Ann. Inst. Henri Poincaré,

Vol. 14, n° 2, 1997, p. 187-209 *Analyse non linéaire*

Nonlinear instability in an ideal fluid

by

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ABSTRACT. – Linearized instability implies nonlinear instability under certain rather general conditions. This abstract theorem is applied to the Euler equations governing the motion of an inviscid fluid. In particular this theorem applies to all $2D$ space periodic flows without stagnation points as well as $2D$ space-periodic shear flows.

Key words: Euler equations, essential spectrum.

RÉSUMÉ. – L'instabilité linéarisée implique l'instabilité non linéaire sous certaines conditions assez générales. Ce théorème abstrait s'applique aux équations d'Euler qui gouvernent le mouvement d'un fluide non visqueux. En particulier ce théorème s'applique à tous les flots périodiques dans le plan, soit sans point de stagnation, soit des écoulements de cisaillement.

INTRODUCTION

In this paper we prove a theorem which states that, under appropriate conditions, linear instability of a steady flow of an ideal fluid implies

A.M.S. Classification: 76, 35.

nonlinear instability. Such a theorem is of course well known for ODE's. For the evolutionary Navier-Stokes equations describing a viscous flow (in a bounded domain with classical boundary conditions) such a result was proved by V. Yudovich (*see* [Y]). The case of the Euler equations addressed in the paper is totally different for at least two reasons. First, in no sense is the dynamics of an ideal fluid effectively finite-dimensional, as it is for the Navier-Stokes equations. The second reason, which is related to the first one, is that the spectral problem associated with the Euler equations is always a degenerate non-elliptic problem with a continuous spectrum. This also makes the question of both linear and nonlinear instability very much dependent on the particular norm used.

The crucial idea underlying the approach in this paper is to use two Banach spaces: a "large" space Z where the spectrum of the linearized operator is studied and a "small" space $X \hookrightarrow Z$ where a *local* existence theorem for the nonlinear equation can be proved. This idea has its origin in the recent paper of Y. Guo and W. Strauss [GS] who proved a similar theorem for the Vlasov-Poisson system which describes collisionless plasmas.

The paper divides into two parts. The first part is an abstract theorem, for which we give two variants (*see* Theorems 2.1 and 2.2). The abstract theorem states, under certain conditions, that spectral instability for the linearized operator L implies nonlinear instability. It is applicable to a wide variety of nonlinear PDEs where a local existence theorem is known. The main difficulty in applying either variant of the abstract theorem lies in proving for a particular PDE that either

(i) e^{tL} satisfies a spectral gap condition which permits projections onto the subspaces of growing and decaying modes (*see* Theorem 2.1)

(ii) e^{tL} has an eigenvalue with absolute value sufficiently close to the spectral radius (*see* Theorem 2.2).

Any problem for which the unstable spectrum of the semigroup is nonempty and purely discrete automatically satisfies both conditions. The second part of the paper studies the specific case of the Euler equation.

In section 1 we describe the spectral gap condition. In section 2 we prove Theorem 2.1 and we state the alternative approach of Theorem 2.2. The proof of Theorem 2.2 is an abstraction of the proof given in Guo and Strauss [GS] for the Vlasov-Poisson system. In section 3 we check that the conditions of Theorem 2.1 other than the spectral gap condition are satisfied for an *arbitrary* smooth Euler equilibrium and for usual functional spaces: Z being the space of solenoidal square integrable vectors, $X = X_s$ is a space of solenoidal vectors with components in the Sobolev space H^s, $s > \frac{n}{2} + 1$.

In section 4 we describe how a dynamical system can be employed to determine the growth rate due to instabilities in the continuous spectrum of the linearised Euler equation. In Vishik [V] an explicit formula is proved for the essential spectral radius as a Lyapunov type exponent. This result is used to prove that certain flows, for example $2D$ shear flows and $2D$ flows without stagnation points, have no unstable continuous spectrum. Hence the results of sections 1-3 prove that any such flow which can be shown to be linearly unstable must be nonlinearly unstable in X_s.

In section 5 we give an example of a linearly unstable $2D$ shear flow, namely the flow with velocity profile $\sin my$, for which all the conditions of Theorem 2.1 (or 2.2) are satisfied. We prove the existence of discrete unstable spectrum following the approach used for the viscous problem by Meshalkin and Sinai [MS] and Yudovich [Y], which utilizes continued fractions to derive and analyze the characteristic equation. It is demonstrated by construction that for $m > 1$ this characteristic equation has at least one root corresponding to a discrete unstable eigenvalue.

1. SPECTRAL GAP CONDITION

Let us fix a pair of Banach spaces $X \hookrightarrow Z$ with a dense embedding. We study the evolution equation

$$\dot{w} = Lw + N(w), \qquad w(0) = w_0 \tag{1.1}$$

where L is a generator of a C_0-group of operators in $\mathcal{L}(Z)$, e^{tL} leaves X invariant for $t \in \mathbb{R}$, $X \subset D(L)$, N being a nonlinear operator $N : X \to Z$. We will list separately our assumptions about L and N.

(H1) The nonlinear term N satisfies the inequality

$$\begin{cases} \|N(w)\|_Z \le c_0 \|w\|_X \|w\|_Z, & \text{for } w \in X \\ \text{with} \\ \|w\|_X < \rho \ \text{ for some } \ \rho > 0. \end{cases} \tag{1.2}$$

(H2) "**Gap condition**". Suppose that for any $t > 0$ the spectrum σ of $e^{tL} \in \mathcal{L}(Z)$ can be represented as follows:

$$\sigma = \sigma(e^{tL}) = \sigma_+ \cup \sigma_-, \qquad \sigma_+ \neq \emptyset \tag{1.3}$$

where

$$\sigma_+ \subset \{z \in \mathbb{C} \mid e^{Mt} < |z| < e^{\Lambda t}\}, \tag{1.4}$$

$$\sigma_- \subset \{z \in \mathbb{C} \mid e^{\lambda t} < |z| < e^{\mu t}\} \tag{1.5}$$

and

$$(1.6)\qquad -\infty < \lambda < \mu < M < \Lambda < \infty$$

We assume moreover that

$$(1.7)\qquad M > 0.$$

No assumptions about the sign of μ are made. The partition of σ is illustrated by Fig. 1. We denote by $P_\pm$ the Riesz projection corresponding to the partition (1.3):

$$(1.8)\qquad P_\pm = \frac{1}{2\pi i}\int_{\gamma_\pm} \frac{1}{z - e^{tL}}\,dz$$

where the contours $\gamma_\pm$ surround $\sigma_\pm$ (*see* Fig. 1). It is clear that $P_\pm$ does not depend on $t > 0$. We now introduce a new norm on Z. For any $x \in Z$ let

$$(1.9)\qquad |||x||| = |||P_+x||| + |||P_-x||| = \int_0^\infty \|e^{-L\tau}P_+x\|_Z e^{M\tau}\,d\tau + \int_0^\infty \|e^{L\tau}P_-x\|_Z e^{-\mu\tau}\,d\tau$$

