ & \le 2 \Lambda \, \omega_N^{-\frac{2}{N}} |\Omega'|^{\frac{1}{N}} \left[|A_1|^{\frac{1}{N}} \int_{A_1 \cap [u \ge v]} |D(u-v)|^2 \, dx \right. \\ & \qquad \left. + |\Omega|^{\frac{1}{N}} \int_{A_2 \cap [u \ge v]} |D(u-v)|^2 \, dx \right] \end{aligned}$$

From this we infer that if $|A_1|$ and M_{A_2} are small or $\Lambda = 0$ we must have, for $i = 1, 2$, $\int_{A_i \cap [u \ge v]} |D(u-v)|^2 = 0$ so that $\|(u-v)^+\|_{W_0^{1,2}(\Omega')} = 0$ and $(u-v)^+ = 0$ in Ω' and we have (a) and (b) for the case $1 < p < 2$.

In the case of $p > 2$, $\Lambda > 0$ we get the same inequalities with M_Ω, M_{A_2} substituted by m_Ω, m_{A_2} from which we deduce (d). $\square$

Before proving the strong comparison principle given by Theorem 1.4 let us recall the statement and the proof (using an Harnack type inequality) of (a version of) the strong maximum principle. We shall see that the differences between the strong maximum and the strong comparison principle are similar to those between the weak maximum and the weak comparison principles. The following theorem is a particular case of a more general result proved by Trudinger (see [12, Theorem 1.2]).

THEOREM 2.1 (Harnack Type Inequality). – *Suppose that* $v \in W_{loc}^{1,p}(\Omega) \cap L_{loc}^{\infty}(\Omega)$ *satisfies*

$$\text{(2-16)} \qquad -\operatorname{div} A(x, Dv) + \Lambda |v|^{p-2} v \ge 0, \quad v \ge 0 \qquad \textit{in } \Omega$$

for a constant $\Lambda \in \mathbb{R}$. *Let* $x_0 \in \Omega$, $\delta > 0$ *with* $\overline{B(x_0, 5\delta)} \subseteq \Omega$ *and* $s > 0$ *with* $s < \frac{N(p-1)}{N-p}$ *if* $p \le N$, $s \le \infty$ *if* $p > N$.

Then there exists a constant $c > 0$ depending on N, p, s, Λ, δ and on the constants γ, Γ in (1-3), (1-4) *such that*

$$\|v\|_{L^s(B(x_0,2\delta))} \leq c\,\delta^{\frac{N}{s}} \inf_{B(x_0,\delta)} v \tag{2-17}$$

Of course here and elsewhere inf means $essinf$ if the functions involved are not continuous. In Trudinger's paper the theorem is proved for operators that satisfy (2-3) and (2-4) (derived in our case from other structural conditions). The following strong maximum principle follows at once from the Harnack inequality.

THEOREM 2.2 (Strong maximum principle). – *Suppose that Ω is connected and $v \in W^{1,p}_{loc}(\Omega) \cap C^0(\Omega)$ satisfies (2-16). Then either $v \equiv 0$ in Ω or $v > 0$ in Ω.*

Proof. – Suppose $v(x_0) = 0$ with $x_0 \in \Omega$. Then the set $O = \{x \in \Omega : v(x) = 0\}$, which is closed relatively to Ω since v is continuous, is nonempty. Since v is continuous, if $v(x) = 0$ and $\delta > 0$ is such that $\overline{B(x, 5\delta)} \subseteq \Omega$, then $\inf_{B(x,\delta)} v = v(x) = 0$. From the Harnack inequality we have that $\int_{B(x,2\delta)} v^s\,dx = 0$ for some $s > 0$ so that $v \equiv 0$ in $B(x, 2\delta)$, because v is continuous and nonnegative. So O is also open and since Ω is connected it must be $O = \Omega$. $\square$

As in the case of the strong maximum principle the strong comparison principle given by Theorem 1.4 follows immediately from the Harnack comparison inequality (Theorem 1.3) whose proof is deferred to the Appendix.

