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The structure of extremals of a class of second order variational problems

by

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ABSTRACT. – We study the structure of extremals of a class of second order variational problems without convexity, on intervals in R_+. The problems are related to a model in thermodynamics introduced in [7]. We are interested in properties of the extremals which are independent of the length of the interval, for all sufficiently large intervals. As in [12, 13] the study of these properties is based on the relation between the variational problem on bounded, large intervals and a limiting problem on R_+. Our investigation employs techniques developed in [10, 12, 13] along with turnpike techniques developed in [16, 17].

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Key words: Turnpike properties, (f)-good functions, periodic minimizers.

RÉSUMÉ. – On étudie la structure des extrêmales d'une classe de problèmes variationnels non convexes du deuxième ordre, sur des intervalles de R_+. Ces problèmes sont reliés à un modèle thérmodynamique introduit dans [7]. Nous nous intéressons aux propriétés des extrêmales qui ne dependent pas de la longeur des intervalles, pourvu que ceux-ci soient assez grands. Comme dans [12,13] l'etude de ces propriétés s'appuie sur la relation entre le problème variationnel sur de grands intervalles bornés et un problème limite sur R_+. Notre travail emploie des techniques développées dans [10,12,13] ainsi que dans [16,17].

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1. INTRODUCTION

In this paper we investigate the structure of optimal solutions of variational problems associated with the functional

$$J^f(D;w) = |D|^{-1} \int_D f(w(t), w'(t), w''(t))dt, \quad \forall w \in W^{2,1}(D),$$

where D is a bounded interval on the real line and $f \in C(R^3)$ belongs to a space of functions to be described below. Specifically we shall consider the problems,

$$(P_D) \qquad \inf\{J^f(D;w) : \ w \in W^{2,1}(D)\}$$

and, for $D = (T_1, T_2)$,

$(P_D^{x,y})$

$$\inf\{J^f(D;w) : \ w \in W^{2,1}(D), \ (w,w')(T_1) = x, \ (w,w')(T_2) = y\}.$$

In connection with these we shall also study the following problem on the half line:

$$(P_\infty) \qquad \inf\{J^f(w) : \ w \in W^{2,1}_{loc}(0,\infty)\},$$

where

$$J^f(w) = \liminf_{T\to\infty} J^f((0,T);w).$$

This can be seen as a limiting problem for (P_D) as $|D| \to \infty$. Variational problems of this type were considered by Leizarowitz and Mizel [10]. Similar *constrained* problems (involving a mass constraint), were studied by Coleman, Marcus and Mizel [7] and by Marcus [12,13]. The constrained problems were conceived as models for determining the thermodynamical equilibrium states of unidimensional bodies involving 'second order' materials (see [7]).

Let $G = G(p,r)$ be a function in $C^4(R^2)$ such that

$$(1.1) \qquad \begin{aligned} &\partial^2 G/\partial r^2(p,r) > 0, \\ &G(p,r) \ge |r|^\gamma - b_1|p|^\beta - b_0, \quad \forall (p,r) \in R^2, \end{aligned}$$

where b_1, b_0 are positive constants, $1 \le \beta \le \gamma$ and $\gamma > 1$. In addition assume that,

$$(1.2) \quad \max\{|G(p,r)|, |\partial G/\partial r(p,r)|, |\partial G/\partial p(p,r)|\} \le M(|p|)(1 + |r|^\gamma),$$

where $M : [0,\infty) \to [0,\infty)$ is a continuous function. A typical example is $G(p,r) = r^2 - bp^2$.

Let α, b_2, b_3 be positive numbers, with $\alpha > \beta$, and let

$$(1.3)\quad \mathfrak{L} = \mathfrak{L}(\alpha, b_2, b_3) = \{\phi \in C^2(R^1) : \ \phi(t) \geq b_3|t|^\alpha - b_2, \quad \forall t \in R^1\}.$$

The space $\mathfrak{L}$ will be equipped with the standard topology of C^2. Finally denote,

$$(1.4)\qquad \mathfrak{L}_G = \mathfrak{L}_G(\alpha, b_2, b_3) = \{F_\phi : \ \phi \in \mathfrak{L}(\alpha, b_2, b_3)\},$$

where,

$$(1.5)\qquad F_\phi(w, p, r) = \phi(w) + G(p, r), \quad \forall (w, p, r) \in R^3.$$

The relation between the minimizers of (P_D) (for large $|D|$) and those of (P_∞) plays a crucial role in our study of their structure. This relation was first investigated by Marcus [12, 13] where it was used in order to derive structural properties of minimizers of problem (P_D) and of related *constrained* problems, in the case $f = r^2 - bp^2 + \phi(w)$. In the present paper we pursue this investigation combining techniques of [12, 13] with turnpike techniques as in Zaslavski [16, 17].

One of our main results is the uniqueness of periodic minimizers of (P_∞) which is generically valid in a very precise sense.

For every potential $\phi \in \mathfrak{L}(\alpha, b_2, b_3)$ there exists a family of arbitrarily small perturbations $\{\phi_s = \phi + s\theta : \ 0 < s < 1\}$, such that problem (P_∞) with $f = F_{\phi_s}$ possesses a unique (up to translation) periodic minimizer.

The function θ can be explicitly constructed in terms of the extremal values of periodic minimizers of (P_∞) with $f = F_\phi$. Combining this result with a recent result of Zaslavski [18], we show that for each potential ϕ_s in this family, the corresponding integrand F_{ϕ_s} possesses an asymptotic turnpike property, which involves the behaviour of the *limit set* of minimizers of (P_∞). Finally, we show that this asymptotic property can be used in order to derive detailed information on the structure of minimizers of problem (P_D) for all sufficiently large intervals D. In this last part the results are valid not only in the generic sense, but apply to every $f \in \mathfrak{L}_G$.

A brief comparison of the present results with those of [13]: In the present work, as in [13], the structure of minimizers of (P_D) is described by observing their behaviour in a 'window' of fixed length (independent of $|D|$) which can be placed anywhere in D. The results of [13] apply to every integrand of the form $f = r^2 - bp^2 + \phi(w)$, for a class of potentials ϕ which includes the standard two–well potentials. The behaviour of minimizers of (P_D) in a 'window' is described by integral estimates, involving 'mass' and 'energy'. The present results are in part generic, but they deal with a very large class of integrands and the behaviour of minimizers in a 'window' is described by pointwise estimates which provide considerably more detailed information.

For a precise statement of the results mentioned above we need some additional notation and definitions.

Let $\mu(f)$ denote the infimum in (P_∞) with $f \in \mathfrak{L}_G$. Leizarowitz and Mizel [10] proved that, if $\mu(f) < \inf_{(w,s)\in R^2} f(w,0,s)$, then (P_∞) possesses a periodic minimizer. Zaslavski [15] showed that the result remains valid for all $f \in \mathfrak{L}_G$.

For $w \in W^{2,1}_{loc}(0,\infty)$ put,

$$\eta^f(T,w) = (J^f((0,T);w) - \mu(f))T,\ T \in (0,\infty). \tag{1.6}$$

Then, either $\sup_{0<T<\infty} |\eta^f(T,w)| < \infty$ or $\lim_{T\to\infty} \eta^f(T,w) = +\infty$. Furthermore, if $\eta^f(\cdot,w)$ is bounded then w and w' are bounded [15, Prop. 3.1].

Let w be an (f)-minimizer of (P_∞). We shall say that w is *(f)-good* if $\eta^f(\cdot,w)$ is bounded. Equivalently, w is (f)-good if and only if there exists a constant $c(w)$ such that,

$$|J^f(D;w) - \mu(f)| \le c(w)/|D| \tag{1.7}$$

for every bounded interval D.

We shall say that w is *optimal on compacts*, or briefly c-optimal, if $w \in W^{2,1}_{loc}(0,\infty) \cap W^{1,\infty}(0,\infty)$ and, for every bounded interval D, the restriction $w|_D$ is a minimizer of $(P_D^{x,y})$, where x, y are the values of (w,w') at the end points of the interval. By a result of Marcus [13, Th. 4.2(vi)], if the integrand f is of the form $f(w,p,r) = r^2 - bp^2 + \phi(w)$, then *every c-optimal minimizer of (P_∞) is (f)-good.* In fact the result remains valid for the more general class of integrands studied here, (see Proposition 2.3 below).

For $w \in W^{2,1}_{loc}(0,\infty) \cap W^{1,\infty}(0,\infty)$ let $\Omega(w)$ denote the set of limiting points of (w,w') as $t \to \infty$.

DEFINITION 1.1. – Let $f \in \mathfrak{L}_G$. We say that f has the asymptotic turnpike property, or briefly (ATP), if there exists a compact set $H(f) \subset R^2$ such that $\Omega(w) = H(f)$ for *every* (f)-good minimizer w.

Clearly, if f has (ATP) and v is a periodic (f)-minimizer of (P_∞) then, $H(f) = \{(v,v')(t) : 0 \le t < \infty\}$.

The asymptotic turnpike property for optimal control problems was studied in [4, 5]. The more standard turnpike property (for problems on finite intervals) is well known in mathematical economics and several variants of it have been studied (see, e.g. [11] and [6, Ch.4 and 6]). Here we shall consider, besides (ATP), the *strong turnpike property*, or briefly (STP), which is defined as follows.

DEFINITION 1.2. – Let $f \in \mathfrak{L}_G$ and let w be a periodic (f)-minimizer of (P_∞) with period $T_w > 0$. We say that f has the strong turnpike property if, for every $\epsilon > 0$ and every bounded set $K \subset R^2$, there exists $L > 0$ such that every minimizer v of $(P^{x,y}_{(0,T)})$, with $x, y \in K$ and $T > T_w + 2L$, satisfies the following:

For every $a \in [L, T - L - T_w]$ there exists $\bar{a} \in [0, T_w)$ such that,

$$(1.8) \qquad |(v, v')(a + t) - (w, w')(\bar{a} + t)| \leq \epsilon, \quad \forall t \in [0, T_w].$$

Note that (STP) implies *uniqueness up to translation* for periodic minimizers of (P_∞). Furthermore, if f has (STP), the structural information contained in (1.8) extends to arbitrary minimizers of the unconstrained problem $(P_{(0,T)})$. More precisely we have,

PROPOSITION 1.1. – *Suppose that $f \in \mathfrak{L}_G$ possesses (STP). Let w be the (unique) periodic minimizer of (P_∞) whose period will be denoted by T_w. Then, given $\epsilon > 0$, there exists $L > 0$ such that every minimizer v of $(P_{(0,T)})$ with $T > T_w + 2L$ satisfies (1.8) for every $a \in [L, T - L - T_w]$ and some $\bar{a} \in [0, T_w)$ depending on v and a.*

This is a consequence of the fact that the set of minimizers of $(P_{(0,T)})$ is bounded in $C^1[0, T]$ by a constant A independent of T, (see [12, Lemma 2.2]).

Our main results are the following.

THEOREM 1.1. – *For $f \in \mathfrak{L}_G$, (STP) holds if and only if (ATP) holds.*

THEOREM 1.2. – *For every $\phi \in \mathfrak{L}$ there exists a non-negative function $\theta \in C^\infty(R^1)$ with $\theta^{(m)} \in L^\infty(R^1)$, $m = 0, 1, \ldots$, such that for every $s \in (0, 1)$, problem (P_∞) with $f = F_{\phi + s\theta}$ possesses a unique (up to translation) periodic minimizer.*

THEOREM 1.3. – *(i) For every $\phi \in \mathfrak{L}$ there exists a function θ as in Theorem 1.2 such that,*

$$F_{\phi+s\theta} \text{ possesses (ATP)}, \forall s \in (0, 1).$$

(ii) (ATP) holds generically in $\mathfrak{L}_G$, in the following sense as well: there exists a countable intersection of open everywhere dense sets in $\mathfrak{L}$, say $\mathfrak{F}_G$, such that

$$\phi \in \mathfrak{F}_G \Longrightarrow F_\phi \text{ possesses (ATP).}$$

A result related to the second part of Theorem 1.3 was obtained by Zaslavski [16], who established the generic validity of (ATP) in a larger space, in a weaker sense.

The proofs of these theorems, in a slightly more general form, are presented in sections 2 (Theorem 1.1) and 3 (Theorems 1.2, 1.3). In addition, in section 3, we establish a number of properties of periodic minimizers of (P_∞) which apply to every $f \in \mathfrak{L}_G$ and may be of independent interest.

2. EQUIVALENCE OF (ATP) AND (STP)

In this section we shall establish Theorem 1.1 for problems involving a larger family of integrands f. Put,

$$\mathfrak{A} = \{f \in C(R^3) : \ |f(x_1, x_2, x_3)| \to \infty \text{ as } |x_3| \to \infty,$$
$$\text{uniformly with respect to } (x_1, x_2) \text{ in compact sets}\}.$$

$\mathfrak{A}$ will be equiped with the uniformity determined by the base,

$$(2.1)\quad \begin{aligned} E(N, \epsilon) = \{(f, g) \in \mathfrak{A} \times \mathfrak{A} :& \\ & |f(x) - g(x)| \le \epsilon, \ \ (x = (x_1, x_2, x_3) \in R^3, \\ & |x_i| \le N, \ i = 1, 2, 3), \\ & 1 - \epsilon \le (|f(x)| + 1)/(|g(x)| + 1) \le 1 + \epsilon, \\ & (x \in R^3, \ |x_1|, |x_2| \le N)\} \end{aligned}$$

where N and ϵ are positive numbers. It is easy to verify that the uniform space $\mathfrak{A}$ is metrizable and complete [8].

