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On the existence of a positive solution of semilinear elliptic equations in unbounded domains

by

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ABSTRACT. – We prove here the existence of a positive solution, under general conditions, for semilinear elliptic equations in unbounded domains with a variational structure. The solutions we build cannot be obtained in general by minimization problems. And even if Palais-Smale condition is violated, precise estimates on the losses of compactness are obtained by the concentration-compactness method which enables us to apply the theory of critical points at infinity.

RÉSUMÉ. – Nous prouvons dans cet article l'existence d'une solution positive, sous des conditions générales, pour des équations semilinéaires elliptiques dans les domaines non bornés avec une structure variationnelle. Les solutions obtenues ne peuvent être en général obtenues par des problèmes de minimisation. Bien que la condition de Palais-Smale n'ait pas lieu, des estimées précises sur les pertes de compacité sont déduites de la méthode de concentration-compacité et nous permettent d'appliquer la théorie des points critiques à l'infini.

I. INTRODUCTION

This paper is concerned with the existence of positive solutions of

$$(1.1)\quad -\Delta u + \lambda_0 u = b(x)\, u^P \text{ in } \Omega, \qquad u \in H_0^1(\Omega), \quad u > 0 \quad \text{in } \Omega$$

where $\lambda_0 > 0$, $\Omega = \overline{O}^c$ and O is a smooth bounded open set in $\mathbb{R}^n$, $n \geq 2$, $1 < p < \frac{n+2}{n-2}$ ($p < \infty$ if $n = 2$) and the weight function b satisfies in all that follows

$$(1.2) \quad b \in C_b(\mathbb{R}^n), \qquad b > 0 \quad \text{on } \mathbb{R}^n, \qquad b \to b^\infty > 0 \quad \text{as } |x| \to \infty.$$

Such problems in unbounded domains arise naturally in various branches of Mathematical Physics and present specific mathematical difficulties. Indeed, if there exist various general methods to solve the analogue (1.1) when Ω is bounded, these argument break down in the above situation because of losses of compactness which can be illustrated by the following well-known fact: the embedding from $H_0^1(\Omega)$ into $L^2(\Omega)$ is no longer compact when Ω is, say, an exterior domain as above. A more precise argument consists in looking at the particular example when $O = \emptyset$, $b \equiv b^\infty$ *i.e.*

$$(1.3) \quad -\Delta u + \lambda_0 u = b^\infty u^p \quad \text{in } \mathbb{R}^n, \qquad u \in H^1(\mathbb{R}^n), \quad u > 0 \quad \text{in } \mathbb{R}^n.$$

This problem being obviously invariant by translations, one deduces immediately that the set of solutions of (1.3) is not compact in any Sobolev space. Let us finally mention that a decisive argument consists in recalling the nonexistence result by M. J. Esteban and P.-L. Lions [17] when $\Omega = \mathbb{R}^n$, $b \in C_b(\mathbb{R}^n)$, $b \geq b > 0$ on $\mathbb{R}^n$ and b is increasing in one direction (notice however that such a b does not satisfy (1.2)).

Many authors have considered the above problem: the first case to be treated was (1.3) by Z. Nehari [29]; G. H. Ryder [31]; M. Berger [8]; C. V. Coffman [11]; S. Coleman, V. Glazer and A. Martin [14]; W. Strauss [23] and H. Berestycki and P.-L. Lions [7] (where general nonlinearities are considered). In all these works dealing with the case $\Omega = \mathbb{R}^n$, $b = b^\infty$, the solution is built through a minimization problem and a reduction to spherically symmetric function which restores the compactness.

Next, some effort to understand precisely this loss of compactness and related ones occuring in various problems was made by various authors (*see* for example P. Sacks and K. Uhlenbeck [39], P.-.L. Lions [21], C. Taubes [35], [36], H. Brezis and J. M. Coron [10], M. Struwe [34]...). In the particular example at hand, this was done via the concentration-compactness method of P.-L. Lions [22] and it led to various existence results for minama of say

$$(1.4) \qquad \operatorname{Inf}\left\{\int_\Omega |\nabla u|^2 + \lambda_0 |u|^2 / u \in H_0^2(\Omega),\ \int_\Omega b\,|u|^{p+1}\,dx = 1\right\},$$

see also W. Y. Ding and W. M. Ni [15], M. J. Esteban and P.-L. Lions [18], P.-L. Lions [23] for related results. In fact (*see* P.-L. Lions [24]), one knows that approximated solutions of (1.1) *i.e.*, Palais-Smale sequences, in the situation when (1.1) has no solutions for instance, break up in a finite number of solutions of (1.3) which roughly speaking are entered at points whose interdistances go to infinity.

To conclude this brief review of known existence results, let us mention that if b, Ω present symmetries some further existence results are known (*see* W. Y. Ding and W. M. Ni [15], C. V. Coffman and M. Marcus [12], P.-L. Lions [25], [26]). Finally, existence is also known in some "perturbation cases": *see* C. V. Coffman and M. Marcus [13], V. Benci and G. Cerami [6].

Let us now state our main result which will use the following assumption on b

$$(1.5)\qquad \begin{cases} b(x) \geq b^{\infty} - C_0 \exp(-\delta\,|x|)\,|x|^{-\frac{N-1}{2}} & \text{if} \quad |x| \geq R_0, \\ \qquad\qquad \text{for some } R_0 < \infty,\ c_0 \geq 0,\ \delta > 0. \end{cases}$$

(Observe that C_0, $|x|^{-\frac{N-1}{2}}$ are not really relevant but we insist on this form for reasons which will be clear later on).

Our argument also requires the uniqueness up to a translation of solutions of (1.3): in view of the general symmetry results of B. Gidas, W. M. Ni and L. Nirenbberg [19], [20], this amounts to the uniqueness of *radial* solutions of (1.3), a fact which has been shown by M. K. Kwong [22] – some partial results in that direction were just given in K. Mac Leod and J. Serrin [27]. We may now state the

THEOREM I.1. – *We assume* (1.5). *Then, there exists a solution of* (1.1). □

The proof of this result is rather long and contains several highly technical aspects. The idea of the proof relies on the method of critical points at infinity [1], [2], [5]. To simplify the presentation, we split the proof in various steps which contain interesting elements by themselves. The last step consists in some crucial "energy balance" (section IV) which is in some sense the key *a priori* estimate required for the analysis of the existence result. Section VII is devoted to various extensions (more general equations and conditions), variants and comments. In particular, we explain how a much easier existence proof can be made if we relax (1.5) to

$$(1.6)\qquad b(x) \geq b^{\infty} - c(\exp(-2\,|x|^{-\frac{N-1}{2}}) \quad \text{as } |x| \to \infty.$$

Indeed, we show that by a careful inspection of the energy balance investigated in section IV, the "interaction of only two solutions at infinity" can be used and this allows to use the idea of J. M. Coron [15] a bit like it was done in V. Benci and G. Cerami [6] (we thus basically refine the analysis of [6]). We also consider in section VII the following equation.

$$(1.7)\qquad \begin{cases} -\Delta u = b(x)\,(u-\lambda_0)^{+p} & \text{in } \Omega, \qquad \nabla u \in L^2(\Omega), \\ u \in L^{\frac{2n}{n-2}}(\Omega),\ u = 0 & \text{on } \partial\Omega, \qquad u > 0 \quad \text{in } \Omega \end{cases}$$

where $\lambda_0 > 0$, b satisfies (1.2), $1 < p < \frac{n+2}{n-2}$ and $n \geq 3$. And we show there exists a solution of (1.7) as soon as b satisfies

$$(1.8)\qquad \begin{cases} b(x) \geq b^\infty - \dfrac{C_0}{|x|^{n-2}} \quad \text{if } |x| \geq R_0, \\ \quad \text{for some } R_0 < \infty,\ C_0 \geq 0. \end{cases}$$

Section VIII contains an existence result when Ω and b have some symmetries which extends the results recalled above; and its proof uses and refines some of the arguments introduced in the course of proving Theorem I.1.

THEOREM I.2. – *We assume that Ω, b are invariant by a subgroup G of the group of orthogonal transforms. Let $R_0 > 0$ be such that $\overline{O} \subset B_{R_0}$, we set for $|\xi| = R_0$, $N(\xi) = \#\{g\cdot\xi/g \in G\}$ and $N = \sup_{|\xi|=R_0} N(\xi)$. We assume $N \geq 2$. Furthermore, if $N < \infty$, we assume there exists ξ such that $|\xi| = R_0$, $N(\xi) = N$ for which the following holds for some $c_0 > 0$*

$$(1.9)\qquad \begin{cases} b^\infty - b(x) \leq c_0\, N^{-1} N_0 \exp(-\sqrt{\lambda_0}\,\Delta\,|x|)\,\Delta^{-\frac{n-1}{2}}\,|x|^{-\frac{n-1}{2}} \\ \qquad\qquad \textit{for } |x| \textit{ large} \\ \quad \textit{where } \Delta = \dfrac{1}{R_0}\operatorname{Min}\{|\xi_i - \xi_j|/i \neq j,\ \{\xi_1, \ldots, \xi_N\}. \\ \qquad = \{g\cdot\xi/g \in G\}, \\ \qquad N_0 = \#\{(i, j)/i \neq j,\ |\xi_i - \xi_j| = \delta\}. \end{cases}$$

Then, there exists a constant $\bar{c} = \bar{c}(p, n) > 0$ such that if $c_0 \leq \bar{c}$, then there exists a solution of (1.1) which is obtained via the following minimization problem

$$\operatorname{Min}\Big\{ \int_\Omega |\nabla u|^2 + \lambda_0\, u^2\, dx/u \in H_0^1(\Omega), \\ u(x) = u(g\cdot x) \quad \text{on } \mathbb{R}^n\ (\forall g \in G)\ \int_\Omega b\,|u|^{p+1}\,dx = 1\Big\}. \quad \square$$

Notice that the above result implies the existence of a solution when $b \equiv 1$ as soon as Ω has "a symmetry group without fixed points" (*i.e.*, $N \geq 2$). Observe also that the above result applies by a simple translation if $\Omega + x_0$, $b(x + x_0)$ satisfy the above assumptions for some $x \in \mathbb{R}^n$.

II. A FEW KNOWN FACTS

We first introduce a few notations. First, the natural functional associated with (1.1) is

$$(2.1)\qquad \begin{cases} I(v) = \displaystyle\int_\Omega \frac{1}{2}|\nabla v|^2 + \frac{1}{2}\lambda_0 v^2 - \frac{1}{p+1} b|v|^{p+1}\,dx, \\ \qquad\qquad \forall\, v \in E_0^1(\Omega). \end{cases}$$

Recall that nonnegative critical points of I are indeed the nonnegative solutions of (1.1). We will also denote by

$$(2.2)\qquad \Sigma = \left\{ u \in H_0^1(\Omega),\ \int_\Omega |\nabla u|^2 + \lambda_0 u^2\,dx = 1 \right\}$$

and we will agree that $H_0^1(\Omega)$ embeds into $H^1(\mathbb{R}^n)$ by extending its elements by 0, while $H_0^1(\Omega) = H^1(\mathbb{R}^n)$ if $\mathcal{O} = \emptyset$. We next introduce

$$(2.3)\qquad J(v) = \sup_{\lambda \geq 0} I(\lambda v), \qquad v \in \Sigma$$

and we denote by $\lambda(v)$ the unique maximum of $I(\lambda v)$ on $[0, \infty)$ for $v \in \Sigma$. Observe that we have here explicit formulae

$$(2.4)\qquad \begin{cases} \lambda(v) = \left(\displaystyle\int_{\mathbb{R}^n} b|v|^{p+1}\,dx \right)^{-1/(p-1)}; \\ J(v) = \dfrac{p-1}{2(p+1)} \left(\displaystyle\int_{\mathbb{R}^n} b|v|^{p+1}\,dx \right)^{-2/(p-1)}; \end{cases}$$

and one checks immediately that λ, J are C^1 on Σ and that if v is a (nonnegative) critical point of J on Σ then $u = \lambda(v)v$ is a nontrivial (nonnegative) critical point of I and conversely any nontrivial critical point of I may be obtained through such a v. Such a reduction to a functional defined on sphere was already used in A. Bahri [3], A. Bahri and H. Berestycki [4] and is in fact valid for more general nonlinearities, a fact that we will not recall in section VII (*see* [3], [4]).

Of course, Σ is the sphere of $H_0^1(\Omega)$ provided we endow $H^1(\mathbb{R}^n)$ with the scalar product

$$(v_1, v_2) = \int_{\mathbb{R}^n} \nabla v_1 \cdot \nabla v_2 + \lambda_0 v_1 v_2 \, dx \tag{2.5}$$

and we will write by $|v|$ the associated norm. The gradient flow of J restricted to Σ (identifying $H^1(\mathbb{R}^n)$ with its dual, and thus considering gradients with respect to the scalar product (2.5)) is the solution of the following differential equation

$$\frac{du}{ds} = -J'(u) = -\lambda^2(u)\, u + K\left(b|\lambda(u)\, u|^{p-1} \lambda^2(u)\, u\right) \quad \text{for } s \geq 0 \tag{2.6}$$

where $z = K f$ is the solution for $f \in H^{-1}(\Omega)$ of

$$-\Delta z + \lambda_0 z = f \quad \text{in } \Omega, \qquad z \in H_0^1(\Omega). \tag{2.7}$$

And one checks that there exists a unique global solution $u(t)$ for (2.6) such that $u(0) = u_0$ where $u_0 \in \Sigma$. Furthermore, $u(t) \in \Sigma$ for all $t \geq 0$ and of course

$$\left| \frac{du}{ds} \right|^2 = -\frac{d}{ds}(J(u)) \qquad \text{for all} \quad s \geq 0. \tag{2.8}$$

Finally, since K is order-preserving (maximum principle), it is possible to show (*see* section IV) that if $u_0 \in \Sigma^+$ then $u(s) \in \Sigma^+$ for all $s \geq 0$ where Σ^+ is given by

$$\Sigma^+ = \{u \in \Sigma,\ u \geq 0 \quad \text{in } \Omega\}. \tag{2.9}$$

Let us also recall the relations between Palais-Smale sequences for I and J (P. S. sequences in short) *i.e.* sequences u_k, v_k satisfying

$$I(u_k) \quad \text{is bounded,} \quad I'(u_k) \xrightarrow{k} 0 \tag{2.10}$$

or

$$J(v_k) \quad \text{is bounded,} \quad J'(v_k) \xrightarrow{k} 0, \quad v_k \in \Sigma. \tag{2.11}$$

Before we do that, we recall that from Sobolev inequalities λ and J are bounded from below away from 0 on Σ.

LEMMA II.1. – 1) *Let* v_k *satisfy* (2.11) *then* $\lambda(v_k)$ *is bounded,* $u_k = \lambda(v_k)\, v_k$ *satisfies* (2.11) *and* $|u_k|$ *is bounded away from* 0.

2) *Let* u_k *satisfy* (2.10) *be such that* $|u_k|$ *is bounded away from* 0 *then* $v_k = \frac{u_k}{|u_k|}$ *satisfies* (2.11) *and* $\lambda(v_k)\,|u_k|^{-1} \underset{k}{\to} 1$.

Remark. – In fact as soon as $J(v_k)$ is bounded, $\lambda(u_k)$ is bounded and thus $|u_k|$ is bounded in $(0, \infty)$ (recall that $v_k \in \Sigma$ and compare with (2.4)).

Proof of Lemma II.1. – 1) The above remark shows that $|u_k|$ is bounded from above and away from 0, hence $I(u_k)$ is bounded. Now, in order to prove that $I'(u_k) \overset{k}{\to} 0$ we just observe that because of (2.3) we always have

$$0 = (I'(u_k),\, v_k) = (u_k - K(b\,|u_k|^{p-1}\, u_k),\, v_k)$$

while (2.11) implies

$$\sup\{|(u_k - K(b\,|u_k|^{p-1}\, u_k),\, w)|/|w| \leq 1,\ (w,\, v_k) = 0\} \underset{k}{\to} 0$$

hence $|u_k - K(b|u_k|^{p-1}\, u_k)| \underset{k}{\to} 0$, proving our claim.

2) If u_k satisfies (2.10) then

$$\varepsilon_k = u_k - K(b\,|u_k|^{p-1}\, u_k) \underset{k}{\to} 0,$$

in particular $||u_k|^2 - \int_\Omega b\,|u_k|^{p+1}\, dx| \leq |\varepsilon_k|\,|u_k|$. And this combined with the bounds on $I(u_k)$ shows that $|u_k|$ is bounded. Hence, $||u_k|^{1-p} - \int_\Omega b\,|v_k|^{p+1}\, dx| \underset{k}{\to} 0$ or $|u_k| - \lambda(v_k) \underset{k}{\to} 0$ and we conclude easily. ■

Using the preceding lemma, we may now show easily that J does not satisfy the P.S. condition on Σ or even Σ^+, *i.e.* that there exist sequences v_k in Σ^+ satisfying (2.11) for which no subsequences converge. In view of Lemma I.1, we just have to build a sequence u_k satisfying (2.10), such that $|u_k|$ is bounded away from 0 and u_k does not have any converging subsequence. To do so we consider a solution ω of (1.3) (whose existence was recalled in the Introduction) and we take any cut-off function $\varphi \in C^\infty \mathbb{R}^n)$ satisfying

(2.12) $$0 \leq \varphi \leq 1,\ \varphi \equiv 0 \text{ near } \overline{\mathcal{O}},\quad \varphi \equiv 1 \quad \text{for} \quad |x| \text{ large}$$

Next, let x_k be any sequence in $\mathbb{R}^n$ going to ∞ and set

(2.13) $$u_k = \varphi\omega(\cdot - x_k).$$

It is a straightforward exercise to check that $u_k \in H_0^1(\Omega)$, satisfies (2.10), $|u_k| \underset{k}{\to} |\omega| > 0$ while $u_k \underset{k}{\to} 0$ weakly in H_0^1...