Proof of Theorem 1.4. – We can suppose that $\Omega \setminus Z$ is connected and, as in the proof of Theorem 2.2, we have to prove that $O = \{x \in \Omega \setminus Z : u(x) = v(x)\}$ is open. If $x \in O$ we have $|Du(x) + |Dv(x)| > 0$ and by continuity there exist $\delta > 0$ and $m > 0$ such that $\overline{B(x, 5\delta)} \subseteq \Omega$ and $|Du| + |Dv| \geq m > 0$ in $\overline{B(x, 5\delta)}$. Since $0 = v(x) - u(x) = \inf_{B(x,\delta)} (v - u)$, by Theorem 1.3 we have $\int_{B(x,2\delta)} (v - u)\,dx = 0$, so that $u \equiv v$ in $B(x, 2\delta)$ and O is open. $\square$

Proof of Corollary 1.1. – Suppose $S \neq \emptyset$. We shall prove that $u < v$ in S, which is a contradiction. If S is compact let B an open set containing S with $\overline{B}$ compact $\subset \Omega$; if S is discrete for each $x \in S$ let $B = B_x$ be a ball such that $\overline{B} \subset \Omega$, $\overline{B} \cap S = \{x\}$. In both cases $\partial B \cap S = \emptyset$ so that $v > u$ on ∂B and there exists $\epsilon > 0$ such that $v - \epsilon \geq u$ on the compact ∂B. Since $v - \epsilon, u \in C^1(\overline{B})$, $v - \epsilon \geq u$ on ∂B, and

$$-\mathrm{div}\,A(x, Du) \leq -\mathrm{div}\,A(x, Dv) = -\mathrm{div} A(x, D(v - \epsilon)) \quad \text{in } B$$

Theorem 1.2 (a) yields $v - \epsilon \geq u$ in B. In particular $v > u$ in S. □

Proof of Corollary 1.2. – Suppose $u \not\equiv v$ in Ω, then $u \not\equiv v$ in $\Omega \backslash Z$. In fact if $u \equiv v$ in $\Omega \backslash Z$ then by continuity $u \equiv v$ on ∂Z. In case (a) $\partial Z = Z$, so that $u \equiv v$ in Ω. In case (b), since $D(u - v) = 0$ in Z, it follows that $u - v$ is constant in each connected component C of $(Z)^o$. For any such component C, we have that in $\overline{C}$ $u - v$ is a constant that must be zero because $\overline{C} \cap \partial Z \neq \emptyset$. So $u \equiv v$ in Ω and this shows that if $u \not\equiv v$ in Ω then $u \not\equiv v$ in $\Omega \backslash Z$.

Since $u \not\equiv v$ in $\Omega \backslash Z$, which is connected (in case (a) because $N \geq 2$) by Theorem 1.4 we have $u < v$ in $\Omega \backslash Z$. So $S = \{x \in \Omega : u(x) = v(x)\} \subseteq Z$ is discrete or compact and hence by the previous corollary it is empty. □

3. PROOF OF SYMMETRY AND MONOTONICITY RESULTS

Proof of Theorem 1.5. – If $\lambda \leq 0$ the functions u, u_λ satisfy the equation

$$-\Delta_p z = f(z) \quad \text{in } \Omega_\lambda$$

with f locally Lipschitz continuous. By Theorem 1.2 (c) (see Remark 1.2) there exist δ, $M > 0$ such that if $\lambda \leq 0$, Ω' is an open subset of Ω_λ with $\Omega' = A_1 \cup A_2$, $|A_1| < \delta$, $M_{A_2} = \sup_{A_2}(|Du| + |Du_\lambda|) < M$ and $u \leq u_\lambda$ on $\partial\Omega'$, then $u \leq u_\lambda$ in Ω'. If $\lambda > -a$ and $\lambda + a$ is small then $|\Omega_\lambda| < \delta$. Moreover if $x \in \partial\Omega_\lambda \cap \partial\Omega$ then $u(x) = 0 \leq u(x_\lambda) = u_\lambda(x)$; if instead $x \in \partial\Omega_\lambda \cap T_\lambda$ then $x_\lambda = x$ and $u = u_\lambda$. So $u \leq u_\lambda$ on $\partial\Omega_\lambda$ and as remarked by Theorem 1.2 (c) (with $A_2 = \emptyset$) we get $u \leq u_\lambda$ in Ω_λ for $\lambda > -a$, λ close to $-a$.

