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Correction to "Properties of pseudoholomorphic curves in symplectisations I: Asymptotics"

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ABSTRACT. – In our paper [1] we derived the proposition 2.1 from Lemma 2.2 which is clearly false and our aim is to give a proof of the proposition. We take the opportunity to remove also a flaw in the proof of Lemma 3.3. The results of [1] remain unaffected.

In the following we shall use the notations, definitions and formulas from [1]. We consider a compact three-manifold M equipped with the contact form λ and a finite energy plane $\tilde{u} = (a, u): \mathbb{C} \to \mathbb{R} \times M$.

PROPOSITION 2.1. – *Let $\tilde{u}$ be a finite energy plane and assume there exists a sequence $R_k \to \infty$ such that $u(R_k e^{2\pi i t}) \to x(Tt)$ in $C^\infty(S^1, M)$. Assume further that x is a non-degenerate T-periodic solution of the Reeb vector field $\dot{x} = X(x)$ associated with the contact form λ. Then given any S^1-invariant C^∞ neighborhood W of the loop $x(T\cdot)$ in $C^\infty(S^1, M)$ there exists an $R_0 > 0$ such that $u(Re^{2\pi i \cdot}) \in W$ for all $R \geq R_0$.*

Proof. – We view M as being embedded in some $\mathbb{R}^n$ and equip the Frechet space $C^\infty(S^1, \mathbb{R}^n)$ with a translation invariant and S^1-invariant metric which we restrict to the subspace $C^\infty(S^1, M)$. The S^1-action on the loop space is the one induced by S^1 itself. Let $\mathcal{T} \subset C^\infty(S^1, M)$ be the

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collection of all loops corresponding to periodic solutions of $\dot{x} = X(x)$. With $x_T(\cdot) = x(T\cdot) \in \mathcal{T}$ we denote the loop corresponding to the distinguished T-periodic solution x of the proposition. Since x is non degenerate we find two disjoint and S^1-invariant open sets V_1 and V_2 in $C^\infty(S^1, M)$ having the properties that $\mathcal{T} \subset (V_1 \cup V_2)$ and $V_1 \cap \mathcal{T} = S^1 * x_T$. In the holomorphic polar coordinates φ the finite energy plane $\tilde{u}$ becomes the finite energy cylinder $\tilde{v} = \tilde{u} \circ \varphi = (b, v): \mathbb{R} \times S^1 \to \mathbb{R} \times M$ and by hypotheses there exists a sequence $s_k \to \infty$ such that

$$v(s_k, \cdot) \to x_T \qquad \text{in } C^\infty(S^1, M).$$

Hence $v_k(s_k, \cdot) \in V_1$ for k large. Recall from the proof of theorem 1.2 that every sequence $\sigma_k \to \infty$ possesses a subsequence σ'_k such that $v(\sigma'_k, \cdot)$ converges in $C^\infty(S^1, M)$ to an element of $\mathcal{T}$. Using this remark we prove proposition 2.1 indirectly. Assuming that $v(s,t)$ does not converge to $S^1 * x_T$ as $s \to \infty$ we find a sequence $\sigma_k \to \infty$ satisfying $v(\sigma_k, \cdot) \in V_2$ for k large, and passing to subsequences, we may assume that $s_k < \sigma_k < s_{k+1}$ for all k.

Since $s \mapsto v(s, \cdot)$ is a continuous path in $C^\infty(S^1, M)$ there is a sequence $s'_k \in (s_k, \sigma_k)$ satisfying $v(s'_k, \cdot) \notin V_1 \cup V_2$. By theorem 2.1 again we deduce a subsequence s''_k of s'_k such that $v(s''_k, \cdot)$ converges to an element $y \in \mathcal{T}$ satisfying $y \notin V_1 \cup V_2$ and hence contradicting $\mathcal{T} \subset (V_1 \cup V_2)$. This finishes the proof of proposition 2.1. □

In the proof of Lemma 3.3 of [1] the L^2-norms have to be replaced by the L^p-norms for $p > 2$.

LEMMA 3.3. – *Define, as in Lemma 3.2,*

$$\xi(s,t) = \frac{z(s,t)}{\|z(s)\|}$$
$$\alpha(s) = \langle A_\infty \xi, \xi \rangle + \langle (S_\infty - S)\xi, \xi \rangle.$$

Then for every $j \in \mathbb{N} = \{0, 1, 2, \ldots\}$ *and every multi index* $\beta \in \mathbb{N} \times \mathbb{N}$

$$\sup_{s,t} |\partial^\beta \xi(s,t)| < \infty \quad \textit{and} \quad \sup_s |\partial^j \alpha| < \infty.$$