Let $a = (a_1, a_2, a_3, a_4) \in R^4, a_i > 0, i = 1, 2, 3, 4$ and let α, β, γ be real numbers such that $1 \le \beta < \alpha$, $\beta \le \gamma$ and $\gamma > 1$. Denote by $\mathfrak{M} = \mathfrak{M}(\alpha, \beta, \gamma, a)$ the family of functions $\{f\}$ such that

$$(2.2)\quad \begin{aligned} & (i) \ \ f \in \mathfrak{A} \cap C^2(R^3), \ \partial f/\partial x_2 \in C^2(R^3), \ \partial f/\partial x_3 \in C^3(R^3), \\ & (ii) \ \ \partial^2 f/\partial x_3^2 > 0, \\ & (iii) \ f(x) \ge a_1|x_1|^\alpha - a_2|x_2|^\beta + a_3|x_3|^\gamma - a_4, \\ & (iv) \ \ (|f| + |\nabla f|)(x) \le M_f(|x_1| + |x_2|)(1 + |x_3|^\gamma), \quad \forall x \in R^3, \end{aligned}$$

where $M_f : [0,\infty) \mapsto [0,\infty)$ is a continuous function depending on f. Finally, let $\bar{\mathfrak{M}}$ denote the closure of $\mathfrak{M}$ in $\mathfrak{A}$. The notations and definitions presented in the introduction with respect to $f \in \mathfrak{L}_G$ apply equally well to $f \in \mathfrak{M}$ and the various statements quoted there remain valid in this context. Put,

$$I^f(T_1, T_2, w) = \int_{T_1}^{T_2} f(w(t), w'(t), w''(t))dt \tag{2.3}$$

where $-\infty < T_1 < T_2 < +\infty$, $w \in W^{2,1}(T_1, T_2)$ and $f \in \bar{\mathfrak{M}}$.

For $T > 0$, $x, y \in R^2$, $f \in \bar{\mathfrak{M}}$, put

$$\begin{aligned} U_T^f(x,y) := \inf\{I^f(0,T,w) :\ & w \in W^{2,1}(0,T), \\ & (w,w')(0) = x,\ (w,w')(T) = y\}. \end{aligned} \tag{2.3a}$$

Let $v \in W^{2,1}(D)$ where $D = (T_1, T_2)$ is a bounded interval. Given $\delta > 0$, we shall say that v is an *(f,δ)–approximate minimizer* in D if,

$$I^f(T_1, T_2, v) \le U_{|D|}^f(X_v(T_1),\ X_v(T_2)) + \delta, \tag{2.3b}$$

$$X_v(t) = (v(t), v'(t)), \quad t \in D.$$

For $x \in R^n$, $B \subset R^n$ put $d(x, B) := \inf\{|x-y| : y \in B\}$ (where $|\cdot|$ is the Euclidean norm) and denote by $\operatorname{dist}(A, B)$ the distance in the Hausdorff metric between two subsets A, B of R^n.

We claim that:

LEMMA 2.1. – *Suppose that $f \in \bar{\mathfrak{M}}$ and that v is an (f)-good function. Then, given $\delta > 0$ there exists $T_\delta > 0$ such that, for every bounded interval (T, T') with $T \ge T_\delta$,*

$$I^f(T, T', v) \le U_{T'-T}^f(X_v(T), X_v(T')) + \delta, \tag{2.4}$$

i.e. v is an (f,δ)–approximate minimizer in (T, T').

Proof. – If the claim is not valid there exists a sequence of disjoint intervals $D_n = (T_n, T'_n)$, $n = 1, 2, \ldots$ with $T_n \to \infty$ such that,

$$I^f(T_n, T'_n, v) - U_{T'_n - T_n}^f(x_n, y_n) \ge \delta, \quad n = 1, 2, \ldots, \tag{2.5}$$

where $x_n = X_v(T_n)$ and $y_n = X_v(T'_n)$. Let h_n denote a minimizer of problem $(P_{D_n}^{x_n, y_n})$ and let $\tilde{v}$ be the function on $[0,\infty)$ defined as follows,

$$\tilde{v}(t) = v(t),\ t \in [0,\infty) \setminus \cup_n D_n, \quad \tilde{v}(t) = h_n(t),\ t \in D_n,\ n = 1, 2, \ldots.$$

Then $\tilde{v} \in W^{2,1}_{loc}(0,\infty)$ and

$$\eta^f(T,\tilde{v}) = (I^f(0,T,\tilde{v}) - I^f(0,T,v)) + \eta^f(T,v).$$

Since $\eta^f(\cdot,v)$ is bounded, say by M, it follows that,

$$\eta^f(T'_n,\tilde{v}) \leq M - \sum_{k=1}^{n}(I^f(T_k,T'_k,v) - U^f_{T'_k-T_k}(x_k,y_k)).$$

This inequality and (2.5) imply that $\eta^f(T'_n,\tilde{v}) \to -\infty$ as $n \to \infty$. However this is impossible because $\eta^f(\cdot,w)$ is bounded from below for every $w \in W^{2,1}_{loc}(0,\infty)$. □

For the next lemma we need the following interpolation inequality (see e.g. Adams [1]):

Assume that $p > 1$ and $\epsilon > 0$. Then there exists a constant $C_\epsilon(p)$ such that, for every $T \geq 1$,

$$\int_0^T |u'|^p\,dt \leq \epsilon \int_0^T |u''|^p\,dt + C_\epsilon(p) \int_0^T |u|^p\,dt, \quad \forall u \in W^{2,p}(0,T). \tag{2.6}$$

LEMMA 2.2. –*(i) For every $\tau > 0$ there exist positive constants b_0, b_1, b_2 (depending on τ) such that, for every $T \geq \tau$,*

(2.7)
$$I^f(0,T,v) \geq \int_0^T \frac{1}{2}(a_3|v''|^\gamma + a_1|v|^\alpha)dt - b_0T \geq b_1\|v\|_{C^1(0,T)} - b_2T,$$

for every $v \in W^{2,1}(0,T)$ and every $f \in \bar{\mathfrak{M}}$. In particular, for every $M > 0$ and $T \geq \tau$ there exists a constant $b_\tau(M,T) > 0$ (depending continuously on M, T) such that, for every $f \in \bar{\mathfrak{M}}$,

$$\text{(2.8)}\quad v \in W^{2,1}(0,T),\ I^f(0,T,v) \leq M \Longrightarrow v \in W^{2,\gamma}(0,T),\ \|v\|_{W^{2,\gamma}(0,T)} \leq b_\tau(M,T).$$

(ii) For every $f \in \bar{\mathfrak{M}}$: if $v \in W^{2,1}_{loc}(0,\infty)$ is an (f)-good function then,

$$\sup_{T\geq 0} \int_T^{T+1} (|v''|^\gamma + |v|^\alpha)dt < \infty. \tag{2.9}$$

Consequently, v and v' are uniformly continuous on $[0,\infty)$.

Proof. – (i) In the proof we shall assume that $\tau = 1$. For arbitrary $\tau > 0$ the result can be obtained by rescaling. By (2.2), every $f \in \mathfrak{M}$ satisfies,

$$f(x) \geq a_1|x_1|^\alpha - a_2|x_2|^\beta + a_3|x_3|^\gamma - a_4. \tag{2.10}$$

Clearly this remains valid for every $f \in \bar{\mathfrak{M}}$. Note that if $\beta = 1$ then $\gamma' = \min(\alpha, \gamma) > 1$ and therefore, if $\beta' \in (1, \gamma')$ we have,

$$f(x) \geq a_1|x_1|^\alpha - a_2|x_2|^{\beta'} + a_3|x_3|^\gamma - (a_2 + a_4).$$

Therefore, without loss of generality, we may assume that $\beta > 1$. Hence, by (2.6) with $p = \beta$ and $\epsilon = \frac{a_3}{2a_2}$, we find that, for $f \in \bar{\mathfrak{M}}$ and $T \geq 1$

$$(2.11)\quad I^f(0,T,v) \geq \int_0^T \frac{1}{2}(a_3|v''|^\gamma + a_1|v|^\alpha)dt - b_0 T, \quad \forall v \in W^{2,1}(0,T)$$

where

$$(2.12)\qquad b_0 = \max_{t \geq 0}(a_2 C_\epsilon(\beta)t^\beta - a_1 t^\alpha/2) + a_4 + a_3/2.$$

(In fact, $v \in C^1[0,T]$. Therefore, by (2.2), $I^f(0,T;v)$ is finite if $v'' \in L^\gamma(0,T)$ and $+\infty$ otherwise.) This proves the first inequality in (2.7). In order to obtain the second inequality in (2.7) observe that,

$$\begin{aligned}\int_s^{s+1}(|v''|^\gamma + |v|^\alpha)dt &\geq \int_s^{s+1}(|v''|^{\gamma'} + |v|^{\gamma'})dt - 1\\ &\geq c_0 \sup_{s \leq t \leq s+1}(|v(t)| + |v'(t)|)^{\gamma'} - 1,\end{aligned}$$

for every $s \in [0, T-1]$, where c_0 is a constant which depends only on $\gamma' = \min(\alpha, \gamma)$. Combining this with the first inequality in (2.7) we obtain,

$$\begin{aligned} I^f(0,T,v) &\geq c_1 \int_0^T (|v''|^\gamma + |v|^\alpha)dt - b_0 T\\ &\geq c_1(c_0 \sup_{0 \leq t \leq T}(|v(t)| + |v'(t)|)^{\gamma'} - 1) - b_0 T\\ &\geq c_1(c_0\|v\|_{C^1(0,T)} - 2) - b_0 T,\end{aligned}$$

where c_1 is a constant which depends only on a_1, a_3. This completes the proof of (2.7). Finally (2.8) follows from (2.7):

$$\int_0^T |v''|^\gamma dt \leq 2(M + Tb_0)/a_3, \qquad \int_0^T |v|^\gamma dt \leq T((M + b_2)/b_1)^\gamma,$$

for every v as in (2.8).

(ii) Since v is (f)-good, (v, v') is bounded in $[0, \infty)$. Clearly, $U_1^f(x,y)$ is bounded for (x,y) in a compact set. Therefore Lemma 2.1 implies that

$I^f(T, T+1, v)$ is bounded by a bound independent of $T \geq 0$. Hence (2.9) follows from (2.7). □

Using these lemmas it is easy to verify that,

LEMMA 2.3. *–For $f \in \mathfrak{M}$, (STP) implies (ATP).*

Proof. – Assume that f has (STP) and let v be an (f)-good function. Pick $\xi \in \Omega(v)$ and let $\{t_k\}$ be a sequence tending to $+\infty$ such that $(v, v')(t_k) \to \xi$. Put $v_k(t) = v(t + t_k)$, $t \geq -t_k$. By Lemma 2.2, for every bounded interval D,

$$\sup_k \int_D (|v_k''|^\gamma + |v_k|^\alpha) dt < \infty.$$

Therefore there exists a subsequence v_{k_n} which converges weakly, say to u, in $W^{2,\gamma}_{loc}(R^1)$. In particular $\{(v_{k_n}, v'_{k_n})\}$ converges uniformly on compact sets. Applying inequality (2.4) to v_{k_n} and taking the limit, we find that (for every bounded interval $D = (0, T)$) $u|_D$ is a minimizer of problem $(P_D^{x,y})$, where x, y are the values of (u, u') at the endpoints of D. This is a consequence of the continuity of $U_T^f(\cdot, \cdot)$ in R^2 and of the weak lower semicontinuity of the functional $I^f(0, T, \cdot)$ in $W^{2,\gamma}(D)$, (see [3]). Since f has (STP) it follows that, for every $\epsilon > 0$, (1.8) holds with v replaced by an arbitrary translate of u, i.e. $u(\cdot + \tau)$, $\tau \in R^1$. Consequently, if w is a periodic minimizer of (P_∞) then, $E := \{(u, u')(t) : t \in R^1\} = \Omega(w)$. In particular, $\xi = (u, u')(0) \in \Omega(w)$ and we conclude that $\Omega(v) \subset \Omega(w)$. On the other hand $E \subset \Omega(v)$, so that $\Omega(v) = \Omega(w)$. Thus f possesses (ATP). □

The fact that (ATP) implies (STP) requires a more delicate argument. Actually we shall prove a more comprehensive result, which will also be used in the proof of Theorem 1.3. Roughly this result states that if $f \in \mathfrak{M}$ has (ATP) then, for every $\epsilon > 0$ there exists $\delta > 0$ such that, if v is an (f, δ)–approximate minimizer in $(0, T)$ and T is sufficiently large, then v satisfies (1.8), which is the condition required for (STP). Furthermore this property persists in a neighborhood of f in $\mathfrak{M}$. The precise formulation follows.