Let us define λ_0 as the sup of those $\lambda \in (-a, 0)$ such that for each $\mu \in (-a, \lambda)$ we have $u \leq u_\mu$ in Ω_μ. If we show that $\lambda_0 = 0$ then by continuity $u \leq u_0$ in Ω_0 with $u(x_1, x')$ nondecreasing for $x_1 < 0$ and the same procedure in the symmetric half Ω^0 yields $u \equiv u_0$. Suppose that $\lambda_0 < 0$. Then by continuity $u \leq u_{\lambda_0}$ in Ω_{λ_0}. Since $u \leq u_{\lambda_0}$ in Ω_{λ_0} by Theorem 1.4 (see Remark 1.4) in each connected component of $\Omega_{\lambda_0} \backslash Z_{\lambda_0}$ we have $u < u_{\lambda_0}$ unless u and u_{λ_0} coincide. If C_{λ_0} is a connected component of Ω_{λ_0} then by the convexity hypothesis on Ω there exists $x \in \partial\Omega \cap \partial C_{\lambda_0}$ such that $x_{\lambda_0} \in \Omega$ (because $\lambda_0 < 0$) so that $0 = u(x) < u(x_{\lambda_0})$. From this we infer that $u \not\equiv u_{\lambda_0}$ in any connected component C_{λ_0} of Ω_{λ_0}. Since $C_{\lambda_0} \backslash Z_{\lambda_0}$ is open and connected by hypothesis and it is a subset of $\Omega_{\lambda_0} \backslash Z_{\lambda_0}$ we deduce that $u < u_{\lambda_0}$ in $C_{\lambda_0} \setminus Z_{\lambda_0}$, unless $u \equiv u_{\lambda_0}$ in $C_{\lambda_0} \setminus Z_{\lambda_0}$. On the other hand arguing as in Corollary 1.2 we have that if $u \equiv u_{\lambda_0}$ in

$C_{\lambda_0} \setminus Z_{\lambda_0}$ then $u \equiv u_{\lambda_0}$ in C_{λ_0}. Since we saw that this is not possible we get $u < u_{\lambda_0}$ in $C_{\lambda_0} \setminus Z_{\lambda_0}$ for each connected component C_{λ_0} of Ω_{λ_0} and we conclude that $u < u_{\lambda_0}$ in $\Omega_{\lambda_0} \setminus Z_{\lambda_0}$.

Let $C = \{x \in \Omega_{\lambda_0} : u(x) = u(x_{\lambda_0})\} \subseteq Z_{\lambda_0}$. Since $Du = Du_{\lambda_0} = 0$ in C there exists an open set A with $C \subseteq A \subseteq \Omega_{\lambda_0}$ such that $M_{A,\lambda_0} = \sup_A(|Du| + |Du_{\lambda_0}|) < \frac{M}{2}$. Let $K \subseteq \Omega_{\lambda_0}$ be compact with $|\Omega_{\lambda_0} \setminus K| < \frac{\delta}{2}$. In the compact $K \setminus A \subseteq \Omega_{\lambda_0} \setminus C$ $u_{\lambda_0} - u$ is positive and it has a positive minimum there. There exists $\epsilon > 0$ such that, by continuity, $\lambda_0 + \epsilon < 0$ and for $\lambda_0 < \lambda < \lambda_0 + \epsilon$ we have $|\Omega_\lambda \setminus K| < \delta$, $M_{A,\lambda} = \sup_A(|Du| + |Du_\lambda|) < M$ and $u_\lambda - u > 0$ in $K \setminus A$ in particular on $\partial(K \setminus A)$. Moreover for such λ we have $u \leq u_\lambda$ on $\partial(\Omega_\lambda \setminus (K \setminus A))$ (if x_0 is a point on that boundary either $x_0 \in T_\lambda$ where $u = u_\lambda$ or $x_0 \in \partial\Omega$ where $0 = u \leq u_\lambda$ or else $x_0 \in \partial(K \setminus A)$ where as observed $u < u_\lambda$). Since $\Omega_\lambda \setminus (K \setminus A)$ is the disjoint union of $A_1 = \Omega_\lambda \setminus K$ and $A_2 = K \cap A$ from Theorem 1.2 (c) we infer as before that $u \leq u_\lambda$ in $\Omega_\lambda \setminus (K \setminus A)$ so that $u \leq u_\lambda$ in Ω_λ for $\lambda_0 < \lambda < \lambda_0 + \epsilon < 0$. This contradicts the definition of λ_0 and ends the proof. □

Proof of Corollary 1.3. – If Z is discrete so is Z_λ for each $\lambda \leq 0$ and from the previous proof we deduce that for each $\lambda \in (-a, 0)$ we have $u < u_\lambda$ in $\Omega_\lambda \setminus Z_\lambda$. If $(x_1, x'), (y_1, x') \in \Omega$ with $x_1 < y_1 < 0$, $\lambda = \frac{x_1+y_1}{2}$ and $(x_1, x') \notin Z_\lambda$ then $u(x_1, x') < u(y_1, x')$. If $Du(x_1, x') = Du(y_1, x') = 0$ since Z is discrete there exist $z_1 \in (x_1, y_1)$ with $Du(z_1, x') \neq 0$. By the previous argument we have $u(x_1, x') < u(z_1, x') < u(y_1, x')$, so $u(x_1, x')$ is strictly increasing for $x_1 < 0$.