Proof. – The scalarproduct and the norm $\| \;\; \|$ above refer to the L^2-space in the t variable, $0 \le t \le 1$. By (40) the smooth function $\xi = \xi(s,t): [s_0, \infty) \times \mathbb{R} \to \mathbb{R}^2$ is 1-periodic in $t \in \mathbb{R}$ and solves the equation

$$\bar{\partial}\xi = -S(s,t)\xi - \alpha(s)\xi, \quad \bar{\partial} := \frac{\partial}{\partial s} + J_0 \frac{\partial}{\partial t}, \tag{48}$$

with smooth functions S and α having the bounds

$$\sup|\partial^{\beta}S| < \infty \quad \text{and} \quad \sup|\alpha| < \infty$$

for all multi indices $\beta \in \mathbb{N}\times\mathbb{N}$. The supremum is taken over all $s \geq s_0$ and $t \in \mathbb{R}$, the function S is 1-periodic in t and the constant matrix J_0 satisfies $J_0^2 = -\mathbf{1}$. In order to derive uniform $W^{k,p}_{\text{loc}}$ bounds for ξ we pick $\delta_0 > 0$ and $s^* > s_0$ and define the sequence $\delta_j \searrow \frac{1}{2}\delta_0$ by $\delta_j = \frac{1}{2}\delta_0(1+2^{-j})$. We choose smooth bump-functions $\beta_j: \mathbb{R} \to [0,1]$ vanishing outside of $(s^*-\delta_{j-1}, s^*+\delta_{j-1})$ and equal to 1 on $[s^*-\delta_j, s^*+\delta_j]$. The shapes of these functions β_j are the same for every choice of s^*. Introducing the nested intervals $I_j = [s^*-\delta_j, s^*+\delta_j] \subset \mathbb{R}$ and $Q_j = I_j \times [0,1] \subset \mathbb{R}^2$ we claim that for every $N \geq 1$ and every $2 < p < \infty$ there exists a constant $C_{N,p} > 0$ such that

$$\|\xi\|_{W^{N,p}(Q_N)} \leq C_{N,p}, \quad \|\alpha\|_{W^{N,p}(I_N)} \leq C_{N,p}, \tag{49}$$

where the constants $C_{N,p}$ are independent of s^*. Lemma 3.3 is an immediate consequence of the local uniform estimates (49) in view of the Sobolev embedding theorem.

In order to prove (49) we proceed inductively making use of the well known a-priori estimate for the $\bar{\partial}$-operator:

$$\|\xi\|_{W^{j,p}(Q_j)} \leq M_p\|\bar{\partial}(\beta_j\xi)\|_{W^{j-1,p}(Q_{j-1})}, \tag{50}$$

where M_p only depends on $\bar{\partial}$. Starting with $j=1$ we first show that ξ is uniformly bounded. Recalling (48) we deduce from (50), setting $p=2$, the estimate $\|\xi\|_{W^{1,2}(Q_1)} \leq c\|\xi\|_{L^2(Q_0)}$. The constant $c>0$ depends on $\sup|S|$ and $\sup|\alpha|$ but not on s^*. Since $\|\xi(s)\|_{L^2(S^1)} = 1$ we have $\|\xi\|^2_{L^2(Q_0)} = 2\delta_0$ so that $\|\xi\|_{W^{1,2}(Q_1)} \leq c_1$ for a constant c_1 independent of s^*. Therefore, using the Sobolev embedding theorem, $\|\xi\|_{L^p(Q_1)} \leq c'_p$, again independent of s^*, for every $1 < p < \infty$. In view of this local uniform L^p-estimate for ξ we deduce from (50) for $p > 1$ the estimate $\|\xi\|_{W^{1,p}(Q_1)} \leq c_p$, the constant being independent of s^*. Hence choosing $p > 2$ we conclude $\sup|\xi| < \infty$ by means of the Sobolev embedding theorem. Recall now equation (42) for α, namely

$$\alpha'(s) = 2\|\xi'(s)\|^2 - \langle\epsilon\xi,\xi'\rangle + \langle\epsilon'\xi,\xi\rangle, \tag{42}$$

where prime denotes the partial derivative in the s-variable and where the smooth function $\epsilon = \epsilon(s,t)$ and all its partial derivatives are uniformly bounded. From (42) we deduce $|\alpha'(s)| \leq c_1\|\xi'(s)\|^2 + c_2$.

Integrating and using Hölder's inequality we find the local L^p-estimate $\|\alpha'\|^p_{L^p(I_1)} \leq c_3\|\xi\|_{W^{1,2p}(Q_1)} + c_4 \leq c_p$ independent of s^*.

We have verified (49) for $N = 1$ and all $p > 2$. Proceeding now inductively, using (48) (50) then (42) and the Sobolev embedding theorems, the desired estimates (49) are verified for all N and the Lemma is proved. □

The above corrections are also a remedy for the same flaws repeated in the follow up paper [2]. We point out that in his work [3] about the asymptotic behaviour of a pseudoholomorphic half plane, C. Abbas presents a more elegant approach to the asymptotic formula in [1]. We would like to thank C. Abbas for valuable discussions.

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