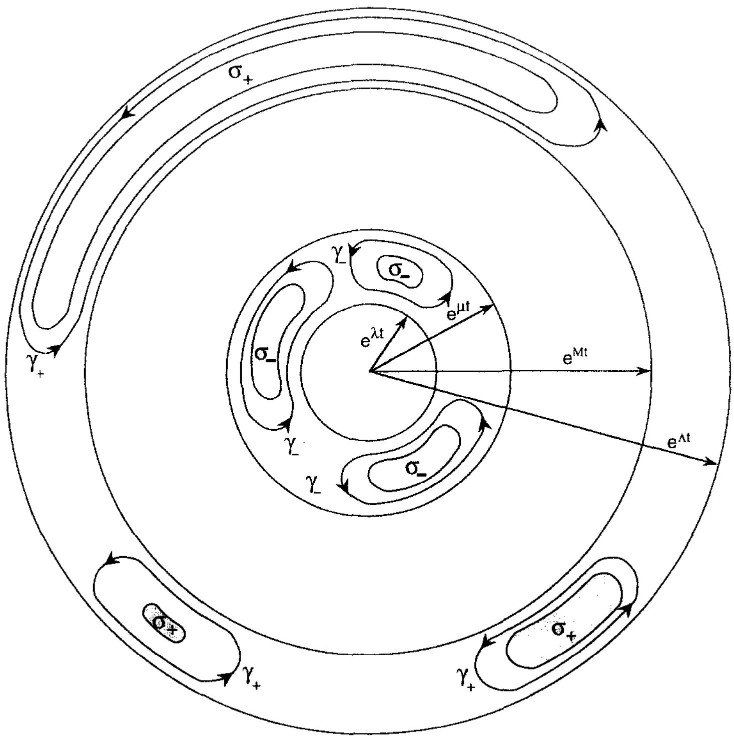

Fig. 1. – Illustration of the partition of a spectrum that satisfies the spectral gap condition.

LEMMA 1.1. – *The norm* $|||\cdot|||$ *is equivalent to* $\|\cdot\|_Z$*: there exists* $C > 0$ *such that*

$$C^{-1}\|x\|_Z \leq |||x||| \leq C\|x\|_Z. \tag{1.10}$$

Proof. – The restriction of e^{tL} to the image of $P_\pm$ is a strongly continuous group and

$$\sigma(e^{tL}|_{\operatorname{Im} P_\pm}) = \sigma_\pm, \qquad t > 0$$

This gives for the second term in the RHS of (1.9) for sufficiently small $\varepsilon > 0$

$$\|e^{L\tau}P_-x\|_Z \leq C_\varepsilon e^{(\mu-\varepsilon)\tau}\|P_-x\|_Z, \qquad \tau \geq 0$$

since $\sigma_- \subset \{z \in \mathbb{C} \mid |z| \leq e^{(\mu-2\varepsilon)t}\}$ for sufficiently small $\varepsilon > 0$. This yields

$$\int_0^\infty \|e^{L\tau}P_-x\|_Z e^{-\mu\tau}\,d\tau \leq \varepsilon^{-1}C_\varepsilon\|P_-x\|_Z. \tag{1.11}$$

Likewise taking negative τ we obtain

$$\|e^{-L\tau}P_+x\|_Z \leq C_\varepsilon e^{-(M+\varepsilon)\tau}\|P_+x\|_Z, \qquad \tau \geq 0$$

for sufficiently small $\varepsilon > 0$. Hence as above

$$\int_0^\infty \|e^{-L\tau}P_+x\|_Z e^{M\tau}\,d\tau \leq \varepsilon^{-1}C_\varepsilon\|P_+x\|_Z. \tag{1.12}$$

From (1.11) and (1.12) $|||x||| \lesssim \|x\|_Z$ *. On the other hand,

$$\|e^{-L\tau}P_+x\|_Z \gtrsim e^{-\Lambda\tau}\|P_+x\|_Z, \qquad \tau \geq 0;$$
$$\|e^{L\tau}P_-x\|_Z \gtrsim e^{\lambda\tau}\|P_-x\|_Z, \qquad \tau \geq 0.$$

Integrating with respect to τ yields

$$|||x||| \gtrsim \frac{1}{\Lambda - M}\|P_+x\|_Z + \frac{1}{\mu-\lambda}\|P_-x\|_Z \gtrsim \|x\|_Z \qquad \square$$

* Here and below $A \lesssim B$ means there exists a $c > 0$ such that $A \leq cB$.

2. NONLINEAR INSTABILITY: ABSTRACT SUFFICIENT CONDITION

We assume that there is a local existence theorem for the equation (1.1). This means that for any $w_0 \in X$ there exists a $T > 0$ and a unique

$$w(t) \in L^\infty((0,T);X) \cap C([0,T],Z) \tag{2.1}$$

which is a solution to (1.1), e.g., in the following sense: for any $\varphi \in \mathcal{D}(0,T)$

$$\int_0^T \{w(\tau)\varphi'(\tau) + (Lw(\tau) + Nw(\tau))\varphi(\tau)\}\, d\tau = 0. \tag{2.2}$$

The initial condition is assumed in the sense of strong convergence in Z:

$$\lim_{\tau \to 0+} \|w(\tau) - w_0\|_Z = 0.$$

Definition of nonlinear stability. – The trivial solution $w_0 = 0$ of the equation (1.1) is called nonlinearly stable in X (Lyapunov stable) if, no matter how small $\varepsilon > 0$ is, there exists a $\delta > 0$ so that $\|w(0)\|_X < \delta$ implies a) we can choose $T = \infty$ in (2.1), and b) $\|w(t)\|_X < \varepsilon$ for a.e. $t \in [0,\infty)$. The trivial solution $w_0 = 0$ is called nonlinearly unstable if it is not stable.

Remark. – By this definition we regard a "blowing up" solution (*i.e.*, there exists a maximal *finite* $T > 0$ in (2.1)) as a particular case of *instability*.

THEOREM 2.1. – *Let N satisfy* (H1) *and L satisfy the inequalities* (1.3)-(1.7) *of the spectral gap condition* (H2). *Let the equation* (1.1) *admit a local existence theorem in the sense described above. Then the trivial solution $w_0 = 0$ to the equation* (1.1) *is nonlinearly unstable.*

Proof. – Suppose the contrary: $w_0 = 0$ is nonlinearly stable. Let $\varepsilon > 0$ be sufficiently small: it will be specified later. Let $w(t)$, $t \in [0,\infty)$ be a *global* solution to (1.1). We know that such a global solution exists for $\|w_0\|_X < \delta$, where δ is constructed from ε using the definition of nonlinear stability.

