

Solitary waves of generalized Kadomtsev-Petviashvili equations

by

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ABSTRACT. – We classify the existence and non-existence cases for localized solitary waves of generalized Kadomtsev-Petviashvili equations according to the sign of the transverse dispersion coefficients and to the nonlinearity. We also prove regularity properties of the solitary waves.

Key words: Kadomtsev-Petviashvili equations, solitary waves.

RÉSUMÉ. – Nous classifions les cas d'existence et de non-existence d'ondes solitaires localisées pour les équations de Kadomtsev-Petviashvili généralisées selon le signe des coefficients de la dispersion transverse et la puissance du terme non-linéaire. Nous montrons également des propriétés de régularité de ces ondes solitaires.

1. INTRODUCTION

Kadomtsev-Petviashvili equations are “universal” models for dispersive, weakly nonlinear waves, which are essentially unidimensional, when weak transverse effects are taken into account [10] [13]. They read, for a general

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nonlinearity $f(u)$,

$$(1.1) \quad \begin{cases} u_t + f(u)u_x + u_{xxx} + \varepsilon v_y = 0, \\ v_x = u_y \end{cases}$$

$u = u(x, y, t)$, $(x, y) \in \mathbb{R}^2$, $t \geq 0$ in dimension 2, and

$$(1.2) \quad \begin{cases} u_t + f(u)u_x + u_{xxx} + av_y + bw_z = 0, \\ v_x = u_y, \\ w_x = u_z, \end{cases}$$

$u = u(x, y, z, t)$, $(x, y, z) \in \mathbb{R}^3$, $t \geq 0$, in dimension 3.

The constants ε , a , b measure the transverse dispersion effects and are normalized to ± 1 . The “usual” Kadomtsev-Petviashvili equations correspond to $f(u) = u$. We will consider therein power nonlinearities. We recall that (1.1) with $f(u) = u$ is integrable by the inverse scattering method and is classically called KPI ($\varepsilon = -1$) or KP II ($\varepsilon = +1$).

Many rigorous results have recently appeared concerning the Cauchy problem associated to (1.1) (1.2) (mainly in the KPI or KP II cases). To quote a few, [5], [7], [8], [9], [17], [19], [21], [22] and the survey [18]. We are interested here in solitary wave solutions of (1.1) (1.2). In order to give a precise definition we need to introduce a few spaces.

We shall denote for $d = 2, 3$, Y the closure of $\partial_x(C_0^\infty(\mathbb{R}^d))$ for the norm

$$\|\partial_x \varphi\|_Y = (\|\nabla \varphi\|_{L^2}^2 + \|\partial_x^2 \varphi\|_{L^2}^2)^{1/2},$$

where $\partial_x(C_0^\infty(\mathbb{R}^d))$ denotes the space of functions of the form $\partial_x \varphi$ with $\varphi \in C_0^\infty(\mathbb{R}^d)$ (i.e. the space of functions ψ in $C_0^\infty(\mathbb{R}^d)$ such that $\int_{-\infty}^{\infty} \psi(x, x') dx = 0$, for every $x' \in \mathbb{R}^{d-1}$).

DEFINITION 1.1. – A solitary wave of (1.1) (resp. (1.2)) is a solution of the type $u(x - ct, y)$ (resp. $u(x - ct, y, z)$) where $u \in Y$ and $c > 0$.

Remark 1.1. – By standard imbedding theorems, if $u \in Y$ and $d = 3$, then $u = \partial_x \varphi$ where $\varphi \in L^6(\mathbb{R}^3)$; if $d = 2$ and $u \in Y$ then $u = \partial_x \varphi$ where $\varphi \in L_{\text{loc}}^q(\mathbb{R}^2)$, $\forall q$, $2 \leq q < +\infty$. Note that for $d = 2$, the choice of $\varphi \in L_{\text{loc}}^q$ such that $u = \partial_x \varphi$ is not unique, but two such φ will differ by a function $\psi(y)$ independent of x . Hence, only one of them (up to a constant) satisfies $v = \partial_y \varphi \in L^2(\mathbb{R}^2)$. We assume in all what follows that when $u \in Y$ and when we take $\varphi \in L_{\text{loc}}^q$ with $\partial_x \varphi = u$, we also have $v = \partial_y \varphi \in L^2$. We then denote $v = \partial_y \varphi$ by $D_x^{-1} u_y$.