THEOREM 2.1. – *Assume that $g \in \mathfrak{M}$ satisfies (ATP). Let w be a periodic minimizer of (P_∞) with integrand g and let $T_w > 0$ be a period of w.*

Given ϵ, $M > 0$ there exists a neighbourhood of g in $\bar{\mathfrak{M}}$, say $\mathfrak{U}_g$, and positive numbers δ, ℓ such that the following statement holds :

Let $f \in \mathfrak{U}_g$ and let $T \geq T_w + 2\ell$. If $v \in W^{2,1}(0, T)$ satisfies,

(2.13)
$$|X_v(0)| \leq M, \ |X_v(T)| \leq M, \quad I^f(0, T, v) \leq U_T^f(X_v(0), X_v(T)) + \delta,$$

then, for each $s \in [\ell, T - T_w - \ell]$ *there exists* $\xi \in [0, T_w]$ *such that,*

$$|X_v(s+t) - X_w(\xi + t)| \leq \epsilon, \quad \forall t \in [0, T_w]. \tag{2.14}$$

Remark. – The conclusion of the theorem can be slightly strengthened as follows:

There exist $\tau_1 \in [0, \ell]$ *and* $\tau_2 \in [T - \ell, T]$ *such that, for every* $s \in [\tau_1, \tau_2 - T_w]$ *there exists* $\xi \in [0, T_w]$ *such that (2.14) holds. Furthermore, if*

$$d(X_v(0), \Omega(w)) \leq \delta, \text{ (respectively } d(X_v(T), \Omega(w)) \leq \delta),$$

the statement holds with $\tau_1 = 0$, *(respectively* $\tau_2 = T$*).*

The proof of the theorem will be based on several lemmas. One of the key ingredients in this proof is provided by the following result due to Leizarowitz and Mizel [10, Sec. 4]. (See also Leizarowitz [9] for a similar result in the context of a discrete model.)

PROPOSITION 2.1. – *Let* $f \in \bar{\mathfrak{M}}$. *Then there exist a continuous function* $\pi^f : R^2 \to R^1$ *given by,*

$$\pi^f(x) = \inf\{\liminf_{T\to\infty}[I^f(0,T,w) - T\mu(f)] : w \in W^{2,1}_{loc}(0,\infty),\ X_w(0) = x\},\quad x \in R^2$$

and a continuous nonnegative function $(T, x, y) \to \theta^f_T(x, y)$ *defined for* $T > 0$ *and* $x, y \in R^2$ *such that,*

$$U^f_T(x, y) = T\mu(f) + \pi^f(x) - \pi^f(y) + \theta^f_T(x, y)$$

for all x, y, T *as above. Furthermore, for every* $T > 0$ *and every* $x \in R^2$ *there is* $y \in R^2$ *such that* $\theta^f_T(x, y) = 0$.

Let $f \in \bar{\mathfrak{M}}$. For $D = (T_1, T_2)$ and $v \in W^{2,1}(D)$ put,

(2.15)

$$\Theta^f(D; v) = \theta^f_{T_2 - T_1}(X_v(T_1), X_v(T_2)),$$
$$\Gamma^f(D; v) = I^f(T_1, T_2, v) - (T_2 - T_1)\mu(f) + \pi^f(X_v(T_2)) - \pi^f(X_v(T_1)).$$

From (2.15) and Proposition 2.1 it follows that

$$\Gamma^f(D; v) \geq \Theta^f(D; v) \geq 0. \tag{2.15a}$$

Clearly, if v is a minimizer of $(P^{x,y}_D)$, $x = X_v(T_1)$, $y = X_v(T_2)$ then $\Gamma^f(D; v) = \Theta^f(D; v)$. However $\Gamma^f(D; v)$ may be positive even in this case. Note that, in the present notation, a function $v \in W^{2,1}(D)$ is an (f,δ)–approximate minimizer in D (see (2.3b)), iff

$$\Gamma^f(D;v) - \Theta^f(D;v) \leq \delta.$$

In this context we introduce the following additional terminology: Let v be a minimizer of (P_∞). We shall say that v is (f)*–perfect* if

$$\text{(2.15}b\text{)} \qquad \Gamma^f(D,u) = 0 \text{ for every bounded interval } D.$$

If $v \in W^{2,1}_{loc}(R) \cap W^{1,\infty}(R)$ and v satisfies (2.15b), then v is a minimizer of (P_∞) and hence it is (f)–perfect. This is an immediate consequence of the definition of Γ^f and the fact that π^f is continuous.

Obviously every (f)–perfect minimizer is c-optimal. Using this fact, it can be shown that every (f)–perfect minimizer is (f)–good (see Proposition 2.3 below). Clearly the converse does not hold, but a partial converse is provided by the following result.

LEMMA 2.4. –*Let* $f \in \bar{\mathfrak{M}}$ *and suppose that* v *is* (f)*-good. Then, for every* $\delta > 0$ *there exists* $T(\delta)$ *such that, for* $D = (T_1, T_2)$,

$$\text{(2.16)} \qquad \Gamma^f(D;v) \leq \delta, \quad \forall T_1 \geq T(\delta).$$

In particular every periodic minimizer of (P_∞) *is* (f)*–perfect.*

Proof. – Since π^f is continuous, if v is an (f)-good function then $\Gamma^f(D;v)$ is bounded. Furthermore, since $D \longrightarrow \Gamma^f(D;v)$ is an additive, non-negative set function, it follows that for every $\delta > 0$ there exists $T(\delta) > 0$ such that (2.16) holds. The last statement of the lemma is a consequence of this inequality. □

The next result shows that every (f)-good function generates a family of perfect minimizers.

LEMMA 2.5. –*Let* $f \in \bar{\mathfrak{M}}$ *and let* $v \in W^{2,1}_{loc}(0,\infty)$ *be an* (f)*-good function. Then, given* $\xi \in \Omega(v)$, *there exists* $u \in W^{2,1}_{loc}(R^1)$ *such that*

$$(*) \qquad \{(u,u')(t) : t \in R^1\} \subset \Omega(v) \text{ and } (u,u')(0) = \xi,$$

and u *is an* (f)*–perfect minimizer.*

Proof. – Let u be constructed as in Lemma 2.3. Then, u satisfies (*) and, in the notation of that lemma,

$$\Gamma^f(D,u) \leq \liminf_{k\longrightarrow\infty} \Gamma^f(D + t_{n_k}, v).$$

This follows from the growth conditions on f (see (2.2)), and the fact that $v_{n_k} \to u$ weakly in $W^{2,\gamma}(D)$. However, by Lemma 2.4, $\Gamma^f(D+\tau, v) \to 0$ as $\tau \to \infty$. Therefore u satisfies (2.15b) and consequently, since $u \in W^{1,\infty}(R)$, it follows that it is (f)–perfect. $\square$

Another useful ingredient in our proof is the following result for which we refer the reader to [10] (proof of Proposition 4.4) and [16].

PROPOSITION 2.2. – *Let $f \in \bar{\mathfrak{M}}$. For every $M_1, M_2, c > 0$ there exists a positive number $A = A_f(M_1, M_2, c)$ such that the following statement holds for every $T \geq c$. If*

$$v \in W^{2,1}(0,T), \quad |X_v(0)| \leq M_1, \quad |X_v(T)| \leq M_1,$$

and if v is an (f, M_2)–approximate minimizer in $(0,T)$ (see (2.3b)) then,

$$|X_v(t)| \leq A, \quad \forall t \in [0,T].$$

Furthermore, for every $g \in \bar{\mathfrak{M}}$ there is a neighbourhood $\mathfrak{U}_g$ in $\bar{\mathfrak{M}}$ such that $A_f(M_1, M_2, c)$ can be chosen uniformly with respect to f in $\mathfrak{U}_g$.

We also need the following lemma.

LEMMA 2.6. –*Let $f \in \bar{\mathfrak{M}}$. Then, for every compact set E there exists a constant $M = M(E) > 0$ such that, for every $T \geq 1$,*

$$U_T^f(x,y) \leq T\mu(f) + M, \quad \forall x, y \in E. \tag{2.17}$$

Proof. – Let w be a periodic minimizer of (P_∞) with period $T_w > 0$. Clearly, for every $A > 0$,

$$\sup\{U_T^f(x,y) : \ x, y \in E, \ 1 \leq T \leq A\} < \infty.$$

Therefore, it is sufficient to show that there exists M such that (2.17) holds for $T \geq 4T_w$. Put $D = (0,T)$. Let τ be the largest integer which does not exceed T/T_w and put $l = 2^{-1}(T - (\tau - 1)T_w)$. Let $D' = (l, T - l)$ so that $|D'| = (\tau - 1)T_w$.

Given $x, y \in E$ let v_1 (resp. v_2) be a minimizer of problem $(P_l^{x,z})$ with $z = (w, w')(l)$ (resp. $(P_l^{\zeta,y})$ with $\zeta = (w, w')(T - l)$). Let $v \in W^{2,\gamma}(D)$ be the function given by,

$$v(t) = \begin{cases} v_1(t), & t \in (0,l) \\ w(t), & t \in D' \\ v_2(t - T + l), & t \in (T - l, T) \end{cases}$$

Since w and w' are bounded and $T_w/2 \leq l \leq T_w$ it follows that there exists a constant M_1 (independent of x, y, T) such that,

$$U_l^f(x, z) = I^f(0, l, v_1) \leq M_1 \text{ and } U_l^f(\zeta, y) = I^f(0, l, v_2) \leq M_1.$$

Since $I^f(l, T - l, w) = (T - 2l)\mu(f)$ it follows that,

$$U_T^f(x, y) \leq I^f(0, T, v) \leq (T - 2l)\mu(f) + 2M_1,$$

which implies (2.17). □

Using these results we can establish the following relation between approximate minimizers and (f)–good functions.

PROPOSITION 2.3. – *Let $f \in \mathfrak{M}$ and $M > 0$. Denote by $\mathbf{A}(f, M)$ the family of minimizers v of (P_∞) such that v is an (f, M)–approximate minimizer in every bounded interval $D \subset R_+$ such that $|D| \geq 1$. Then*

$$v \in \mathbf{A}(f, M) \Longrightarrow v \text{ is } (f)\text{–good}.$$

In particular, every c-optimal function is (f)-good. Furthermore, the family of periodic minimizers is uniformly bounded in the norm $\|v\|_{(1)} := \sup_{R_+} |X_v|$.

Proof. – Let v be a minimizer of (P_∞). Then, for every $T > 0$, $\lim_{T' \to \infty} \frac{1}{T' - T} I^f(T, T', v) = \mu(f)$. Hence there exists $T_0 > T$ such that

$$I^f(T_0, T_0 + 1, v) \leq M := \mu(f) + 1.$$

Consequently there exists a monotone sequence $\{T_n\}$ tending to $+\infty$ such that,

$$I^f(T_n, T_n + 1, v) \leq M, \quad n = 1, 2, \ldots.$$

By Lemma 2.2 there exists a constant M_1 (independent of v) such that,

$$(*) \qquad \sup\{|X_v(t)| : T_n \leq t \leq T_n + 1\} \leq M_1, \quad n = 1, 2, \ldots.$$

Now suppose that, $v \in \mathbf{A}(f, M)$. Then inequality (*) and Proposition 2.2 imply that there exists a constant M_2 (independent of v) such that,

$$(**) \qquad \sup\{|X_v(t)| : T_1 \leq t\} \leq M_2, \quad n = 1, 2, \ldots.$$

Thus $|X_v| \in L^\infty(R_+)$. (Note that in general T_1 depends on v so that $\sup_{R_+} |X_v|$ may not be uniformly bounded relative to $v \in \mathbf{A}(f, M)$.)

Further, inequality (2.3b), the boundedness of X_v and Lemma 2.6 imply that,

$$I^f(0,T,v) \le U_T^f(X_v(0),X_v(T)) + M \le T\mu(f) + M + M', \quad \forall T \ge 1,$$

where $M' = M(E)$ is as in (2.17) with $E = cl\{X_v(t) : t \in R_+\}$. Thus $\eta^f(\cdot, v)$ is bounded on $(1,\infty)$ and hence on R_+, i.e. v is (f)-good.

If v is a c-optimal function then, by definition, X_v is bounded and therefore, by the previous part of the proof, v is (f)-good.

Finally, if v is a periodic minimizer then inequality (**) implies that $\sup_{R_+} |X_v| \le M_2$, which proves the last assertion of the proposition. □

The next lemma will be needed in order to establish the stability of (ATP).

LEMMA 2.7. –*Let $g \in \bar{\mathfrak{M}}$ and let $D = (0,T)$. For $M > 0$ put,*

$$\mathfrak{V}_M(D) = \{v \in W^{2,1}(D) : \int_0^T (|v''|^\gamma + |v|^\alpha)\, dt \le M\}.$$

Then for every $\epsilon, M > 0$ there exists a neighbourhood $\mathfrak{N}_g$ of g in $\bar{\mathfrak{M}}$ such that, for every $f \in \mathfrak{N}_g$,

$$|I^f(0,T,v) - I^g(0,T,v)| < \epsilon, \quad \forall v \in \mathfrak{V}_M(D), \tag{2.18}$$

and

$$x,y \in R^2,\ |x|,|y| < M \Longrightarrow |U_T^f(x,y) - U_T^g(x,y)| < \epsilon. \tag{2.19}$$

The neighborhood $\mathfrak{N}_g$ can be chosen independently of T for T in compact sets of $(0,\infty)$.