Let for $t \geq 0$

$$v(t) = e^{-Lt} w(t). \tag{2.3}$$

Differentiating (2.3) and using (1.1) we obtain

$$\dot{v}(t) = e^{-Lt} N(w(t)). \tag{2.4}$$

From definition (1.9) and (2.3)

$$(2.5)\qquad \begin{aligned}|||P_+w(t)||| &= \int_0^\infty \|e^{-L(s-t)}P_+v(t)\|_Z e^{Ms}\,ds\\ &= e^{Mt}\int_{-t}^\infty \|e^{-L\tau}P_+v(t)\|_Z e^{M\tau}\,d\tau\\ &\stackrel{\text{def}}{=} e^{Mt}F(t).\end{aligned}$$

We compute the right derivative of $F(t)$. Letting $h > 0$, we have

$$(2.6)\qquad \begin{aligned}&\frac{F(t+h)-F(t)}{h}\\ &= \frac{1}{h}\int_{-t-h}^{-t} \|e^{-L\tau}P_+v(t+h)\|_Z e^{M\tau}\,d\tau\\ &+ \frac{1}{h}\int_{-t}^\infty \{\|e^{-L\tau}P_+v(t+h)\|_Z - \|e^{-L\tau}P_+v(t)\|_Z\}e^{M\tau}\,d\tau\\ &\stackrel{\text{def}}{=} I_1(h)+I_2(h)\end{aligned}$$

We have, because of the strong continuity $t \to v(t)$ in Z,

$$(2.7)\qquad \lim_{h\to 0} I_1(h) = \|e^{Lt}P_+v(t)\|_Z e^{-Mt} = e^{-Mt}\|P_+w(t)\|_Z.$$

On the other hand,

$$(2.8)\qquad \begin{aligned}|I_2(h)| &\le \int_{-t}^\infty \left\|e^{-L\tau}P_+\frac{v(t+h)-v(t)}{h}\right\|_Z e^{M\tau}\,d\tau\\ &= e^{-Mt}\int_0^\infty \left\|e^{-L\tau}P_+e^{Lt}\frac{v(t+h)-v(t)}{h}\right\|_Z e^{M\tau}\,d\tau\\ &= e^{-Mt}\left|\left|\left|e^{Lt}P_+\frac{v(t+h)-v(t)}{h}\right|\right|\right|\end{aligned}$$

From (2.4), (2.5)-(2.8) we obtain

$$(2.9)\qquad \begin{aligned}\frac{d}{dt^+}|||P_+w(t)||| &= M|||P_+w(t)||| + e^{Mt}\frac{d}{dt^+}F(t)\\ &\ge M|||P_+w(t)||| + \|P_+w(t)\|_Z - \overline{\lim_{h\to 0}} I_2(h)\\ &\ge M|||P_+w(t)||| + \|P_+w(t)\|_Z - |||P_+N(w(t))|||\end{aligned}$$

But, because $|||\cdot|||$ is a convex functional, the right derivative of $|||P_+w(t)|||$ exists for all $t \geq 0$. From (2.9) and (1.10)

$$(2.10) \quad |||P_+w(t_1)||| - |||P_+w(t_2)||| \geq \int_{t_2}^{t_1} \{(M + C^{-1})|||P_+w(\tau)||| - |||P_+N(w(\tau))|||\}\, d\tau$$

for any $t_1 \geq t_2 \geq 0$. Likewise for the minus component $P_-w(t)$,

$$(2.11) \quad \begin{aligned} |||P_-w(t)||| &= \int_0^\infty \|e^{L\tau}P_-w(t)\|_Z e^{-\mu\tau}\, d\tau \\ &= \int_0^\infty \|e^{L(t+\tau)}P_-v(t)\|_Z e^{-\mu\tau}\, d\tau \\ &= e^{\mu t}\int_t^\infty \|e^{L\tau}P_-v(t)\|_Z e^{-\mu\tau}\, d\tau. \end{aligned}$$

Therefore,

$$(2.12) \quad \frac{d}{dt^+}|||P_-w(t)||| = \mu|||P_-w(t)||| + e^{\mu t}\frac{d}{dt^+}G(t)$$

where

$$(2.13) \quad G(t) = \int_t^\infty \|e^{L\tau}P_-v(t)\|_Z e^{-\mu\tau}\, d\tau$$

We have for $h > 0$

$$(2.14) \quad \begin{aligned} &\frac{G(t+h) - G(t)}{h} \\ &= -\frac{1}{h}\int_t^{t+h} \|e^{L\tau}P_-v(t)\|_Z e^{-\mu\tau}\, d\tau \\ &+ \int_t^\infty \frac{1}{h}\{\|e^{L\tau}P_-v(t+h)\|_Z - \|e^{L\tau}P_-v(t)\|_Z\}e^{-\mu\tau}\, d\tau \\ &\stackrel{\text{def}}{=} I_1(h) + I_2(h) \end{aligned}$$

As above, from the strong continuity in Z of $t \to v(t)$,

$$(2.15) \quad \begin{aligned} \lim_{h\to 0} I_1(h) &= -\|e^{Lt}P_-v(t)\|_Z e^{-\mu t} \\ &= -\|P_-w(t)\|_Z e^{-\mu t} \end{aligned}$$

We have for the second term in the RHS of (2.14)

$$(2.16)\quad |I_2(h)| \leq \int_t^\infty \left\| e^{L\tau} P_- \frac{v(t+h)-v(t)}{h} \right\|_Z e^{-\mu\tau}\, d\tau$$
$$= e^{-\mu t} \int_0^\infty \left\| e^{L\tau} P_- e^{Lt} \frac{v(t+h)-v(t)}{h} \right\|_Z e^{-\mu\tau}\, d\tau$$
$$= e^{-\mu t} \left\| P_- e^{Lt} \frac{v(t+h)-v(t)}{h} \right\|$$

From (2.12)-(2.16) and (2.4)

$$(2.17)\qquad \frac{d}{dt^+} \| P_- w(t) \| \leq \mu \| P_- w(t) \| - \| P_- w(t) \|_Z + \| P_- N(w(t)) \|$$

Integrating both sides of (2.17) from t_2 to t_1 where $t_1 \geq t_2 \geq 0$ and using (1.10) we obtain

$$(2.18)\qquad \| P_- w(t_1) \| - \| P_- w(t_2) \|$$
$$\leq \int_{t_2}^{t_1} \{ (\mu - C^{-1}) \| P_- w(\tau) \| + \| P_- N(w(\tau)) \| \}\, d\tau,$$
$$t_1 \geq t_2 \geq 0$$

Subtracting (2.18) from (2.10) yields

$$(2.19)\qquad (\| P_+ w(t) \| - \| P_- w(t) \|) \Big|_{t=t_2}^{t=t_1}$$
$$\geq \int_{t_2}^{t_1} \{ M \| P_+ w(\tau) \| - \mu \| P_- w(\tau) \|$$
$$+ C^{-1} \| w(\tau) \| - \| N(w(\tau)) \| \}\, d\tau$$

We will use inequality (2.19) to prove nonlinear instability of the trivial solution $w_0 = 0$ of equation (1.1). Let $\bar{w}_0 \in X$ be an arbitrary vector satisfying

$$(2.20)\qquad \| P_+ \bar{w}_0 \| > \| P_- \bar{w}_0 \|, \qquad \|\bar{w}_0\|_X < 1$$