Having thus shown that the P.S. condition is violated in general, we now explain the precise mechanism involved. This result is derived from P.-L. Lions [23], [22] and its proof is given in the Appendix for the reader's convenience.

PROPOSITION II.1. – *Let u_k be a sequence in $H_0^1(\Omega)$ satisfying* (2.10). *Then, there exists a subsequence (still denoted by u_k) for which the following holds: there exist an integer $m \geq 0$, sequences x_k^i for $1 \leq i \leq m$, functions $\overline{u}$, ω_i for $1 \leq i \leq m$ such that*

$$-\Delta \overline{u} + \lambda_0 \overline{u} = b\,|\overline{u}|^{p-1}\,\overline{u} \quad in\ \Omega, \qquad \overline{u} \in H_0^1(\Omega) \tag{2.14}$$

$$\begin{cases} -\Delta \omega_i + \lambda_0\, \omega_i = b^\infty\, |\omega_i|^{p-1}\, \omega_i \quad in\ \mathbb{R}^n, \\ \omega_i \in H^1(\mathbb{R}^n), \quad \omega_i \not\equiv 0 \end{cases} \tag{2.15}$$

$$\begin{cases} u_k - \left(\overline{u} + \sum_{i=1}^{m} \omega_i\,(\,\cdot\, - x_k^i)\right) \underset{k}{\rightarrow} 0, \\ I(u_k) \underset{k}{\rightarrow} I(\overline{u}) + \sum_{i=1}^{m} I^\infty(\omega_i) \end{cases} \tag{2.16}$$

$$|x_k^i| \underset{k}{\rightarrow} +\infty, \quad |x_k^i - x_k^j| \underset{k}{\rightarrow} +\infty \qquad for \quad 1 \leq i \neq j \leq m, \tag{2.17}$$

where we agree that in the case $m = 0$ the above holds without ω_i, x_k^i and

$$I^\infty(u) = \int_{\mathbb{R}^n} \frac{1}{2}\,|\nabla u|^2 + \lambda_0\, u^2 - \frac{b^\infty}{p+1}\,|u|^{p+1}\,dx \qquad for\ all \quad u \in H^1(\mathbb{R}^n)$$

In addition, if $u_k \geq 0$ then $\overline{u} \geq 0$, and ω_i may be taken to be for all $2 \leq i \leq m$ the unique positive radial solution *cf.* (1.3).

This result immediately implies the

COROLLARY II.1. – *Under the assumptions of Theorem* I.1, *we denote by ω the unique radial solution of* (1.3) *and by $S = \frac{p-1}{2\,(p+1)}\,|\omega|^2$. Then, if v_k is a sequence in Σ^+ satisfying* (2.11), *there is a subsequence of v_k still denoted by v_k, an integer $m \geq 1$ and sequences x_k^i of points in $\mathbb{R}^n$ for $1 \leq i \leq m$ such that* (2.17) *holds and*

$$\begin{cases} \lambda(v_k)\, v_k - \sum_{i=1}^{m} \omega\,(\,\cdot\, - x_k^i) \underset{k}{\rightarrow} 0, \qquad \lambda(v_k) \underset{k}{\rightarrow} m^{1/2}\,|\omega|, \\ J(v_k) \underset{k}{\rightarrow} m\,S. \end{cases} \tag{2.18}$$

Remark. – Of course, we may replace in (2.18) $\sum_{i=1}^{m} \omega(\cdot - x_k^i)$ by $\varphi(\sum_{i=1}^{m} \omega(\cdot - x_k^i))$ where $\varphi \in C^\infty(\mathbb{R}^n)$ satisfies (2.12).

To conclude this section of preliminaries, we recall briefly a few informations on ω (taken out from [20], [32], [7] for instance): $\omega \in C^\infty(\mathbb{R}^n)$ is radial ($\omega = \omega(r)$) and satisfies

$$\omega(x)\,|x|^{\frac{n-1}{2}} \exp(\sqrt{\lambda_0}\,|x|) \to c > 0 \quad \text{as} \quad |x| \to \infty \tag{2.19}$$

$$\omega'(r)\, r^{\frac{n-1}{2}} \exp(\sqrt{\lambda_0}\, r) \to -c\sqrt{\lambda_0} \quad \text{as} \quad r = |x| \to \infty. \tag{2.20}$$

In fact, it is possible to show that $c = c_n \int_{\mathbb{R}^n} b^\infty \omega^p (\sqrt{\lambda_0}\, r)^{-1} \operatorname{sh}(\sqrt{\lambda_0}\, r)\, dx$ for some constant c_n depending only on n. This may be deduced from the following lemma that we will use later on

LEMMA II.2. – *Let $\varphi \in C_b(\mathbb{R}^n)$, $\psi \in C(\mathbb{R}^n)$ satisfy for some $\alpha \geq 0$, $\beta \geq C$, $c \in \mathbb{R}$*

$$\varphi(x) \exp(\alpha\,|x|)\,|x|^\beta \to c \quad \textit{as} \quad |x| \to \infty \tag{2.21}$$

$$\int_{\mathbb{R}^n} |\psi(x)| \exp(\alpha\,|x|)(1 + |x|^\beta)\, dx < \infty \tag{2.22}$$

Proof. – We just have to bound $|\varphi(x+y)\psi(x)| \exp(\alpha\,|y|)\,|y|^\beta$ by an L^1 function to conclude by the dominated convergence theorem. In order to do so, we prove that

$$\lim_{|y|\to\infty} \left[\left\{ \int_{\mathbb{R}^n} \varphi(x+y)\,\psi(x)\, dx \right\} \exp(\alpha\,|y|)\,|y|^\beta \right.$$
$$\left. - c \int_{\mathbb{R}^n} \exp\left(-\frac{\alpha(x, y)}{|y|} \right) \psi(z)\, dx \right] = 0.$$

In particular, if ψ is radial, we deduce that

$$\lim_{|y|\to\infty} \left\{ \int_{\mathbb{R}^n} \varphi(x+y)\,\psi(x)\, dx \right\} \exp(\alpha\,|y|)\,|y|^\beta = c \int_{\mathbb{R}^n} \frac{\operatorname{sh} \alpha\,|x|}{\alpha\,|x|}\, \psi(x)\, dx$$

This follows from the study of various cases. First of all, if $|x + y| \leq 1$, since $|y| \leq 1 + |x|$, we obtain

$$|\varphi(x+y)\,\psi(x)| \exp(\alpha\,|y|)\,|y|^\beta \leq C\,|\psi(x)| \exp(\alpha\,|x|)(1 + |x|^\beta)$$

where C denotes various constants independent of x, y. Next, if $1 \leq |x+y| \leq \frac{|y|}{2}$ remarking that $|y| \leq 2|x|$, we deduce

$$\begin{aligned}|\varphi(x+y)\psi(x)|\exp(\alpha|y|)|y|^{\beta} &\leq C|\psi(x)||x|^{\beta}\exp(\alpha|y|)\exp(-\alpha|x+y|)\\ &\leq C|\psi(x)||x|^{\beta}\exp(\alpha|x|).\end{aligned}$$

Finally, if $|x+y| \geq |y|/2$, we obtain

$$\begin{aligned}|\varphi(x+y)\psi(x)|\exp(\alpha|y|)|y|^{\beta} &\leq C\exp(-\alpha|x+y|)|\psi(x)|\exp(\alpha|y|)\\ &\leq C|\psi(x)|\exp(\alpha|x|).\end{aligned}$$

And we obtain the desired bound by summing up the three bounds we obtained. ■

III. CONTINUOUS SELECTION OF PARAMETERS

Proposition II.1 shows that "almost critical" points are close to a finite sum of "elementary solutions at infinity" ω centered at points infinitely away from 0 and from each other. For later purposes, it will be useful to project such configurations on weighted sums of such elementary solutions. To this end, we first choose $m \geq 1$, $\varphi \in C^{\infty}(\mathbb{R}^n)$ satisfying (2.12) and for $\varepsilon \in (0, 1)$ we consider

$$V(m,\varepsilon)=\Big\{v\in\Sigma^{+}/(x_1,\dots,x_m)\in\Omega^m,\ |x_i-x_j|>\frac{1}{\varepsilon}\ \text{if } I\leq i\neq j\leq m,\ |x_i|>\frac{I}{\varepsilon},\ \Big|\lambda(v)v-\varphi\Big(\sum_i^m\omega(\cdot-x_j)\Big)\Big|<\varepsilon\Big\}. \tag{3.1}$$

and we want to solve for $v \in V(m, \varepsilon)$ the following minimization problem

$$\operatorname{Min}\Big\{\Big|\lambda(v)v-\varphi\Big(\sum_i^m\alpha_j\,\omega(\cdot-x_j)\Big)\Big|\Big/\ \alpha_j\geq 0,\ x_j\in\Omega,\ \forall\, 1\leq j\leq m\Big\}. \tag{3.2}$$

PROPOSITION III.1. – *There exists* $\varepsilon_0 (= \varepsilon_0(m)) > 0$ *such that Problem* (3.2) *has, for any* $v \in V(m, \varepsilon_0)$, *a unique solution* $(\overline{\alpha}_j, \overline{x}_j)_{1\leq j\leq m}$

up to a permutation and there exist constants (independent of v*)* $\underline{\alpha}$, $\overline{\alpha}$, R_0 *such that (up to a permutation) if* $|\lambda(v)v - (\sum_i^m \omega(\cdot - x_j)| < \varepsilon$ *for some* $(x_j)_j$ *satisfying* $x_j \in \Omega$, $|x_i - x_j| > 1/\varepsilon$, $|x_j| > 1/\varepsilon$ *for* $1 \leq i \neq j \leq m$, *then we have*

$$|\overline{x}_j - x_j| < R_0, \quad 0 < \underline{\alpha} \leq \alpha_j \leq \overline{\alpha} \qquad \textit{for all} \quad 1 \leq j \leq m. \tag{3.3}$$

Furthermore, if $v \in V(m, \varepsilon)$ *with* $0 < \varepsilon \leq \varepsilon_0$, (3.3) *holds for some constants* $R_0(\varepsilon)$, $\underline{\alpha}(\varepsilon)$, $\overline{\alpha}(\varepsilon)$ *which satisfy*

$$R_0(\varepsilon) \to 0, \qquad \underline{\alpha}(\varepsilon) \to 1, \qquad \overline{\alpha}(\varepsilon) \to 1 \qquad as \quad \varepsilon \to 0_+. \tag{3.4}$$

Proof. – The proof will be made in several stages. We first show (3.3) for minimizing sequences (3.2) and the existence of a minimum satisfying (3.3)-(3.4). Next, we show some local strict convexity of the solution of (α_j, x_j) involved in (3.2). Finally, we prove the uniqueness by a simple continuation argument.

To simplify the presentation, we will only make the proof in the case when $\Omega = \mathbb{R}^n$ *i.e.* $\varphi \equiv 1$ and then the conditions $|x_i| > \frac{1}{\varepsilon}$, $x_j \in \Omega$ will play no role. The general case follows immediately by easy adaptations. Let us also remark that we just have to work with $u = \lambda(v)v$.

To prove the first claims made above, we observe that if $v \in V(m, \varepsilon)$ there exist $x_1, \ldots, x_m \in \mathbb{R}^n$ such that $|x_i - x_j| > \frac{1}{\varepsilon}$ for $1 \leq i \neq j \leq m$ and $|u - \sum_i^m \omega(\cdot - x_j)| < \varepsilon$; and thus the minimization problem (3.2) may be restricted to those $\tilde{\alpha}_j \geq 0$, $\tilde{x}_j$ such that

$$\left| u - \sum_{j=1}^m \tilde{\alpha}_j \, \omega(\cdot - \tilde{x}_j) \right| < \varepsilon. \tag{3.5}$$

We first claim that this implies that the $\tilde{x}_j$ are bounded from above by a fixed constant. Indeed, we deduce from (3.5)

$$\left| \sum_{j=1}^m \tilde{\alpha}_j \, \omega(\cdot - \tilde{x}_j) \right| < 2\varepsilon + \left| \sum_{j=1}^m \omega(\cdot - x_j) \right| < 2\varepsilon + mC$$

where C denotes various constants independent of v, u, ε. Then, this implies obviously, using the fact that ω is nonnegative,

$$\lambda_0 \left(\sum_{j=1}^m \tilde{\alpha}_j^2 \right) |\omega|_{L^2}^2 \leq \left| \sum_{j=1}^m \tilde{\alpha}_j \, \omega(\cdot - \tilde{x}_j) \right|_{L^2}^2 \leq (2\varepsilon + mC)^2$$

Therefore there exists $\overline{\alpha}$ (independent of v, u, ε) such that

$$0 \le \tilde{\alpha}_j \le \overline{\alpha} \quad \text{for} \quad 1 \le j \le m. \tag{3.6}$$

Next, we remark that (3.5) implies

$$\left| \sum_i^m \omega(\cdot - x_j) - \sum_i^m \tilde{\alpha}_j \, \omega(\cdot - \tilde{x}_j) \right| < 2\varepsilon. \tag{3.7}$$

Hence, if we fix $i \in \{1, ..., m\}$, we deduce

$$\begin{aligned} \frac{4}{\lambda_0}\varepsilon^2 &\ge |\omega(\cdot - x_i) - (-\sum_{j \ne i} \omega(\cdot - x_j)) + \sum_{j=1}^m \tilde{\alpha}_j \, \omega(\cdot - \tilde{x}_j)|_{L^2}^2 \\ &\ge |\omega|_{L^2}^2 + 2 \sum_{j \ne i} \int_{\mathbb{R}^n} \omega(x - x_i)\, \omega(x - x_j)\, dx - 2 \sum_{j=1}^m \tilde{\alpha}_j \times \\ &\quad \times \int_{\mathbb{R}^n} \omega(x - x_i)\, \omega(x - \tilde{x}_j)\, dx \\ &\ge |\omega|_{L^2}^2 - 2\,\overline{\alpha} \sum_{j=1}^m \int_{\mathbb{R}^n} \omega(x - x_i)\, \omega(x - \tilde{x}_j)\, dx \end{aligned}$$

where we use the positivity of ω and (3.6). Therefore, for ε small enough, there exists at least one index $j = j(i) \in \{1, ..., m\}$ such that

$$\int_{\mathbb{R}^n} \omega(x - x_i)\, \omega(x - \tilde{x}_j)\, dx \ge \frac{1}{4\,\overline{\alpha}} |\omega|_{L^2}^2.$$

And Lemma II.2 implies that there exists R_0 such that

$$|\tilde{x}_j - x_i| \le R_0, \quad \text{for} \quad j = j(i). \tag{3.8}$$

Then, in particular

$$|\tilde{x}_{j(i_1)} - \tilde{\alpha}_{j(i_2)}| \ge \frac{1}{\varepsilon} - 2 R_0$$

and for ε small enough, up to the permutation $(i \to j(i))$, we deduce finally

$$|x_i - \tilde{x}_i| \le R_0, \quad \text{for all} \quad 1 \le i \le m \tag{3.9}$$

for ε small enough. But, the same argument as above then shows

$$2\,\tilde{\alpha}_i \int_{\mathbb{R}^n} \omega(x - x_i)\,\omega(x - \tilde{x}_i)\,dx \geq |\omega|_{L^2}^2 - \frac{4\,\varepsilon^2}{\lambda_0} - 2\,\overline{\alpha} \sum_{j \neq i} \int_{\mathbb{R}^n} \omega(x - \tilde{x}_i)\,\omega(x - \tilde{x}_j)\,dx$$

and Lemma II.2 implies in particular that the right-hand side goes to $|\omega|_{L^2}^2$ as ε goes to 0 since $|\tilde{x}_i - \tilde{x}_j| \geq \frac{1}{\varepsilon} - 2\,R_0$ for $i \neq j$. This yields a uniform lower bound on $\tilde{\alpha}_j$ $(\forall j)$. In conclusion, we have shown that for ε small enough, a minimum with the properties claimed in Proposition III.1 exists. In addition, the above argument shows that

$$\begin{aligned}\frac{4}{\lambda_0}\,\varepsilon^2 &\geq |\omega(\,\cdot\, - x_i) - \tilde{\alpha}_i\,\omega(\,\cdot\, - \overline{x}_i)|_{L^2}^2 \\ &\quad + 2 \sum_{j \neq i} \int_{\mathbb{R}^n} \{\omega(x - x_i) - \overline{\alpha}_i\,\omega(x - \overline{x}_i)\} \\ &\quad \times \{\omega(x - x_j) - \overline{\alpha}_j\,\omega(x - \overline{x}_j)\}\,dx \\ &\geq |\omega(\,\cdot\, - x_i) - \overline{\alpha}_i\,\omega(\,\cdot\, - \overline{x}_i)|_{L^2}^2 - m(\varepsilon)\end{aligned}$$

where $m(\varepsilon) \to 0$ as $\varepsilon \to 0_+$ $(m(\varepsilon) \sim \exp(-\sqrt{\lambda_0}\,\frac{1}{\varepsilon})\,\varepsilon^{\frac{n-1}{2}}$ by Lemma II.2). Therefore, $\overline{\alpha}_i\,\omega(\,\cdot\, + x_i - \overline{x}_i) \underset{L^2}{\to} \omega$ as $\varepsilon \to 0_+$ uniformly in v. Hence, $\overline{\alpha}_i \to 1$ as $\varepsilon - 0_+$ uniformly in v and then one checks easily that $x_i - \overline{x}_i \to 0$ as $\varepsilon - 0_+$ uniformly in v.