Proof. – Put $M_0(T) = \sup\{||v||_{C^1[0,T]} : v \in \mathfrak{V}_M(D)\}$. By Lemma 2.2, if $T \in (T_1,T_2)$, with $0 < T_1 < T_2 < \infty$, then $M_1 = \sup_{T\in[T_1,T_2]} M_0(T) < \infty$. For every $N,\delta > 0$ let $B_g(N,\delta) = \{f \in \bar{\mathfrak{M}} : (f,g) \in E(N,\delta)\}$ (see (2.1)). Now, given $\delta > 0$ choose $N > 2M_1$ sufficiently large so that, for every $f \in B_g(N,\delta)$,

$$x \in R^3,\ |x_1|,|x_2| \le M_1,\ |x_3| \ge N \Longrightarrow g(x) > 0, \qquad 1 - 2\delta < f(x)/g(x) < 1 + 2\delta. \tag{2.20}$$

Assume that $f \in B_g(N,\delta)$ and $v \in \mathfrak{V}_M(D)$. Then,

$$|I^f(0,T,v) - I^g(0,T,v)| \le \int_{E(v,N)} |(f-g)(v,v,'v'')|dt + \int_{E'(v,N)} |(f-g)(v,v,'v'')|dt, \tag{2.21}$$

where $E(v,N) = \{t \in D: |v''(t)| \leq N\}$ and $E'(v,N) = D \setminus E(v,N)$. The first term on the right is bounded by $T\delta$ and the second by $2\delta \int_D |g(v,v',v'')|$. The last integral is uniformly bounded for $v \in \mathfrak{V}_M(D)$. This follows from the inequality,

$$|f|(x) \leq M_f(|x_1| + |x_2|)(1 + |x_3|^\gamma), \quad \forall x \in R^3,$$

which, by (2.2), holds for $f \in \mathfrak{M}$ and remains valid also for $f \in \bar{\mathfrak{M}}$. Therefore, choosing δ sufficiently small so that the right hand side of (2.21) is smaller than ϵ and then choosing N sufficiently large as indicated before, we obtain (2.18).

Finally, (2.19) is a consequence of (2.18) and the fact that (by Proposition 2.2) the family of minimizers of $(P_D^{x,y})$, $|x|, |y| \leq M$ is bounded by a bound independent of f for f in a neighbourhood of g. □

The next lemma plays an important role in the proof of Theorem 2.1 and the results following it.

LEMMA 2.8. *–Let $f \in \mathfrak{M}$ and let $D = (T_1, T_2)$ be a bounded interval. Suppose that $w_1, w_2 \in W^{2,1}(D)$ and that $\Gamma^f(D, w_1) = \Gamma^f(D, w_2) = 0$. If there exists $\tau \in (T_1, T_2)$ such that $(w_1, w_1')(\tau) = (w_2, w_2')(\tau)$ then $w_1 = w_2$ everywhere in D.*

Proof. – Put

$$u(t) = w_1(t), \; t \in [T_1, \tau], \quad u(t) = w_2(t), \; t \in (\tau, T_2].$$

Evidently $u \in W^{2,1}(D)$ and $\Gamma^f(D, u) = 0$. Since u, w_1, w_2 satisfy the Euler-Lagrange equation we conclude that $u = w_1, w_2$ everywhere in D. □

To complete the proof of Theorem 2.1 we need two more auxilliary results, stated below as Lemmas A and B. The proofs of these lemmas, which are more technical than the previous ones, will be given in Appendixes A and B respectively. In both of these lemmas we consider an integrand f possessing (ATP) and study the relation between a fixed periodic minimizer of (P_∞), say w, and approximate minimizers of $(P_{(0,T)})$. In Lemma A it is shown that (given $\epsilon, M > 0$) there exists $\ell > T_w =$ (period of w) such that every (f, M)–approximate minimizer in $(0,T)$, $T > \ell$, whose endvalues are bounded by M, is intermittently close to w in the following sense. Every interval $D \subset (0,T)$, $|D| = \ell$ contains a subinterval D^* of length T_w such that $\sup_{D^*} |X_v - X_{w^*}| < \epsilon$ where w^* is a translate of w. In Lemma B it is shown that if in addition to the above, the endvalues of v are sufficiently close to $\Omega(w)$ (=the limit set of w), and if M is sufficiently small, then the relation described above holds in *every subinterval* D^* of length T_w. (In general the translate w^* will depend on D^*.) Finally, these properties persist in a neighborhood of the given integrand. The precise formulation follows.

LEMMA A. –*Suppose that* $g \in \mathfrak{M}$ *possesses (ATP). Let* w *be a periodic minimizer of* (P_∞) *with integrand* g *and let* $T_w > 0$ *be a period of* w. *Given* $M_0, M_1, \epsilon > 0$ *there exists an integer* $q_1 \geq 1$ *and a neighbourhood* $\mathfrak{U}$ *of* g *in* $\mathfrak{M}$ *such that the following statement holds.*

Let $f \in \mathfrak{U}$ *and* $T \geq q_1 T_w$. *If* $v \in W^{2,1}(0,T)$ *satisfies*

(2.24)
$$|X_v(t)| \leq M_0 \ \textit{for}\ t = 0, T, \qquad I^f(0,T,v) \leq U_T^f(X_v(0), X_v(T)) + M_1,$$

then, for every $\tau \in [0, T - q_1 T_w]$ *there exist* $\xi \in [0, T_w)$ *and* $s \in [\tau, \tau + (q_1 - 1)T_w]$ *such that*

$$|X_v(s+t) - X_w(\xi + t)| \leq \epsilon, \quad t \in [0, T_w]. \tag{2.25}$$

LEMMA B. –*Let* g, w, T_w *be as in Lemma A. Given* $\epsilon > 0$ *there exist* $\delta \in (0,1)$ *and* $Q_0 > T_w$, *such that for every* $Q > Q_0$ *there exists a neighbourhood* $\mathfrak{U}_Q$ *of* g *in* $\mathfrak{M}$ *such that the following statement holds.*

Let $f \in \mathfrak{U}_Q$ *and* $\tau \in [Q_0, Q]$. *If* $v \in W^{2,1}(0,\tau)$ *satisfies,*

(2.26)
$$d(X_v(t), \Omega(w)) \leq \delta \ \textit{for}\ t = 0, \tau, \qquad I^f(0,\tau,v) \leq U_\tau^f(X_v(0), X_v(\tau)) + \delta,$$

then, for every $s \in [0, \tau - T_w]$ *there exists* $\xi \in [0, T_w)$ *such that (2.25) holds.*

Proof of Theorem 2.1. – It is sufficient to prove the theorem for all sufficiently large M. Therefore we may assume that

$$M > 2\|X_w\|_{L^\infty(R)} + 8.$$

By Proposition 2.2 there exist a neighborhood of g in $\mathfrak{M}$, say $\mathfrak{N}(M)$, and a number $S > M + 1$ such that for each $f \in \mathfrak{N}(M)$ and each $T \geq \inf\{1, T_w\}$:

$$v \in W^{2,1}(0,T), \ |X_v(0)|, |X_v(T)| \leq M + 1, \ I^f(0,T,v) \leq U_T^f(X_v(0), X_v(T)) + 4 \tag{2.27}$$

implies that,

$$|X_v(t)| \leq S, \ t \in [0,T]. \tag{2.28}$$

Given ϵ as in the theorem, there exist $\delta \in (0,1)$ and $Q_0 > T_w$ such that the statement of Lemma B holds.

By Lemma A there exist a positive integer q_1 and a neighborhood of g in $\bar{\mathfrak{M}}$, say $\mathfrak{N}(S,\delta)$, such that for each f in this neighborhood and each $T \geq q_1 T_w$:

$$(2.29)\qquad v \in W^{2,1}(0,T),\ |X_v(t)| \leq S+1 \text{ for } t=0,T,$$
$$I^f(0,T,v) \leq U_T^f(X_v(0),X_v(T))+4$$

implies that for every $\tau \in [0, T - q_1 T_w]$ there exist $\xi \in [0, T_w)$ and $s \in [\tau, \tau + (q_1 - 1)T_w]$ such that

$$(2.30)\qquad |X_v(s+t) - X_w(\xi+t)| \leq \delta, \quad t \in [0,T_w].$$

Choose

$$(2.31)\qquad Q_1 > 8(Q_0 + q_1 T_w).$$

By Lemma B there exists a neighborhood of g in $\bar{\mathfrak{M}}$, say $\mathfrak{N}_\epsilon$ such that for each $f \in \mathfrak{N}_\epsilon$ and each $\tau \in [Q_0, Q_1]$:

If $v \in W^{2,1}(0,\tau)$ satisfies (2.26) then for every $s \in [0, \tau - T_w]$ there is $\xi \in [0, T_w)$ such that,

$$(2.32)\qquad |X_v(s+t) - X_w(\xi+t)| \leq \epsilon, \quad t \in [0,T_w].$$

We claim that the statement of the theorem holds with $\mathfrak{U}_g = \mathfrak{N}(M) \cap \mathfrak{N}(S,\delta) \cap \mathfrak{N}_\epsilon$, with δ as above and $\ell = 2q_1 T_w + 4(Q_1 + 4)$.

Assume that $f \in \mathfrak{U}_g$, $T \geq 2\ell + T_w$ and v satisfies (2.13). Then v satisfies (2.28) and consequently (2.29). Therefore, for each $\tau \in [0, T - q_1 T_w]$ there exist $\xi \in [0, T_w)$ and $s \in [\tau, \tau + (q_1 - 1)T_w]$ such that (2.30) holds. Let m be the largest integer such that $(m+1)q_1 T_w \leq T$. Put $\tau_k = kq_1 T_w$, $k = 0, \cdots, m+1$. Then, for $k = 0, \cdots, m$, τ_k is in $[0, T - q_1 T_w]$ and consequently there exists $\xi_k \in [0, T_w)$ and $s_k \in [\tau_k, \tau_k + (q_1 - 1)T_w] \subset [\tau_k, \tau_{k+1})$ such that,

$$(2.33)\qquad |X_v(s_k+t) - X_w(\xi_k+t)| \leq \delta, \quad t \in [0,T_w],\ k = 0, \cdots, m.$$

This implies,

$$(2.34)\qquad d(X_v(s_k), \Omega(w)) \leq \delta,\ k = 0, \cdots, m.$$

Let ν_0 be the smallest integer such that $\nu_0 \geq Q_0/(q_1 T_w)$ and let ν_1 be the largest integer such that $\nu_1 \leq Q_1/(q_1 T_w)$. Since $Q_1 - Q_0 > 8q_1 T_w$ we have $\nu_1 - \nu_0 > 6$. an interval Put $D_{j,k} := [s_j, s_k]$ where $0 \le j < k \le m$ and observe that if $\nu_0 + 1 < k - j \le \nu_1 - 1$ then,

$$Q_0 \le \nu_0 q_1 T_w < \tau_k - \tau_{j+1} \le |D_{j,k}| \le \tau_{k+1} - \tau_j \le \nu_1 q_1 T_w \le Q_1.$$

Further observe that the last inequality in (2.13) implies that,

$$I^f(s_j, s_k, v) \le U_T^f(X_v(s_j), X_v(s_k)) + \delta. \tag{2.35}$$

Indeed this holds for every subinterval of $[0, T]$ because,

$$I^f(a, b, v) \text{ is additive and } U_{b-a}^f(X_v(a), X_v(b)) \text{ is subadditive}$$

on finite partitions of $(0, T)$ consisting of subintervals and because

$$I^f(a, b, v) \ge U_{b-a}^f(X_v(a), X_v(b)).$$

Therefore we may apply Lemma B to the function v restricted to $D_{j,k}$ where $\nu_0 + 1 < k - j \le \nu_1 - 1$, and conclude that for every $s \in [s_j, s_k - T_w]$ there exists $\xi \in [0, T_w)$ such that (2.32) holds. Finally this implies that for every $s \in [s_0, s_m - T_w]$ there exists $\xi \in [0, T_w)$ such that (2.32) holds. Since $s_0 \le q_1 T_w$ and $T - s_m \ge 2q_1 T_w$, we find that the theorem holds with ℓ as above. □

The following result is an immediate consequence of Theorem 2.1, Proposition 2.2 and Lemma 2.1. Roughly it states that if f has (ATP) and w is a periodic minimizer of (P_∞) then every (f)–good function is eventually 'close' to w.

THEOREM 2.2. – *Assume that $g \in \mathfrak{M}$ has (ATP) and $w \in W_{loc}^{2,1}(R^1)$ is a periodic (g)-minimizer with a period $T_w > 0$. Then, for every $\epsilon > 0$, there exists a neighborhood $\mathfrak{U}$ of g in $\bar{\mathfrak{M}}$ such that for each $f \in \mathfrak{U}$:*

If v is an (f)-good function, there exists t_ϵ (depending on ϵ, v) such that, for every $s \ge t_\epsilon$, there exists $\xi \in [0, T_w)$ such that,

$$|X_v(s + t) - X_w(\xi + t)| \le \epsilon, \quad t \in [0, T_w].$$

COROLLARY 2.1. – *If $f \in \mathfrak{M}$ has (ATP) then problem (P_∞) possesses a unique (up to translation) periodic minimizer.*

Finally we observe that Theorem 1.1 can be easily deduced from Theorem 2.1. Suppose that G satisfies (1.1) and (1.2) and let $\mathfrak{L}(\alpha, b_2, b_3)$ and $\mathfrak{L}_G(\alpha, b_2, b_3)$ be defined as in (1.3),(1.4). Clearly, for G and $\mathfrak{L}$ as in (1.1)–(1.4) and an appropriate choice of a,

$$\mathfrak{L}_G(\alpha, b_2, b_3) \subset \mathfrak{M}(\alpha, \beta, \gamma, a),$$

and the operator

$$\phi \to F_\phi \in \mathfrak{M}(\alpha, \beta, \gamma, a), \quad \phi \in \mathfrak{L}(\alpha, b_2, b_3)$$

is continuous. Therefore Theorem 2.1 implies Theorem 1.1.