If $\Omega = B(0, R)$ and Z is discrete we can repeat the proof for any direction, so u is radial and radially strictly decreasing. □

Proof of Theorem 1.6. – The proof is similar to that of Theorem 1.5 but simpler. If the points where the gradient of u vanishes are contained in T_0 then for any $\lambda \in (-a, 0)$ we have $Z_\lambda = \emptyset$ so that, if we know that $u \leq u_\lambda$ in Ω_λ for $\lambda < 0$, by Theorem 1.4 we get, as in Theorem 1.5, that $u < u_\lambda$ in Ω_λ. Moreover, since for any $\lambda < 0$ we have $|Du| + |Du_\lambda| \geq m > 0$ in $\overline{\Omega}_{\frac{\lambda}{2}}$, we can use Theorem 1.2 (d) to get the weak inequality $u \leq u_\lambda$ in small domains contained in $\overline{\Omega}_{\frac{\lambda}{2}}$ provided $\lambda < 0$.

More precisely if $m_1 = \inf_{\Omega_{\frac{-a}{2}}} |Du| > 0$ then for each $\lambda \in (-a, \frac{-a}{2})$ we have $|Du| + |Du_\lambda| \geq m_1$ and by Theorem 1.2 (d) there exists $\delta_1 > 0$ depending also on m_1 such that $u \leq u_\lambda$ in Ω_λ provided $|\Omega_\lambda| < \delta_1$ and $u \leq u_\lambda$ on $\partial\Omega_\lambda$. Since for $\lambda \in (-a, \frac{-a}{2})$ close to $-a$ this conditions are satisfied we get, using also Theorem 1.4, that $u < u_\lambda$ in Ω_λ if λ is close to $-a$.

Let λ_0 be the sup of the $\lambda < 0$ such that for each $\mu \in (-a, \lambda)$ we have $u < u_\mu$ in Ω_μ and suppose that $\lambda_0 < 0$. If we define $m_2 = \inf_{\Omega_{\frac{\lambda_0}{2}}} |Du| > 0$ we have $|Du| + |Du_\lambda| \geq m_2$ in Ω_λ for any $\lambda < \frac{\lambda_0}{2}$ and as before there exists $\delta_2 > 0$ such that for $\lambda \in (\lambda_0, \frac{\lambda_0}{2})$ if Ω' is an open subset of Ω_λ with measure less than δ_2 then $u \leq u_\lambda$ in Ω' provided $u \leq u_\lambda$ on $\partial\Omega'$.

Proceeding as in the proof of Theorem 1.5 (with $Z_\lambda = A = \emptyset$) we conclude the proof. □

Proof of Theorem 1.7. – Let us observe that if $0 < \tau < \tau_1$ with $\tau_1 - \tau$ small then $u < u_\tau$ in D_τ. In fact if this were not the case there would exist a sequence $\tau_n \to \tau_1$ and a sequence x_n such that $x_n \in D_{\tau_n}$ (*i.e.* $x_n, x_n + \tau_n e_1 \in \Omega$) and $u(x_n) \geq u_{\tau_n}(x_n)$. For a subsequence, that we still denote by x_n, there exists $x_1 \in \overline{\Omega}$ such that $x_n \to x_1$ and $x_n + \tau_n e_1 \to x_1 + \tau_1 e_1$. By continuity $u(x_1) \geq u(x_1 + \tau_1 e_1)$, which contradicts (1-13), since by the definition of τ_1 necessarily $x_1, x_1 + \tau_1 e_1 \in \partial\Omega$.

Let us define τ_0 as the inf of those $\tau \in (0, \tau_1)$ such that for each $\sigma \in (\tau, \tau_1)$ we have $u \leq u_\sigma$ in D_σ. The theorem will be proved if we show that $\tau_0 = 0$. Suppose that $\tau_0 > 0$, then by continuity $u \leq u_{\tau_0}$ in D_{τ_0}. By hypothesis $C_{\tau_0} \setminus Z_{\tau_0}$ is connected for each connected component C_{τ_0} of D_{τ_0} and as in the proof of Theorem 1.5 we get, using Theorem 1.4, $u < u_{\tau_0}$ in $D_{\tau_0} \setminus Z_{\tau_0}$. Moreover by (1-13),(1-14), we have that $u < u_{\tau_0}$ on ∂D_{τ_0}, so that the set $S = \{x \in D_{\tau_0} : u(x) = u_{\tau_0}(x)\}$ is compact in D_{τ_0} and for each $x \in S$ we have $Du(x) = Du_{\tau_0}(x) = 0$.