At this stage, all the statements of Proposition III.1 but the uniqueness have been proved and in fact we also proved the existence for each $\delta \in (0,\,1)$ of $\varepsilon = \varepsilon(\delta)$ small enough such that if $|\lambda(v)\,v - \sum_i^m \omega(\,\cdot\, - x_j)| < \varepsilon$ for some $(x_j)_j$ satisfying $|x_j - x_i| > \frac{1}{\varepsilon}$ $(\forall\, i \neq j)$ then a minimum of (3.2) exists and any minimum $\sum_i^m \overline{\alpha}_j\,\omega(\,\cdot\, - \overline{x}_j)$ satisfies, up to a permutation of the indices,

$$\text{(3.10)} \qquad \begin{cases} |x_i - \overline{x}_i| \leq \delta, \;\; |\overline{x}_i - \overline{x}_j| \geq \dfrac{1}{\varepsilon} - 2, \;\; 1 - \delta \leq \overline{\alpha}_i \leq 1 + \delta \\ \qquad\qquad \text{for all} \quad 1 \leq i \neq j \leq m. \end{cases}$$

The second step consists in showing that the functional

$$\Phi(\overline{\alpha}, \ldots, \overline{\alpha}_m;\, \overline{x}_1, \ldots, \overline{x}_m) = \frac{1}{2}\,\left|\, u - \sum_i^m \overline{\alpha}_j\,\omega(\,\cdot\, - \overline{x}_j)\right|^2$$

has a definite positive second derivative at all points satisfying (3.10) provided ε is small enough *i.e.* $\varepsilon < \varepsilon_0 \le \varepsilon_1$. Indeed, the quadratic form obtained through the second derivative of Φ at $(\overline{\alpha}_1, ..., \overline{\alpha}_m; \overline{x}_1, ..., \overline{x}_m)$ acting on variations $(h_1, ..., h_m; \xi_1, ..., \xi_m) \in \mathbb{R}^m \times (\mathbb{R}^n)^m$ is given by

$$Q = \left| \sum_{j=1}^{m} h_j \, \omega(\cdot - \overline{x}_j) - \overline{\alpha}_j \nabla\omega(\cdot - \overline{x}_j) \cdot \xi_j \right|^2 + \left(\sum_{j=1}^{m} \overline{\alpha}_j \, \omega(\cdot - \overline{x}_j) - u, -2 \sum_{j=1}^{m} h_j \nabla\omega(\cdot - \overline{x}_j) \cdot \xi_j + \sum_{j=1}^{m} \overline{\alpha}_j D^2 \omega(\cdot - \overline{x}_j) [\xi_j, \xi_j] \right).$$

And, because of (3.10), Lemma II.2, (2.19), (2.20) we deduce

$$Q \ge \sum_{j=1}^{m} h_j^2 |\omega|^2 - \mu(\varepsilon) \left(\sum_{j=1}^{m} h_j^2 + |\xi_j|^2 \right) + \left(u, 2 \sum_{j=1}^{m} h_j \nabla\omega(\cdot - \overline{x}_j) \cdot \xi_j \right) - \left(u, \sum_{j=1}^{m} \overline{\alpha}_j D^2 \omega(\cdot - \overline{x}_j) [\xi_j, \xi_j] \right). \tag{3.11}$$

Here and below, μ denotes various positive constants (depending only on ε) such that $\mu \underset{\varepsilon}{\to} 0$. Indeed, observe that we have for all $1 \le \alpha, \beta, \gamma \le n$

$$\int_{\mathbb{R}^n} \omega \frac{\partial \omega}{\partial x_\alpha} dx = 0,$$

$$\int_{\mathbb{R}^n} \omega \frac{\partial^2 \omega}{\partial x_\alpha \partial x_\beta} = -\delta_{\alpha\beta} \int_{\mathbb{R}^n} \left(\frac{\partial \omega}{\partial x_\alpha} \right)^2 dx,$$

$$\int_{\mathbb{R}^n} \frac{\partial \omega}{\partial x_\alpha} \frac{\partial^2 \omega}{\partial x_\alpha \partial x_\beta} dx = 0,$$

$$\int_{\mathbb{R}^n} \frac{\partial \omega}{\partial x_\alpha} \frac{\partial^2 \omega}{\partial x_\alpha \partial x_\beta \partial x_\gamma} = -\delta_{\beta\gamma} \int_{\mathbb{R}^n} \left(\frac{\partial^2 \omega}{\partial x_\alpha \partial x_\beta} \right)^2 dx.$$

Next, we observe that we may replace u by $\sum_i^m \omega(\cdot - x_j)$ and using the above rules we finally obtain

$$(3.12)\quad Q \geq \sum_{j=1}^m h_j^2 |\omega|^2 + \sum_{j=1}^m \overline{\alpha}_j (\nabla\omega(\cdot - x_j)\cdot\xi_j, \nabla\omega(\cdot - \overline{x}_j)\cdot\xi_j) - \mu(\varepsilon)\left(\sum_{j=1}^m h_j^2 + |\xi_j|^2\right).$$

And using once more (3.10) we deduce

$$(3.13)\quad Q \geq \sum_{j=1}^m h_j^2 |\omega|^2 + \sum_{j=1}^m (1-\delta)|\nabla\omega\cdot\xi_j|^2 - (\mu(\varepsilon)+\mu(\delta))\left(\sum_{j=1}^m h_j^2 + |\xi_j|^2\right).$$

To conclude, we just observe that for all $1 \leq \alpha, \beta, \gamma \leq n$

$$\int_{\mathbb{R}^n} \frac{\partial\omega}{\partial x_\alpha}\frac{\partial\omega}{\partial x_\beta}\,dx = \delta_{\alpha\beta}\int_{\mathbb{R}^n}\left(\frac{\partial\omega}{\partial x_\beta}\right)^2 dx$$

$$\int_{\mathbb{R}^n} \frac{\partial^2\omega}{\partial x_\alpha\,\partial x_\beta}\frac{\partial^2\omega}{\partial x_\alpha\,\partial x_\gamma}\,dx = \delta_{\beta\gamma}\int_{\mathbb{R}^n}\left|\frac{\partial^2\omega}{\partial x_\alpha\,\partial x_\beta}\right|^2 dx.$$

And this implies that there exists $\nu > 0$ (independent of u) such that for ε, δ small

$$(3.14)\quad Q \geq \nu\sum_{j=1}^m h_j^2 + |\xi_j|^2.$$

To conclude the proof of Proposition III.1, we use a simple continuation argument. Indeed, we have just shown that any minimum (up to a permutation) is nondegenerate and since all the above estimates are uniform along the paths $(t \in [0, 1] \to tu + (1-t)\sum_i^m \omega(\cdot - x_j))$, we just have to show the uniqueness (up to a permutation) when $u = \sum_i^m \omega(\cdot - x_j)$. But this amounts to check that if $\sum_j^m \overline{\alpha}_j\,\omega(\cdot - \overline{x}_j) = \sum_j^m \omega(\cdot - x_j)$ for some $\overline{\alpha}_j \geq 0$, $\overline{x}_j \in \mathbb{R}^n$, $x_j \in \mathbb{R}^n$ where the x_j are distinct then $\{(\overline{\alpha}_j, \overline{x}_j), 1 \leq j \leq m\} = \{(1, x_j), 1 \leq j \leq m\}$. This is in particular insured by the following lemma which thus concludes the proof of Proposition III.1.

LEMMA III.1. – *Let $N \geq 1$, $x_1, ..., x_N$ be N distinct points in $\mathbb{R}^n$ and let $\gamma_1, ..., \gamma_N \in \mathbb{R}$. Assume that*

$$(3.15)\quad \sum_i^N \gamma_j\,\omega(\cdot - x_j) = 0,$$

then $\gamma_1 = ... = \gamma_N = 0$.

Proof. – Denoting by ω the Fourier transform of ω, we deduce from (3.15)

$$\left(\sum_i^N \gamma_j e^{i\xi \cdot x_j}\right) \omega(\xi) = 0 \qquad \forall \xi \in \mathbb{R}^n. \tag{3.16}$$

Furthermore, using the decay of ω, one can show easily that ω vanishes at most on a countable set so (3.16) implies

$$\sum_i^N \gamma_j e^{i\xi \cdot x_j} = 0 \quad \text{on } \mathbb{R}^n \tag{3.17}$$

and we conclude since the points x_j are distinct. ■

IV. A LOCAL DEFORMATION ARGUMENT

This section is devoted to an important technical point, namely the analysis of deformation of the level sets of the functional J. This deformation argument is quite typical in Liusternik-Schnirelman type arguments (*see* Milnor [27], P. H. Rabinowitz [29]...) and even if, by opposition to the rather sharp Morse deformation lemma, it is a rough deformation we will have to analyse it very precisely.

We fix $m \in \mathbb{N}$ and we consider two positive constants θ_m, $\bar{\varepsilon}_m$. We will denote by

$$b_n = (n+1)S, \qquad \text{for all} \quad n \geq 0; \tag{4.1}$$

and we set $\overline{W}_n = \{u \in \Sigma^+ / J(u) \leq b_n\}$ $(\forall n \geq 0)$ and

$$J^c = \{u \in \Sigma / J(u) \leq c\}, \qquad \forall c \in \mathbb{R} \tag{4.2}$$

We will use a modification of the "true gradient flow" (2.6) namely

$$\frac{du}{ds} = -\frac{J'(u)}{(1 + |J'(u)|^2)^{1/2}} \qquad \text{for} \quad s \geq 0, \quad u|_{s=0} = u_0 \tag{4.3}$$

where u_0 is any initial condition in Σ^+. The result which follows will give in particular the existence of a global unique solution $u(s, u_0)$ which depends continuously upon u_0. We then consider for all $\delta \in \mathbb{R}$

$$T_\delta(u_0) = \inf(s \geq 0, \ J(u(s, u_0)) \leq b_{m-1} + \delta) \tag{4.4}$$

if no such s exists we set $T_\delta(u_0) = +\infty$. We finally denote, assuming that (4.3) admits a global solution and that T_δ is finite on the set considered – all points which will be answered in the result below –, by

$$\tilde{W}^\delta_{m-1} = \{u(T, u_0)/u_0 \in \overline{W}_m\}, \qquad \text{for} \quad m \geq 2, \quad 0 < \delta < \frac{S}{2} \tag{4.5}$$

where T is given by

$$T = T(u_0) = (T_\delta(u_0) + \sqrt{\delta}) \wedge T_0(u_0). \tag{4.6}$$

Finally, if $m = 1$, we set $\overline{W}^\delta_{m-1} = J^{b_0+\delta} \cap \Sigma^+$, for $0 < \delta < \frac{S}{2}$.

We then have the

LEMMA IV.1. – *The differential equation* (4.3) *has a unique global solution* $u(s) = u(s, u_0)$, *which depends continuously upon* u_0 *and maps* Σ^+ *into itself. Next, if we assume that* (1.1) *has no solutions or equivalently that* J *has no critical points on* Σ^+ *and that* (1.3) *has a unique radial solution, then* $T_\delta(u_0)$ *is continuous on* $\overline{W}_m$ *(with values in* $]0, +\infty]$ *if* $\delta = 0$*), for* $\delta \geq 0$. *Therefore, for* $0 < \delta < \frac{S}{2}$, *the pair* $(\overline{W}_m, \overline{W}_{m-1})$ *retracts by (this) deformation onto the pair* $(\tilde{W}^\delta_{m-1}, \overline{W}_{m-1})$ *and for any* $\varepsilon > 0$, *we may choose* $\delta > 0$ *small enough such that*

$$\overline{\tilde{W}^\delta_{m-1} \backslash \overline{W}_{m-1}} \subset V(m, \varepsilon). \tag{4.7}$$

Proof. – The fact that (4.3) is a well-posed ordinary differential equation is easily deduced from the explicit formulas giving $\lambda(u)$, $J(u)$. Indeed, $\lambda(u)$, $J(u)$, $J'(u)$, $\lambda'(u)$ are clearly locally Lipschitz and the Lipschitz bounds depend only on a bound from above of J (or λ). This, of course, immediately implies the existence of a maximal solution of (4.3) which is global provided one bounds from above J on this trajectory. But since J is non-increasing along the trajectory, the upper bound on J is obvious and the global existence follows as well the continuous dependence upon the initial condition u_0.

We now proceed to prove the remaining assertions on the semiflow. We first show that the flow preserves Σ^+ (as announced in section II). Indeed observe first that by a change of clock, we just have to show that the "true" gradient flow (2.6) (which exists for the same reasons as above) preserves also Σ^+. To this end, we modify, for $u_0 \in \Sigma^+$, the Equation (2.6) as follows: if $u_0 \in \Sigma^+$, there exists by the same arguments as above a maximal solution on $[0, \overline{T}[$ of

$$\frac{du}{ds} = -\lambda^2(u)\, u + K(b\, \lambda^{p+1}(u)\, u^{+p}) \tag{2.6'}$$

where $\lambda(u)$ is still defined by (2.4) on $H_0^1(\Omega)\backslash\{0\}$, and $u^+ = \max(u, 0)$. If we show that u remains nonnegative, then u solves in fact (2.6) and our claim is proved. To do so, we fix $t_0 < \overline{T}$.

Then, denoting by $u^- = u^+ - u$, we multiply (2.6′) (recall that we always use the scalar product of $H_0^1(\Omega)$) by ku^- and we obtain

$$-\frac{1}{2}\frac{d}{ds}\int_\Omega |u^-|^2\,dx \geq \lambda^2(u)\int_\Omega |u^-|^2\,dx + \int_\Omega K\,(b\,\lambda^{p+1}(u)\,u^{+p})\,u^-\,dx$$

or

$$\frac{d}{ds}\int_\Omega |u^-|^2\,dx \leq 0$$

and we conclude easily since $\int_\Omega |u_0^-|^2\,dx = 0$.

Next, in all the remainder of the proof, we assume that J has no critical points on Σ^+. Hence, $J(u(s, u_0))$ is decreasing for all $s \geq 0$. Next, we claim that, for each $h > 0$, there exists $\gamma > 0$ such that

$$(4.8) \qquad |J'(v)| \geq \gamma \qquad \text{if} \quad v \in \Sigma^+, \quad J(v) \in [b_{m-1} + h, b_m - h].$$

This is indeed an immediate consequence of Corollary II.1, arguing by contradiction. Then, since we have for all $t \geq s \geq 0$, $u_0 \in \Sigma^+$

$$(4.9) \qquad J(u(t, u_0)) - J(u(s, u_0)) = +\int_s^t |J'(u(\sigma, u_0))|^2\,(1 + |J'(u(\sigma, u_0))|^2)^{-1/2}\,d\sigma$$

we deduce easily that for all $u_0 \in \overline{W}_m$, $T_\delta(u_0) < \infty$ for all $\delta > 0$.

We next show that T_δ is continuous on $\overline{W}_n$ if $\delta > 0$, or continuous with values in $[0, +\infty]$ if $\delta = 0$. The proof being quite similar in both cases, and in fact a bit simpler when $\delta > 0$, we just prove the continuity of T_0. To this end, we take a sequence $(u_0^n)_n$ in $\overline{W}_m$ such that $u_0^n \underset{n}{\rightarrow} u_0 \in \overline{W}_m$, $T_0(u_0^n) \underset{n}{\rightarrow} T_0 \in [0, \infty]$ and we want to show that $T_0 = T_0(u_0)$. First of all, if $t < T_0(u_0)$, $J(u(t, u_0)) > b_{m-1}$ and by continuity we still have for n large enough $J(u(t, u_0^n)) > b_{m-1}$, hence $t < T_0(u_0^n)$ and $T_0 \geq T_0(u_0)$. Next, assume that $T_0(u_0) < T_0$ and thus in particular $T_0(u_0) < \infty$. Therefore, for all $h > 0$

$$J(u(T_0(u_0) + h, u_0)) < b_{m-1}$$

and again by continuity the same inequality holds for n large *i.e.*

$$J\left(u\left(T_0\left(u_0\right)+h,\, u_0^n\right)\right)<b_{m-1}.$$

Therefore, $T_0\left(u_0^n\right)<T_0\left(u_0\right)+h$ for n large, and we reach the contradiction which proves our claim. In conclusion, $T\left(u_0\right)$ is continuous on $\overline{W}_m$ and clearly $T(u_0)\equiv 0$ on $\overline{W}_{m-1}$. The deformation is now clear: consider the map

$$[0,1]\times\overline{W}_m\ni\left(t,\, u_0\right)\rightarrow u\left(t\, T\left(u_0\right),\, u_0\right).$$

In the case $m=1$, the situation is much simpler and the deformation is immediate.