3. PROOF OF THEOREMS 1.2, 1.3

First we establish a more general version of Theorem 1.2:

THEOREM 3.1. – *Let $f \in \mathfrak{M}$. Then there exists a nonnegative function $\phi \in C^\infty(R^1)$ such that $\phi(t) > 0$ for all large $|t|$, $\phi^{(m)}$ is bounded for every $m \geq 0$, and the following statement holds.*
Denote

$$f_\rho(x_1, x_2, x_3) = f(x_1, x_2, x_3) + \rho\phi(x_1), \quad (\rho, x_1, x_2, x_3) \in R^4. \tag{3.1}$$

Then for each $\rho \in (0, 1)$, $f_\rho \in \mathfrak{M}$, $\mu(f_\rho) = \mu(f)$ and problem (P_∞) with $f = f_\rho$ possesses a unique (up to translation) periodic minimizer.

We start with a brief description of the strategy of the proof, which will be presented through several lemmas. Given $f \in \mathfrak{M}$, denote by $\mathfrak{C}(f)$ the set of all *periodic* (f)-minimizers of (P_∞). If $w \in \mathfrak{C}(f)$ is not a constant, we denote by $\tau(w)$ the minimal period of w. In the first lemma we show that every non-constant periodic minimizer w has precisely two extremal points in each interval $[a, a + \tau(w))$ and is strictly monotone between two consecutive extremal points. Using this fact we show that if $\mu(f) < \inf\{f(t, 0, 0) : t \in R^1\}$, then the set $\{\tau(w) : w \in \mathfrak{C}(f)\}$ is bounded. Next we show that there exists $w^* \in \mathfrak{C}(f)$ whose range $D_{w^*}$is minimal in the sense that it is either disjoint from or stricty contained inthe range of any other element $w \in \mathfrak{C}(f)$, unless w is a translate of w^*. Finally we observe that if there exists $\phi \in C^\infty(R)$ which vanishes on D_{w^*} and is positive everywhere else, then the assertion of Theorem 3.1 holds. Since D_{w^*} is a closed bounded interval, such a function is easily constructed.

LEMMA 3.1. –*Assume that $w \in \mathfrak{C}(f)$ and w is not constant. Applying an appropriate translation we may assume that $w(0) = \min_{R^1} w$. Then there exists $\bar{\tau} \in (0, \tau(w))$ such that w is strictly increasing in $[0, \bar{\tau}]$ and strictly decreasing in $[\bar{\tau}, \tau(w)]$.*

Remark. – In the special case $f(v, v', v'') = |v''|^2 - q|v'|^2 - (v^2 - 1)^2$, this lemma was independently established by Mizel, Peletier, Troy [14]. Their proof uses the special symmetries of the integrand.

Proof. – Let $E = \{\tau \in [0, \infty) : w'(\tau) = 0\}$. We claim that $E \cap [0, \tau(w)]$ is a finite set. Otherwise there exists a sequence of positive numbers $\{t_n\}$ converging to a point $t^* \in [0, \tau(w)]$, such that $w'(t_n) = 0,\ n = 1, 2, \ldots$. By the mean value theorem, this implies that for $m = 1, \cdots, 4$, there exists a sequence $\{t_{m,n}\}_{n=1}^\infty$ converging to t^*, such that $w^{(m)}(t_{m,n}) = 0$ for all n. Therefore $w^{(m)}(t^*) = 0,\ m = 1, \cdots, 4$. Since w satisfies the Euler–Lagrange equation corresponding to our variational problem this implies that w is a constant, contrary to our assumption. (Note that, for $f \in \mathfrak{M}$ the Euler–Lagrange equation is a regular, fourth order equation.)

Put,

$$\tau_1 = \sup\{\tau \in E \cap [0, \tau(w)] : w'(t) \geq 0,\ \forall t \in [0, \tau]\}.$$

Clearly $\tau_1 \in (0, \tau(w))$ and w is strictly increasing in $(0, \tau_1)$. Similarly we define

$$\tau_2 = \sup\{\tau \in E \cap (\tau_1, \tau(w)] : w'(t) \leq 0,\ \forall t \in [\tau_1, \tau]\}.$$

Proceeding in this manner we obtain a strictly increasing sequence $\{\tau_j : j = 0, \cdots, k\}$ such that $\tau_0 = 0,\ \tau_k = \tau(w),\ w'(\tau_j) = 0,\ j = 0, \cdots, k$ and w' does not change sign in each of the intervals $D_j = [\tau_j, \tau_{j+1}],\ j = 0, \cdots, k-1$. More precisely, w is strictly increasing in D_j, if j is even, and strictly decreasing in D_j, if j is odd. Obviously k is even.

Let D_j^* denote the interval $[w(\tau_j), w(\tau_{j+1})]$ (resp. $[w(\tau_{j+1}), w(\tau_j)]$) when j is even (resp. odd).

Evidently, for each integer j, $0 \leq j < k$ the function $t \to w(t)\ t \in D_j$ is invertible. Composing the inverse function thus obtained with the function $t \to w'(t),\ t \in D_j$, we obtain a function $h_j \in C(D_j^*)$ such that $w'(t) = h_j(w(t))$ for every $t \in D_j$.

Now we claim that for $i < j$, $w(\tau_j) \neq w(\tau_i)$, unless $i = 0$ and $j = k$. Suppose that there exists $(i, j) \neq (0, k)$ such that $0 \leq i < j \leq k$ and $w(\tau_j) = w(\tau_i)$. Then let u be the periodic function, with period $\tau_j - \tau_i$, such that $u(t) = w(t),\ t \in [\tau_i, \tau_j]$. Recall that $w'(\tau_m) = 0$ for $m = 0, \cdots, k$. Hence $u \in W^{2,1}_{loc}(R^1)$. Furthermore, by Lemma 2.4, $\Gamma^f(D; u) = \Gamma^f(D; w) = 0$ in every bounded interval D. (Recall that the function $D \to \Gamma^f(D; v)$ is additive.) Therefore by Lemma 2.8, $u \equiv w$, which contradicts the assumption that the period of u is strictly smaller than $\tau(w)$.

Next, we claim that, if $k > 2$ then $D_j^* \subset D_{j-1}^*$ for $j = 1, \cdots, k$. We verify this claim by induction. For $j = 1$, we have $w(0) \leq w(\tau_2) < w(\tau_1)$. (Recall that $w(0)$ is the minimum of w.) Furthermore, since $k > 2$, the previous argument yields $w(0) < w(\tau_2) < w(\tau_1)$. Now suppose that the claim holds for $j = 1, \cdots, m-1$. To fix ideas assume that m is even. Then we know that w is strictly increasing in D_m so that $w(\tau_{m+1}) > w(\tau_m)$. We must show that $w(\tau_{m+1}) < w(\tau_{m-1})$. Suppose the contrary. Since, by assumption, $D_{m-1}^* \subset D_{m-2}^*$ it follows that,

$$w(\tau_{m-2}) < w(\tau_m) < w(\tau_{m-1}) < w(\tau_{m+1}).$$

Therefore the functions h_{m-2} and h_m defined in D^*_{m-2} and D^*_m respectively must intersect somewhere in $[w(\tau_m), w(\tau_{m-1})]$. (Recall that both functions are non-negative in their intervals of definition and vanish at the end points of these intervals.) This means that there exist $s_1 \in D_{m-2}$ and $s_2 \in D_m$ such that $(w, w')(s_1) = (w, w')(s_2)$. However, applying once again Lemma 2.8, the argument used before shows that this is impossible and proves our claim.

Combining the last two claims we conclude that, if $k > 2$, the inclusion $D^*_j \subset D^*_{j-1}$, $j = 1, \cdots, k$ is strict. But this is impossible because $w(\tau_0) = w(\tau_k)$. □

COROLLARY 3.1. – *Suppose that $f \in \mathfrak{M}$ and that $f(x_1, x_2, x_3) = f(x_1, -x_2, x_3)$, for every $x \in R^3$. Let w and $\bar{\tau}$ be as in the statement of the lemma. Then $w' > 0$ in $(0, \bar{\tau})$ and $w' < 0$ in $(\tau, \tau(w))$. Furthermore, $\bar{\tau} = \tau(w)/2$ and w is even.*

Proof. – Since f is even in the second argument, it follows that the function $\bar{w}$ given by $\bar{w}(t) = w(-t)$ is also a periodic minimizer. Recall that we assume that $w(0) = \min_R w$ so that $w'(0) = 0$. Consequently, $X_w(0) = X_{\bar{w}}(0)$. Hence, by Lemma 2.8, $w \equiv \bar{w}$ i.e. w is even. Further this implies that $w(t) = w(\tau(w) - t)$ for every real t. Now suppose that $s \in (0, \tau(w))$ and $w'(s) = 0$. Then $X_w(s) = X_w(\tau(w) - s)$. Using again Lemma 2.8 we deduce that $w(t) = w(t + 2s - \tau(w))$, for every $t \in R^1$. Thus $2s - \tau(w)$ is a period of w and therefore it must be equal to $k\tau(w)$ for some integer k. Since $s \in (0, \tau(w))$ it follows that $k = 0$. This proves our assertion. □

LEMMA 3.2. –*Assume that $f \in \mathfrak{M}$ satisfies the condition,*

$$\mu(f) < \inf\{f(t, 0, 0) : t \in R^1\}. \tag{3.2}$$

Then no element of $\mathfrak{E}(f)$ is constant and

$$\sup\{\tau(w) : w \in \mathfrak{E}(f)\} < \infty. \tag{3.3}$$

Remark. – This result was established by Marcus [13] in the special case $f(v, v', v'') = |v''|^2 - \mu|v'|^2 + \psi(v)$, for a large class of potentials ψ.

Proof. Step 1 – Suppose that $\{T_i\}_{i=0}^{\infty}$ is a sequence of positive numbers tending to infinity, and that $\{w_i : w_i \in W^{2,1}(0, T_i)\}$ is a sequence of functions such that,

(3.4

(i) $I^f(0, T_i, w_i) = T_i\mu(f) + \pi^f(X_{w_i}(0)) - \pi^f(X_{w_i}(T_i)),\ i = 0, 1, 2, \ldots.$

(ii) $\{|X_{w_i}(0)|\}_{i=0}^{\infty}$ and $\{|X_{w_i}(T_i)|\}_{i=0}^{\infty}$, are bounded,

(iii) $w_i'(t) \geq 0, \quad t \in (0, T_i),\ i = 0, 1, 2, \ldots.$

We claim that,

$$\mu(f) = \inf\{f(z, 0, 0) : z \in R^1\}. \tag{3.5}$$

The same conclusion holds if in (3.4), the condition "$w_i'(t) \geq 0$" is replaced by the condition "$w_i'(t) \leq 0$".

Assumption (3.4)(i) implies that $I^f(0, T_i, w_i) = U_{T_i}^f(X_{w_i}(0), X_{w_i}(T_i))$ and consequently, Proposition 2.2 and assumption (3.4)(ii) imply that there exists $M > 0$ such that,

$$\sup_{t \in [0, T_i]} |X_{w_i}(t)| \leq M, \quad i = 0, 1, 2, \ldots, \tag{3.6}$$

and

$$||w_i||_{W^{2,\gamma}(T, T+1)} \leq M, \quad \forall T \in (0, T_i - 1), \ i = 0, 1, 2, \ldots. \tag{3.7}$$

Therefore there exits a subsequence (which we shall continue to denote by $\{w_i\}$) and a function $v \in W_{loc}^{2,\gamma}(0, \infty)$ such that, for every $T \geq 1$,

$$w_i \to v \text{ weakly in } W^{2,\gamma}(0, T) \text{ as } i \to \infty.$$

By the lower semicontinuity of integral functionals [3] and Proposition 2.1,

$$I^f(0, T, v) = T\mu(f) + \pi^f(X_v(0)) - \pi^f(X_v(T)), \quad \forall T \geq 1.$$

By (3.7),

$$||v||_{W^{2,\gamma}(T, T+1)} \leq M, \quad \forall T \in (1, \infty) \tag{3.8}$$

and by (3.4)(iii), $v' \geq 0$ in $(0, \infty)$. Consequently $v(t)$ possesses a finite limit, say d_0, and $v'(t) \to 0$ as $t \to \infty$.

Let v_j, $j = 0, 1, 2, \ldots$ be the function defined in $[0, 1]$ by $v_j(t) = v(j+t)$. By (3.8) the sequence $\{v_j\}$ is bounded in $W^{2,\gamma}(0, 1)$ and therefore a subsequence will converge weakly in this space to a function u. Clearly u is the constant function $u \equiv d_0$. Since $I^f(0, 1, v_j) = \mu(f) + \pi^f(X_v(j)) - \pi^f(X_v(j+1))$ and $X_v(j)$ converges, we conclude (by the lower semicontinuity of integral functionals) that $I^f(0, 1, u) = \mu(f)$. This implies (3.5). It is obvious that the conclusion remains valid if the sign in (3.4)(iii) is inverted.