By Theorem 1.2 (c) (see Remark 1.2 (iii)) there exists $M > 0$ depending on Λ_2 and $|\Omega|$ such that for each $\tau \in (0, \tau_1)$ and each open $A \subseteq D_\tau$ with $|Du| + |Du_\tau| < M$ in A we have $u \leq u_\tau$ in A provided $u \leq u_\tau$ on ∂A. Choose an open set A with $S \subseteq A \subseteq D_{\tau_0}$ and $|Du| + |Du_{\tau_0}| < \frac{M}{2}$ in A. In the compact $\overline{D}_{\tau_0} \setminus A$ the minimum of $u_{\tau_0} - u$ is positive and, for τ less than τ_0 and close to τ_0, $u_\tau - u$ is positive in $\overline{D}_\tau \setminus A$ (in particular on ∂A). On the other hand for τ less than τ_0 and close to τ_0 we have $|Du| + |Du_\tau| < M$ in A with $u \leq u_\tau$ on ∂A which by the previous remark implies $u \leq u_\tau$ in A. So there exists $\tau' \in (0, \tau_0)$ such that for each $\tau \in (\tau', \tau_0)$ we have $u \leq u_\tau$ in D_τ. This contradiction shows that $\tau_0 = 0$. Finally for the case of Z discrete the proof is completely analogous to that of Corollary 1.3. □

Proof of Theorem 1.8. – The proof is very simple and it is based only on Corollary 1.1. As in the proof of Theorem 1.7 we see that if $\tau < \tau_1$ with $\tau_1 - \tau$ small then $u < u_\tau$ in D_τ. Let τ_0 be the inf of those $\tau > 0$ such that for each $\sigma \in (\tau, \tau_1)$ we have $u < u_\sigma$ in D_σ. As before the Theorem is proved if we show that $\tau_0 = 0$. Suppose the contrary, then $\tau_0 > 0$ and by continuity $u \leq u_{\tau_0}$ in D_{τ_0}. From (1-13), (1-14) we know that $u < u_{\tau_0}$ on ∂D_{τ_0} (because $\tau_0 > 0$) and, since f is nondecreasing and $u \leq u_{\tau_0}$, we have by Corollary 1.1 (see Remark 1.4) that $u < u_{\tau_0}$ in D_{τ_0} and, by (1-13), (1-14), also in $\overline{D}_{\tau_0}$. So the minimum of $u_{\tau_0} - u$ in $\overline{D}_{\tau_0}$ is positive and by continuity $u < u_\tau$ in D_τ for τ less than τ_0 and close to τ_0 contradicting the definition of τ_0.

Finally if v is another solution to the problem the same reasoning made before, with u substituted by v, shows that for any $\tau \in (0, \tau_1)$ we have $v < u_\tau$ in D_τ and by continuity $v \leq u_0 = u$ in $D_0 = \Omega$. Interchanging the roles of u, v we obtain $u = v$. □

APPENDIX

In this Appendix we prove Theorem 1.3, using (2-1), (2-2) to get an estimate for the difference $v - u$ analogous to the estimate for v used by Trudinger in [12] to prove Theorem 2.1 when $p = 2$. Then we can follow his proof (based on Moser's iterative technique) closely.

In the proof we shall use the following theorem, which is a particular case of Theorem 7.21 in [6].

THEOREM A.1. – *Let $u \in W^{1,1}(B)$, where B is a ball in $\mathbb{R}^N$, and suppose that there exists a constant K such that*

$$\int_{B \cap B_R} |Du|\, dx \leq K R^{N-1} \quad \textit{for all balls} B_R \tag{A-1}$$

Then there exist positive constants σ and C depending only on N such that

$$\int_B \exp\left(\frac{\sigma}{K}|u - u_B|\right) dx \leq C|B| \tag{A-2}$$

where $u_B = \frac{1}{|B|} \int_B u\, dx$.