To complete the proof of Lemma IV.1, we have to show (4.7). We first consider the case when $m\geq 2$ and we will then treat the case when $m=1$. Again, in view of Corollary II.1, we just have to show that there exists a positive constant $C\geq 0$ (independent of δ) such that

$$\text{(4.10)}\qquad \left|J'\left(v\right)\right|\leq C\,\delta^{1/2},\qquad \text{for all}\quad v\in\overline{W}_{m-1}^{\delta}\backslash\overline{W}_{m-1}.$$

To prove this bound, we take $v\in\overline{W}_{m-1}^{\delta}\backslash\overline{W}_{m-1}$ *i.e.* $J\left(v\right)>b_{m-1}$ and therefore $v=u\left(T_\delta\left(u_0\right)+\sqrt{\delta},\, u_0\right)$ for some $u_0\in\overline{W}_m$. To simplify notations we will denote by $\tilde{v}=u\left(T_\delta\left(u_0\right),\, u_0\right)$, $v\left(s\right)=u\left(s,\, u_0\right)$ for $s\geq 0$. And we deduce from (4.9)

$$\text{(4.11)}\qquad \int_{T_\delta\left(u_0\right)}^{T_\delta\left(u_0\right)+\sqrt{\delta}}\left|J'\left(v\left(s\right)\right)\right|^2\left(1+\left|J'\left(v\left(s\right)\right)\right|^2\right)^{-1/2}ds\leq\delta.$$

Hence, there exists $\overline{s}\in\left]T_\delta\left(u_0\right),\, T_\delta\left(u_0\right)+\sqrt{\delta}\right[$ such that

$$\left|J'\left(v\left(\overline{s}\right)\right)\right|^2\left(1+\left|J'\left(v\,\overline{s}\right)\right)\right|^2\right)^{-1/2}\leq\delta^{1/2}$$

and thus there exists a constant independent of $\delta\in\left(0,\,\frac{s}{2}\right)$ such that

$$\text{(4.12)}\qquad \left|J'\left(v\left(\overline{s}\right)\right)\right|\leq C\,\delta^{1/2}.$$

Now, in view of (4.3), we deduce

$$\text{(4.13)}\qquad \left|v-v\left(\overline{s}\right)\right|\leq\delta^{1/2}.$$

To conclude, we use the fact that J' is Lipschitz on $\Sigma^+\cap J^R$ for all $R<\infty$ and (4.10) follows from combining (4.12) and (4.13).

In the case when $m = 1$, that is we consider $v \in (J^{b_0+\delta} \cap \Sigma^+)\backslash \overline{W}_0$: observe that since $b_0 = \inf_{\Sigma} J$ is not achieved then $W_0 = \emptyset$ and by I. Ekeland's variational principle [16], we can find $w \in \Sigma^+$ such that

$$J(w) \leq J(v), \qquad |J'(w)| \leq \sqrt{\delta}, \qquad |v - w| \leq \sqrt{\delta}$$

and we conclude easily since for δ small enough this implies that $w \in V(1, \varepsilon)$. ■

In particular, we deduce from Lemma IV.1 that there exists $\gamma_m(\cdot)$ continuous, nondecreasing and nonnegative such that $\gamma_m(0) = 0$ and

$$\tilde{W}^{\delta}_{m-1} \subset \overline{W}_{m-1} \cup \overline{V(m, \gamma_m(\delta))}.$$

We then set

$$\overline{W}^{\delta}_{m-1} = \overline{W}_{m-1} \cup \overline{V(m, \gamma_m(\delta))}.$$

V. A TOPOLOGICAL ARGUMENT

This section is devoted to a topological argument which will imply the existence result (Theorem I.1) provided we admit an important "energy-balance" type result (Proposition V.1 below) that we prove in the next section. Throughout this section we will assume that (1.3) has a unique positive radial solution and that (I.1) has no solution and we will reach a contradiction proving Theorem I.1. The topological argument we use is quite close to the ones introduced in A. Bahri [1], [2]. A. Bahri and J. M. Coron [5].

We will need a few notations: S^{n-1} is an $(n-1)$ dimensional standard sphere embedded in $\overline{\Omega}$ so that $\lambda S^{n-1} \subset \overline{\Omega}$ for all $\lambda \geq 1$ and $H_{n-1}(S^{n-1})$ embeds in $H_{n-1}(\overline{\Omega})$. We may assume without loss of generality that $S^{n-1} = \{x \in \mathbb{R}^n / |x| = 1\}$: indeed, this may be achieved by a simple scaling. For $m \geq 1$, we denote by $(S^{n-1})^m$ its m-th power, by γ_m the embedding of $(S^{n-1})^m$ into $\overline{\Omega}^m$, and by $\lambda\gamma_m$ the corresponding embedding of $(\lambda S^{n-1})^m$ into $\overline{\Omega}^m$.

We will also denote by $\Delta_{m-1} = \{(t_1, ..., t_m) / \sum_i^m t_i = 1,\ t_i \geq 0$ for all $i\}$ the standard $(m-1)$-simplex, by $\partial\Delta_{m-1}$ its boundary, by $\Delta^{\varepsilon}_{m-1} = \{(t_1, ..., t_m) \in \Delta_{m-1} / \sup_i |t_i - \frac{1}{m}| \leq \varepsilon\}$, by $\partial\Delta^{\varepsilon}_{m-1}$ its boundary. Notice that $\Delta^{\varepsilon}_{M-1} \subset \Delta_{m-1}$ and $(\Delta_{m-1}, \partial\Delta_{m-1})$ retracts by deformation on $(\Delta^{\varepsilon}_{m-1}, \partial\Delta^{\varepsilon}_{m-1})$.

Next, we denote by $D_m = \{(x_1, ..., x_m) \in (S^{n-1})^m / \exists\, i \neq j \; x_i = x_j\}$, by σ_m the group of permutation of $\{1, ..., m\}$, by V_m a σ_m-invariant tubular neighbourhood of D_m. V_m may be considered as a $(n-1)\,m$ dimensional manifold with boundary, which retracts by deformation on D_m (*see* e.g. Bredon [9]). We will denote by $(S_0^{n-1})^m$ the $(n-1)\,m$ dimensional manifold with boundary ∂V_m given by $\overline{(S^{n-1})^m \backslash V_m}$. Of course, σ_m acts on $(S^{n-1})^m \times \Delta_{m-1}$, $(S^{n-1})^m \times \partial\Delta_{m-1}) \backslash (D_m \times \Delta_{m-1})$, $(S_0^{n-1})^m \times \Delta_{m-1}$, $((S_0^{n-1})^m \times \partial\Delta_{m-1}) \cup (\partial V_m \times \Delta_{m-1})$. The quotient of these sets under the action of σ_m will be denoted with a subscript σ_m under the product or union signs; for instance, the quotient of $(S^{n-1})^m \times \Delta_{m-1}$ under the action of σ_m will be denoted by $(S^{n-1})^m \underset{\sigma_m}{\times} \Delta_{m-1}$. We will consider five main pairs

$$((S^{n-1})^m \underset{\sigma_m}{\times} \Delta_{m-1},\ ((S^{n-1})^m \times \partial\Delta_{m-1}) \bigcup_{\sigma_m} (D_m \times \Delta_{m-1}))) \tag{5.1}$$

$$((S_0^{n-1})^m \underset{\sigma_m}{\times} \Delta_{m-1},\ ((S_0^{n-1})^m \times \partial\Delta_{m-1}) \bigcup_{\sigma_m} (\partial V_m \times \Delta_{m-1}))) \tag{5.2}$$

$$((S^{n-1})^m / \sigma_m,\ (S_0^{n-1})^m / \sigma_m) \tag{5.3}$$

$$(\overline{W}_m,\ \overline{W}_{m-1}) \tag{5.4}$$

$$(\overline{W}^{\delta}_{m-1},\ \overline{W}_{m-1}) \tag{5.5}$$

where $\overline{W}_{m-1}$, $\overline{W}_m$, $\overline{W}^{\delta}_{m-1}$ have been defined in the preceding section.

We denote by s_m the map from $V(m, \varepsilon_0)$ into $\overline{\Omega}^m / \sigma_m$ which maps $v \in V(m, \varepsilon_0)$ into $(\overline{x}_1, ..., \overline{x}_m)$ solution of (3.2) (as given in Proposition III.1), by i_m the embedding from $(S_0^{n-1})^m / \sigma_m$ into $(S^{n-1})^m / \sigma_m$, by k_m the embedding from

$$((S_0^{n-1})^m \underset{\sigma_m}{\times} \Delta_{m-1},\ (S_0^{n-1})^m \times \partial\Delta_{m-1}) \bigcup_{\sigma_m} (\partial V_m \times \Delta_{m-1}))$$

into

$$((S^{n-1})^m \underset{\sigma_m}{\times} \Delta_{m-1},\ ((S^{n-1})^m \times \partial\Delta_{m-1}) \bigcup_{\sigma_m} (D_m \times \Delta_{m-1}))).$$

Finally, we denote by

$$B_m(S^{n-1}) = \left\{ \sum_i^m t_i\, \delta_{x_i} / (x_1, ..., x_m) \in (S^{n-1})^m,\ (t_1, ..., t_m) \in \Delta_{m-1} \right\}$$

where δ_x denotes the Diract mass at x. $B_m(S^{n-1})$ is endowed of the weak $\star$ topology of measures on S^{n-1}. One may also think of $B_m(S^{n-1})$ as the quotient of $(S^{n-1})^m \underset{\sigma_m}{\times} \Delta_{m-1}$ (endowed of its natural topology) through the following equivalence relation: $(x_1, \ldots, x_m, t_1, \ldots, t_n) \sim (x_1', \ldots, x_m', t_1', \ldots, t_m')$ if for any x_i such that $t_i \neq 0$ we have

$$\sum_{k:x_k=x_i} t_k = \sum_{k:x_k'=x_i} t_k'$$

and if for any x_i' such that $t_i' \neq 0$ we have

$$\sum_{k:x_k'=x_i'} t_k' = \sum_{k:x_k=x_i'} t_k.$$

Let θ_m be the corresponding projection from $(S^{n-1})^m \underset{\sigma_m}{\times} \Delta_{m-1}$ onto $B_m(S^{n-1})$. Let ℓ_m be the projection on the x-component of $(x, t) \in (S^{n-1})^m \underset{\sigma_m}{\times} \Delta_{m-1}$ from $(S^{n-1})^m \underset{\sigma_m}{\times} \Delta_{m-1}$ onto $(S^{n-1})^m / \sigma_m$.

For $\lambda \geq 1$, we introduce a continuous map from $B_m(S^{n-1})$ into Σ^+ given by

$$f_m(\lambda) : \sum_1^m t_i \, \delta_{x_i} \to \varphi\left(\sum_1^m t_i \, \omega(\cdot - \lambda \, x_i)\right) \left| \varphi\left(\sum_1^m t_i \, \omega(\cdot - \lambda \, x_i)\right) \right|^{-1}$$

where φ is a fixed cut-off function in $C^\infty(\mathbb{R}^n)$ satisfying (2.12).

The following result will be proved in section VI.

PROPOSITION V.1. – *For any $m \geq 1$, $\varepsilon_1 \in (0, \varepsilon_0(m))$ ($\varepsilon_0(m)$ has been defined in Proposition* III.1), *$\delta > 0$ such that* (4.7) *holds with $\varepsilon = \varepsilon_1$, there exists $\lambda_m \geq 1$, $\varepsilon_1' > 0$ such that for $\lambda \geq \lambda_m$ we have*

(i) *$f_m(\lambda)$ maps $(B_m(S^{n-1}), B_{m-1}(S^{n-1}))$ into $(\overline{W}^\delta_{m-1}, \overline{W}_{m-1}) \subset (\overline{W}_m, \overline{W}_{m-1})$ and $f_m(\lambda) \circ \theta_m$ maps $((S^{n-1})^m \underset{\sigma_m}{\times} \Delta_{m-1}), ((S^{n-1})^m \times \partial\Delta_{m-1}) \bigcup_{\sigma_m} (D_m \times \Delta_{m-1}))$ into $(\overline{W}^\delta_{m-1}, \overline{W}_{m-1})$.*

(ii) *$f_m(\lambda) \circ \theta_m$ maps $((S_0^{n-1})^m \underset{\sigma_m}{\times} \Delta^{\varepsilon_1'}_{m-1}, ((S_0^{n-1})^m \times \partial\Delta^{\varepsilon_1'}_{m-1}) \bigcup_{\sigma_m} (\partial V_m \times \Delta^{\varepsilon_1'}_{m-1}))$ into $(\overline{W}^\delta_{m-1} \cap V(m, \varepsilon_1), \overline{W}_{m-1} \cap V(m, \varepsilon_1))$ and* *the following diagram is commutative*

$$\begin{array}{ccc} ((S_0^{n-1})^m \underset{\sigma_m}{\times} \Delta_{m-1}^{\varepsilon_1''}),\ ((S_0^{n-1})^m \times \partial\Delta_{m-1}^{\varepsilon_1'}) & & \\ \times \bigcup_{\sigma_m} (\partial V_m \times \Delta_{m-1}^{\varepsilon_1'})) & \xrightarrow{f_m(\lambda)\circ\theta_m} & (\overline{W}^{\delta}_{m-1} \cap V(m, \varepsilon_1),\ \overline{W}_{m-1} \cap V(m, \varepsilon_1)) \\ \downarrow \ell_m & & \downarrow s_m \\ (S_0^{n-1})^m/\sigma_m & \xrightarrow{\lambda\gamma_m\circ i_m} & \overline{\Omega}^m/\sigma_m \end{array}$$

(iii) $\exists\, m \geq 1$ *such that for* $m' \geq m$, $f_{m'}(\lambda)\,[B_{m'}(S^{n-1})] \subset \overline{W}_{m'-1}$. ■

We may now conclude the *proof of Theorem* I.1. Let us recall that the argument below is a repetition of the argument introduced by A. Bahri and J. M. Coron [5], [2]. We first mention that all homologies below are with $\mathbb{Z}_2$-coefficients. Next, observe that θ_m defines a homeomorphism from

$$(S^{n-1})^m \underset{\sigma_m}{\times} \Delta_{m-1} \backslash ((S^{n-1})^m \times \partial\Delta_{m-1}) \bigcup_{\sigma_m} (D_m \times \Delta_{m-1})$$

$$\text{onto } B_m(S^{n-1})/B_{m-1}(S^{n-1})$$

and that $((S^{n-1})^m \times \partial\Delta_{m-1}) \bigcup_{\sigma_m} (D_m \times \Delta_{m-1})$ is a retract by deformation of $((S^{n-1})^m \times \hat{\Delta}_{m-1}) \bigcup_{\sigma_m} (V_m \times \Delta_{m-1})$ where $\hat{\Delta}_{m-1} = \{(t_1, ..., t_m) \in \Delta_{m-1} / \sum_i^m |t_i - \frac{1}{m}| \geq \frac{1}{2m}\}$. Furthermore, $((S^{n-1})^m \times \hat{\Delta}_{m-1}) \bigcup_{\sigma_m} (V_m \times \Delta_{m-1})$ is a closed neighborhood of $((S^{n-1})^m \times \partial\Delta_{m-1}) \bigcup_{\sigma_m} (D_m \times \Delta_{m-1})$. Therefore, we deduce by excision

$$(5.6) \quad \begin{aligned} & H_\star(B_m(S^{n-1}), B_{m-1}(S^{n-1})) \\ & = H_\star((S^{n-1})^m \underset{\sigma_m}{\times} \Delta_{m-1}, (S^{n-1})^m \times \partial\Delta_{m-1}) \\ & \times \bigcup_{\sigma_m} (D_m \times \Delta_{m-1})). \end{aligned}$$

As V_m retracts by deformation equivariantly on D_m, we also have

$$(5.7) \quad \begin{aligned} & H_\star((S^{n-1})^m \underset{\sigma_m}{\times} \Delta_{m-1}, ((S^{n-1})^m \times \partial\Delta_{m-1}) \bigcup_{\sigma_m} (D_m \times \Delta_{m-1})) \\ & = H_\star((S^{n-1})^m \underset{\sigma_m}{\times} \Delta_{m-1}, ((S^{n-1})^m \times \partial\Delta_{m-1}) \\ & \bigcup_{\sigma_m} (V_m \times \Delta_{m-1})) \end{aligned}$$

Hence, by excision

$$(5.8)\quad H_\star\big(((S^{n-1})^m \underset{\sigma_m}{\times} \Delta_{m-1},\ ((S^{n-1})^m \times \partial\Delta_{m-1})\bigcup_{\sigma_m}(D_m \times \Delta_{m-1})\big)$$

$$= H_\star\big(((S_0^{n-1})^m \underset{\sigma_m}{\times} \Delta_{m-1},\ ((S_0^{n-1})^m \times \partial\Delta_{m-1})$$

$$\bigcup_{\sigma_m}(\partial V_m \times \Delta_{m-1})\big)$$

$$= H_\star\big(((S_0^{n-1})^m \underset{\sigma_m}{\times} \Delta_{m-1},\ \partial((S_0^{n-1})^m \underset{\sigma_m}{\times} \Delta_{m-1})\big)$$

(Observe here that $(S_0^{n-1})^m/\sigma_m$ is a retract by deformation of some neighbourhood of this set in $(S^{n-1})^m/\sigma_m$).