The condition (2.20) defines an open set in Z. Since X is dense in Z there has to be a solution $\bar{w}_0$ to (2.20). Let $w(t)$, $t \in [0, \infty)$ be a *global* solution to (1.1) with $w(0) = w_0 = \delta \bar{w}_0$. Since $\|w_0\|_X < \delta$ and because of our assumption that the trivial solution is nonlinearly stable in X we have

$$(2.21)\qquad \|w(t)\|_X < \varepsilon \qquad \text{a.e. } t \in [0, \infty).$$

Using (1.2), (1.10) we obtain from (2.21)

$$\|N(w(t))\| \leq Cc_0\varepsilon\|w(t)\|_Z \leq C^2c_0\varepsilon\|w(t)\| \quad \text{a.e.} \quad t \in [0,\infty) \tag{2.22}$$

Let $0 < T_1 \leq \infty$ be defined as

$$T_1 = \sup\{T > 0 \mid \|P_+w(t)\| - \|P_-w(t)\| > 0 \quad \text{for} \quad 0 \leq t \leq T\}$$

Because of (2.20) and strong continuity of $w : [0,\infty) \to Z$ we have $T_1 > 0$. We now assert that $T_1 = \infty$. Indeed, if $T_1 < \infty$, then

$$\|P_+w(T_1)\| - \|P_-w(T_1)\| = 0 \tag{2.23}$$

We apply (2.19) for $t_1 = T_1$, $t_2 = 0$ and conclude using (2.23), (2.22) and (1.6) that

$$\begin{aligned} \delta(\|P_-\bar{w}_0\| - \|P_+\bar{w}_0\|) &\geq \int_0^{T_1} M\|P_+w(\tau)\| - \mu\|P_-w(\tau)\| \\ &\qquad + C^{-1}\|w(\tau)\| - C^2c_0\varepsilon\|w(\tau)\|\,d\tau \\ &\geq \int_0^{T_1} C^{-1}\|w(\tau)\| - C^2c_0\varepsilon\|w(\tau)\|\,d\tau \end{aligned} \tag{2.24}$$

because $\|P_+w(\tau)\| > \|P_-w(\tau)\|$ for $\tau \in [0, T_1)$ by definition of T_1. Suppose ε is small enough so that $\varepsilon < \min(C^{-3}c_0^{-1}, \rho)$, then the RHS of (2.24) is positive while the LHS is negative (*see* (2.20)). This contradiction proves that $T_1 = \infty$.

Applying (2.19) again we have

$$\begin{aligned} &\|P_+w(t)\| - \|P_-w(t)\| \\ &\quad \geq \delta\|P_+\bar{w}_0\| - \delta\|P_-\bar{w}_0\| + \int_0^t \{M(\|P_+w(\tau)\| - \|P_-w(\tau)\|) \\ &\qquad + (M-\mu)\|P_-w(\tau)\| + C^{-1}\|w(\tau)\| - N\|w(\tau)\|\}\,d\tau \\ &\quad \geq \delta(\|P_+\bar{w}_0\| - \|P_-\bar{w}_0\|) + M\int_0^t \{\|P_+w(\tau)\| - \|P_-w(\tau)\|\}\,d\tau \\ &\qquad t \in [0,\infty) \end{aligned}$$

provided $\varepsilon < \min(c_0^{-1}C^{-3}, \rho)$ as above. Using Gronwalls' inequality we get

$$\|P_+w(t)\| - \|P_-w(t)\| \geq \delta(\|P_+\bar{w}_0\| - \|P_-\bar{w}_0\|)\exp Mt, \quad t \in [0,\infty) \tag{2.25}$$

For sufficiently large t (2.25) contradicts our assumption that $\|w(t)\|_X < \varepsilon$. □

We now formulate the second abstract theorem applicable to the Euler equation. We replace the conditions (H1), (H2) by a different set of conditions. Consider the equation (1.1) where L and N have all the properties stated at the beginning of §1. We assume the following.

(H1′) There exist $\eta \in (0,1]$, $c_0 > 0$, $\rho > 0$ so that

$$\|N(w)\|_Z \le c_0 \|w\|_X^{1-\eta} \|w\|_Z^{1+\eta}, \qquad w \in X, \quad \|w\|_X < \rho. \tag{2.26}$$

(H2′) Let

$$\Lambda_1 = \lim_{t\to\infty} \frac{1}{t} \log \|e^{tL}\|_Z > 0$$

Assume there exist $\lambda_1 > \frac{\Lambda_1}{1+\eta}$, $c_1 > 0$, $c_2 > 0$ and $w_0 \in X$, $w_0 \neq 0$ so that

$$c_1 e^{t\lambda_1} \|w_0\|_Z \le \|e^{tL} w_0\|_Z \le c_2 e^{t\lambda_1} \|w_0\|_Z, \qquad 0 \le t < \infty.$$

THEOREM 2.2. – *Let the conditions* (H1′) *and* (H2′) *be satisfied. Suppose the equation* (1.1) *admits a local existence theorem. Then the trivial solution to the equation* (1.1) *is nonlinearly unstable.*

We omit the proof since the proof of a similar result appeared in [GS].

3. THE EULER EQUATION

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^∞-smooth boundary $\partial\Omega$ so that $\overline{\Omega}$ has a structure of an n-dimensional manifold with boundary. Let u_0 be a C^∞-vector field on $\bar{\Omega}$ which satisfies the steady Euler equations with classical boundary conditions

$$\begin{cases} (u_0, \nabla) u_0 + \nabla p_0 = 0 \\ \operatorname{div} u_0 = 0, \qquad (u_0, n)|_{\partial\Omega} = 0 \end{cases} \tag{3.1}$$

where $p_0 : \bar{\Omega} \to \mathbb{R}$ is a C^∞-smooth pressure. Here n denotes the unit outward normal vector on the boundary $\partial\Omega$. Alternatively let $\Omega = T^n = \mathbb{R}^n / 2\pi\mathbb{Z}^n$ and

$$u_0 \in (C^\infty(T^n))^n, \qquad p_0 \in C^\infty(T^n)$$

satisfies 3.1 (in this case $\partial\Omega = \emptyset$). We take arbitrary $s > \frac{n}{2} + 1$ and denote

$$X = X_s = \{w \in (H^s(\Omega))^n \mid \operatorname{div} w = 0 \text{ in } \Omega;\ (w, n)|_{\partial\Omega} = 0\} \tag{3.2}$$

$$Z = \{w \in (L^2(\Omega))^n \mid \operatorname{div} w = 0 \text{ in } \Omega;\ (w, n)|_{\partial\Omega} = 0\} \tag{3.3}$$

In the case $\Omega = T^n$ of course there is no boundary condition imposed.