We are thus looking for “localized” solutions to the systems

$$(1.3) \quad \begin{cases} -cu_x + f(u)u_x + u_{xxx} + \varepsilon v_y = 0 \\ v_x = u_y \end{cases}$$

$$(1.4) \quad \begin{cases} -cu_x + f(u)u_x + u_{xxx} + av_y + bw_z = 0 \\ v_x = u_y \\ w_x = u_z \end{cases}$$

Except for the KPI equation (where the existence of “lumps” solitary waves is well known (*see* [1] [2]) no general results seemed to exist so far concerning solitary wave solutions to equations (1.1) and (1.2). The aim of the present paper is to solve completely this problem for power nonlinearities $f(u)$. Throughout the paper, we will assume that $f(u) = u^p$, with $p = m/n \geq 1$, m and n relatively prime, and n odd, except in Section 4, where p is a positive integer.

Remark 1.2. – Note that we may from now on assume that $c = 1$, since the scale change $\tilde{u}(x, x') = c^{-1/p}u(\frac{x}{c^{1/2}}, \frac{x'}{c})$, where $x' = y$ (resp. $x' = (y, z)$) transforms the system (1.3) (resp. (1.4)) in u , into the same in \tilde{u} , but with $c = 1$.

We now describe our results. In Section 2 we use Pohojaev type identities to prove nonexistence of solitary waves. In Section 3 we prove the existence of solitary waves in the remaining cases. Our strategy is to consider the minimization problem

$$(1.5) \quad I_\lambda = \text{Inf} \left\{ \|u\|_Y^2, u \in Y, \text{ with } \int_{\mathbb{R}^d} u^{p+2} dx dx' = \lambda \right\},$$

where $x' = y$ if $d = 2$, $x' = (y, z)$ if $d = 3$ and $\lambda > 0$. We shall use the concentration-compactness principle of P.L. Lions [14]. There are some difficulties due to the functional setting of the Kadomtsev-Petviashvili equations. In particular the minimizing sequence u_n is not bounded in H^1 and we have to prove a compactness lemma in L^2_{loc} for bounded sequences in Y . In Section 4 we show that solitary waves are smooth; namely, they belong to $H^\infty(\mathbb{R}^d) = \bigcap_{m \in \mathbb{N}} H^m(\mathbb{R}^d)$ where $H^m(\mathbb{R}^d)$ is the classical Sobolev space of order m . The difficulty arises from the nonisotropy of the symbol of the underlying elliptic operator $-\Delta + \partial_x^4$. We argue by “bootstrapping”, by using the imbedding theorems for anisotropic Sobolev spaces [4], and a variant due to Lizorkin [15] of the Mihlin-Hörmander multiplier theorem. Finally Section 5 is devoted to an extension to physically minded equations with other dispersions and to some concluding remarks.

The results of this paper were announced in [6]. After this work has been completed we have been aware of the paper [20] where an existence theorem for solitary waves to (1.1) is presented.

2. NONEXISTENCE OF SOLITARY WAVES

The main result in this section is the

THEOREM 1.1. – (i) *Assume that $d = 2$. The equation (1.1) does not admit any nontrivial solitary wave satisfying $u = \partial_x \varphi \in Y$, $u \in H^1(\mathbb{R}^2) \cap L^\infty_{\text{loc}}(\mathbb{R}^2)$, $\partial_x^2 u$ and $\partial_y^2 \varphi \in L^2_{\text{loc}}(\mathbb{R}^2)$, if*

$$(2.1) \quad \varepsilon = -1 \quad \text{and} \quad p \geq 4$$

or

$$(2.2) \quad \varepsilon = +1 \quad \text{and} \quad p \text{ is arbitrary.}$$

(ii) *Assume that $d = 3$. The equation (1.2) does not admit any nontrivial solitary wave satisfying $u = \partial_x \varphi \in Y$, $u \in H^1(\mathbb{R}^3) \cap L^{2(p+1)}(\mathbb{R}^3) \cap L^\infty_{\text{loc}}(\mathbb{R}^3)$, $\partial_x^2 u$, $\partial_y^2 \varphi$ and $\partial_z^2 \varphi \in L^2_{\text{loc}}(\mathbb{R}^3)$ if*

$$(2.3) \quad ab = -1 \quad (\text{resp. } a = b = 1) \quad \text{and} \quad p \text{ is arbitrary}$$

or

$$(2.4) \quad a = b = -1, \quad p \geq \frac{4}{3}$$

Proof. – It is based on Pohojaev type identities. The regularity assumptions of Theorem 1.1 are needed to justify them by the following standard truncation argument. Let $\chi_0 \in C^\infty_0(\mathbb{R})$, $0 \leq \chi_0 \leq 1$, $\chi_0(t) = 1$ if $0 \leq |t| \leq 1$, $\chi_0(t) = 0$, $|t| \geq 2$. We set $\chi_j = \chi_0(\frac{|\cdot|^2}{j^2})$, $j = 1, 2, \dots$