Step 2. – Assume that the assertion of the lemma is not valid. Then there exists a sequence $\{w_i\}_{i=1}^{\infty}$ in $\mathfrak{E}(f)$ such that

$$\tau(w_i) \to \infty \text{ as } i \to \infty. \tag{3.9}$$

Without loss of generality we may assume that $w_i(0) = \min_R w_i, i = 1, 2, \ldots$.

By Lemma 3.1, for each integer $i \geq 1$ there exists a number $\bar{\tau}_i \in (0, \tau(w_i))$ such that w_i is strictly increasing in $[0, \bar{\tau}_i]$ and strictly decreasing in $[\bar{\tau}_i, \tau(w_i)]$. In view of (3.9) either $\bar{\tau}_i \to \infty$ or $\tau(w_i) - \bar{\tau}_i \to \infty$ or both. In the first case put $T_i = \bar{\tau}_i$ and $v_i = w_i|_{[0, \bar{\tau}_i]}$; in the second case put $T_i = \tau(w_i) - \bar{\tau}_i$ and define v_i in $[0, T_i]$ by, $v_i(t) = w_i(t + \bar{\tau}_i)$ for $i = 1, 2, \cdots$. Then the sequence $\{T_i\}$ tends to infinity and the sequence $\{v_i\}$ satisfies conditions (i), (iii) of Step 1, possibly with a negative sign in (iii). Furthermore, by Proposition 2.3 there exists a number $S > 0$ such that

$$\sup\{|X_v(t)| : t \in R^1, v \in \mathfrak{E}(f)\} \leq S. \tag{3.10}$$

Thus the sequence $\{v_i\}$ satisfies also condition (ii).

Consequently, the statement established in Step 1 implies that (3.5) holds, which contradicts the assumptions of the lemma. □

LEMMA 3.3. *–Let $f \in \mathfrak{M}$. If $w_1, w_2 \in \mathfrak{E}(f)$ then the sets*

$$D_i := \{w_i(t) : t \in R\}, \quad i = 1, 2$$

are either disjoint or one of them is contained in the other. Furthermore if, say, $D_1 \subseteq D_2$ then either w_1 is a translate of w_2 or D_1 is contained in the interior of D_2.

Proof. – We may assume that $w_i(0) = \min_R w_i$, $i = 1, 2$. By Lemmas 2.8 and 2.4, if $w_1 \not\equiv w_2$, then for any two points $s_i \in (0, \tau(w_i))$, $i = 1, 2$ we have $(w_1, w_1')(s_1) \neq (w_2, w_2')(s_2)$. Therefore, if one of the two functions (say w_1) is a constant, then the value of this constant must be different from both the minimum and the maximum of w_2 so that our claim holds. Thus we assume that neither of the two functions is a constant. Hence, by Lemma 3.1, there exists exactly one point $\bar{\tau}_i$ in $(0, \tau(w_i))$ such that w_i is strictly increasing in $[0, \bar{\tau}_i]$ and strictly decreasing in $[\bar{\tau}_i, \tau(w_i)]$. Consequently the function w_i, $i = 1, 2$ is represented in the phase plane (w, w') by a simple closed curve Λ_i consisting of two branches stretching between the points $(w_i(0), 0)$ and $(w_i(\bar{\tau}_i), 0)$ and $\Lambda_1 \cap \Lambda_2 = \emptyset$. Since $D_i = [w_i(0), w_i(\bar{\tau}_i)]$ this proves our claim. □

Define

$$\mathfrak{D} = \{\{w(t) : t \in R^1\} : w \in \mathfrak{E}(f)\}. \tag{3.11}$$

LEMMA 3.4. *–Let $f \in \mathfrak{M}$. The set $\mathfrak{D}$, ordered according to set inclusion, possesses a minimal element D_0 such that, for every $D \in \mathfrak{D}$ either $D_0 \subseteq D$ or $D_0 \cap D = \emptyset$.*

Furthermore, if

$$\mu(f) < \inf\{f(z,0,0) : z \in R^1\}, \tag{3.12}$$

then $\mathfrak{D}$ possesses only finitely many minimal elements.

Proof. – If $\mu(f) = \inf\{f(z,0,0) : z \in R^1\}$ then there exists a periodic minimizer which is a constant so that $\mathfrak{D}$ contains an element D_0 consisting of one point. Obviously D_0 is a minimal element of $\mathfrak{D}$. Therefore we may assume that (3.12) is valid. We claim that under this assumption,

$$\alpha := \inf\{|D| : D \in \mathfrak{D}\} > 0, \tag{3.13}$$

and that there exists $v \in \mathfrak{E}(f)$ such that $\max v - \min v = \alpha$.

Let $\{w_n\}$ be a sequence in $\mathfrak{E}(f)$ such that $\alpha_n := \max w_n - \inf w_n \to \alpha$. We may assume that each function w_n attains its minimum at zero. Put $b_n := \min_R w_n$, $c_n := \max_R w_n$ and $\tau_n := \tau(w_n)$. By Lemma 3.2 the sequence of periods $\{\tau(w_n)\}$ is bounded and, by Proposition 2.3, the set $\mathfrak{E}(f)$ is uniformly bounded. Therefore, by taking a subsequence if necessary, we may assume that $\{b_n\}$, $\{c_n\}$ and $\{\tau_n\}$ converge. We denote their limits by b^*, c^*, τ^* respectively. By Lemma 2.2, $\{w_n\}$ is bounded in $W^{2,\gamma}_{loc}(R)$ and consequently there exists a subsequence $\{w_{n_j}\}$ which converges weakly in $W^{2,\gamma}(0,T)$ and strongly in $C^1[0,T]$, for any $T > 0$. Its limit v satisfies $b^* = v(0) = \min_{R_+} v$ and $c^* = \max_{R_+} v$. By the weak lower semicontinuity of the functionals, v is (f)–perfect (see (2.15b)). If $\tau^* = 0$ then $b^* = c^*$, i.e. v is a constant. However, by (3.12), this is impossible. Thus $\tau^* > 0$ and v is a periodic minimizer with period τ^*. Hence $D^* = [b^*, c^*] \in \mathfrak{D}$ and $c^* - b^* = \alpha$. Since v is not a constant $\alpha > 0$. Therefore (3.13) holds and our claim is proved. In view of Lemma 3.3 this implies that D^* is a minimal element.

In order to verify the last statement of the lemma, observe that if D_1, D_2 are two distinct minimal elements of $\mathfrak{D}$ then, by Lemma 3.3, $D_1 \cap D_2 = \emptyset$. Therefore, the uniform boundedness of $\mathfrak{E}(f)$ and (3.13) imply that the number of minimal elements is finite. □

Proof of Theorem 3.1. – Let w_0 be a function in $\mathfrak{E}(f)$ such that

$$[b,c] = \{w_0(t) : \quad t \in R\}$$

is a minimal element of $\mathfrak{D}$. Let ϕ be a function in $C^\infty(R)$ such that,

$$\phi(x) = 0, \ \forall x \in [b,c], \quad \phi(x) > 0, \ \forall x \in R \setminus [b,c],$$

and $\phi^{(m)} \in L^\infty(R),\ m = 0, 1, 2, \dots$. In the present case such a function is easily constructed. In a more general context the existence of such functions was established in [2, Ch. 2, Sec.3].

With ϕ as above, let f_ρ be defined as in the statement of the theorem. Then

$$J^{f_\rho}(v) \geq J^f(v), \quad \forall v \in W^{2,1}_{loc}(0, \infty). \tag{3.14}$$

If v is a periodic function, equality holds in (3.14) if and only if

$$\{v(t) : t \in [0, \infty)\} \subseteq [b, c].$$

Hence

$$\mu(f_\rho) \geq \mu(f) = J^f(w_0) = J^{f_\rho}(w_0) \geq \mu(f_\rho). \tag{3.15}$$

Consequently, $\mu(f) = \mu(f_\rho)$ and w_0 is a minimizer of (P_∞) with integrand f_ρ. We claim that w_0 is the unique (up to translation) periodic minimizer of this problem. Indeed, if w is another periodic minimizer of this problem then, by (3.14), (3.15), $w \in \mathfrak{C}(f)$ and $\{w(t) : t \in R\} \subseteq [b, c]$. Since $[b, c]$ is a minimal element of $\mathfrak{D}$ it follows that $\{w(t) : t \in R\} = [b, c]$. However, by Lemma 3.3, this implies that w is a translate of w_0. □

Next we prove a slightly stronger formulation of Theorem 1.3 (i):

THEOREM 3.2. – *Let $f \in \mathfrak{M}$. If $\phi \in C^\infty(R)$ and f_ρ are as in Theorem 3.1 then, for each $\rho \in (0, 1)$, f_ρ possesses (ATP).*

Proof. – First suppose that $\mu(f) < \inf_R f(\cdot, 0, 0)$. In this case the statement of the theorem is an immediate consequence of Theorem 3.1 and the following result of Zaslavski [18]:

Assume that $h \in \mathfrak{M}$ and that $\mu(h) < \inf_R h(\cdot, 0, 0)$. Then h has (ATP) if and only if there exists a unique (up to translation) periodic (h)-minimizer.

Next suppose that $\mu(f) = \inf_R f(\cdot, 0, 0)$. Then there exists $\xi_0 \in R^1$ such that $f(\xi_0, 0, 0) = \mu(f)$ and ϕ is positive everywhere except at ξ_0. By Theorem 3.1, for every $\rho \in (0, 1)$, problem (P_∞) with integrand f_ρ has a unique periodic minimizer, namely the constant function with value ξ_0. In order to prove that (f_ρ) possesses (ATP) we must prove that,

$$v \in W^{2,1}_{loc}(0, \infty) \text{ and } v \text{ is } (f_\rho)\text{-good} \Longrightarrow \lim_{t \to \infty} (v, v')(t) = (\xi_0, 0). \tag{3.16}$$

Let v satisfy the assumptions of (3.16) for some $\rho \in (0, 1)$. Then, in view of (3.14), $J^f(v) = \mu(f)$. Since

$$0 \leq \eta^{f_\rho}(T, v) - \eta^f(T, v) = \int_0^T \rho\phi(v(t))\, dt,$$

and $\eta^{f_p}(\cdot, v)$ is bounded on $(0, \infty)$ it follows that $\eta^f(\cdot, v)$ is bounded, i.e. v is an (f)-good function, and $\lim_{T\to\infty} \int_0^T \phi(v(t))dt < \infty$. We claim that

$$\lim_{t\to\infty} v(t) = \xi_0. \tag{3.17}$$

Indeed by Lemma 2.2 v and v' are uniformly continuous on $(0, \infty)$. Therefore, if there exists a sequence $\{t_n\}$ tending to infinity such that $v(t_n) \to \xi_1 \neq \xi_0$ then there exists a positive δ such that

$$\liminf_{n\to\infty} \operatorname{dist}(\xi_0, \{v(t) : t_n - \delta \leq t \leq t_n + \delta\}) > 0.$$

Since v is bounded and ϕ is positive except at ξ_0 this contradicts the integrability of $\phi(v(\cdot))$ on $(0, \infty)$.

Next we claim that $\lim_{t\to\infty} v'(t) = 0$. If not, assume for instance that $\limsup v'(t) = \zeta > 0$. Then, because of the uniform continuity of v', it follows that there exists a sequence $\{t_n\}$ tending to infinity and a positive δ such that $\inf\{v'(t) : t_n - \delta \leq t \leq t_n + \delta\} > \zeta/2$ for all sufficiently large n. Therefore $v(t_n + \delta) - v(t_n) > \delta\zeta/2$ for all sufficiently large n, which contradicts (3.17). Thus $\lim_{t\to\infty}(v, v')(t) = (\xi_0, 0)$ and (3.16) is proved. □

Finally we turn to,

Proof of Theorem 1.3 (ii). – Denote by E the set of all functions $\phi \in \mathfrak{L}(\alpha, b_2, b_3)$ such that F_ϕ has (ATP). By Theorem 3.2 the set E is everywhere dense in $\mathfrak{L}(\alpha, b_2, b_3)$. For each $\phi \in E$ there exist $v_\phi \in W^{2,1}_{loc}(R^1), T_\phi > 0$ such that

$$v_\phi(t + T_\phi) = v_\phi(t),\ t \in R^1,\ I^{F_\phi}(0, T_\phi, v_\phi) = \mu(F_\phi)T_\phi. \tag{3.18}$$

Let $\phi \in E$, $n \geq 1$ be an integer. By (3.18), the definition of the set E, the continuity of the operator

$$\phi \to F_\phi, \quad \phi \in \mathfrak{L}(\alpha, b_2, b_3).$$

and Theorem 2.2 there exist an open neighborhood $U(\phi, n)$ of ϕ in $\mathfrak{L}(\alpha, b_2, b_3)$ such that for each $\psi \in U(\phi, n)$ and each (F_ψ)-good function $w \in W^{2,1}_{loc}(0, \infty)$

$$\operatorname{dist}(\Omega(w), \{X_{v_\phi}(t) : t \in R^1\}) \leq (2n)^{-1}. \tag{3.19}$$

Define

$$\mathfrak{F} = \cap_{n=1}^{\infty} \cup \{U(\phi, n) : \phi \in E\}.$$

Let $h \in \mathfrak{F}$, w_1, w_2 be (F_h)-good functions. To complete the proof of the theorem it is sufficient to show that $\Omega(w_1) = \Omega(w_2)$. Let $\epsilon \in (0, 1)$. There exist an integer $n \geq 8\epsilon^{-1}$ and $\phi \in E$ such that $h \in U(\phi, n)$. It follows from the definition of $U(\phi, n)$ that

$$\operatorname{dist}(\Omega(w_i), \{X_{v_\phi}(t) : t \in R^1\}) \leq (2n)^{-1}, i = 1, 2,\ \operatorname{dist}(\Omega(w_1), \Omega(w_2)) \leq \epsilon.$$

This completes the proof of the theorem. □

APPENDIX A

This appendix is devoted to the proof of Lemma A, which will be based on several additional lemmas.