We define L as the generator of the group (linearized Euler equation) which is given by

$$(3.4) \qquad Lw = -(u_0, \nabla)w - (w, \nabla)u_0 - \nabla p, \qquad w \in Z \cap (C^\infty(\bar{\Omega}))^n.$$

Then $e^{Lt}w_0 = w(t)$ for $w_0 \in Z$ where $w(t)$ is the solution to the linearized Euler equation,

$$(3.5) \qquad \begin{cases} \dot{w}(t) = -(u_0, \nabla)w - (w, \nabla)u_0 - \nabla p \\ \operatorname{div} w = 0, \qquad w(0) = w_0 \in Z \end{cases}$$

We now define $N : X \to Z$ as follows

$$(3.6) \qquad N(w) = -(w, \nabla)w - \nabla q.$$

We note that for this choice of L, N equation (1.1) becomes the standard Euler equation

$$(3.7) \qquad \begin{cases} (u_0 + w)^\cdot = -(u_0 + w, \nabla)(u_0 + w) - \nabla P \\ \operatorname{div}(u_0 + w) = 0 \end{cases}$$

PROPOSITION 3.1. – *The operator N satisfies*

$$\|N(w)\|_Z \leq c_0 \|w\|_X \|w\|_Z \qquad \textit{for all} \quad w \in X.$$

Proof. – According to the Weyl decomposition lemma,

$$\begin{aligned} \|N(w)\|_Z &= \| - (w, \nabla)w - \nabla q\|_{(L^2(\Omega))^n} \\ &\leq \|(w, \nabla)w\|_{(L^2(\Omega))^n} \lesssim \|w\|_{(L^2(\Omega))^n} \|\nabla w\|_{(L^\infty(\Omega))^{n\times n}} \\ &\lesssim \|w\|_{(L^2(\Omega))^n} \|w\|_{(H^s(\Omega))^n} \\ &= \|w\|_Z \|w\|_X \end{aligned}$$

We used the Sobolev embedding theorem on the last step ($s > \frac{n}{2} + 1$). □

All the general conditions we imposed on the pair $X \hookrightarrow Z$ are clearly satisfied by (3.2), (3.3); the spectral gap condition, on the contrary, needs to be checked in each case separately. The local existence theorem for the Euler equation in X_s for $s > \frac{n}{2} + 1$ is well known (for example, *see* [T] following earlier results in [L], [G], [W], [K]).

Remark. – To our knowledge a local existence theorem for the Euler equation is unknown in X_s for $s \leq \frac{n}{2} + 1$.

Remark. – As follows from [BB] the results of this section are valid also for

$$X = X_s = \{w \in (W^{s,p}(\Omega))^n \mid \operatorname{div} w = 0,\ (w,n)|_{\partial\Omega} = 0\};$$
$$Z = \{w \in (L^p(\Omega))^n \mid \operatorname{div} w = 0,\ (w,n)|_{\partial\Omega} = 0\}$$
$$\text{where } s > \frac{n}{p} + 1, \qquad 1 < p < \infty.$$

They are also valid in Hölder classes.

We will indicate now that for appropriate choice of X, Z the condition (2.26) is satisfied for the Euler equation.

LEMMA 3.2. – *Let* $s > \frac{n}{2} + 1$, $X = X_s$ *as in* (3.2), Z *is defined by* (3.3). *Then the inequality* (2.26) *for the nonlinear term* $N(w)$ *defined by* (3.6) *is satisfied for* $\rho = \infty$, *appropriate* $c_0 > 0$ *and* $\eta = \frac{1}{2} - \frac{2+n}{4s} > 0$.

Proof. – Following the proof of Proposition 3.1 and choosing $r = \frac{1}{2}(s + \frac{n}{2} + 1) > \frac{n}{2} + 1$,

$$\begin{aligned}
\|N(w)\|_Z &\lesssim \|w\|_{(L^2(\Omega))^n} \|\nabla w\|_{(L^\infty(\Omega))^{n\times n}} \\
&\lesssim \|w\|_{(L^2(\Omega))^n} \|w\|_{(H^r(\Omega))^n} \\
&\lesssim \|w\|^{1+\eta}_{(L^2(\Omega))^n} \|w\|^{1-\eta}_{(H^s(\Omega))^n}
\end{aligned}$$

On the last step we noticed that $r = (1-\eta)s$ and used the interpolation inequality

$$\|w\|_{(H^r(\Omega))^n} \leq \|w\|^{\eta}_{(L^2(\Omega))^n} \|w\|^{1-\eta}_{(H^s(\Omega))^n} \qquad \square$$

4. THE SPECTRUM OF THE EVOLUTION OPERATOR FOR THE LINEARIZED EULER EQUATION

Here we analyze $\sigma(e^{tL})$ where L is defined as in (3.4). In general we do not have a recipe to check whether for a given smooth flow u_0 the spectral gap condition (1.3)-(1.7) is satisfied. Little information is known in general.

We first define the essential spectrum (following Browder [B]). For any Banach space B and an operator $T \in \mathcal{L}(B)$ we use the following classification of spectral points.

A point $z \in \sigma(T)$ is called a point of *discrete spectrum* if it satisfies the following conditions:

1. z is an isolated point in $\sigma(T)$.
2. z has finite multiplicity; that is, $\bigcup_{r=1}^{\infty} \ker(z - T)^r = N$ is a finite-dimensional subspace in B.

3. The range of $z - T$ is closed, which implies that there is a complementary subspace $Q \subset B$ such that $B = N \oplus Q$, $TQ \subset Q$ and $(z - T)$ is invertible on Q. On the contrary, if z does not satisfy (1.3), it is called a point of *essential spectrum*. Thus

$$\sigma(T) = \sigma_{\text{ess}}(T) \cup \sigma_{\text{disc}}(T) \tag{4.1}$$

and the union in (4.1) is disjoint. We define for any $T \in \mathcal{L}(B)$, following Nussbaum [N],

$$r_{\text{ess}}(T) = \sup\{|z| \bigm| z \in \sigma_{\text{ess}}(T)\}. \tag{4.2}$$

We will use below the following theorem proved recently by Vishik [V].

THEOREM 4.1. – *Let $\Omega = T^n = \mathbb{R}^n/2\pi\mathbb{Z}^n$ and u_0 be a C^∞ steady solution to the Euler equation $(u_0, \nabla)u_0 + \nabla p_0 = 0$, $\operatorname{div} u_0 = 0$, $u_0 \in (C^\infty(\Omega))^n$, $p_0 \in C^\infty(\Omega)$. Then for any $t > 0$,*

$$r_{\text{ess}}(e^{tL}) = e^{\omega t} \tag{4.3}$$

where

$$\omega = \lim_{t\to\infty} \frac{1}{t} \log \sup_{\substack{x_0,\xi_0,b_0 \\ (b_0,\xi_0)=0 \\ |\xi_0|=1,\ |b_0|=1}} |b(x_0, \xi_0, b_0; t)| \tag{4.4}$$

Here (x, ξ, b) satisfies the following system of ODE's (which we call the bicharacteristic amplitude equations)

$$\begin{cases} \dot{x} = u_0(x) \\ \dot{\xi} = -\left(\dfrac{\partial u_0}{\partial x}\right)^T \xi \\ \dot{b} = -\left(\dfrac{\partial u_0}{\partial x}\right) b + 2|\xi|^{-2}\left(\xi, \dfrac{\partial u_0}{\partial x} b\right)\xi \end{cases} \tag{4.5}$$

$$x(0) = x_0, \quad \xi(0) = \xi_0, \quad b(0) = b_0$$

The quantity on the RHS of (4.4) is the Lyapunov exponent of the cocycle over the dynamical system in the projectivization of the cotangent bundle defined by the b-equation. The dynamical system describes the evolution of a point x and a direction $\pm\xi/|\xi|$ at this point. It is given by the first two equations (4.5).