To begin with we treat the 2-dimensional case. We multiply (1.3)₁ by $x\chi_j u$ and we integrate over \mathbb{R}^2 to get (note that the third integral has to be interpreted as a $H^1 - H^{-1}$ duality)

$$(2.5) \quad - \int x\chi_j \partial_x \left(\frac{u^2}{2} \right) dx dy + \frac{1}{p+2} \int x\chi_j \partial_x (u^{p+2}) dx dy + \int x\chi_j uu_{xxx} dx dy + \varepsilon \int x\chi_j v_y u dx dy = 0,$$

and after several integrations by parts we obtain

$$\begin{aligned}
 (2.6) \quad & \frac{1}{2} \int \chi_j u^2 dx dy - \frac{1}{p+2} \int \chi_j u^{p+2} dx dy + \frac{3}{2} \int \chi_j u_x^2 dx dy \\
 & + \frac{\varepsilon}{2} \int \chi_j v^2 dx dy + \frac{1}{j^2} \int x \chi_0' \left(\frac{r^2}{j^2} \right) u^2 dx dy \\
 & - \frac{2}{j(p+2)} \int x^2 \chi_0' \left(\frac{r^2}{j^2} \right) u^2 dx dy - \frac{3}{j^2} \int \chi_0' \left(\frac{r^2}{j^2} \right) u^2 dx dy \\
 & - \frac{6}{j^4} \int \chi_0'' \left(\frac{r^2}{j^2} \right) u^2 dx dy - \frac{6}{j^4} \int x \chi_0'' \left(\frac{r^2}{j^2} \right) u^2 dx dy \\
 & - \frac{4}{j^6} \int x^3 \chi_0''' \left(\frac{r^2}{j^2} \right) u^2 dx dy + \frac{3}{j^2} \int x \chi_0' \left(\frac{r^2}{j^2} \right) u_x^2 dx dy \\
 & - \frac{2\varepsilon}{j^2} \int xy \chi_0' \left(\frac{r^2}{j^2} \right) uv dx dy + \frac{1}{j^2} \int x \chi_0' \left(\frac{r^2}{j^2} \right) v^2 dx dy = 0
 \end{aligned}$$

where $r^2 = x^2 + y^2$. By Lebesgue dominated convergence theorem, we infer from (2.6) that

$$(2.7) \quad \int \left[-\frac{1}{2} u^2 + \frac{u^{p+2}}{p+2} - \frac{3}{2} u_x^2 - \frac{\varepsilon}{2} v^2 \right] dx dy = 0.$$

From now on, we will proceed formally, the rigorous proofs following by the same truncation argument as above. We multiply (1.3)₁ by yv and integrate (the 2 last integrals are understood as a $H^1 - H^{-1}$ duality). After several integrations by parts and using (1.3)₂ we obtain finally

$$(2.8) \quad \int \left[\frac{1}{2} u^2 - \frac{1}{(p+1)(p+2)} u^{p+2} + \frac{u_x^2}{2} + \frac{\varepsilon}{2} v^2 \right] dx dy = 0.$$

To prove the third identity, we first remark that if $u \in Y \cap L^{2(p+1)}$ satisfies (1.3) in $\mathcal{D}'(\mathbb{R}^2)$, and if Y' is the dual space of Y , then u satisfies

$$-u + u_{xx} + \frac{u^{p+1}}{p+1} + \varepsilon D_x^{-1} v_y = 0 \quad \text{in } Y'$$

where $v = D_x^{-1} u_y \in L^2(\mathbb{R}^2)$ and $D_x^{-1} v_y \in Y'$ is defined by $\langle D_x^{-1} v_y, \psi \rangle_{Y, Y'} = (v, D_x^{-1} \psi_y)$ for any $\psi \in Y$. Taking then the $Y - Y'$ duality product of this last equation with $u \in Y$, we obtain

$$(2.9) \quad \int \left[-u^2 + \frac{u^{p+2}}{p+1} - u_x^2 + \varepsilon v^2 \right] dx dy = 0.$$

By subtracting (2.7) from (2.8) we get

$$(2.10) \quad \int \left[u^2 - \frac{1}{p+1} u^{p+2} + 2u_x^2 + \varepsilon v^2 \right] dx dy = 0.$$

Adding (2.9) and (2.10) yields

$$(2.11) \quad \int [u_x^2 + 2\varepsilon v^2] dx dy = 0,$$

which rules out (2.2). The identity (2.11) for $\varepsilon = -1$, namely $\int u_x^2 dx dy = 2 \int v^2 dx dy$, gives when inserted in (2.7), (2.9),

$$\int \left[-\frac{1}{2} u^2 + \frac{u^{p+2}}{p+2} - \frac{5}{2} v^2 \right] dx dy = 0$$

$$\int \left[-u^2 + \frac{u^{p+2}}{p+1} - 3v^2 \right] dx dy = 0.$$

Eliminating v^2 leads to

$$(2.12) \quad \int \left[u^2 + \frac{p-4}{2(p+1)(p+2)} u^{p+2} \right] dx dy = 0.$$

On the other hand adding (2.7) and (2.8) yields

$$\int u_x^2 dx dy = \frac{p}{(p+1)(p+2)} \int u^{p+2} dx dy,$$

and (2.1) follows from this equality reported in (2.12).