LEMMA A.1. *–Let $\epsilon, M > 0$. Then there exist $\delta > 0$ and an integer $q_1 \geq 1$ such that for each $v \in W^{2,1}(0, q_1 T_w)$ which satisfy*

$$|X_v(s)| \leq M, \quad s = 0, q_1 T_w, \tag{A.1}$$

$$I^g(0, q_1 T_w, v) \leq q_1 T_w \mu(g) + \pi^g(X_v(0)) - \pi^g(X_v(q_1 T_w)) + \delta$$

there exist $\xi \in [0, T_w), \tau \in [0, (q_1 - 1)T_w]$ such that

$$|X_v(\tau + t) - X_w(\xi + t)| \leq \epsilon, \quad t \in [0, T_w].$$

Proof. – Let us assume the converse. Then for each integer $p \geq 1$ there exists $v_p \in W^{2,1}(0, pT_w)$ such that

$$|X_{v_p}(s)| \leq M, \quad s = 0, pT_w, \tag{A.2}$$

$$I^g(0, pT_w, v_p) \leq pT_w \mu(g) + \pi^g(X_{v_p}(0)) - \pi^g(X_{v_p}(pT_w)) + 2^{-p}$$

and for each $\xi \in [0, T_w)$, each $\tau \in [0, (p-1)T_w]$

$$\sup\{|X_{v_p}(\tau + t) - X_w(\xi + t)| : t \in [0, T_w]\} > \epsilon. \tag{A.3}$$

By (A.2) and Proposition 2.2 there exists $M_1 > 0$ such that for each integer $p \geq 1$

$$|X_{v_p}(t)| \leq M_1, \quad t \in [0, pT_w]. \tag{A.4}$$

(A.2), (A.4) and (2.2) imply that for any integer $n \geq 1$ the sequence $\{v_p''\}_{p=n}^{\infty}$ is bounded in $L^\gamma[0, nT_w]$. It is easy to verify that there are $v \in W_{loc}^{2,\gamma}(0, \infty)$ and a strictly increasing subsequence of natural numbers $\{p_k\}_{k=1}^{\infty}$ such that for every integer $n \geq 1$

$$v_{p_k} \to v \text{ as } k \to \infty \text{ weakly in } W^{2,\gamma}(0, nT_w). \tag{A.5}$$

By (A.2) and the lower semicontinuity of integral functionals [3] for each integer $n \geq 1$

$$I^g(0, nT_w, v) = nT_w \mu(g) + \pi^g(X_v(0)) - \pi^g(X_v(nT_w)). \tag{A.6}$$

Clearly

(A.7) $$|X_v(t)| \le M_1, \quad t \in [0, \infty).$$

It follows from (A.5) and the definition of $\{v_p\}_{p=1}^{\infty}$ (see (A.2), (A.3)) that for each $\tau \in [0, \infty)$ and each $\xi \in [0, T_w)$

(A.8) $$\sup\{|X_v(\tau + t) - X_w(\xi + t)| : t \in [0, T_w]\} > 2^{-1}\epsilon.$$

(A.6) and (A.7) imply that the function v is (g)-good. Then

(A.9) $$\Omega(v) = \Omega(w).$$

There exists a sequence of numbers $\{t_j\}_{j=1}^{\infty} \subset (0, \infty)$ such that
(A.10)
$$t_1 \ge 8T_w + 8, \; t_{j+1} - t_j \ge 8T_w, \; j = 1, 2, \ldots, \; X_v(t_j) \to X_w(0) \text{ as } j \to \infty.$$

For each integer $j \ge 1$ we define $u_j \in W^{2,1}(-4T_w, 4T_w)$ as follows

(A.11) $$u_j(t) = v(t_j + t), \quad t \in [-4T_w, 4T_w].$$

By (2.2), (A.11), (A.6) and (A.7) the sequence $\{u_j''\}_{j=1}^{\infty}$ is bounded in $L^{\gamma}[-4T_w, 4T_w]$. It is easy to verify that there are $u \in W^{2,1}(-4T_w, 4T_w)$ and a strictly increasing subsequence of natural numbers $\{j_p\}_{p=1}^{\infty}$ such that
(A.12)
$$u_{j_p}(t) \to u(t), \; u'_{j_p}(t) \to u'(t) \text{ as } p \to \infty \text{ uniformly in } [-4T_w, 4T_w],$$
$$u''_{j_p} \to u'' \text{ as } p \to \infty \text{ weakly in } L^{\gamma}[-4T_w, 4T_w].$$

By (A.6) and the lower semicontinuity of integral functionals [3]

(A.13) $$I^g(-4T_w, 4T_w, u) = 8T_w\mu(g) + \pi^g(X_u(-4T_w)) + \pi^g(X_u(4T_w)).$$

Clearly

(A.14) $$X_u(0) = X_w(0).$$

It follows from (A.11), (A.12) and (A.8) which holds for each $\tau \in [0, \infty)$ and each $\xi \in [0, T_w)$, that

$$\sup\{|X_u(t) - X_w(t)| : t \in [0, T_w]\} > 4^{-1}\epsilon.$$

On the other hand (A.13), (A.14) and Lemma 2.8 imply that $u(t) = w(t)$ for all $t \in [-4T_w, 4T_w]$. The obtained contradiction proves the lemma. □

LEMMA A.2. *–Let $M_0, M_1, \epsilon > 0$. Then there exists an integer $q \geq 1$ such that for each $v \in W^{2,1}(0, qT_w)$ which satisfies*

(A.15)

$$|X_v(s)| \leq M_0, \ s = 0, qT_w, \ I^g(0, qT_w, v) \leq U^g_{qT_w}(X_v(0), X_v(qT_w)) + M_1$$

there exist $\xi \in [0, T_w)$, $\tau \in [0, (q-1)T_w]$ such that

$$|X_v(\tau + t) - X_w(\xi + t)| \leq \epsilon, \quad t \in [0, T_w]. \tag{A.16}$$

Proof. – By Proposition 2.2 there is $S_0 > M_0 + M_1 + 2$ such that for each $\tau \geq 2^{-1} \inf\{T_w, 1\}$, each $v \in W^{2,1}(0, \tau)$ which satisfies

$$|X_v(0)|, |X_v(\tau)| \leq M_0, \quad I^g(0, \tau, v) \leq U^g_\tau(X_v(0), X_v(\tau)) + M_1 + 1$$

the following relation holds

$$|X_v(t)| \leq S_0, \quad t \in [0, \tau]. \tag{A.17}$$

By Lemma A.1 there exists an integer $q_1 \geq 1$ and a number $\delta > 0$ such that for each $v \in W^{2,1}(0, q_1 T_w)$ which satisfies

$$|X_v(t)| \leq S_0, \quad t = 0, q_1 T_w, \tag{A.18}$$

$$I^g(0, q_1 T_w, v) \leq q_1 T_w \mu(g) + \pi^g(X_v(0)) - \pi^g(X_v(q_1 T_w)) + \delta$$

there exist $\xi \in [0, T_w)$, $\tau \in [0, (q_1 - 1)T_w]$ such that (A.16) holds. By Lemma 2.6 there exists $K_0 > 0$ such that for each $\tau \geq 4T_w$, each $x, y \in R^2$ satisfying $|x|, |y| \leq M_0 + S_0 + 1$ the following relation holds

$$U^g_\tau(x, y) \leq \tau \mu(g) + \pi^g(x) - \pi^g(y) + K_0. \tag{A.19}$$

Here we use the fact that π^g is bounded on compact sets. Fix an integer

$$q > [(M_1 + K_0 + 1)\delta^{-1} + 4] q_1. \tag{A.20}$$

Assume that $v \in W^{2,1}(0, qT_w)$ and (A.15) holds. It follows from (A.15) and the definition of K_0 (see (A.19)) that

$$\begin{aligned} I^g(0, qT_w, v) &\leq U^g_{qT_w}(X_v(0), X_v(qT_w)) + M_1 \\ &\leq qT_w \mu(g) + \pi^g(X_v(0)) - \pi^g(X_v(qT_w)) + M_1 + K_0. \end{aligned} \tag{A.21}$$

By the definition of S_0 (see (A.17)) and (A.15)

$$|X_v(t)| \leq S_0, \quad t \in [0, qT_w]. \tag{A.22}$$

There exists a sequence $\{t_i\}_{i=0}^{s} \subset [0, qT_w]$ such that

(A.23) $t_0 = 0,\ t_{i+1} = t_i + q_1 T_w$ if $0 \le i \le s-1,\ t_s \in [qT_w - q_1 T_w, qT_w]$.

Clearly

$$s \ge q q_1^{-1} - 1 \ge 3 + \delta^{-1}(M_1 + K_0 + 1). \tag{A.24}$$

Together with (A.21) this implies that there is $j \in \{0, \dots s-1\}$ for which

(A.25) $I^g(t_j, t_{j+1}, v) \le (t_{j+1} - t_j)\mu(g) + \pi^g(X_v(t_j)) - \pi^g(X_v(t_{j+1})) + \delta.$

It follows from this relation, (A.22), (A.23) and the definition of δ, q_1 (see (A.18)) that there exist $\xi \in [0, T_w)$, $\tau \in [t_j, t_{j+1} - T_w]$ such that (A.16) holds. This completes the proof of the lemma. □

Proof of Lemma A. – By Proposition 2.2 there are a neighborhood $\mathfrak{U}_1$ of g in $\mathfrak{M}$ and a number $M_2 > M_0 + M_1$ such that for each $f \in \mathfrak{U}_1$, each $T \ge \inf\{T_w, 1\}$ and each $v \in W^{2,1}(0, T)$ satisfying (2.24) the following relation holds

$$|X_v(t)| \le M_2, \quad t \in [0, T]. \tag{A.26}$$

By Lemma A.2 there exists an integer $q_1 \ge 1$ such that for each $v \in W^{2,1}(0, q_1 T_w)$ which satisfies

$$|X_v(0)|, |X_v(q_1 T_w)| \le M_2, \tag{A.27}$$

$$I^g(0, q_1 T_w, v) \le U^g_{q_1 T_w}(X_v(0), X_v(q_1 T_w)) + 2M_1 + 8$$

there exist $\xi \in [0, T_w)$, $s \in [0, (q_1 - 1)T_w]$ such that (2.25) holds.

There exists a number $\Gamma_0 > 0$ for which

$$\sup\{|U^g_{q_1 T_w}(x, y)| : x, y \in R^2,\ |x|, |y| \le M_2\} \le \Gamma_0. \tag{A.28}$$

By Lemma 2.7 there exists a neighborhood $\mathfrak{U}_2$ of g in $\mathfrak{M}$ such that for each $f \in \mathfrak{U}_2$, each $x, y \in R^2$ satisfying $|x|, |y| \le M_2$ the relation $|U^f_{q_1 T_w}(x, y) - U^g_{q_1 T_w}(x, y)| \le 2^{-1}$ holds.

By Lemma 2.7 there exists a neighborhood $\mathfrak{U}_3$ of g in $\mathfrak{M}$ such that for each $f \in \mathfrak{U}_3$, each $v \in W^{2,1}(0, q_1 T_w)$ satisfying

$$\inf\{I^f(0, q_1 T_w, v), I^g(0, q_1 T_w, v)\} \le 2\Gamma_0 + 2M_1 + 4$$

the relation $|I^f(0, q_1 T_w, v) - I^g(0, q_1 T_w, v)| \le 2^{-1}$ holds. Set $\mathfrak{U} = \mathfrak{U}_1 \cap \mathfrak{U}_2 \cap \mathfrak{U}_3$.

Assume that $f \in \mathfrak{U}$, $T \geq q_1 T_w$, $v \in W^{2,1}(0,T)$ satisfies (2.24) and $\tau \in [0, T - q_1 T_w]$. By the definition of $\mathfrak{U}_1$ and M_2 relation (A.26) holds. It follows from (2.24), (A.26), the definition of $\mathfrak{U}_2$ and (A.28) that

$$\text{(A.29)} \qquad I^f(\tau, \tau + q_1 T_w, v) \leq U^f_{q_1 T_w}(X_v(\tau), X_v(\tau + q_1 T_w)) + M_1$$

$$\leq U^g_{q_1 T_w}(X_v(\tau), X_v(\tau + q_1 T_w)) + 2^{-1} + M_1 \leq \Gamma_0 + 2^{-1} + M_1.$$

By this relation and the definition of $\mathfrak{U}_3$

$$|I^f(\tau, \tau + q_1 T_w, v) - I^g(\tau, \tau + q_1 T_w, v)| \leq 2^{-1},$$

$$I^g(\tau, \tau + q_1 T_w, v) \leq U^g_{q_1 T_w}(X_v(\tau), X_v(\tau + q_1 T_w)) + 1 + M_1.$$

It follows from this relation, (A.26) and the definition of q_1 (see (A.27)) that there exist $\xi \in [0, T_w)$, $s \in [\tau, \tau + q_1 T_w - T_w]$ such that (2.25) holds. The lemma is proved. □

APPENDIX B

Here we establish Lemma B whose proof is based on several auxilliary results.