This theorem implies in particular that *any* $z \in \sigma(e^{tL})$ with $|z| > e^{\omega t}$ is a point of *discrete spectrum*. Any accumulation point of $\sigma_{\text{disc}}(e^{tL})$

necessarily belongs to $\sigma_{\text{ess}}(e^{tL})$. Thus if $\sigma(e^{tL}) \cap \{|z| > e^{\omega t}\} \neq \emptyset$ then there exists a partition

$$\sigma(e^{tL}) = \sigma_+ \cup \sigma_-,$$

satisfying the gap condition (1.3)-(1.7). We note that for any flow u_0, $\omega \geq 0$ (*see* [V]) and therefore M can be chosen to be positive.

COROLLARY 4.2. – *Assume the conditions of the Theorem 4.1. Suppose there is a $z \in \sigma(e^{tL})$ with $|z| > e^{\omega t}$ then the flow u_0 is nonlinearly unstable in $X = X_s$ for every $s > \frac{n}{2} + 1$.*

Proof. – We combine either Theorem 2.1 or Theorem 2.2 with Theorem 4.1. □

The Lyapunov exponent ω can be effectively computed in a number of examples [FV]. In [FV] it is proved that exponential stretching (positivity of the Lyapunov exponent along at least one Lagrangian trajectory) implies that $\omega > 0$. In the present paper we show that $\omega = 0$ for several classes of flows *without exponential stretching*. We present here two examples of this situation.

PROPOSITION 4.3. – *Let $\Omega = T^2$, u_0 as in the formulation of the theorem 4.1. Let $u_0(x) \neq 0$ for all $x \in T^2$. Then $\omega = 0$ and hence $r_{\text{ess}}(e^{tL}) = 1$ for $t > 0$.*

Proof. – We first point to the following general feature of the system (4.5). We claim that $(b_0, \xi_0) = 0$ implies

$$(b(t), \xi(t)) = 0, \qquad t \in \mathbb{R} \tag{4.6}$$

Indeed, differentiating and using (4.5)

$$(b, \xi)^{\cdot} = \left(b, -\left(\frac{\partial u_0}{\partial x}\right)^T \xi\right) - \left(\frac{\partial u_0}{\partial x} b, \xi\right) + 2\left(\frac{\partial u_0}{\partial x} b, \xi\right) = 0.$$

We next assert that for $n = 2$

$$|b(t)| \cdot |\xi(t)| = \text{const} \tag{4.7}$$

along any trajectory of (4.5). Indeed,

$$\frac{d}{dt}(b \times \xi) = \left(-\frac{\partial u_0}{\partial x} b\right) \times \xi - b \times \left(\left(\frac{\partial u_0}{\partial x}\right)^T \xi\right)$$

But $(\frac{\partial u_0}{\partial x}\xi - (\frac{\partial u_0}{\partial x})^T \xi, \xi) = 0$. Since, according to (4.6) $(b, \xi) = 0$, it also follows that $\frac{\partial u_0}{\partial x}\xi - (\frac{\partial u_0}{\partial x})^T \xi = c(t) b$ for some function $c(t)$.

Hence

$$\frac{d}{dt}(b \times \xi) = -\operatorname{tr}\left(\frac{\partial u_0}{\partial x}\right) \cdot (b \times \xi) = 0 \tag{4.8}$$

since $\operatorname{div} u_0 = 0$. Thus (4.7) is proved.

Now we assert that, given the trajectory $x(t)$, the equation for ξ can be solved explicitly for $n = 2$ and u_0 non vanishing as follows. Let k be a unit vector in the directions x_3. We may decompose

$$\xi_0 = c_1 \frac{u_0(x_0)}{|u_0(x_0)|^2} + c_2\, k \times u_0(x_0). \tag{4.9}$$

Then

(4.10)

$$\xi(t) = c_1 \frac{u_0(x(t))}{|u_0(x(t))|^2}$$
$$+ \Big(c_2 - c_1 \int_0^t \frac{\left(\frac{\partial u_0}{\partial x} u_0 + \left(\frac{\partial u_0}{\partial x}\right)^T u_0,\ k \times u_0\right)}{|u_0|^4}(x(\tau))\, d\tau\Big)\, k \times u_0(x(t))$$

In order to prove this identity we differentiate the right side of (4.10) and use (4.5) to get

(411)

$$c_1 \frac{\frac{\partial u_0}{\partial x} u_0}{|u_0|^2} - 2c_1 \frac{u_0\left(\frac{\partial u_0}{\partial x} u_0, u_0\right)}{|u_0|^4}$$
$$- c_1 \frac{\left(\frac{\partial u_0}{\partial x} u_0 + \left(\frac{\partial u_0}{\partial x}\right)^T u_0,\ k \times u_0\right)}{|u_0|^4}\ k \times u_0$$
$$+ \Big(c_2 - c_1 \int_0^t \frac{\left(\frac{\partial u_0}{\partial x} u_0 + \left(\frac{\partial u_0}{\partial x}\right)^T u_0,\ k \times u_0\right)}{|u_0|^4}\, d\tau\Big)\, k \times \left(\frac{\partial u_0}{\partial x} u_0\right)$$

We claim that the last factor $k \times \frac{\partial u_0}{\partial x} u_0 = -(\frac{\partial u_0}{\partial x})^T (k \times u_0)$. Indeed, for every $\eta \in \mathbb{R}^2$

$$\left(-\left(\frac{\partial u_0}{\partial x}\right)^T (k \times u_0), \eta\right) = -\left(k \times u_0, \frac{\partial u_0}{\partial x}\eta\right)$$
$$= \left(\frac{\partial u_0}{\partial x}\ k \times u_0, \eta\right) + \left(k \times \frac{\partial u_0}{\partial x} u_0, \eta\right)$$
$$= \left(k \times \frac{\partial u_0}{\partial x} u_0, \eta\right)$$

since $\frac{\partial u_0}{\partial x}k = 0$ and $\operatorname{div} u_0 = 0$ as claimed. Thus in order to check that (4.11) equals $-(\frac{\partial u_0}{\partial x})^T$ times the right side of (4.10), we need only check the first three terms of (4.11):

$$\begin{aligned}(4.12)\qquad & \frac{\frac{\partial u_0}{\partial x}u_0}{|u_0|^2} - 2\frac{u_0\left(\frac{\partial u_0}{\partial x}u_0, u_0\right)}{|u_0|^4} \\ & \quad - \frac{\left(\frac{\partial u_0}{\partial x}u_0 + \left(\frac{\partial u_0}{\partial x}\right)^T u_0,\ k \times u_0\right)}{|u_0|^4} k \times u_0 \\ & = -\left(\frac{\partial u_0}{\partial x}\right)^T \frac{u_0}{|u_0|^2}\end{aligned}$$

The left and right sides of (4.12) have the same scalar product with u_0 and $k \times u_0$ which proves (4.12). Thus (4.10) follows.