Let us now consider the case $d = 3$. Again we give a formal proof which can be justified by the aforementioned truncation process. We multiply successively (1.4)₁ by xu , yv and zw and integrate to get

$$(2.13) \quad \int \left[-\frac{1}{2} u^2 + \frac{1}{p+2} u^{p+2} - \frac{3}{2} u_x^2 - \frac{a}{2} v^2 - \frac{b}{2} w^2 \right] dx dy dz = 0,$$

$$(2.14) \quad \int \left[\frac{1}{2} u^2 - \frac{1}{(p+1)(p+2)} u^{p+2} + \frac{1}{2} u_x^2 + \frac{a}{2} v^2 - \frac{b}{2} w^2 \right] dx dy dz = 0,$$

$$(2.15) \quad \int \left[\frac{1}{2} u^2 - \frac{1}{(p+1)(p+2)} u^{p+2} + \frac{1}{2} u_x^2 + \frac{b}{2} w^2 - \frac{a}{2} v^2 \right] dx dy dz = 0.$$

Integrating (1.4)₁ once in x , and taking the duality product of the resulting equation with $u \in Y$ as in dimension 2, one obtains

$$(2.16) \quad \int \left[-u^2 + \frac{u^{p+2}}{p+1} - u_x^2 + av^2 + bw^2 \right] dx dy dz = 0.$$

Subtracting (2.15) from (2.14) yields

$$(2.17) \quad \int [av^2 - bw^2] dx dy dz = 0,$$

which rules out the case (2.3) when $ab = -1$. Now adding (2.16) and twice (2.13) implies

$$(2.18) \quad \int \left[-2u^2 - 4u_x^2 + \frac{3p+4}{(p+1)(p+2)} u^{p+2} \right] dx dy dz = 0.$$

Adding (2.18) and $(3p+4)$ times (2.14), and using (2.17), we obtain

$$\int \left[\frac{3p}{2} u^2 + \frac{3p-4}{2} u_x^2 \right] dx dy dz = 0,$$

which rules out (2.4). On the other hand, from (2.14) and (2.17) we infer

$$\frac{1}{(p+1)(p+2)} \int u^{p+2} dx dy dz = \frac{1}{2} \int [u^2 + u_x^2] dx dy dz.$$

This identity plugged in (2.16) yields

$$\int \left[\frac{p}{2} u^2 + \frac{p}{2} u_x^2 + 2av^2 \right] dx dy dz = 0,$$

which proves (2.3) for $a = b = 1$.

3. EXISTENCE OF SOLITARY WAVES

In this section, we prove the existence of solitary waves solutions of equations (1.1) and (1.2) by using the minimization problem I_λ defined in Section 1. The existence results are the following.

THEOREM 3.1. – *Let $d = 2$, $\varepsilon = -1$ and p be such that $1 \leq p < 4$. Then equation (1.3) possesses a solution (u, v) with $u \in Y$, $u \neq 0$.*

THEOREM 3.2. – *Let $d = 3$, $a = b = -1$ and $1 \leq p < 4/3$, then equation (1.4) possesses a solution (u, v, w) with $u \in Y$, $u \neq 0$.*

Remark 3.1. – The uniqueness of solitary waves to (1.1) or (1.2) (when they exist !) is an open problem.

As said previously, Theorems 3.1 and 3.2 will be proved by considering the minimization problem (1.5). More precisely, we will show that under

the conditions of each theorem, I_λ has a nontrivial solution $u \in Y$. This will be done by using the concentration-compactness principle (see [14]). Then, when $d = 2$ for example, if $v = D_x^{-1}u_y$, there is a Lagrange multiplier θ such that

$$(3.1) \quad -u_{xx} + u + D_x^{-1}v_y = \frac{\theta}{p+1}u^{p+1} \quad \text{in } Y'(\mathbb{R}^2)$$

where $D_x^{-1}v_y$ is the element of Y' (the dual space of Y in the L^2 -duality) such that for any $\psi \in Y$,

$$\langle D_x^{-1}v_y, \psi \rangle_{Y',Y} = (v, D_x^{-1}\psi_y).$$

By taking the x -derivative of (3.1) in $\mathcal{D}'(\mathbb{R}^2)$, using the definition of $v = D_x^{-1}u_y$, and performing the scale change $\underline{u} = \text{sgn}(\theta)|\theta|^{1/p}u$ and $\underline{v} = \text{sgn}(\theta)|\theta|^{1/p}v$, one then easily see that $(\underline{u}, \underline{v})$ satisfies the system (1.3) (with $c = 1$) in $\mathcal{D}'(\mathbb{R}^2)$.