The following lemma shows that given $\epsilon > 0$ and a (g)-good function v, for sufficiently large T the restriction of (v, v') to $[T, T + T_w]$ is within ϵ of a translation of (w, w').

LEMMA B.1. *–Assume that $v \in W^{2,1}_{loc}(0, \infty)$ is a (g)-good function and $\epsilon > 0$. Then there exists $T(\epsilon) > 0$ such that for each $T \geq T(\epsilon)$ there is $\xi \in [0, T_w)$ such that*

$$|X_v(T + t) - X_w(\xi + t)| \leq \epsilon, \quad t \in [0, T_w].$$

Proof. – Since v is a (g)-good function for each $\delta > 0$ there exists $T(\delta) > 0$ such that

$$\text{(B.1)} \qquad I^g(\tau_1, \tau_2, v) \leq (\tau_2 - \tau_1)\mu(g) + \pi^g(X_v(\tau_1)) - \pi^g(X_v(\tau_2)) + \delta$$

for each $\tau_1 \geq T(\delta)$ and each $\tau_2 > \tau_1$ (see Lemma 2.4).

Assume that the lemma is wrong. Then there exists a sequence of numbers $\{t_i\}_{i=1}^{\infty} \subset (0, \infty)$ such that

$$\text{(B.2)} \qquad t_i \geq T(2^{-i}) + 2i + 2, \quad i = 1, 2, \ldots$$

and for each integer $i \geq 1$ and each $\xi \in [0, T_w)$

$$\text{(B.3)} \qquad \sup\{|X_v(t_i + t) - X_w(\xi + t)| : t \in [0, T_w]\} > \epsilon.$$

For each integer $i \geq 1$ we define $u_i \in W^{2,1}_{loc}(-t_i, \infty)$ as follows

$$\text{(B.4)} \qquad u_i(t) = v(t_i + t), \quad t \in [-t_i, \infty).$$

It follows from the definition of $T(\delta)$, $\delta > 0$ (see (B.1)), (B.2), (B.4) and (2.2) that for any integer $n \geq 1$ the sequence $\{u_i''\}_{i=n}^{\infty}$ is bounded in $L^\gamma[-n, n]$.

It is easy to see that there exist $u \in W^{2,\gamma}_{loc}(R^1)$ and a strictly increasing subsequence of natural numbers $\{i_p\}_{p=1}^{\infty}$ such that for every integer $n \geq 1$

$$\text{(B.5)} \qquad u_{i_p} \to u \text{ as } p \to \infty \text{ weakly in } W^{2,\gamma}(-n, n).$$

By the definition of $T(\delta)$, $\delta > 0$ (see (B.1)), (B.2), (B.4), (B.5) and the lower semicontinuity of integral functionals [3]

$$\text{(B.6)} \qquad I^g(\tau_1, \tau_2, u) = (\tau_2 - \tau_1)\mu(g) + \pi^g(X_u(\tau_1)) - \pi^g(X_u(\tau_2))$$

for each $\tau_1 \in R^1, \tau_2 > \tau_1$.

It is easy to see that for each $t \in R^1$

$$X_u(t) \in \Omega(v) = \{X_w(s) : s \in R^1\}.$$

Together with (B.6), Lemma 2.8 this implies that there exists $\xi_0 \in [0, T_w)$ such that $u(t) = w(t + \xi_0)$, $t \in R^1$. It follows from this relation and (B.5), (B.4) that there exists an integer $p_0 \geq 1$ such that for each integer $p \geq p_0$

$$|X_v(t_{i_p} + t) - X_w(\xi_0 + t)| \leq 2^{-1}\epsilon, \quad t \in [0, T_w].$$

This is contradictory to the definition of $\{t_i\}_{i=1}^{\infty}$ (see (B.3)). The obtained contradiction proves the lemma. □

LEMMA B.2. –*Let $\epsilon > 0$. Then there exists $\delta > 0$ such that for each $\tau \geq T_w$ and each $s \in [0, \tau - T_w]$, if v is a function in $W^{2,1}(0, \tau)$ such that*

$$\text{(B.7)} \qquad d(X_v(s), \{X_w(t) : t \in R^1\}) \leq \delta, \quad s = 0, \tau,$$

$$I^g(0, \tau, v) \leq \tau\mu(g) + \pi^g(X_v(0)) - \pi^g(X_v(\tau)) + \delta$$

then there is $\xi \in [0, T_w)$ for which

$$\text{(B.8)} \qquad |X_v(s + t) - X_w(\xi + t)| \leq \epsilon, \quad t \in [0, T_w].$$

Proof. – By Proposition 2.1 and the continuity of π^g, $U^g_{T_w}$ for each integer $i \geq 1$ there exists $\delta_i \in (0, 4^{-i})$ such that for each $x, y \in R^2$ satisfying $|x - y| \leq \delta_i$, $d(x, \{X_w(t) : t \in R^1\}) \leq \delta_i$ the following relation holds

$$U^g_{T_w}(x, y) \leq \pi^g(x) - \pi^g(y) + T_w\mu(g) + 2^{-i}. \tag{B.9}$$

Assume that the lemma is wrong. Then for each integer $i \geq 1$ there exist $\tau_i \geq T_w$, $v_i \in W^{2,1}(0, \tau_i)$ such that

$$d(X_{v_i}(s), \{X_w(t) : t \in R^1\}) \leq \delta_i, \; s = 0, \tau_i, \tag{B.10}$$

$$I^g(0, \tau_i, v_i) \leq \tau_i\mu(g) + \pi^g(X_{v_i}(0)) - \pi^g(X_{v_i}(\tau_i)) + \delta_i$$

and there exists $s_i \in [0, \tau_i - T_w]$ such that for each $\xi \in [0, T_w)$

$$\sup\{|X_{v_i}(s_i + t) - X_w(\xi + t)| : t \in [0, T_w]\} > \epsilon. \tag{B.11}$$

For each integer $i \geq 1$ there exist $\xi_i^1, \xi_i^2 \in [0, T_w)$ such that

$$|X_{v_i}(0) - X_w(\xi_i^1)|, |X_{v_i}(\tau_i) - X_w(\xi_i^2)| \leq \delta_i. \tag{B.12}$$

For each integer $i \geq 1$ there exists a function $u_i \in W^{2,1}(0, \tau_i + 2T_w)$ such that

(B.13)

$$X_{u_i}(0) = X_w(\xi_i^1), \; u_i(t) = v_i(t - T_w), \; t \in [T_w, T_w + \tau_i], \; X_{u_i}(\tau_i + 2T_w)$$

$$= X_w(\xi_i^2), \; I^g(s, s + T_w, v) = U^g_{T_w}(X_{u_i}(s), X_{u_i}(s + T_w)), \; s = 0, \tau_i + T_w.$$

It follows from (B.13), (B.12) and the definition of $\{\delta_i\}_{i=1}^{\infty}$ (see (B.9)) that for each integer $i \geq 1$

$$I^g(s, s + T_w, u_i) \leq T_w\mu(g) + \pi^g(X_{u_i}(s))$$

$$-\pi^g(X_{u_i}(s + T_w)) + 2^{-i}, \; s = 0, \tau_i + T_w.$$

Together with (B.13), (B.10) this implies that for each integer $i \geq 1$

(B.14)

$$I^g(0, \tau_i + 2T_w, u_i) \leq (\tau_i + 2T_w)\mu(g) + \pi^g(X_{u_i}(0)) - \pi^g(X_{u_i}(\tau_i + 2T_w)) + 3 \cdot 2^{-i}$$

For each integer $i \geq 1$ there exists $\xi_i^3 \in [T_w, 2T_w]$ such that

$$T_w^{-1}[\xi_i^2 + \xi_i^3 - \xi_{i+1}^1] \text{ is an integer.} \tag{B.15}$$

We define sequences of numbers $\{b_i\}_{i=1}^{\infty}$, $\{c_i\}_{i=1}^{\infty}$ as follows

$$b_1 = 0, \; c_i = b_i + \tau_i + 2T_w, \; b_{i+1} = c_i + \xi_i^3, \; i = 1, 2, \ldots. \tag{B.16}$$

It is easy to verify that there exists $u \in W^{2,1}_{loc}(0, \infty)$ such that for each integer $i \geq 1$

(B.17)
$$u(b_i + t) = u_i(t), \ t \in [0, \tau_i + 2T_w], \ u(c_i + t) = w(\xi_i^2 + t), \ t \in [0, \xi_i^3].$$

For each integer $i \geq 1$ we set

$$s_i^0 = b_i + T_w + s_i.$$

It follows from (B.16), (B.17), (B.13), (B.11) that for each integer $i \geq 1$, for each $\xi \in [0, T_w)$

$$\sup\{|X_u(s_i^0 + t) - X_w(\xi + t)| : t \in [0, T_w]\} > \epsilon. \tag{B.18}$$

(B.17), (B.14), (B.16) imply that u is a (g)-good function. By Lemma B.1 there exists a number $T_* > 0$ such that for each $T \geq T_*$ there is $\xi \in [0, T_w)$ such that

$$|X_u(T + t) - X_w(\xi + t)| \leq 2^{-1}\epsilon, \quad t \in [0, T_w].$$

This is contradictory to (B.18) which holds for each integer $i \geq 1$ and each $\xi \in [0, T_w)$. The obtained contradiction proves the lemma.

Analogously to Lemma 3.7 in [17] we can establish the following result.

LEMMA B.3. –*Let* $f \in \mathfrak{M}$, $w \in W^{2,1}_{loc}(R^1)$, $T > 0$, $w(t + T) = w(t)$, $t \in R^1$, $I^f(0, T, w) = T\mu(f)$, $\epsilon > 0$. *Then there exists an integer* $q \geq 1$ *such that for any* $\xi \in [0, T)$ *there is a function* $v \in W^{2,1}(0, qT)$ *such that* $X_v(0) = X_w(0)$, $X_v(qT) = X_w(\xi)$, $I^f(0, qT, v) \leq qT\mu(f) + \pi^f(X_w(0)) - \pi^f(X_w(\xi)) + \epsilon$.

Lemma B.3 implies the following result.

LEMMA B.4. –*Let* $\epsilon > 0$. *Then there exists a number* $q(\epsilon) > 0$ *such that for each* $\tau \geq q(\epsilon)$, *each* $\xi_1, \xi_2 \in [0, T_w)$ *there exists* $v \in W^{2,1}(0, \tau)$ *which satisfies* $X_v(0) = X_w(\xi_1)$, $X_v(\tau) = X_w(\xi_2)$,

$$I^g(0, \tau, v) \leq \tau\mu(g) + \pi^g(X_v(0)) - \pi^g(X_v(\tau)) + \epsilon.$$

Lemma B.4, Proposition 2.1 and the continuity of π^g and U_T^g imply the following extension of Lemma B.3.

LEMMA B.5. –*Let* $\epsilon > 0$. *Then there exist numbers* $\delta, q(\epsilon) > 0$ *such that for each* $\tau \geq q(\epsilon)$, *each* $x, y \in R^2$ *satisfying*

$$d(x, \{X_w(t) : t \in R^1\}) \leq \delta, \ d(y, \{X_w(t) : t \in R^1\}) \leq \delta \tag{B.19}$$

there exists $v \in W^{2,1}(0, \tau)$ *which satisfies*

$$X_v(0) = x,\ X_v(\tau) = y,\ I^g(0, \tau, v) \le \tau\mu(g) + \pi^g(X_v(0)) - \pi^g(X_v(\tau)) + \epsilon.$$

COROLLARY B.1. – *Let* $\epsilon > 0$ *and let* $\delta, q(\epsilon) > 0$ *be as guaranteed in Lemma B.5. Then for each* $\tau \ge q(\epsilon)$, *each* $x, y \in R^2$ *satisfying (B.19) the following relation holds*

$$U^g_\tau(x, y) \le \tau\mu(g) + \pi^g(X_v(0)) - \pi^g(X_v(\tau)) + \epsilon.$$

Corollary B.1 and Lemma B.2 imply the following result.

LEMMA B.6. –*Let* $\epsilon > 0$. *Then there exist* $\delta > 0, Q > T_w$ *such that for each* $\tau \ge Q$, *each* $v \in W^{2,1}(0, \tau)$ *which satisfies* $d(X_v(s), \{X_w(t) : t \in R^1\}) \le \delta$, $s = 0, \tau$, $I^g(0, \tau, v) \le U^g_\tau(X_v(0), X_v(\tau)) + \delta$ *and each* $s \in [0, \tau - T_w]$ *there is* $\xi \in [0, T_w)$ *for which*

$$|X_v(s + t) - X_w(\xi + t)| \le \epsilon, \quad t \in [0, T_w].$$

Lemmas B.6 and 2.6 imply Lemma B.

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