From (4.9) and using $u_0(x) \neq 0$ for all $x \in T^2$, we get

$$|c_1| \leq C|\xi_0| = C; \qquad |c_2| \leq C$$

for some constant C. Hence from (4.10)

$$|\xi(t)| \lesssim (1+t)|\xi_0| = 1+t, \qquad t \geq 0.$$

Changing $t \mapsto -t$ we find also

$$|\xi(t)| \gtrsim \frac{|\xi_0|}{1+t} = \frac{1}{1+t}, \qquad t \geq 0$$

Indeed if we start at point $x(t)$ and apply the previous argument to $-u_0$, then the equation for ξ has the same solution just run in the reverse direction. Thus we obtain $|\xi(0)| \lesssim (1+t)|\xi(t)|$ as claimed. Thus from (4.7)

$$|b(t)| \lesssim (1+t)|b_0| = 1+t, \qquad t \geq 0$$

Hence, from (4.4), $\omega = 0$. □

In the following proposition we consider a very simple class of $2D$ shear flows that may vanish.

PROPOSITION 4.4. – *Let* $\Omega = T^2$, $u(x_1, x_2) = (U(x_2), 0)$. *Then* $\omega = 0$.

Proof. – From (4.7) (we were not making use of the assumption $u_0 \neq 0$ in deriving (4.7))

$$|b(t)| \lesssim \frac{1}{|\xi(t)|} \lesssim (1+t), \qquad t \geq 0$$

because ξ is a function linear in t. Hence $\omega = 0$. □

5. AN EXAMPLE OF A SHEAR FLOW THAT IS NONLINEARLY UNSTABLE

For the case of classical boundary conditions which are not addressed in this section, the Rayleigh criterion states that any shear flow without inflexion points is linearly stable. Of course for periodic boundary condition there must be an inflexion point.

In this section we prove the existence of an unstable eigenvalue $z \in \sigma_{\text{disc}}(e^{tL})$, $|z| > 1$ for a particular shear flow. In fact we will construct a C^∞ (even C^ω) eigenfunction of L with an eigenvalue of positive real part. According to proposition 4.2 and theorem 4.1 the spectral gap condition is satisfied for such a flow. Thus according to Theorem 2.1 and the results of §3 showing that all the other conditions (besides the spectral gap condition) of this theorem are satisfied, this flow is *nonlinearly* unstable in say X_s, $s > 2$ (*see* (3.2)).

In this section $n = 2$, $u_0 = (U(y), 0)$, $R^2 = (x, y)$, $T^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2$. We choose the flow

$$U(y) = \sin my, \qquad m \in \mathbb{Z}_+ \setminus \{0\}. \tag{5.1}$$

For perturbations w satisfying $(2\pi)^{-2} \int_{T^2} w\, dx\, dy = 0$, we may introduce a stream function $\psi : T^2 \to \mathbb{R}$, $(2\pi)^{-2} \int \psi\, dx\, dy = 0$,

$$w_1 = -\frac{\partial \psi}{\partial y}, \qquad w_2 = \frac{\partial \psi}{\partial x} \tag{5.2}$$

The equation (3.5) written for the stream function ψ instead of w is

$$(\Delta\psi)^{\cdot} + \sin my \frac{\partial}{\partial x}(\Delta\psi + m^2\psi) = 0 \tag{5.3}$$

We will construct a solution to (5.3) of the form

(5.4)
$$\psi(x, y, t) = e^{\sigma t + ikx} \sum_{j=-\infty}^{\infty} c_j e^{ijy} = e^{\sigma t + ikx} \chi(y) \qquad \operatorname{Re} \sigma > 0, \quad k \in \mathbb{Z}$$

The coefficients c_j decay exponentially as $j \to \pm\infty$, thus the eigenfunction for L constructed using (5.2) is in $(C^\omega(T^2))^2$. For fixed k our eigenvalue problem is described by the Rayleigh equation which follows from (5.3), (5.4)

$$\sigma\left(-k^2 + \frac{d^2}{dy^2}\right)\chi + ik \sin my \left(m^2 - k^2 + \frac{d^2}{dy^2}\right)\chi = 0 \tag{5.5}$$

Here $k \in \mathbb{Z}$ is fixed and we are looking for a solution $X : T^1 \to \mathbb{R}$ with $\operatorname{Re} \sigma > 0$.

PROPOSITION 5.1. – *Let $m > 1$ and $m^2 \neq m_1^2 + m_2^2$ where $m_1, m_2 \in \mathbb{Z}$, $m_1 \cdot m_2 \neq 0$. Then there exists a real eigenvalue σ of the operator L with $\sigma > 0$ with a smooth (analytic) eigenfunction. Therefore this flow is nonlinearly unstable in X_s with $s > 2$.*

We follow the approach of [MS] for the Navier-Stokes equations who investigated stability of *viscous* shear flow with a profile like (5.1) using the techniques of continued fractions. Their elegant paper was followed by [Yu] and recently by [Li]. To our knowledge no previous proof of instability for the inviscid flow (5.1) appears in the literature. In 1935 Tollmien [To] gave a heuristic demonstration of instability of an inviscid shear flow $U(y) = \sin y$ with the boundary condition $w_2 = 0$ in a sufficiently wide channel. This is a classical result widely quoted in engineering literature although no mathematical proof has been given to our knowledge.

Proof. – The recurrence relation equivalent to (5.5) is

$$\begin{aligned} \frac{2\sigma}{k}(j^2 + k^2)c_j &+ (k^2 - m^2 + (j - m)^2)c_{j-m} \\ &- (k^2 - m^2 + (j + m)^2)c_{j+m} = 0 \end{aligned} \tag{5.6}$$

We assume $k \neq 0$ since otherwise (5.5) does not have nontrivial solutions with $\operatorname{Re} \sigma > 0$. We may moreover assume $k > 0$ since the equation (5.5) remains valid under substitutions $k \mapsto -k$, $\sigma \mapsto \bar{\sigma}$, $X \mapsto \bar{X}$. We assumed for simplicity that the diophantine equation

$$m^2 = m_1^2 + m_2^2$$

does not have solutions with $m_1 m_2 \neq 0$. We will construct below a solution with $k < m$. We define

$$a_j = \frac{2\sigma}{k} \frac{j^2 + k^2}{j^2 + k^2 - m^2}, \qquad j \in \mathbb{Z} \tag{5.7}$$

Note that denominator in (5.7) is not vanishing. Let

$$d_j = c_j(j^2 + k^2 - m^2), \qquad j \in \mathbb{Z}. \tag{5.8}$$

Then (5.6)-(5.8) imply

$$d_{j+m} = a_j d_j + d_{j-m}, \qquad j \in \mathbb{Z}. \tag{5.9}$$

We shall construct a sequence $d_j \neq 0$, $j \in m\mathbb{Z}$. Define

$$\rho_j = \frac{d_j}{d_{j-m}}, \qquad j > 0 \tag{5.10}$$

$$\tilde{\rho}_j = \frac{d_{j-m}}{d_j}, \qquad j \leq 0. \tag{5.11}$$

It follows from (5.9), (5.10) that

$$\begin{aligned} \rho_m &= \frac{-1}{a_m - \rho_{2m}} = \frac{-1}{a_m + \frac{1}{a_{2m} - \rho_{3m}}} = \cdots \\ &= \frac{-1}{a_m + \frac{1}{a_{2m} + \cdots + \frac{1}{a_{pm} - \rho_{(p+1)m}}}} \\ &\stackrel{\text{def}}{\equiv} \frac{-1}{[a_m, \ldots, a_{(p-1)m}, a_{pm} - \rho_{(p-1)m}]} \end{aligned} \tag{5.12}$$