A similar argument works for the 3-dimensional case. We now turn to the proof of the existence of a minimum for I_λ .

Proof of Theorem. 3.1. – First, observe that $I_\lambda > 0$ for any $\lambda > 0$: this follows from the imbedding theorem for anisotropic Sobolev spaces (see [4] p. 323) which gives

$$\|u\|_{L^q} \leq C\|u\|_Y \quad \text{for any } u \in Y \text{ and } 2 \leq q \leq 6.$$

Hence $|\int u^{p+2} dx dy| \leq C\|u\|_Y^{p+2}$ for any $u \in Y$ and $I_\lambda \geq (\frac{\lambda}{C})^{2/(p+2)} > 0$ for any positive λ .

Now, let $\lambda > 0$ and let u_n be a minimizing sequence for (1.5). Then, as was noticed in Remark 1.1, there is a sequence of functions φ_n which belong to $L^q_{\text{loc}}(\mathbb{R}^2)$ for any positive and finite q , satisfying $u_n = \partial_x \varphi_n$. Let $v_n = \partial_y \varphi_n = D_x^{-1}u_{n,y}$; we apply the concentration-compactness lemma of [14] to $\rho_n = |u_n|^2 + |v_n|^2 + |\partial_x u_n|^2$ (note that $\int \rho_n dx dy = \|u_n\|_Y^2 \xrightarrow{n \rightarrow \infty} I_\lambda > 0$).

(i) Assume first that “vanishing” occurs, i.e. that for any $R > 0$,

$$(3.2) \quad \lim_{n \rightarrow +\infty} \sup_{(x,y) \in \mathbb{R}^2} \int_{(x,y) + B_R} (|u_n|^2 + |v_n|^2 + |\partial_x u_n|^2) = 0,$$

where B_R is the ball of radius R centered at 0. Let q such that $2 < q < 6$; then from the Sobolev inequalities in anisotropic Sobolev spaces (see [4]),

there is a positive constant C independent of $(x, y) \in \mathbb{R}^2$ such that if $\varphi_x \in Y$,

$$\begin{aligned} \int_{(x,y)+B_1} |\varphi_x|^q &\leq C \left(\int_{(x,y)+B_1} (|\varphi_x|^2 + |\varphi_y|^2 + |\varphi_{xx}|^2) \right)^{q/2} \\ &\leq C \left(\sup_{(x,y) \in \mathbb{R}^2} \int_{(x,y)+B_1} (|\varphi_x|^2 + |\varphi_y|^2 + |\varphi_{xx}|^2) \right)^{\frac{q-2}{2}} \\ &\quad \times \int_{(x,y)+B_1} (|\varphi_x|^2 + |\varphi_y|^2 + |\varphi_{xx}|^2). \end{aligned}$$

Now, covering \mathbb{R}^2 by balls of radius 1, in such a way that each point of \mathbb{R}^2 is contained in at most 3 balls, we have

$$\int_{\mathbb{R}^2} |\varphi_x|^q \leq 3C \left(\sup_{(x,y) \in \mathbb{R}^2} \int_{(x,y)+B_1} (|\varphi_x|^2 + |\varphi_y|^2 + |\varphi_{xx}|^2) \right)^{\frac{q-2}{2}} \|\varphi_x\|_Y^2.$$

for any φ such that $\varphi_x \in Y$. From this, we conclude that under assumption (3.2), $u_n \rightarrow 0$ in L^q for any q such that $2 < q < 6$, which contradicts the constraint in I_λ .

(ii) Assume now that “dichotomy” occurs, *i.e.* that

$$(3.3) \quad \begin{cases} \lim_{t \rightarrow +\infty} Q(t) = \alpha \in]0, I_\lambda[. & \text{where for } t \geq 0, \\ Q(t) = \lim_{n \rightarrow +\infty} \sup_{(x_0, y_0) \in \mathbb{R}^2} \int_{(x_0, y_0)+B_1} \rho_n \, dx dy. \end{cases}$$

Note that the sub-additivity condition of [14] holds here, since we have for $\lambda > 0$: $I_\lambda = \lambda^{2/(p+2)} I_1$. Assumption (3.3) will then give a contradiction provided that it leads to the splitting of u_n into two sequences u_n^1 and u_n^2 with disjoint supports. In order to get u_n^1 and u_n^2 in Y , we have to localize φ_n instead of u_n ; but since φ_n is not in $L^2(\mathbb{R}^2)$, the splitting property of u_n is not a direct consequence of [14]. To prove this splitting property, we first need to show the following lemma.