Therefore, we have

$$(5.9)\quad H_*\big(B_m(S^{n-1}),\ B_{m-1}(S^{n-1})\big)$$
$$= H_*\big(((S_0^{n-1})^m \underset{\sigma_m}{\times} \Delta_{m-1},\ \partial((S_0^{n-1})^m \underset{\sigma_m}{\times} \Delta_{m-1})\big)$$

The cap product

$$(5.10)\quad H^\star\big((S_0^{n-1})^m \underset{\sigma_m}{\times} \Delta_{m-1}\big) \otimes H\big((S_0^{n-1})^m \underset{\sigma_m}{\times} \Delta_{m-1},$$
$$\partial((S_0^{n-1})^m \underset{\sigma_m}{\times} \Delta_{m-1})\big)$$
$$\to H_\star\big((S_0^{n-1})^m \underset{\sigma_m}{\times} \Delta_{m-1},\ \partial((S_0^{n-1})^m \underset{\sigma_m}{\times} \Delta_{m-1})\big)$$

equips $H_\star\big(((S_0^{n-1})^m \underset{\sigma_m}{\times} \Delta_{m-1}),\ \partial((S_0^{n-1})^m \underset{\sigma_m}{\times} \Delta_{m-1})\big)$ with a structure of $H^\star((\overline{\Omega})^m/\sigma_m)$-moduli via the homomorphism

$$(5.11)\quad i_m^\star \circ \gamma_m^\star : H^\star((\overline{\Omega})^m/\sigma_m) \to H^\star((S_0^{n-1})^m/\sigma_m)$$
$$= H^\star\big((S_0^{n-1})^m \underset{\sigma_m}{\times} \Delta_{m-1}\big).$$

In the absence of a solution to (1), the deformation Lemma IV.1 states that $(\overline{W}_m, \overline{W}_{m-1})$ retracts by deformation onto $(\tilde{W}_{m-1}^\delta, \overline{W}_{m-1})$ and that we have $\tilde{W}_{m-1}^\delta \subset \overline{W}_{m-1}^\delta \subset \overline{W}_m$ and

$$(5.12)\quad \overline{\overline{W}_{m-1}^\delta \backslash \overline{W}_{m-1}} \subset V(m, \varepsilon_1).$$

Therefore, we have on one hand

$$(5.13)\quad H_\star(\overline{W}_m, \overline{W}_{m-1}) = H_\star(\overline{W}^\delta_{m-1}, \overline{W}_{m-1}) = H_\star(\overline{\overline{W}^\delta_{m-1}\backslash \overline{W}_{m-1}}, \overline{\overline{W}^\delta_{m-1}\backslash \overline{W}_{m-1}} \cap \overline{W}_{m-1})$$

and on the other hand, we have a well-defined homeomorphism, via the map s_m.

$$(5.14)\qquad i^\star \circ s^\star_m : H^\star((\overline{\Omega})^m/\sigma_m) \to H^\star(\overline{\overline{W}^\delta_{m-1}\backslash \overline{W}_{m-1}})$$

where i is the inclusion (5.12).

(5.13) and (5.14) imply that, in the absence of a solution to (1.1), $H^\star(\overline{W}_m, \overline{W}_{m-1})$ is naturally equipped with a structure of $H^\star((\overline{\Omega})^m/\sigma_m)$-module.

Using the commutativity of the diagram in (ii) of Proposition V.1, the map

$$(5.15)\quad (f_m(\lambda)\circ\theta_m)_\star : H_\star((S_0^{n-1})^m \underset{\sigma_m}{\times} \Delta^{\varepsilon_1'}_{m-1}, ((S_0^{n-1})^m \times \partial\Delta^{\varepsilon_1'}_{m-1}) \bigcup_{\sigma_m}(\partial V_m \times \Delta^{\varepsilon_1'}_{m-1})) \to H_\star(\overline{W}^\delta_{m-1} \cap V(m, \varepsilon_1), \overline{W}_{m-1} \cap V(m, \varepsilon_1))$$

is $H^\star((\overline{\Omega})^m/\sigma_m)$-linear.

Using the commutativity of the diagram

(5.16)

$$\begin{array}{ccc} ((S^{n-1})^m \underset{\sigma_m}{\times} \Delta_{m-1}), ((S^{n-1})^m \times \partial\Delta_{m-1}) \bigcup_{\sigma_m}(D_m \times \Delta_{m-1})) & \to & (\overline{W}^\delta_{m-1}, \overline{W}_{m-1}) \\ \uparrow & & \uparrow \\ ((S_0^{n-1})^m \underset{\sigma_m}{\times} \Delta^{\varepsilon_1'}_{m-1}), \partial((S_0^{n-1})^m \times \Delta^{\varepsilon_1'}_{m-1})) & \to & (\overline{W}^\delta_{m-1} \cap V(m, \varepsilon_1), \overline{W}_{m-1} \cap V(m, \varepsilon_1)) \end{array}$$

and the fact that the vertical arrows are, by (5.8), (5.12) and the equivariant retraction by deformation of $(\Delta_{m-1}, \partial\Delta_{m-1})$ onto $(\Delta^{\varepsilon_1'}_{m-1}, \partial\Delta^{\varepsilon_1'}_{m-1})$, isomorphisms, we derive that the map

$$(5.17)\quad (f_m(\lambda)\circ\theta_m)_\star : H_\star\Big((S^{n-1})^m \underset{\sigma_m}{\times} \Delta_{m-1}, \big((S^{n-1})^m \times \partial\Delta_{m-1}\big) \bigcup_{\sigma_m} (D_m \times \Delta_{m-1})\Big) \to H_\star(\overline{W}^{\delta}_{m-1}, \overline{W}_{m-1})$$

is $H^\star((\overline{\Omega})^m/\sigma_m)$-linear.

Therefore, via (5.8), (5.2) and (5.13), the map

$$(5.18)\quad (f_m(\lambda))_\star : H_\star(B_m(S^{n-1}), B_{m-1}(S^{n-1})) \to H_\star(\overline{W}_m, \overline{W}_{m-1})$$

is $H^\star((\Omega)^m/\sigma_m)$-linear.

Let now

$$(5.19)\quad \begin{cases} O_{S^{n-1}} \text{ be the orientation class in } H^{n-1}(S^{n-1}) \\ \text{and let } O_{\overline{\Omega}} \text{ be in } H^{n-1}(\overline{\Omega}) \text{ such that } \gamma_1^\star(O_{\overline{\Omega}}) = O_{S^{n-1}}, \\ \text{when } \gamma_1 \text{ is the embedding of } S^{n-1} \text{ in } \overline{\Omega}. \end{cases}$$

Let $\sigma_1 \times \sigma_{m-1}$ be the subgroup of σ_m of permutations leaving 1 stable (σ_m permutes $\{1, ..., m\}$).

The transfer homeomorphism (*see* e.g. Bredon [9]) will be denoted by $\mu^\star$ defines a map from $H^\star(\overline{\Omega} \underset{\sigma_1\times\sigma_{m-1}}{\times} (\overline{\Omega})^{m-1})$ into $H^\star((\overline{\Omega})^m/\sigma_m)$ and, similarly, a map from $H^\star(S^{n-1} \underset{\sigma_1\times\sigma_{m-1}}{\times} (S^{n-1})^{m-1})$ into $H((S^{n-1})^m/\sigma_m)$. Let

$$(5.20)\quad \begin{cases} q : \overline{\Omega} \underset{\sigma_1\times\sigma_{m-1}}{\times} (\overline{\Omega})^{m-1} \to \overline{\Omega} \\ \text{be the projection on the first component,} \end{cases}$$

$$(5.21)\quad \begin{cases} \tilde{q} : S^{n-1} \underset{\sigma_1\times\sigma_{m-1}}{\times} (S^{n-1})^{m-1} \to S^{n-1} \\ \text{be the projection on the first component also.} \end{cases}$$

Taking λ larger than $\operatorname{Sup}(\lambda_m, \lambda_{m-1})$, the following diagram is obviously commutative

(5.22)

$$\begin{array}{ccc} H_\star(B_m(S^{n-1}), B_{m-1}(S^{m-1})) & \xrightarrow{f_m(\lambda)_\star} & H_\star(\overline{W}_m, \overline{W}_{m-1}) \\ \downarrow \partial & & \downarrow \partial_1 \\ H_{\star-1}(B_{m-1}(S^{n-1}), B_{m-2}(S^{n-1})) & \xrightarrow{f_{m-1}(\lambda)_\star} & H_{\star-1}(\overline{W}_{m-1}, \overline{W}_{m-2}). \end{array}$$

where ∂ and ∂_1 are connecting homomorphisms.

We claim that we have

$$\partial\left(\gamma_m^\star \circ \mu^\star \circ q^\star(O_{\overline{\Omega}}) \cap [B_m(S^{n-1}),\, B_{m-1}(S^{n-1})]\right) = [B_{m-1}(S^{n-1}),\, B_{m-2}(S^{n-1})] \tag{5.23}$$

where $\cap$ is the cap-product and $[B_m(S^{n-1}), B_{m-1}(S^{n-1})]$ is the orientation class, via (5.9), of the manifold with boundary $(S_0^{n-1})^m \underset{\sigma_m}{\times} \Delta_{m-1}$ (respectively for $[B_{m-1}(S^{n-1}), B_{m-2}(S^{n-1})]$, use $(S_0^{n-1})^{m-1} \underset{\sigma_{m-1}}{\times} \Delta_{m-2}$). (5.23) will be proved later on.

Using now the $H^\star((\overline{\Omega})^m/\sigma_m)$-linearity of $(f_m(\lambda))_\star$ and the commutativity of (5.22), we derive:

$$\begin{aligned} f_{m-1}(\lambda)_\star&([B_{m-1}(S^{n-1}),\, B_{m-2}(S^{n-1})]) \\ &= (f_{m-1}(\lambda))_\star \circ \partial\left(\gamma_m^\star \circ \mu^\star \circ q^\star(O_{\overline{\Omega}}) \right.\\ &\qquad \left. \cap [B_m(S^{n-1}),\, B_{m-1}(S^{n-1})]\right) \\ &= \partial_1 \circ (f_m(\lambda))_\star\left(\gamma_m^\star \circ \mu^\star \circ q^\star(O_{\overline{\Omega}}) \right.\\ &\qquad \left. \cap [B_m(S^{n-1}),\, B_{m-1}(S^{n-1})]\right) \\ &= \partial_1\left(\mu^\star \circ q^\star(O_{\overline{\Omega}}) \cap f_m(\lambda)_\star([B_m(S^{n-1}),\, B_{m-1}(S^{m-1})])\right) \end{aligned} \tag{5.24}$$

since the $H^*((\overline{\Omega})^m/\sigma_m)$-structure of module of $H_*(B_m(S^{n-1}), B_{m-1}(S^{n-1}))$ is via $\gamma_m^\star$ (*see* (5.11)) the cap-product action of $H^\star((S_0^{n-1})^m \underset{\sigma_m}{\times} \Delta_{m-1})$ onto

$$\begin{aligned} H_\star&((S_0^{n-1})^m \underset{\sigma_m}{\times} \Delta_{m-1},\, \partial(S_0^{n-1})^m \underset{\sigma_m}{\times} \Delta_{m-1}) \\ &= H_\star(B_m(S^{n-1}),\, B_{m-1}(S^{n-1})). \end{aligned}$$

Therefore, we have the induction

$$\begin{cases} f_m(\lambda)_\star([B_m(S^{n-1}),\, B_{m-1}(S^{n-1})]) \text{ is non zero if} \\ f_{m-1}(\lambda)_\star([B_{m-1}(S^{n-1}),\, B_{m-2}(S^{n-1})]) \text{ is non zero.} \end{cases} \tag{5.25}$$

Observe now that by (iii) of Proposition V.1, $f_m(\lambda)$ maps $B_m(S^{n-1})$ into $\overline{W}_{m-1}$ for m large enough. Therefore $f_m(\lambda)_\star(B_m(S^{n-1}), B_{m-1}(S^{n-1}))$ is zero. This, together with (5.25) implies

$$f_1(\lambda)_\star([B_1(S^{n-1}),\, B_0(S^{n-1})]) = 0. \tag{5.26}$$

Now we have:

$$B_1(S^{n-1}) = S^{n-1}; \qquad B_0(S^{n-1}) = \emptyset \tag{5.27}$$

$f_1(\lambda)$ maps $B_1(S^{n-1})$ by (i) of Proposition V.1, into $\overline{W}_0^{\delta}$. As $\overline{W}_0 = \emptyset$ and as $\overline{W}_0^{\delta} \backslash \overline{W}_0$ is contained in $V(1, \varepsilon_1)$ by Lemma IV.1, we have

$$\overline{W}_0^{\delta} \subset V(1, \varepsilon_1)$$

We therefore have a map

$$\begin{aligned} f_1(\lambda) : S^{n-1} &\to V(1, \varepsilon_1) \\ y &\to \varphi\omega(-\lambda y)\, |\varphi\omega(\cdot - \lambda y)|^{-1}. \end{aligned} \tag{5.28}$$

Then

$$s_1 \circ f_1(\lambda) \text{ maps } y \in S^{n-1} \quad \text{into } \lambda y \in \lambda S^{n-1} \subset \overline{\Omega}, \quad \text{with } \lambda \geq \lambda_1 \tag{5.29}$$

as the solution of the minimization problem (3.2) for $v = \frac{\varphi\omega(\cdot - \lambda y)}{|\varphi\omega(\cdot - \lambda y)|}$ is $(|\varphi\omega(\cdot - \lambda y)|^{-1} \lambda(v), \lambda y)$.

Next, the map

$$\begin{aligned} \left[\frac{1}{\lambda}, 1\right] \times S^{n-1} &\to \mathbb{R}^n \\ (t, y) &\to t s_1 \circ f_1(\lambda) \end{aligned} \tag{5.30}$$

is valued in $\overline{\Omega}$ and defines there a homotopy of $s_1 \circ f_1(\lambda)$ to the embedding γ_1 of S^{n-1} into $\overline{\Omega}$.

But $\gamma_{1\star}(O_{S^{n-1}})$ is non zero. Therefore $(s_1 \circ f_1)_\star (O_{S^{n-1}})$ is non zero contradicting (5.26). Hence, the proof of Theorem 1.

We nos prove the remaining claim (5.23). The following diagram commutes

$$\begin{array}{ccc} S^{n-1} \underset{\sigma_1 \times \sigma_{m-1}}{\times} (S^{n-1})^{m-1} & \xrightarrow{(\gamma_1, \gamma_{m-1})} & \overline{\Omega} \underset{\sigma_1 \times \sigma_{m-1}}{\times} (\overline{\Omega})^{m-1} \\ \downarrow \tilde{q} & & \downarrow q \\ S^{n-1} & \xrightarrow{\gamma_1} & \overline{\Omega} \end{array} \tag{5.31}$$

and γ_m is σ_m-equivariant.

Therefore, by naturality of the transfer homeomorphism (*see* e.g. Bredon [9]), we have:

$$\gamma_m^\star \circ \mu^\star \circ q^\star (O_{\overline{\Omega}}) = \mu^\star \circ \tilde{q}^\star \circ \gamma_1^\star (O_{\overline{\Omega}}) = \mu^\star \circ \tilde{q}^\star (O_{S^{n-1}}). \tag{5.32}$$

Taking into account (5.27), (5.23) becomes:

$$\begin{aligned} \partial (\mu^\star \circ \tilde{q}^\star (O_{S^{n-1}}) \cap [B_m (S^{n-1}), B_{m-1} (S^{n-1})]) \\ = [B_{m-1} (S^{n-1}), B_{m-2} (S^{n-1})]. \end{aligned} \tag{5.33}$$

In order to prove (5.43), we pick a point ξ in S^{n-1}. The map

$$\begin{cases} [0, 1] \times B_{m-1} (S^{n-1}) \to B_m (S^{n-1}) \\ \left(t, \sum_{i=1}^{m-1} t_i \, \delta_{x_i}\right) \to t \, \delta_\xi + \sum_{i=1}^{m-1} t_i \, \delta_{x_i} \end{cases} \tag{5.34}$$

induces a map τ from the cone $CB_{m-1} (S^{n-1})$ over $B_{m-1} (S^{n-1})$ into $B_m (S^{n-1})$. τ, in fact, maps

$$\begin{aligned} \tau :(CB_{m-1} (S^{n-1}), B_{m-1} (S^{n-1})) \\ \to (B_m (S^{n-1}), B_{m-1} (S^{n-1})) \end{aligned} \tag{5.35}$$

and we have the following commutative diagram

$$\begin{array}{ccc} H_\star (CB_{m-1} (S^{n-1}), B_{m-1} (S^{n-1})) & \xrightarrow{\partial_2} & \tilde{H}_{\star-1} (B_{m-1} (S^{n-1})) \\ \tau_\star \downarrow & & \nearrow \partial_3 \\ H_\star (B_m (S^{n-1}), B_{m-1} (S^{n-1})). & & \end{array} \tag{5.36}$$

∂_2 and ∂_3 are connecting homeomorphisms and $\tilde{H}_\star$ is the reduced homology. Observe that ∂_2 is injective as $B_{m-1} (S^{n-1})$ is contractible.

Let Π be the restriction of $\tilde{q}$ to $(S_0^{n-1})^m \underset{\sigma_1 \times \sigma_{m-1}}{\times} \Delta_{m-1}$

$$\Pi : (S_0^{n-1})^m \underset{\sigma_1 \times \sigma_{m-1}}{\times} \Delta_{m-1} \to S^{n-1} \tag{5.37}$$

$\Pi^{-1} (\xi)$ is a submanifold (with boundary $\partial \Pi^{-1} (\xi)$) of $(S_0^{n-1})^m \underset{\sigma_1 \times \sigma_{m-1}}{\times} \Delta_{m-1}$. Let $[\Pi^{-1} (\xi), \partial \Pi^{-1} (\xi)]$ denote its orientation class.