Proof of Theorem 1.3. – If (1-8) is satisfied for $\Lambda < 0$ then it is satisfied with $\Lambda = 0$, because $u \leq v$. So we can suppose $\Lambda \geq 0$. In this case if $\tau > 0$ then $u, v + \tau$ satisfy (1-8) and we can suppose $v - u \geq \tau > 0$ (substituting if necessary v with $v + \tau$ and then letting $\tau \to 0$). Let $B = B(x_0, 5\delta)$ and $\eta \in C_c^1(B)$, with $0 \leq \eta \leq 1$. Testing (1-8) with $\phi = \eta^2(v - u)^\beta$, $\beta < 0$ yields

$$\begin{aligned} -|\beta| &\int_B \eta^2 (v-u)^{\beta-1} [A(x, Du) - A(x, Dv)] \cdot (Dv - Du)\, dx \\ &+ 2 \int_B \eta (v-u)^\beta [A(x, Du) - A(x, Dv)] \cdot D\eta\, dx \\ &+ \Lambda \int_B (u - v) \eta^2 (v-u)^\beta\, dx \leq 0 \end{aligned}$$

Using estimates (2-1), (2-2) we get, if $1 < p \leq 2$

$$\begin{aligned} c_2|\beta|M^{p-2} & \int_B \eta^2(v-u)^{\beta-1}|Dv - Du|^2\,dx \\ & \leq 2c_1 m^{p-2} \int_B \eta(v-u)^{\beta}|D(v-u)||D\eta|\,dx + \Lambda \int_B \eta^2(v-u)^{\beta+1}\,dx \end{aligned}$$

where c_1, c_2 are the constants in (2-1) and (2-2), depending on p and on the constants γ, Γ in (1-3), (1-4). If $p > 2$ we obtain the same inequality with the roles of m, M interchanged. In any case we have for any $\beta < 0$:

$$\begin{aligned} |\beta| & \int_B \eta^2(v-u)^{\beta-1}|D(v-u)|^2\,dx \\ & \leq C_1 \left(\int_B \eta(v-u)^{\beta}|D(v-u)||D\eta|\,dx + \int_B \eta^2(v-u)^{\beta+1}\,dx \right) \end{aligned}$$

for a constant C_1 that depends on p, γ, Γ, Λ and, if $p \neq 2$, also on m and M. By Young inequality we have

$$\begin{aligned} \eta(v-u)^{\beta}|D(v-u)||D\eta| &= (v-u)^{\beta-1}\eta|D(v-u)|(v-u)|D\eta| \\ & \leq (v-u)^{\beta-1}\left[\frac{|\beta|}{2C_1}\eta^2|D(v-u)|^2 + \left(\frac{|\beta|}{2C_1}\right)^{-1}(v-u)^2|D\eta|^2 \right] \end{aligned}$$

so that we get

$$\begin{aligned} |\beta| & \int_B \eta^2(v-u)^{\beta-1}|D(v-u)|^2\,dx \\ & \leq C_2^2\left(1 + \frac{1}{|\beta|}\right)\int_B (\eta^2 + |D\eta|^2)(v-u)^{\beta+1}\,dx \end{aligned}$$

and finally

$$\begin{aligned} \text{(A-3)} \qquad & \int_B \eta^2(v-u)^{\beta-1}|D(v-u)|^2\,dx \\ & \leq C_2^2\left(1 + \frac{1}{|\beta|}\right)^2 \int_B (\eta^2 + |D\eta|^2)(v-u)^{\beta+1}\,dx \end{aligned}$$

with C_2 depending on p,γ, Γ, Λ and if $p \neq 2$ also on m and M.

Now (A-3) is an estimate for $v - u$ analogous to the estimate for v used in Trudinger's proof of Theorem 2.1 when $p = 2$. The proof is then concluded using the Moser's iterative technique as in the proof of [12, Theorem 1.2]. For convenience of the reader we recall the details of the procedure.

Let us put, if $h > 0$ and $-\infty < t < \infty$, $t \neq 0$:

$$\Phi(t,h) = \left[\int_{B(x_0,h)} (v-u)^t \, dx\right]^{\frac{1}{t}}$$

so that

$$\sup_{B(x_0,h)} (v-u) = \Phi(+\infty,h)\,, \qquad \inf_{B(x_0,h)} (v-u) = \Phi(-\infty,h)$$

We put in (A-3) $\beta = -1$ and for $y \in B(x_0, \frac{5\delta}{2})$, $r < \frac{5\delta}{2}$ we choose $\eta \in C_c^1(B)$ with $\eta = 1$ in $B(y,r)$, supp $\eta \subseteq B(y,2r)$ and $|D\eta| \leq \frac{2}{r}$. We obtain, with $w = \log(v-u)$

$$\left[\int_{B(y,r)} |Dw|^2 \, dx\right]^{\frac{1}{2}} \leq 2C_2\left(1+\frac{2}{r}\right)|B(y,2r)|^{\frac{1}{2}} \leq C_3 \frac{5\delta+2}{r} r^{\frac{N}{2}}$$