We note that

$$a_j = \frac{2\sigma}{k}(1 + O(j^{-2})), \qquad j \to \pm\infty \tag{5.13}$$

We *define* now $d_j = 0$ for $j \not\equiv 0 \pmod{m}$ and *define* for $p \geq 1$, $p \in \mathbb{Z}$, σ real and positive

$$\rho_{pm} = \rho_{pm}(\sigma) = \frac{-1}{[a_{pm}, a_{(p+1)m}, \ldots]} \tag{5.14}$$

Obviously the continued fraction in the RHS of (5.14) is convergent. Indeed the partial denominators grow exponentially because the elements are positive and bounded away from zero. Likewise, from (5.9), (5.11)

$$\begin{aligned} \tilde{\rho}_0 &= \frac{1}{a_{-m} + \tilde{\rho}_{-m}} = \frac{1}{a_{-m} + \frac{1}{a_{-2m} + \tilde{\rho}_{-2m}}} \\ &= \frac{1}{[a_{-m}, a_{-2m}, \ldots, a_{-pm} + \tilde{\rho}_{-pm}]} \end{aligned}$$

We define for $p \geq 0$, $p \in \mathbb{Z}$, σ real and positive

$$\tilde{\rho}_{-pm} = \tilde{\rho}_{-pm}(\sigma) = \frac{1}{[a_{-(p+1)m}, a_{-(p+2)m}, \ldots]} \tag{5.15}$$

Again, convergence of the continued fraction in the RHS of (5.15) is evident.

It is easy to check that as $p \to \infty$

$$(5.16) \qquad \rho_{pm} \to \rho_\infty = \frac{\sigma}{k} - \sqrt{1 + \frac{\sigma^2}{k^2}};$$

$$(5.17) \qquad \tilde{\rho}_{pm} \to -\rho_\infty = -\frac{\sigma}{k} + \sqrt{1 + \frac{\sigma^2}{k^2}}.$$

We see that

$$(5.18) \qquad -1 < \rho_\infty < 0$$

From (5.9)-(5.11) we obtain the characteristic equation

$$(5.19) \qquad \rho_m = a_0 + \tilde{\rho}_0 = a_0 + \frac{1}{a_{-m} + \tilde{\rho}_{-m}}$$

Since $a_{-pm} = a_{pm}$ (*see* (5.7)), (5.19) is equivalent to

$$(5.20) \qquad -\frac{a_0}{2} = \frac{1}{[a_m, a_{2m}, \ldots]} \equiv F(\sigma)$$

Suppose there exists σ real and positive such that (5.20) is satisfied. Choose

$$(5.21) \qquad \begin{cases} d_0 = 1; \\ d_{pm} = \rho_m \ \rho_{2m} \ \cdots \ \rho_{pm}; \\ d_{-pm} = \tilde{\rho}_0 \ \tilde{\rho}_{-m} \ \cdots \ \tilde{\rho}_{(p-1)m}. \end{cases}$$

Then this sequence satisfies (5.9) by construction and $d_{pm}, d_{-pm} \to 0$ exponentially because of (5.16)-(5.18).

We are now going to study the characteristic equation (5.20). The RHS $F(\sigma)$ defined for $\sigma \in (0, \infty)$ is a continuous function because the sequence of partial fractions is uniformly convergent on $[\varepsilon, \infty)$ for any $\varepsilon > 0$ (with exponential estimate). Because the elements are positive, we have

$$(5.22) \qquad g(\sigma) = \frac{1}{a_m + \frac{1}{a_{2m}}} < F(\sigma) < \frac{1}{a_m} = f(\sigma), \qquad 0 < \sigma < \infty$$

Figure 2 shows the graphs of $g(\sigma)$ and $f(\sigma)$ together with the graph of $-\frac{a_0}{2} = \frac{\sigma}{k} \frac{k^2}{m^2 - k^2}$.

Here

$$f(\sigma) = \frac{k}{2\sigma} \frac{k^2}{m^2 + k^2};$$

$$g(\sigma) = \frac{1}{\frac{2\sigma}{k} \frac{m^2+k^2}{k^2} + \frac{k}{2\sigma} \frac{3m^2+k^2}{4m^2+k^2}}$$

As is obvious from Figure 2 equation (5.20) is guaranteed to have a positive solution provided that $g'(0) > -\frac{a_0'(0)}{2}$. This condition is

$$2\frac{4m^2 + k^2}{3m^2 + k^2} > \frac{k^2}{m^2 - k^2} \tag{5.23}$$

which is guaranteed for, say,

$$\frac{m^2}{k^2} > \frac{23}{16} \tag{5.24}$$

□

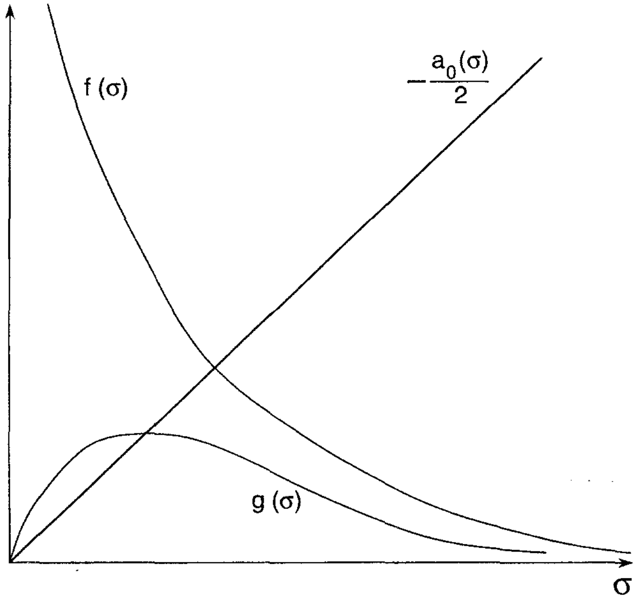

Fig. 2. – Graphs of curves showing the existence of a solution to the characteristic equation (5.20).

ACKNOWLEDGEMENTS

S. Friedlander acknowledges support from NSF grants DMS-9123946 and GER-9350069. M. Vishik acknowledges support from NSF grant DSM-9301172. Their collaborative work is also partially supported by DMS-9300752. W. Strauss acknowledges support from NSF DMS-9322116 and ARO DAAH 04-93-G0198. The authors thank Margaret Combs for her excellent typing of the manuscript.

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*(Manuscript received November 8, 1994;
revised January 30, 1996.)*