Introducing the quotient map

$$(5.38)\quad r : ((S_0^{n-1})^m \underset{\sigma_1 \times \sigma_{m-1}}{\times} \Delta_{m-1}, \partial((S_0^{n-1})^m \underset{\sigma_1 \times \sigma_{m-1}}{\times} \Delta_{m-1})) \to ((S_0^{n-1})^m \underset{\sigma_m}{\times} \Delta_{m-1}, \partial((S_0^{n-1})^m \underset{\sigma_m}{\times} \Delta_m))$$

we have the following diagram

(5.39)

$$\begin{array}{ccc} ((S_0^{n-1})^m \underset{\sigma_m}{\times} \Delta_{m-1}, \partial((S_0^{n-1})^m \underset{\sigma_m}{\times} \Delta_{m-1})) & \overset{j=roi}{\longleftarrow} & (\Pi^{-1}(\xi), \partial\Pi^{-1}(\xi)) \\ \uparrow r & & \swarrow i \\ ((S_0^{n-1})^m \underset{\sigma_1 \times \sigma_{m-1}}{\times} \Delta_{m-1}, \partial((S_0^{n-1})^p \underset{\sigma_1 \times \sigma_{m-1}}{\times} \Delta_{m-1})) & & \\ \downarrow \Pi & & \\ S^{n-1} & & \end{array}$$

In (5.9), we pointed out an isomorphism, which we denote by $I_\star$

$$(5.40)\quad \begin{cases} H_\star(B_m(S^{n-1}), B_{m-1}(S^{n-1})) \xrightarrow{I_\star} H_\star((S_0^{n-1})^m \underset{\sigma_m}{\times} \Delta_{m-1}), \\ \qquad \partial((S_0^{n-1})^m \underset{\sigma_m}{\times} \Delta_{m-1}). \end{cases}$$

Similarly, there is an isomorphism $\nu_\star$

$$(5.41)\quad \nu_\star : H_\star(CB_{m-1}(S^{n-1}), B_{m-1}(S^{n-1})) \to H_\star(\Pi^{-1}(\xi), \partial\Pi^{-1}(\xi))$$

and we readily have

$$(5.42)\quad I_\star \circ \tau_\star = j_\star \circ \nu_\star$$

Furthermore, as $B_{m-2}(S^{n-1})$ is contractible in $B_{m-1}(S^{n-1})$, the quotient map

$$(5.43)\quad J_\star : \tilde{H}_\star(B_{m-1}(S^{n-1})) \to H_\star(B_{m-1}(S^{n-1}), B_{m-2}(S^{n-1}))$$

is injective.

Now $H_{(m-1)(n-1)+m-2}(B_{m-1}(S^{n-1}),\ B_{m-2}(S^{n-1}))$ is, by (5.23), uniquely generated by the orientation class of $((S_0^{n-1})^{m-2} \underset{\sigma_{m-1}}{\times} \Delta_{m-2},$ $\partial((S_0^{n-1})^{m-2} \underset{\sigma_{m-1}}{\times} \Delta_{m-2})$. Therefore

$$(5.44) \qquad H_{(m-1)(n-1)+m-2}(B_{m-1}(S^{n-1}),\ B_{m-2}(S^{n-1})) = \mathbb{Z}_2.$$

The generator was denoted (*see* (5.23) by $[B_{m-1}(S^{n-1}),\ B_{m-2}(S^{n-1})]$. We then have recalling that ∂_2 is defined in (5.36), and $[\Pi^{-1}(\xi),\ \partial\Pi^{-1}(\xi)]$ is the orientation class of $(\Pi^{-1}(\xi),\ \partial\Pi^{-1}(\xi))$

$$(5.45) \qquad \begin{aligned} &J_\star \circ \partial_2 \circ J_\star^{-1}([\Pi^{-1}(\xi),\ \partial\Pi^{-1}(\xi)]) \\ &\quad = [B_{m-1}(S^{n-1}),\ B_{m-2}(S^{n-1})]. \end{aligned}$$

On the other hand, the transfer map

$$(5.46) \quad \mu^\star : H^\star((S_0^{n-1})^m \underset{\sigma_1 \times \sigma_{m-1}}{\times} \Delta_{m-1}) \to H^\star((S_0^{n-1})^m \underset{\sigma_m}{\times} \Delta_{m-1})$$

is equal to the Gysin-homomorphism as the map $r_\star$, defined in (5.38), is a covering.

Therefore, we have for any u in $H^\star((S_0^{n-1})^m \underset{\sigma_1 \times \sigma_{m-1}}{\times} \Delta_{m-1})$

$$(5.47) \qquad \begin{aligned} &\mu^\star(u) \cap [(S_0^{n-1})^m \underset{\sigma_m}{\times} \Delta_{m-1},\ \partial((S_0^{n-1})^m \underset{\sigma_m}{\times} \Delta_{m-1})] \\ &\quad = r_\star(u \cap [(S_0^{n-1})^m \underset{\sigma_1 \times \sigma_{m-1}}{\times} \Delta_{m-1},\ \partial((S_0^{n-1})^m \\ &\qquad \underset{\sigma_1 \times \sigma_{m-1}}{\times} \Delta_{m-1}\}]). \end{aligned}$$

In particular

$$(5.48) \quad \begin{aligned} &\mu^\star(\Pi^\star(O_{S^{n-1}})\} \cap [(S_0^{n-1})^m \underset{\sigma_m}{\times} \Delta_{m-1},\ \partial((S_0^{n-1})^m \underset{\sigma_m}{\times} \Delta_{m-1})] \\ &\quad = r_\star(\Pi^\star(O_{S^{n-1}})\} \cap [(S_0^{n-1})^m \underset{\sigma_1 \times \sigma_{m-1}}{\times} \Delta_{m-1},\ \partial((S_0^{n-1})^m \\ &\qquad \underset{\sigma_1 \times \sigma_{m-1}}{\times} \Delta_{m-1})] \end{aligned}$$

Now, we readily have

$$(5.49) \quad \begin{aligned} &\Pi^\star(O_{S^{n-1}}) \cap [(S_0^{n-1})^m \underset{\sigma_1 \times \sigma_{m-1}}{\times} \Delta_{m-1},\ \partial((S_0^{n-1})^m \underset{\sigma_m}{\times} \Delta_{m-1})] \\ &\quad = i_\star([\Pi^{-1}(\xi),\ \partial\Pi^{-1}(\xi)]). \end{aligned}$$

Therefore

$$(5.50)\quad \mu^\star(\Pi^\star(O_{S^{n-1}}))\cap[(S_0^{n-1})^m \underset{\sigma_m}{\times} \Delta_{m-1}, \partial((S_0^{n-1})^m \underset{\sigma_m}{\times} \Delta_{m-1})]$$
$$= j_\star([\Pi^{-1}(\xi), \partial\Pi^{-1}(\xi)]).$$

We use now (5.45). As $J_\star$ is injective as well as ∂_2 (∂_2 defined in (5.36)) (5.45) reads

$$(5.51)\qquad \nu_\star^{-1}([\Pi^{-1}(\xi), \partial\Pi^{-1}(\xi)])$$
$$= \partial_2^{-1} \circ J_\star^{-1}([B_{m-1}(S^{n-1}), B_{m-2}(S^{n-1})])$$

Applying $\Gamma_\star \circ \tau_\star$ and using (5.42), we have

$$(5.52)\quad j_\star([\Pi^{-1}(\xi), \partial\Pi^{-1}(\xi)])$$
$$= I_\star \circ \tau_\star \circ \partial_2^{-1} \circ J_\star^{-1}([B_{m-1}(S^{n-1}), B_{m-2}(S^{n-1})]$$

(5.48) and (5.51) yield

$$(5.53)\quad I_\star \circ \tau_\star \circ \partial_2^{-1} \circ J_\star^{-1}([B_{m-1}(S^{n-1}), B_{m-2}(S^{n-1})])$$
$$= \mu^\star(\Pi^\star(O_{S^{n-1}}))\cap[(S_0^{n-1})^m$$
$$\underset{\sigma_m}{\times} \Delta_{m-1}, \partial((S_0^{n-1})^m \underset{\sigma_m}{\times} \Delta_{m-1})].$$

By the commutativity of (5.36), we have

$$(5.54)\qquad \tau_\star \circ \partial_2^{-1} = \partial_3^{-1}.$$

On the other hand, $I_\star$ is an isomorphism. Therefore, we have

$$(5.55)\qquad [B_{m-1}(S^{n-1}), B_{m-2}(S^{n-1})]$$
$$= I_\star \circ \partial_3 \circ I_\star^{-1}(\mu^\star(\Pi^\star(O_{S^{n-1}}))\cap[(S_0^{n-1})^m$$
$$\underset{\sigma_m}{\times} \Delta_{m-1}, \partial((S_0^{n-1})^m \underset{\sigma_m}{\times} \Delta_{m-1})]$$

and (5.55) yields (5.23) immediately.

VI. MAIN ENERGY ESTIMATE

We now prove Proposition V.1 which, in view of the arguments given in sections III, IV, reduces to the following assertions

(6.1) $$\text{If } m \text{ is large, } f_m(\lambda)\,[B_m(S^{n-1})] \subset \overline{W}_{m-1}$$

(6.2) $$\begin{cases} \text{If } m \geq 1, \; f_m(\lambda) \circ \theta_m \, [(S_0^{n-1})^m \underset{m}{\times} \Delta_{m-1}^{\varepsilon_1'}] \subset \overline{W}^{\delta}_{m-1} \cap V(m, \varepsilon_1) \\ \text{and} \qquad f_m(\lambda)\,[B_m(S^{n-1})] \subset \overline{W}^{\delta}_{m-1} \end{cases}$$

for convenient choices of the parameters. In fact, (6.2) is very easy if $m = 1$ and we will first prove

(6.3) $$\text{If} \quad m \geq 2, \qquad f_m(\lambda) \circ \theta_m \, [(S_0^{n-1})^m \underset{m}{\times} \Delta_{m-1}^{\varepsilon_1'}] \subset V(m, \varepsilon_1).$$

Indeed, the commutativity of the diagram given in (ii) for instance then follows from the uniqueness of the selection shown in section III.

We first prove (6.3). Of course, we may consider in all the remainder of this section that m is fixed ≥ 2, $\varepsilon_1 < \varepsilon_0(m)$. In fact, the proof of (6.3) consists only in looking precisely at what really are the various objects we are using. Indeed, if $\xi \in (S_0^{n-1})^m \underset{\sigma_m}{\times} \Delta_{m-1}^{\varepsilon_1'}$, $u = f_m(\lambda) \circ \theta_m(\xi)$ is given by

(6.4) $$\begin{cases} u = \varphi\left(\sum_i^m t_i\, \omega(\,\cdot\, - \lambda\, x_i)\right) \left| \varphi\left(\sum_i^m t_i\, \omega(\,\cdot\, - \lambda\, x_i)\right) \right|^{-1}; \\ \qquad |x_i| = 1 \quad (\forall\, i) \end{cases}$$

and there exists $\gamma > 0$ (independent of ξ) such that

(6.5) $$\begin{cases} |x_i - x_j| \geq \gamma \qquad \text{if} \quad i \neq j, \\ \sum_1^m t_i = 1, \qquad t_i \geq 0, \qquad \left| t_i - \dfrac{1}{m} \right| \leq \varepsilon_1' \quad (\forall\, i). \end{cases}$$

At this stage the remainder of the proof of (6.3) is quite easy. Indeed, observe that by explicit computations $\lambda(u)|\varphi(\sum_i^m t_i\, \omega(\,\cdot\, - \lambda\, x_i))|^{-1} \to m$ as $\lambda \to \infty$, $\varepsilon_1' \to 0$, hence $\lambda(u)\, u - \varphi(\sum_i^m \omega(\,\cdot\, - \lambda\, x_i)) \to 0$ as $\lambda \to \infty$, $\varepsilon_1' \to 0$, uniformly in ξ and (6.3) is proved.

We now turn to the really important estimate (6.1). Recalling that if $v \in \Sigma$,

$$J(v) = \frac{p-1}{2(p+1)} \left(\int_{\mathbb{R}^n} b\,|v|^{p+1} \right)^{-\frac{2}{p-1}}$$

while

$$S = \frac{p-1}{2(p+1)} |\omega|^2 = \frac{p-1}{2(p+1)} \left(\int_{\mathbb{R}^n} b^\infty \left(\frac{\omega}{|\omega|} \right)^{p+1} \right)^{-2/(p-1)},$$

(6.1) is obviously deduced from the following

$$(6.7) \quad \left| \varphi \left(\sum_1^m t_i\, \omega(\cdot - \lambda x_i) \right) \right|^2 \times \left(\int_{\mathbb{R}^n} b\, \varphi^{p+1} \left(\sum_1^m t_i\, \omega(x - \lambda x_i) \right)^{p+1} dx \right)^{-\frac{2}{p+1}} < m^{\frac{p+1}{p+1}} \frac{|\omega|^2}{\left(\int_{\mathbb{R}^n} \omega^{p+1}\, dx \right)^{-\frac{2}{p+1}}}$$

for all $x_1, \ldots, x_m \in S^{n-1}$, $t_1, \ldots, t_m \geq 0$, $\sum_i^m t_i = 1$, provided λ is large and φ is chosen conveniently. The proof of (6.7) will require some careful analysis and to keep the ideas clear we first prove (6.7) in the particular case when $O = \emptyset$ *i.e.* $\Omega = \mathbb{R}^n$ and thus no cut-off function φ is required (or in other words we may take $\varphi \equiv 1$).

We begin by estimating

$$(6.8) \quad \left| \sum_1^m t_j\, \omega_j \right|^2 = \left(\sum_1^m t_j^2 \right) |\omega|^2 + \sum_{i \neq j} t_i\, t_j\, (\omega_i, \omega_j)$$

where we denote by $\omega_i = \omega(\cdot - \lambda x_i)$. Next, recalling that ω solves

$$(6.9) \quad -\Delta\omega + \lambda_0\, \omega = b^\infty\, \omega^p \quad \text{in } \mathbb{R}^n, \quad \omega > 0 \quad \text{in } \mathbb{R}^n, \quad \omega \in H^1(\mathbb{R}^n)$$

we deduce

$$(6.10) \quad \left| \sum_1^m t_j\, \omega_j \right|^2 = \left(\sum_1^m t_j^2 \right) |\omega|^2 + \sum_{i \neq j} t_i\, t_j \int_{\mathbb{R}^n} b^\infty\, \omega_i^p\, \omega_j\, dx.$$

And by the results of section II, we have for all $1 \leq i \neq j \leq m$

$$
(6.11) \quad \left(\int_{\mathbb{R}^n} b^\infty \omega_i^p \omega_j \, dx\right) \times \exp(\lambda_0^{1/2} \lambda \Delta_{ij}) \lambda^{\frac{n-1}{2}} \Delta_{ij}^{\frac{n-1}{2}} \to C_0 > 0 \quad \text{as } \lambda \to \infty
$$

where C_0 is independent of $x_1, ..., x_m, \lambda_1, ..., \lambda_m$ and $\Delta_{ij} = |x_i - x_j|$.

Next, we remark

$$
(6.12) \quad \int_{\mathbb{R}^n} b \left(\sum_1^m t_i \omega_i\right)^{p+1} dx \geq \int_{\mathbb{R}^n} b^\infty \left(\sum_1^m t_i \omega_i\right)^{p+1} dx - \int_{\mathbb{R}^n} (b^\infty - b)^+ \left(\sum_1^m t_i \omega_i\right)^{p+1} dx.
$$

And since we may always assume that in (1.5) $\delta < p + 1$, we deduce from the results of section II that for $\lambda \geq 1$

$$
(6.13) \quad \int_{\mathbb{R}^n} (b^\infty - b)^+ \left(\sum_1^m t_i \omega_i\right)^{p+1} \leq C_2 \exp(-\delta \lambda \lambda_0^{1/2}) \lambda^{-\frac{n-1}{2}}
$$

where $C_2 > 0$ is independent of $\lambda, x_1, ..., x_m, t_1, ..., t_m$.