with C_3 depending on C_2 and N. It follows, using Hölder's inequality, that

$$\int_{B(y,r)} |Dw| \, dx \; \leq C_4 r^{\frac{N}{2}} r^{-1} r^{\frac{N}{2}} = C_4 r^{N-1}$$

with C_4 depending on C_2, N and δ. By Theorem A.1 there exist $r_0 > 0$ ($r_0 = \frac{\sigma}{C_4}$ with $\sigma = \sigma(N)$) and $C = C(N) > 0$ such that

$$\int_{B'} \exp(r_0|w - w_{B'}|) \, dx \; \leq C|B'|$$

where $B' = B(x_0, \frac{5\delta}{2})$. As a consequence we have

$$\int_{B'} \exp(r_0 w) \, dx \; \int_{B'} \exp(-r_0 w) \, dx \; \leq C|B'|^2 = C' \, \delta^{2N}$$

Recalling that $w = \log(v-u)$ and taking the $\frac{1}{r_0}$ power we obtain

$$\Phi\left(r_0, \frac{5\delta}{2}\right) \leq C' \, \delta^{\frac{2N}{r_0}} \Phi\left(-r_0, \frac{5\delta}{2}\right) \tag{A-4}$$

where C' depends on N and r_0 depends on C_2, N and δ.

Next we consider (A-3) when $\beta < 0$, $\beta \neq -1$. Let us put for $-1 \neq \beta < 0$

$$q = \frac{\beta+1}{2} \qquad r = 2q = \beta + 1$$

Observe that $\beta < -1$ iff $q, r < 0$ while $-1 < \beta < 0$ iff $0 < q, r$; $r < 1$.

For $\delta \le h' < h'' \le 5\delta$ we take $\eta \in C_c^1(B)$ with $\eta = 1$ in $B(x_0, h')$, supp $\eta \subseteq B(x_0, h'')$ and $|D\eta| \le \frac{2}{(h''-h')}$. If $w = (v-u)^q$ from (A-3) we get

$$\begin{aligned}\|\eta\, Dw\|_{2;h''} &\le C_2|q|\left(1+\frac{1}{|\beta|}\right)\sqrt{1+\frac{4}{(h''-h')^2}}\|w\|_{2;h''}\\ &\le C_5|q|\left(1+\frac{1}{|\beta|}\right)(h''-h')^{-1}\|w\|_{2,h''}\end{aligned}$$

where $\| \quad \|_{t;h}$ is the norm in $L^t(\,B(x_0,h)\,)$ and C_5 depends on C_2 and δ. It follows that

$$\|D(\eta\, w)\|_{2,h''} \le \left[2 + C_5\,|q|\left(1+\frac{1}{|\beta|}\right)\right](h''-h')^{-1}\|w\|_{2,h''}$$

Since $\eta\, w \in W_0^{1,2}(B_{h''})$ and $\|w\|_{2\chi,h'} \le \|\eta w\|_{2\chi,h''}$ we obtain by Sobolev inequality that if $\chi = \frac{N}{N-2}$ (χ arbitrary if $N = 2$)

$$\|w\|_{2\chi,h'} \le C_6\left[1+|q|\left(1+\frac{1}{|\beta|}\right)\right](h''-h')^{-1}\|w\|_{2,h''}$$

for a constant C_6 depending on C_2, δ and N. By the definition of w, q and r this is equivalent to

$$\begin{aligned}&\left[\int_{B(x_0,h')}(v-u)^{\chi r}\,dx\right]^{\frac{q}{\chi r}}\\ &\quad\le C_6\left[1+|q|\left(1+\frac{1}{|\beta|}\right)\right](h''-h')^{-1}\left[\int_{B(x_0,h'')}(v-u)^r\,dx\right]^{\frac{q}{r}}\end{aligned}$$

Taking the $\frac{1}{q}$ power we obtain

$$\text{(A-5)}\qquad \Phi(\chi r, h') \le C_6^{\frac{2}{r}}\left[1+|\frac{r}{2}|\left(1+\frac{1}{|\beta|}\right)\right]^{\frac{2}{r}}(h''-h')^{\frac{-2}{r}}\,\Phi(r,h'')$$

if $q > 0$ *i.e.* $-1 < \beta < 0$ and $0 < r < 1$.