LEMMA 3.1. – *Let q be such that $2 \leq q < +\infty$; then there exists a positive constant C such that for all $f \in L^1_{loc}(\mathbb{R}^2)$ with $\nabla f \in L^2_{loc}(\mathbb{R}^2)$, for all $R > 0$ and for all $\mathbf{x}_0 \in \mathbb{R}^2$*

$$\begin{aligned} &\left(\int_{R \leq |\mathbf{x} - \mathbf{x}_0| \leq 2R} |f(\mathbf{x}) - m_R(f)|^q \, d\mathbf{x} \right)^{1/q} \\ &\leq CR^{2/q} \left(\int_{R \leq |\mathbf{x} - \mathbf{x}_0| \leq 2R} |\nabla f|^2 \, d\mathbf{x} \right)^{1/2} \end{aligned}$$

where

$$m_R(f) = \frac{1}{\text{vol}(\Omega_{\mathbf{x}_0, R})} \int_{R \leq |\mathbf{x} - \mathbf{x}_0| \leq 2R} f(\mathbf{x}) d\mathbf{x}, \quad \mathbf{x} = (x, y) \in \mathbb{R}^2$$

and

$$\Omega_{\mathbf{x}_0, R} = \{ \mathbf{x} \in \mathbb{R}^2, R < |\mathbf{x} - \mathbf{x}_0| < 2R \}.$$

Proof of Lemma 3.1. – The lemma is proved by applying Poincaré inequality for zero mean-value H^1 functions on the bounded open set $\Omega_{\mathbf{x}_0, R}$. Then, using Sobolev imbedding theorem, we obtain the existence of a positive constant $C(\mathbf{x}_0, R)$ such that

$$\begin{aligned} & \left(\int_{R \leq |\mathbf{x} - \mathbf{x}_0| \leq 2R} |f(\mathbf{x}) - m_R(f)|^q d\mathbf{x} \right)^{1/q} \\ & \leq C(\mathbf{x}_0, R) \left(\int_{R \leq |\mathbf{x} - \mathbf{x}_0| \leq 2R} |\nabla f|^2 d\mathbf{x} \right)^{1/2}. \end{aligned}$$

Then the translation invariance of Lebesgue’s measure, and the scale change $f \mapsto f(\frac{\cdot}{R})$ show that $C(\mathbf{x}_0, R) = CR^{2/q}$ where C is independent of \mathbf{x}_0 and R . ■

We are now able, with the use of Lemma 3.1, to prove the following Lemma 3.2.

LEMMA 3.2. – Assume that (3.3) holds. Then for all $\varepsilon > 0$, there is a $\delta(\varepsilon)$ (with $\delta(\varepsilon) \xrightarrow{\varepsilon \rightarrow 0} 0$), such that we can find u_n^1 and u_n^2 in Y satisfying for $n \geq n_0$:

$$\begin{aligned} & \|u_n^1 + u_n^2 - u_n\|_Y \leq \delta(\varepsilon) \\ & \|u_n^1\|_Y - \alpha \leq \delta(\varepsilon) \\ & \|u_n^2\|_Y - (I_\lambda - \alpha) \leq \delta(\varepsilon) \\ & \left| \int_{\mathbb{R}^2} [(u_n^1)^{p+2} + (u_n^2)^{p+2} - u_n^{p+2}] \right| \leq \delta(\varepsilon) \end{aligned}$$

and

$$\text{dist}(\text{supp } u_n^1, \text{supp } u_n^2) \xrightarrow{n \rightarrow +\infty} +\infty.$$

Proof of Lemma 3.2. – The proof is adapted from [14] by using Lemma 3.1. For the reader’s convenience, we give the details. Assume that (3.3) holds, and fix $\varepsilon > 0$. Then we can find $R_0 > 0$, $R_n > 0$ with $R_n \nearrow +\infty$ and $\mathbf{x}_n \in \mathbb{R}^2$ such that

$$\alpha \geq \int_{\mathbf{x}_n + B_{R_0}} (|u_n|^2 + |v_n|^2 + |\partial_x u_n|^2) \geq \alpha - \varepsilon \quad \text{and} \quad Q_n(2R_n) \leq \alpha + \varepsilon$$

for $n \geq n_0$, where

$$Q_n(t) = \sup_{(x_0, y_0) \in \mathbb{R}^2} \int_{(x_0, y_0) + B_t} (|u_n|^2 + |v_n|^2 + |\partial_x u_n|^2).$$

It follows that

$$\int_{R_0 \leq |\mathbf{x} - \mathbf{x}_n| \leq 2R_n} (|u_n|^2 + |v_n|^2 + |\partial_x u_n|^2) \leq 2\varepsilon.$$