We next observe that if $a_1, ..., a_m$ are arbitrary nonnegative reals then there exists a constant $C_3 \geq 0$ (independent of $a_1, ..., a_m$) such that

$$
\left(\sum_1^m a_i\right)^{p+2} \geq \sum_1^m a_i^{p+1} + (p+1) \sum_{i \neq j} a_i^p a_j - C_3 \sum_{i \neq j} a_i^{\frac{p+1}{2}} a_j^{\frac{p+1}{2}}
$$

(in fact, if $p \geq 2$, we way take $C_3 = 0$ and in general we only need this inequality to hold for $0 \leq a_i \leq \sup_{\mathbb{R}^n} \omega$). Hence, (6.12)-(6.13) yield

$$
(6.14) \quad \int_{\mathbb{R}^n} b \left(\sum_1^m t_i \omega_i\right)^{p+1} dx \geq \left(\sum_1^m t_i^{p+1}\right) \int_{\mathbb{R}^n} b^\infty \omega^{p+1} dx + (p+1) \sum_{i \neq j} t_i^p t_j \int_{\mathbb{R}^n} b^\infty \omega_i^p \omega_j \, dx - C_2 \times \exp(-\delta \lambda \lambda_0^{1/2}) \lambda^{-\frac{n-1}{2}} - C_3 \sum_{i \neq j} t_i^{\frac{p+1}{2}} t_j^{\frac{p+1}{2}} \int_{\mathbb{R}^n} b^\infty \omega_i^{\frac{p+1}{2}} \omega_j^{\frac{p+1}{2}} dx.
$$

And in view of the results of section II and of (6.14) we deduce finally

$$(6.15)\quad \int_{\mathbb{R}^n} b\Big(\sum_i^m t_i\,\omega_i\Big)^{p+1} dx \geq \Big(\sum_i^m t_i^{p+1}\Big)\int_{\mathbb{R}^n} b^\infty\,\omega^{p+1}\,dx$$
$$+ (p+1)\sum_{i\neq j} t_i^p\,t_j \int_{\mathbb{R}^n} b^\infty\,\omega_i^p\,\omega_j\,dx - C_2\,\exp\big(-\delta\,\lambda_0^{1/2}\,\lambda\big)\,\lambda^{-\frac{n-1}{2}}$$
$$- C_4\sum_{i\neq j} t_i^{\frac{p+1}{2}}\,t_j^{\frac{p+1}{2}}\,\exp\big(-q\,\lambda_0^{1/2}\,\lambda\Delta_{ij}\big)\,\Delta_{ij}^{-\frac{n+1}{2}}\,\lambda^{-\frac{n-1}{2}}$$

where $1 < q < (p+1)$, and C_4 is a positive constant independent of $\lambda \geq 1,\ x_1, \ldots, x_m,\ t_1, \ldots, t_m$.

Now, we have to deduce (6.7) from (6.10), (6.11) and (6.15). We first observe that all the parameters t_i may be assumed to be close to $\frac{1}{m}$ since we have

$$(6.16)\quad \begin{cases} \Big(\sum_1^m t_i^2\Big)\Big(\sum_1^m t_i^{p+1}\Big)^{-\frac{2}{p+1}} \leq m^{\frac{p-1}{p+1}} - \alpha\,\max_i\,\Big|t_i - \dfrac{1}{m}\Big|^2, \\ \forall t_i \geq 0,\ \sum_1^m t_i = 1 \end{cases}$$

for some $\alpha > 0$. Therefore, (6.7) holds immediately if $\max_i |t_i - \frac{1}{m}| \geq \delta_0(\lambda)$ where $\delta_0(\lambda) \to 0$ as $\lambda \to +\infty$. Hence, we may assume that for $\lambda \geq \lambda_0$, $\max_i |t_i - \frac{1}{m}| < \delta_0(\lambda) < \frac{1}{2m}$. We now rewrite (6.15) as follows

$$(6.17)\quad \int_{\mathbb{R}^n} b\Big(\sum_1^m t_i\,\omega_i\Big)^{p+1} dx \geq \Big(\sum_1^m t_i^{p+1}\Big)\int_{\mathbb{R}^n} b^\infty\,\omega^{p+1}\,dx$$
$$+ (p+1-\mu(\lambda))\sum_{i=j}^m t_i^p\,t_j\,C_0\,\exp\big(-\lambda_0^{1/2}\lambda\,\Delta_{ij}\big)\,\lambda^{-\frac{n-1}{2}}\,\Delta_{ij}^{-\frac{n-1}{2}}$$
$$- C_2\,\exp\big(-\delta\lambda_0^{1/2}\,\lambda\big)\,\lambda^{-\frac{n-1}{2}}$$

where $\mu(\lambda) \to 0$ as $\lambda \to +\infty$.

To conclude, we just have to observe that we have

$$(6.18)\quad \lim_{m\to\infty}\ \mathrm{Max}\,\{\mathop{\mathrm{Min}}_{i\neq j}\,|x_i - x_j|/(x_1, \ldots, x_n) \in (S^{n-1})^m,\ x_i \neq x_j\,\forall\, i \neq j\} = 0$$

hence, if m is large enough, there always exist $i \neq j$ such that

(6.19) $$\Delta_{ij} < \delta/2.$$

And the combination of (6.17) and (6.19) easily yields (6.7).

We now explain how we modify the above argument in the general case when $\Omega = \mathbb{R}^n$. We first observe that (6.15) still holds in this case. On the other hand, (6.8)-(6.10) become now

$$\begin{aligned}\left|\varphi\left(\sum_1^m t_i\,\omega_i\right)\right|^2 &= \int_{\mathbb{R}^n} \varphi^2 \left\{\left|\nabla\left(\sum_1^m t_i\,\omega_i\right)\right|^2 + \lambda_0\left(\sum_1^m t_i\,\omega_i\right)^2\right\} dx \\ &+ 2\int_{\mathbb{R}^n} \varphi\left(\sum_1^m t_i\,\omega_i\right) \nabla\varphi\cdot\nabla\left(\sum_1^m t_j\,\omega_j\right) dx \\ &+ \int_{\mathbb{R}^n} |\nabla\varphi|^2 \left(\sum_1^m t_i\,\omega_i\right)^2 dx \\ &\leq \left|\sum_1^m t_i\,\omega_i\right|^2 + \int_{\mathbb{R}^n} |\Delta\varphi| \left(\sum_1^m t_i\,\omega_i\right)^2 dx\end{aligned}$$

where we integrated by parts the second term and used the fact that $0 \leq \varphi \leq 1$. And we deduce easily

(6.20) $$\begin{aligned}\left|\varphi\left(\sum_1^m t_i\,\omega_i\right)\right|^2 &\leq \left(\sum_1^m t_i^2\right)|\omega|^2 + \sum_{i\neq j} t_i\,t_j \int_{\mathbb{R}^n} b^\infty\,\omega_{ij}^p\,dx \\ &+ C_5\left(\int_{\mathbb{R}^n} |\Delta\varphi|\,dx\right) \exp\left(-2\,\lambda_0^{1/2}\,\lambda\right)(2\,\lambda)^{-\frac{n-1}{2}},\end{aligned}$$

where C_5 is independent of $x_1, \ldots, x_m, t_1, \ldots, t_m, \lambda \geq 1$ and φ. We may now repeat the above argument and conclude the proof.

To conclude the proof of the existence theorem, we still have to prove the second part of (6.2). We first observe that the above estimates show that there exists a continuous, positive function $K(\lambda)$ vanishing for $\lambda = 0$ such that

(6.22) $$J\left(\varphi\left(\sum_{i=1}^m t_i\,\omega\left(\cdot + \lambda\,x_i\right)\right)\right) < m\,S$$

as soon as $\max_i |t_i - \frac{1}{m}| > K(\lambda)$, or $\min_{i \neq j} |x_i - x_j| < 1/K(\lambda)$. On the other hand, if $\max_i |t_i - \frac{1}{m}| \leq K(\lambda)$ and $\min_{i \neq j} |x_i - x_j| \geq 1/K(\lambda)$, then $f_m(\lambda)(\sum_{i=1}^m t_i \delta_{x_i}) \in \overline{V(m, \gamma_m(\delta))}$ for λ large enough.

And combining these two facts we conclude the proof of (6.2) and of the existence theorem. ■

We would like to conclude by mentioning that the proof of the existence we gave in fact yields the existence of a solution u such that

$$I(u) < m I^{\infty}(\omega), \tag{6.21}$$

where m is the least integer such that (iii) in Proposition V.1 holds.

VII. EXTENSIONS, VARIANTS AND COMMENTS

We first give an extension of Theorem I.1 where we relax assumption (1.5) and where we consider a more general equation than (1.1) namely

$$-\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + a(x)u = g(x, u) \quad \text{in } \Omega, \qquad u \in H_0^1(\Omega), \quad u > 0 \quad \text{in } \Omega \tag{7.1}$$

where Ω, n, p are as in the Introduction and where a_{ij}, $a \in C_b(\overline{\Omega})$ $(\forall i, j)$ satisfy

$$\exists \nu > 0, \quad \forall \xi \in \mathbb{R}^n, \quad \forall x \in \mathbb{R}^n, \quad \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \nu|\xi|^2 \tag{7.2}$$

$$\exists \underline{a} > 0, \qquad a(x) \geq \underline{a} \quad \text{on } \mathbb{R}^n \tag{7.3}$$

$$\begin{cases} a_{ij}(x) \to a_{ij}^{\infty}, \qquad a(x) \to a^{\infty} \qquad \text{as} \quad |x| \to \infty, \\ \qquad\qquad \text{for all} \quad 1 \leq i, \ j \leq n \end{cases} \tag{7.4}$$

Furthermore, g is continuous on $\overline{\Omega} \times \mathbb{R}_+$, continuously differentiable in s, $s\frac{\partial g}{\partial s}(x, s)$ is continuously differentiable in s and g satisfies

$$\begin{cases} \lim_{s \to 0^+} \sup_{x \in \overline{\Omega}} g(x, s)s^{-1} = 0, \\ \inf\left\{\frac{\partial^2 g}{\partial s^2}(x, s)/x \in \overline{\Omega}, \ s \in \left[\delta, \frac{1}{\delta}\right]\right\} > 0 \end{cases} \tag{7.5}$$

for all $\delta \in (0, 1)$,

$$\exists\, \theta \in]0, 1[, \qquad \theta \frac{\partial g}{\partial s}(x, s)\, s \geq g(x, s) \quad \forall s \geq 0, \quad \forall x \in \overline{\Omega} \tag{7.6}$$

$$\begin{cases} \exists\, q < \dfrac{4}{n-2}, & \dfrac{\partial}{\partial s}\left(\dfrac{\partial g}{\partial s}(x, s)\, s\right) \leq C(s^q + 1), \\ & \forall (x, s) \in \overline{\Omega} \times (C, \infty) \end{cases} \tag{7.7}$$

for some $C \geq 0$,

$$\begin{cases} g \text{ converges, } as\ |x| \to \infty, \ to\ b^\infty s^{+p} \text{ uniformly on } [0, R], \\ \qquad \text{for all } R < \infty \end{cases} \tag{7.8}$$

for some $b^\infty > 0$.

Next, we denote by $G(x, s) = \int_0^s g(x, \sigma)\, d\sigma$ and we assume

$$\begin{cases} \exists\, q \geq p,\ q > 2,\ \exists\, c_0 \geq 0,\ \forall R < \infty,\ \exists\, \alpha_R(x),\ \beta_R(x) \geq 0, \\ \dfrac{b^\infty}{p+1} s^{p+1} - G(x, s) \leq \alpha_R\, s^2 + \beta_R\, s^q \quad \text{on } \overline{\Omega} \times [0, R] \\ \limsup\limits_{|x| \to \infty} \beta_R \exp(\delta (a^\infty)^{1/2} |x|)\, |x|^{\frac{n-1}{2}} \leq c_0, \\ \alpha_R \exp(\delta (a^\infty)^{1/2} |x|)\, |x|^{\frac{n-1}{2}} \in L^1, \quad \text{for some } \delta > 0 \end{cases} \tag{7.9}$$

$$\begin{cases} (a(x) - a^\infty)^+ \exp(\delta (a^\infty)^{1/2} |x|)\, |x|^{\frac{n-1}{2}} \in L^1, \\ \qquad \text{for some } \delta > 0 \end{cases} \tag{7.10}$$

$$\begin{cases} \lambda_1^+ (a_{ij}(x) - a_{ij}^\infty) \exp(\delta (a^\infty)^{1/2} |x|)\, |x|^{\frac{n-1}{2}} \in L^1, \\ \qquad \text{for some } \delta > 0 \end{cases} \tag{7.11}$$

where λ_1^+ denotes the positive part of the maximal eigenvalue of the matrix considered. Observe in particular that all conditions hold if $g(x, s) = b(x)\, s^p$ and

$$\overline{\lim_{|x|}}\, (b^\infty - b)^+ \exp(\delta (a^\infty)^{1/2} |x|)\, |x|^{\frac{n-1}{2}} \leq c_0.$$

Of course, (1.3) is now replaced by

$$\begin{cases} -\displaystyle\sum_{i, j=1}^n a_{ij}^\infty \frac{\partial^2 \omega}{\partial x_i\, \partial x_j} + a^\infty \omega = b^\infty \omega^p \quad \text{in } \mathbb{R}^n, \\ \qquad \omega > 0 \quad \text{in } \mathbb{R}^n, \qquad \omega \in H^1(\mathbb{R}^n) \end{cases} \tag{7.12}$$

but up to a rotation, a change of scales and a multiplication by a positive scalar, ω is still the solution of (1.3).

Inspecting closely the proof of Theorem I.1, one sees that the following result – whose detailed proof we leave to the reader – holds.

THEOREM VII.1. – *We assume* (7.2)-(7.11) *and that* (1.3) *admits a unique radial function. Then, if* c_0 *is small enough, there exists a solution of* (7.1).

Remark. – If in (7.9) we may take $c_0 = 0$ in (7.9) then we may replace in (7.9) $q \geq p$, $q > 2$ by $q > 2$.

It is also quite clear that the method presented in the preceding sections can be adapted to treat other situations such as, for instance, other nonlinearities at infinity (replace $b^\infty s^p$ by another nonlinearity $g^\infty(s)$ with appropriate convexity properties) or other unbounded domains such as strip-like domains: for instance, take $\Omega = (Q \times \mathbb{R}^m)\backslash\overline{\mathcal{O}}$ where Q is a bounded, smooth open set in $\mathbb{R}^n$, $\mathcal{O}$ is a bounded, smooth open set in $\mathbb{R}^{n+m}$ (in fact we may even consider domains which "approach at infinity" domains of the form $Q \times \mathbb{R}^m$). Then, the analysis given in the preceding sections remains valid essentially replacing S^{n-1} by S^{m-1} provided $\underline{m \geq 2}$. We will not give here more details about such variants and extensions.

Next, we observe that in general the solution built in Theorem I.1 is not equivalent to a minimum of $J|_\Sigma$. Indeed, in the case for instance when $b \equiv b^\infty$, $J|_\Sigma$ does not have a minimum if $\Omega \neq \mathbb{R}^n$ (*see* for example [21]).

We would like now to explain how the proof of Theorem I.1 may be simplified if we relax (1.5) and we replace (1.5) by (1.6). Indeed, we claim that if (1.6) holds then part (iii) of Proposition V.1 holds with $m = 2$ *i.e.*

$$\sup_{\xi \in B_2(S^{n-1})} J(f_2(\lambda)[\xi]) < 2S, \qquad \text{for } \lambda \text{ large enough} \tag{7.13}$$

provided we choose conveniently φ (the cut-off function). Once this claim is proven, the existence follows from an easy adaptation of J. M. Coron's argument [15] (*see* also V. Benci and G. Cerami [6]). Indeed, exactly as in the preceding section, we have (this is essentially (6.15))

$$\begin{aligned} \int_{\mathbb{R}^n} b\,\varphi^{p+1}\left(\sum_1^2 t_i\,\omega_i\right)^{p+1} dx \geq \left(\sum_1^2 t_i^{p+1}\right) \int_{\mathbb{R}^n} b^\infty\,\omega^{p+1}\,dx \\ + (p+1)\sum_{i\neq j}^2 t_i^p\,t_j\,C_0\,\exp(-\lambda\lambda_0^{1/2}\,\Delta_{ij})\,(\lambda\Delta_{ij})^{-\frac{n-1}{2}} \\ - 0\,(\exp(-2\,\lambda_0^{1/2}\,\lambda)\,\lambda^{-\frac{n-1}{2}}) \\ - C_4\sum_{i\neq j}^2 t_i^{\frac{p+1}{2}}\,t_j^{\frac{p+1}{2}}\,\exp(-q\,\lambda_0^{1/2}\,\lambda\,\Delta_{ij})\,(\Delta_{ij}\,\lambda)^{-\frac{n-1}{2}} \end{aligned} \tag{7.14}$$

while (6.20) still holds of course. The proof of (7.13) is then straightforward by observing first that $\Delta_{12} = \Delta_{21} = |x_1 - x_2| \leq 2$ and that replacing φ by $\varphi\left(\frac{\cdot}{\sigma}\right)$ $(\sigma \geq 1)$ we can make $\int_{\mathbb{R}^n} |\Delta\varphi|\, dx$ as small as possible.

We may now conclude this section with another existence result concerning another model equation namely (1.7).

THEOREM VII.2. – *If we assume* (1.8), *then there exists a solution of* (1.7).