If instead $q < 0$ *i.e.* $\beta < -1$ and $r < 0$ we obtain

$$\text{(A-6)}\qquad \Phi(\chi r, h') \ge C_6^{\frac{2}{r}}\left[1+|\frac{r}{2}|\left(1+\frac{1}{|\beta|}\right)\right]^{\frac{2}{r}}(h''-h')^{\frac{-2}{r}}\,\Phi(r,h'')$$

For $r_0 > 0$ given by (A-4) and $k = 0, 1, \ldots$ let us define $r'_k = (-r_0)\,\chi^k$ and $h_k = \delta\,[1+\frac{3}{2}(\frac{1}{2})^k]$. We have that $r'_k \to -\infty$, $\beta_k = r'_k - 1 \to -\infty$ and $\frac{1}{|\beta_k|} \le 1$; $h_0 = \frac{5\delta}{2}$, $h_k \to \delta$ and $h_k - h_{k+1} = (\frac{3\delta}{2})\frac{1}{2^{k+1}}$.

Iterating (A-6) (where we can suppose $C_6 \geq 1$) we get

$$\begin{aligned}
&\Phi(r'_{k+1}, h_{k+1}) \\
&\geq \left(C_6^{-\frac{2}{r_0}}\right)^{\frac{1}{\chi^k}} [(1+|r_0|\chi^k)^{-\frac{2}{r_0}}]^{\frac{1}{\chi^k}} (\delta^{\frac{2}{r_0}})^{\frac{1}{\chi^k}} \left(\frac{1}{2}\right)^{\frac{2(k+1)}{r_0(\chi)^k}} \Phi(r'_k, h_k) \\
&\geq C_7^{\sum_{k\geq 0}\frac{1}{\chi^k}} [(2\chi)^{-\frac{2}{r_0}}]^{\sum_{k\geq 0}\frac{k}{\chi^k}} (\delta^{\frac{2}{r_0}})^{\sum_{0\leq j\leq k}\frac{1}{\chi^j}} \Phi(-r_0, h_0)
\end{aligned}$$

with C_7 depending on C_6 and r_0.

If $k \to \infty$ since $r'_k \to -\infty$, $h_k \to \delta$ and $\sum_{k\geq 0}\frac{1}{\chi^k} = \frac{N}{2}$ we obtain

$$\Phi(-\infty, \delta) \geq C_8\, \delta^{\frac{N}{r_0}}\, \Phi\left(-r_0, \frac{5\delta}{2}\right) \tag{A-7}$$

where C_8 depends on C_6, N and r_0.

If $0 < s \leq r_0$ we have by Hölder's inequality that

$$\Phi(s, 2\delta) \leq \Phi\left(s, \frac{5\delta}{2}\right) \leq c_N\, \delta^{(\frac{N}{s}-\frac{N}{r_0})}\, \Phi\left(r_0, \frac{5\delta}{2}\right) \tag{A-8}$$

which combined with (A-4) and (A-7) yields

$$\Phi(s, 2\delta) \leq C_9\, \delta^{\frac{N}{s}}\, \Phi(-\infty, \delta) \tag{A-9}$$

where C_9 depends on N and C_8, so it depends on p,γ,Γ,Λ,δ, N and if $p \neq 2$ also on m and M. This is exactly (1-9).

If instead $r_0 < s < \frac{N}{N-2}$ to get (A-8) we proceed as in the deduction of (A-7) but taking a finite number of iterations and using (A-5) instead of (A-6).

More precisely if $r_0 < s < \chi = \frac{N}{N-2}$ then $\frac{s}{\chi^{k_0+1}} = r_1 \leq r_0$ for a natural number k_0. If we put, for $k = 0, \ldots k_0 + 1$, $r'_k = r_1\chi^k$ and $h_0 = \frac{5\delta}{2} > h_1 > \ldots h_{k_0+1} = 2\delta$, with $h_k - h_{k+1} = \frac{1}{k_0+1}\,\frac{\delta}{2}$, then for $k \leq k_0$ we have $r'_k < 1$ and (A-5) is true.

After k_0 iterations of (A-5) we obtain as in the deduction of (A-7)

$$\Phi(s, 2\delta) \leq C_{10}\, \delta^{\frac{N}{s}-\frac{N}{r_1}}\, \Phi\left(r_1, \frac{5\delta}{2}\right) \tag{A-10}$$

where C_{10} depends not only on C_6 and r_1 but also on s through the bound $\frac{1}{|\beta_k|} \leq \frac{1}{|\beta_{k_0}|}$, $k = 0, \ldots, k_0$, with $|\beta_{k_0}| = |r'_{k_0} - 1| = 1 - \frac{s}{\chi}$.

Since (A-4) is certainly true with r_1 instead of r_0 and (A-7) can be deduced exactly in the same way with r_1 instead of r_0, putting together (A-10) and (the modified) (A-4) and (A-7) we obtain again (1-9). $\square$

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