Let ξ and $\eta \in C_0^\infty(\mathbb{R}^2)$ be as in [14], i.e. $0 \leq \xi \leq 1$, $0 \leq \eta \leq 1$. $\xi \equiv 1$ on B_1 , $\text{supp } \xi \subset B_2$, $\eta \equiv 1$ on $\mathbb{R}^2 \setminus B_2$, $\text{supp } \eta \subset \mathbb{R}^2 \setminus B_1$. We set $\xi_n = \xi(\frac{\cdot - \mathbf{x}_n}{R_1})$, $\eta_n = \eta(\frac{\cdot - \mathbf{x}_n}{R_n})$, and we consider

$$u_n^1 = \partial_x(\xi_n(\varphi_n - a_n)), \quad u_n^2 = \partial_x(\eta_n(\varphi_n - b_n))$$

where (a_n) and (b_n) are sequences of real numbers which will be chosen later. Lastly we set

$$v_n^1 = D_x^{-1}(u_n^1)_y = \partial_y(\xi_n(\varphi_n - a_n))$$

and

$$v_n^2 = D_x^{-1}(u_n^2)_y = \partial_y(\eta_n(\varphi_n - b_n)).$$

Then we have for example

$$\begin{aligned} & \|u_n^1 + u_n^2 - u_n\|_{L^2} \\ & \leq \|(\partial_x \xi_n)(\varphi_n - a_n)\|_{L^2} + \|(\partial_x \eta_n)(\varphi_n - b_n)\|_{L^2} + \sqrt{2\varepsilon} \end{aligned}$$

and

$$\begin{aligned} \|(\partial_x \xi_n)(\varphi_n - a_n)\|_{L^2} &= \left(\int_{R_1 \leq |\mathbf{x} - \mathbf{x}_n| \leq 2R_1} |\partial_x \xi_n|^2 |\varphi_n - a_n|^2 \right)^{1/2} \\ &\leq \|\partial_x \xi_n\|_{L^p} \left(\int_{R_1 \leq |\mathbf{x} - \mathbf{x}_n| \leq 2R_1} |\varphi_n - a_n|^q \right)^{1/q} \end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$. Now choosing

$$a_n = \frac{1}{\text{vol}(\Omega_{\mathbf{x}_n, R_1})} \int_{R_1 \leq |\mathbf{x} - \mathbf{x}_n| \leq 2R_1} \varphi_n(\mathbf{x}) d\mathbf{x} = m_{R_1}(\varphi_n),$$

and applying Lemma 3.1, we get

$$\begin{aligned} & \|(\partial_x \xi_n)(\varphi_n - a_n)\|_{L^2} \\ & \leq CR_1^{\frac{2}{p} + \frac{2}{q} - 1} \left(\int_{R_1 \leq |\mathbf{x} - \mathbf{x}_n| \leq 2R_1} |u_n|^2 + |v_n|^2 \right)^{1/2} \leq C' \sqrt{\varepsilon}. \end{aligned}$$

In the same way, choosing $b_n = m_{R_n}(\varphi_n)$ leads to the bound

$$\begin{aligned} & \|(\partial_x \eta_n)(\varphi_n - b_n)\|_{L^2} \\ & \leq C \left(\int_{R_n \leq |x-x_n| \leq 2R_n} (|u_n|^2 + |v_n|^2) \right)^{1/2} \leq C\sqrt{\varepsilon}. \end{aligned}$$

This implies the desired estimate on $\|u_n^1 + u_n^2 - u_n\|_{L^2}$; the bound on $\|v_n^1 + v_n^2 - v_n\|_{L^2}$ is obtained in the same way. Now, consider

$$\begin{aligned} & \|\partial_x u_n^1 + \partial_x u_n^2 - \partial_x u_n\|_{L^2} \\ & = \|\partial_x^2(\xi_n(\varphi_n - a_n)) + \partial_x^2(\eta_n(\varphi_n - b_n)) - \partial_x^2 \varphi_n\|_{L^2} \\ & \leq \|(\partial_x^2 \xi_n)(\varphi_n - a_n)\|_{L^2} + \|(\partial_x^2 \eta_n)(\varphi_n - b_n)\|_{L^2} + \|(1 - \xi_n - \eta_n)\partial_x u_n\|_{L^2} \\ & \quad + 2\|(\partial_x \xi_n)u_n\|_{L^2} + 2\|(\partial_x \eta_n)u_n\|_{L^2}. \end{aligned}$$

The three first terms in the right hand side of the above inequality are bounded as the preceding ones. For the two last terms, one may use for example

$$\|(\partial_x \xi_n)u_n\|_{L^2} \leq \|\partial_x \xi_n\|_{L^\infty} \left(\int_{R_1 \leq |x-x_n| \leq 2R_1} |u_n|^2 \right)^{1/2} \leq C\sqrt{\varepsilon}.$$

All the other terms of Lemma 3.2 are bounded in a similar way; the last bound follows from the first one, the fact that $\text{supp } u_n^1 \cap \text{supp } u_n^2 = \emptyset$ and the injection of Y into $L^{p+2}(\mathbb{R}^2)$.