We will not give the proof of this result which is very much similar to the proof of Theorem I.1. Let us only mention that the problem at infinity becomes in this case

$$(7.15)\qquad \begin{cases} -\Delta\omega = b^\infty (\omega - \lambda_0)^{+p} \quad \text{in } \mathbb{R}^n, \quad \omega \in L^{\frac{2n}{n-2}}(\mathbb{R}^n), \\ \nabla\omega \in L^2(\mathbb{R}^n), \qquad \omega > 0 \quad \text{in } \mathbb{R}^n. \end{cases}$$

By [19] we know that any solution of (7.15) is radial up to a translation and then the uniqueness of a radial solution of (7.15) may be deduced from [27]. Furthermore, we have

$$(7.16)\qquad \omega(x)\,|x|^{n-2} \to C_0 > 0 \quad \text{as } |x| \to \infty$$

$$(7.17)\qquad (-\omega'(r))\, r^{n-1} \to (n-2)\, C_0 \quad \text{as } r = |x| \to \infty$$

$$(7.18)\qquad |D^2\omega(x)| \leq \frac{C}{|x|^n} \qquad \text{for } \ |x| \geq 1, \qquad \text{for some} \quad C > 0.$$

Finally, up to tedious verifications, the existence follows from the main energy balance we sketch now. We consider

$$(7.19)\qquad I(v) = \int_\Omega \frac{1}{2}|\nabla v|^2 - \frac{b}{p+1}(v - \lambda_0)^{+p+1}\, dx,$$

for all $v \in L^{\frac{2n}{n-2}}(\Omega)$, $\nabla v \in L^2(\Omega)$, $v = 0$ on $\partial\Omega$ and we denote by

$$S = I^\infty(\omega) = \int_{\mathbb{R}^n} \frac{1}{2}|\nabla\omega|^2 - \frac{b^\infty}{p+1}(\omega - \lambda_0)^{+p+1}\, dx.$$

Then, we claim that for m large the following inequality holds

$$(7.20)\qquad \begin{cases} I\left(\left(\sum_1^m \omega(\cdot + \lambda x_i)\right)\right) < m\, S, \\ \forall (x_1, \ldots, x_m) \in (S^{n-1})^m / \sigma_m \end{cases}$$

for λ large. Indeed, we obtain by computations similar to those made in section VI.

$$I\left(\varphi\left(\sum_1^m \omega(\cdot + \lambda x_i)\right)\right) \leq m S + \sum_{i \neq j} \int_{\mathbb{R}^n} b^\infty (\omega_i - \lambda_0)^{+p} \omega_j \, dx$$

$$+ C_5 \left(\int_{\mathbb{R}^n} |\Delta\varphi| \, dx\right) \lambda^{-(n-2)} - (p+1) \sum_{i \neq j} \int_{\mathbb{R}^n} b^\infty (\omega_i - \lambda_0)^{+p} \omega_j \, dx$$

$$+ m C_2 \lambda^{-(n-2)} + 0 (\lambda^{-(n-2)})$$

where C_2 depends only on (1.8) and b^∞, λ_0, p, n and C_5 is independent of $x_1, \ldots, x_m$, $t_1, \ldots, t_m$, $\lambda \geq 1$ and φ. Next, we observe that $(\omega_i - \lambda_0)^+$ has compact support, hence we deduce (from section II)

$$I\left(\varphi\left(\sum_1^m \omega(\cdot + \lambda x_i)\right)\right) \leq m S + (m C_2 + C_5 |\Delta\varphi|_{L^1}) \lambda^{-(n-2)}$$

$$- \nu \lambda^{-(n-2)} \sum_{i \neq j} \frac{1}{|x_i - x_j|^{n-2}} + O(\lambda^{-(n-2)}).$$

And we conclude easily since for all $(x_1, \ldots, x_m) \in (S^{n-1})^m / \sigma_m$ we have

$$\sum_{i \neq j} \frac{1}{|x_i - x_j|^{n-2}} \geq \frac{1}{2^{n-1}} m(m-1)$$

and we conclude (7.20) taking m large.

VIII. A RELATED EXISTENCE RESULT

We now prove Theorem I.2 considering the following minimization problem

$$(8.1) \qquad I^G = \operatorname{Inf}\left\{\left(\int_\Omega |\nabla u|^2 + \lambda_0 u^2 \, dx\right) \times \left(\int_\Omega b |u|^{p+1} dx\right)^{-\frac{2}{p+1}} / u \in H_0^1(\Omega), \; u \not\equiv 0, \; u(x) = u(g \cdot x) \text{ in } \Omega, \; \forall g \in G\right\}$$

Then, by the results of [26], existence will follow immediately if we show

$$(8.2) \qquad \begin{cases} I^G < N^{\frac{p-1}{p+1}} I^\infty, \\ I^\infty = \left(\int_{\mathbb{R}^n} |\nabla \omega|^2 + \lambda_0 \omega^2 \, dx\right) \left(\int_{\mathbb{R}^n} b^\infty \omega^{p+1} dx\right)^{-\frac{2}{p+1}} \end{cases}$$

where ω is any ground-state (radial) solution of (1.3) that is minimizing

$$\text{(8.3)}\quad \operatorname{Min}\left\{\left(\int_{\mathbb{R}^n} |\nabla u|^2 + \lambda_0 u^2\, dx\right) \times \left(\int_{\mathbb{R}^n} b^\infty |u|^{p+1}\, dx\right)^{-\frac{2}{p+1}} \Big/ u \in H^1(\mathbb{R}^n),\, u \not\equiv 0\right\}$$

In order to show (8.2), we only have to consider the case when $N < \infty$ and we choose $|\xi| = R_0$, $N(\xi) = N$ such that (1.9) holds for some constant to be determined later on. Then, we denote by $\{\xi_1, \ldots, \xi_N\} = \{g \cdot \xi / g \in G\}$ and we consider

$$\text{(8.4)}\qquad u_\lambda = \varphi\left(\sum_1^N \frac{1}{N}\, \omega(\cdot - \lambda\xi_i)\right) \qquad \text{for} \quad \lambda \geq 1$$

where $\varphi \in C^\infty(\mathbb{R}^n)$ is some radial cut-off function to be determined satisfying (2.12). Observe that since ω is radial, u_λ given by (8.4) is invariant by G. Hence, showing (8.2) is equivalent to showing (6.7) with $t_i = \frac{1}{N}$ for $1 \leq i \leq N$, $m = N$. Then, it is easy to adapt the proof of (6.7) given in section VI and to show that (8.2) holds if c_0 is small enough. One only needs to observe that

$$\begin{aligned}
&\frac{1}{N^{p+1}} \int_{\mathbb{R}^n} b\, \varphi^{p+1} \left(\sum_i^N \omega(x - \lambda\xi_i)\right)^{p+1} dx \\
&\geq \frac{1}{N^{p+1}} \int_{\mathbb{R}^n} b^\infty \left(\sum_i^N \omega(x - \lambda\xi_i)\right)^{p+1} \\
&\quad - \frac{1}{N^{p+1}} \int_{\mathbb{R}^n} (b^\infty - b\, \varphi^{p+1})^+ \left(\sum_i^N \omega(x - \lambda\xi_i)\right)^{p+1} \\
&\geq \frac{1}{N^p} \int_{\mathbb{R}^n} b^\infty\, \omega^{p+1}\, dx + \frac{1}{N^{p+1}} (p + 1 - \varepsilon(\lambda)) \\
&\quad \times \int_{\mathbb{R}^n} \sum_{i \neq j} \omega^p(x - \lambda\xi_i)\, \omega(x - \lambda\xi_j) \\
&\quad - \frac{1}{N^{p+1}} N \left(\int_{\mathbb{R}^n} \omega^{p+1}\, dx\right) c_0\, N^{-1} \int_{\mathbb{R}^n} \sum_{i \neq j} \omega^p(x - \lambda\xi_i)\, \omega(x - \lambda\xi_i)
\end{aligned}$$

where $\varepsilon(\sigma) \to 0$ as $\sigma \to 0$. Here, we used (1.9) and the results of section II. ■

Remark. – In fact, if we inspect closely the above argument and the bounds obtained in section VI, we see that (1.9) may be extended as follows: there exist λ_0 (depending on b through $\sup b$), $\overline{c} = \overline{c}(p, N)$ such that if b satisfies for some $\lambda \geq \lambda_0$, $c_0 \leq \overline{c}$ the following condition

$$\sup_{|e|=1} \int_{\mathbb{R}^n} (b^\infty - b)^+ \omega^{p+1} (x + \lambda e)\, dx \leq \frac{c_0}{N} \sum_{i \neq j} \exp(-\sqrt{\lambda_0}\, \Delta_{ij}\, \lambda)\, \Delta_{ij}^{\frac{n-1}{2}}\, \lambda^{-\frac{n-1}{2}} \tag{8.5}$$

then the conclusion of Theorem I.2 still holds.

APPENDIX

On Palais-Smale sequences

We prove here Proposition II.1; in fact, the arguments which follow are taken from [24] and a more general proof than the one we present here can also be directly deduced using the full stregnth of concentration-compactness lemma as in [24], [22]. We thus consider a sequence $(u_k)_k$ bounded in $H_0^1(\Omega)$ satisfying

$$-\Delta u_k + \lambda_0 u_k - b(x) |u_k|^{p-1} u_k = \varepsilon_k \underset{k}{\rightarrow} 0 \quad \text{in } H^{-1}(\Omega). \tag{A.1}$$

Following [22], [24], we introduce for an arbitrary sequence $(w_k)_k$ bounded in $L^2(\mathbb{R}^n)$ the concentration function of $|w_k|^2$

$$Q_k(t) = \sup_{y \in \mathbb{R}^n} \int_{y + B_t} |w_k|^2\, dx, \qquad \text{for all} \quad t \geq 0. \tag{A.2}$$

We first recall a few preliminary results whose proofs we postpone.

LEMMA A.1. – *Let $(w_k)_k$ be bounded in $H^1(\mathbb{R}^n)$ and assume that for some $t_0 > 0$*

$$Q_k(t_0) \underset{k}{\rightarrow} 0. \tag{A.3}$$

Then, $w_k \underset{k}{\rightarrow} 0$ in $L^q(\mathbb{R}^n)$ for all $2 < q < \frac{2n}{n-2}$. If in addition w_k satisfies (A.1), then $w_k \underset{k}{\rightarrow} 0$ in $H^1(\mathbb{R}^n)$.

LEMMA A.2. – *Let $(\varphi_k)_k$ converge weakly to φ in $H^1(\mathbb{R}^n)$ then we have*

$$b |\varphi_k|^{p-1} \varphi_k - b |\varphi|^{p-1} \varphi \underset{k}{\rightarrow} 0 \quad \textit{in } H^{-1} \tag{A.4}$$

LEMMA A.3. – *For each* $C_0 \geq 0$, *there exists* $\delta > 0$ *such that if* $v \in H^1(\mathbb{R}^n)$ *solves*

$$-\Delta v + \lambda_0 v = b^\infty |v|^{p-1} v \quad in\ \mathbb{R}^n, \quad v \in H^1(\mathbb{R}^n) \tag{A.5}$$

and $|v|_{H^1} \leq c_0$, $|v|_{L^2} \leq \delta$, *then* $v \equiv 0$.

We may now *prove Proposition* II.1: to be precise, several subsequences should be extracted in the arguments below but we will always denote by the same sequence all the extracted subsequences... First of all, with these conventions, we may assume that u_k converges weakly to some $u \in H_0^1(\Omega)$. It is a standard exercise to check that u solves

$$-\Delta u + \lambda_0 u = b |u|^{p-1} u \quad \text{in } \Omega, \qquad u \in H_0^1(\Omega). \tag{A.6}$$

Because of Lemma A.2, we see that we may always assume that u_k converges weakly to 0 replacing if necessary $(u_k)_k$ by $(u_k - u)_k$.

Next, in view of Lemma A.1, either $u_k \underset{k}{\to} 0$ in H^1 and the proof is over or these exists $\alpha > 0$ such that we have (up to a subsequence...)

$$Q_k(1) > \alpha > 0 \tag{A.7}$$

and thus there exists $(y_k)_k$ in $\mathbb{R}^n$ such that

$$\int_{y_k + B_1} |u_k|^2 \, dx \geq \alpha > 0. \tag{A.8}$$

Therefore, by Rellich-Kondrekov theorem, $u_k(y_k + \cdot) = \tilde{u}_k$ converges weakly in $H^1(\mathbb{R}^n)$ to some $\tilde{u} \not\equiv 0$. Since $u_k \underset{k}{\to} 0$ in H^1, we deduce

$$|y_k| \underset{k}{\to} +\infty. \tag{A.9}$$

But then from Lemma A.2 we deduce that $v_k = \tilde{u}_k - \tilde{u}$ satisfies

$$-\Delta v_k + \lambda_0 v_k - b^\infty |v_k|^{p-1} v_k \underset{k}{\to} 0 \qquad \text{in} \quad H^{-1}(\mathbb{R}^n) \tag{A.10}$$

while $\tilde{u}$ solves

$$-\Delta \tilde{u} + \lambda_0 \tilde{u} = b^\infty |\tilde{u}|^{p-1} \tilde{u} \quad \text{in } \mathbb{R}^n, \qquad \tilde{u} \in H^1(\mathbb{R}^n). \tag{A.11}$$

Furthermore, we have

$$\text{(A.12)}\quad \begin{cases} \left| \int_{\mathbb{R}^n} |\nabla u_k|^2 \, dx - \int_{\mathbb{R}^n} |\nabla \tilde{u}|^2 \, dx - \int_{\mathbb{R}^n} |\nabla v_k|^2 \, dx \underset{k}{\to} 0 \right. \\ \left| \int_{\mathbb{R}^n} |u_k|^2 \, dx - \int_{\mathbb{R}^n} |\tilde{u}|^2 \, dx - \int_{\mathbb{R}^n} |v_k|^2 \, dx \underset{k}{\to} 0 \right. \end{cases}$$

To conclude, we just iterate the above argument and this iteration pocedure has to stop in a finite number of steps since, if $\tilde{u}_1, \ldots, \tilde{u}_n$ denote the limit solutions of (A.11) obtained through this procedure, we have

$$\sum_{i=1}^{m} \int_{\mathbb{R}^n} |\tilde{u}_i|^2 \, dx \le \underline{\lim}_{k} \int_{\mathbb{R}^n} |u_k|^2 \, dx.$$

Thus, m cannot go to ∞ in view of Lemma A.2.

Proof of Lemma A.1. – We cover $\mathbb{R}^n$ by balls of radius t_0 centered at integer coordinates points. Hence, we have (denoting by Q such a generic ball) for any $2 < q < r < \frac{2n}{n-2}$

$$\begin{aligned} \int_{\mathbb{R}^n} |w_k|^q \, dx &\le \sum_Q \int_Q |w_k|^q \, dx \\ &\le \sum_Q \left(\int_Q |w_k|^2 \, dx \right)^{\alpha} \left(\int_Q |w_k|^r \, dx \right)^{\beta} \\ &\qquad \text{by Hölder's inequalities} \end{aligned}$$

where $\alpha = \frac{r-q}{r-2}$, $\beta = \frac{q-2}{r-2}$, and then by Sobolev's inequalities we deduce

$$\begin{aligned} \int_{\mathbb{R}^n} |w_k|^q \, dx &\le C\, Q_k(t_0)^{\alpha} \sum_Q \left(\int_Q |\nabla w_k|^2 + |w_k|^2 \, dx \right)^{\frac{r\beta}{2}} \\ &\le C\, Q_k(t_0)^{\alpha} \end{aligned}$$

if $\frac{r\beta}{2} \ge 1$, where C denotes various constants independent of k. But, since $\frac{r\beta}{2} \to \frac{q}{2} > 1$ as $r \to q$, we may now conclude easily. ■

Proof of Lemma A.2. – We denote by $\psi_k = \varphi_k - \varphi$ and we observe that $\psi_k \to 0$ weakly in H^1, a.e. and strongly in L^q_{loc} for all $q < \frac{2n}{n-2}$. Hence,

$$b\,|\varphi + \psi_k|^{p-1} (\varphi + \psi_k) - b\,|\varphi|^{p-1} \varphi \underset{k}{\to} 0 \quad \text{a.e.,} \quad \text{in } L^{\frac{p+1}{p}}_{\text{loc}}.$$

It is then easy to conclude by observing that for all $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|b\,|\varphi + \psi_k|^{p-1}(\varphi + \psi_k) - b\,|\varphi|^{p-1}\varphi|^{\frac{p+1}{p}} \leq \varepsilon\,|\psi_k|^{p+1} + C_\varepsilon\,|\varphi|^{p+1}.$$

Indeed, this immediately yields that $b\,|\varphi+\psi_k|^{p-1}(\varphi+\psi_k) - b\,|\varphi|^{p-1}\varphi \underset{k}{\to} 0$ in $L^{\frac{p+1}{p}}$ and thus in $H^{-1}(\mathbb{R}^n)$ by Sobolev embeddings. ■

Proof of Lemma A.3. – Multiplying (A.5) by v and integrating by parts, we find

$$\|v\|_{H^1}^2 \leq C \int_{\mathbb{R}^n} |v|^{p+1}\,dx \leq C\,|v|_{L^2}^{\alpha}\left(\int_{\mathbb{R}^n} |v|^{\frac{2n}{n-2}}\,dx\right)^{\beta}$$
$$\leq C\,\delta^{\alpha}\,\|v\|_{H^1}^{\gamma}, \qquad \text{with} \quad \gamma = \frac{2n}{n-2}\,\beta$$

by Hölder and Sobolev inequalities, where C denotes various nonnegative constants independent of C_0, δ and where $\alpha = (\frac{2n}{n-2} - (p+1))\,(\frac{2n}{n-2} - 2)^{-1}$, $\beta = (p-1)\,(\frac{2n}{n-2} - 2)^{-1}$ (at least if $n \geq 3$, when $n \leq 2$ the argument is easily adapted...). Now, if $p \geq 1 + \frac{4}{n}$ then $\gamma \geq 2$ and we conclude easily if δ is small enough. On the other hand if $p < 1 + \frac{4}{n}$, we deduce

$$\|v\|_{H^1} \leq C\,\delta^2 \qquad \text{for some} \quad a > 0 \quad (a = \alpha\,(2-\gamma)^{-1}).$$

While the first inequality also implies

$$\|v\|_{H^1}^2 \leq C\,\|v\|_{H^1}^{p+1} \quad ■$$

and we conclude easily.

Remark. – The proofs of Lemma A.2 and A.3 seem to be highly dependent on the power type behaviour of the nonlinearity but it is not so. Indeed, appropriate modifications show that the results are still valid for large classes of non-linearities: only, the behaviours of the nonlinearity at 0 and at ∞ matter.

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