We now continue the proof of Theorem 3.1. Taking subsequences if necessary, we may assume that

$$\int_{\mathbb{R}^2} (u_n^1)^{p+2} \xrightarrow{n \rightarrow \infty} \lambda_1(\varepsilon), \quad \int_{\mathbb{R}^2} (u_n^2)^{p+2} \xrightarrow{n \rightarrow \infty} \lambda_2(\varepsilon)$$

with $|\lambda_1(\varepsilon) + \lambda_2(\varepsilon) - \lambda| \leq \delta(\varepsilon)$.

• Assume first that $\lim_{\varepsilon \rightarrow 0} \lambda_1(\varepsilon) = 0$; then choosing ε sufficiently small, we have for n large enough $\int_{\mathbb{R}^2} (u_n^2)^{p+2} dx dy > 0$. Hence by considering

$$\left(\frac{\lambda_2(\varepsilon)}{\int_{\mathbb{R}^2} (u_n^2)^{p+2}} \right)^{1/(p+2)} u_n^2,$$

we get

$$I_{\lambda_2(\varepsilon)} \leq \liminf_{n \rightarrow +\infty} \|u_n^2\|_Y \leq I_\lambda - \alpha + \delta(\varepsilon)$$

but this is a contradiction since $\lim_{\varepsilon \rightarrow 0} \lambda_2(\varepsilon) = \lambda$.

• Thus, we may assume that $\lim_{\varepsilon \rightarrow 0} |\lambda_1(\varepsilon)| > 0$ and $\lim_{\varepsilon \rightarrow 0} |\lambda_2(\varepsilon)| > 0$. In the same way as before we then obtain

$$I_{|\lambda_1|} + I_{|\lambda_2|} \leq \liminf_{n \rightarrow +\infty} \|u_n^1\|_Y + \liminf_{n \rightarrow +\infty} \|u_n^2\|_Y \leq I_\lambda + \delta(\varepsilon).$$

We reach a contradiction by letting ε tend to zero, and by using the fact that $I_\mu = \mu^{2/(p+2)} I_1$ for any positive μ . This ends to rule out the “dichotomy” case.

(iii) The only remaining possibility is the following : there is a sequence (\mathbf{x}_n) with $\mathbf{x}_n \in \mathbb{R}^2$ such that for all $\varepsilon > 0$, there exists a finite $R > 0$, and $n_0 > 0$, with

$$\int_{\mathbf{x}_n + B_R} (|u_n|^2 + |v_n|^2 + |\partial_x u_n|^2) dx dy \geq I_\lambda - \varepsilon \quad \text{for } n \geq n_0.$$

Note that this implies, for n large enough,

$$\int_{\mathbf{x}_n + B_R} |u_n|^2 \geq \int_{\mathbb{R}^2} |u_n|^2 - 2\varepsilon.$$

Since u_n is bounded in Y , we may assume that $u_n(\cdot - \mathbf{x}_n)$ converges weakly in Y to some $u \in Y$. We then have

$$\int_{\mathbb{R}^2} |u|^2 dx dy \leq \liminf_{n \rightarrow +\infty} \int_{\mathbb{R}^2} |u_n|^2 dx dy \leq \liminf_{n \rightarrow +\infty} \int_{\mathbf{x}_n + B_R} |u_n|^2 dx dy + 2\varepsilon.$$

The following lemma shows that the injection $Y \subset L^2_{\text{loc}}(\mathbb{R}^2)$ is compact.

LEMMA 3.3. – Let u_n be a bounded sequence in Y , and let $R > 0$. Then there is a subsequence u_{n_k} which converges strongly to u in $L^2(B_R)$.

We first end the proof of Theorem 3.1, afterwhat we prove Lemma 3.3. By Lemma 3.3, we may assume that $u_n(\cdot - \mathbf{x}_n)$ converges to u strongly in L^2_{loc} . But then, the inequality preceding Lemma 3.3 shows that in fact $u_n(\cdot - \mathbf{x}_n)$ converges to u strongly in $L^2(\mathbb{R}^2)$, and by interpolation, using the imbedding $Y \subset L^6(\mathbb{R}^2)$, $u_n(\cdot - \mathbf{x}_n)$ also converges to u strongly in L^{p+2} so that $\int u^{p+2} = \lambda$. Since $\|u\|_Y \leq \liminf \|u_n\|_Y = I_\lambda$, this shows that u is a solution of I_λ .

Proof of Lemma 3.3. – Let u_n be a bounded sequence in Y , with $u_n = \partial_x \varphi_n$, $\varphi_n \in L^2_{\text{loc}}(\mathbb{R}^2)$, and let $v_n = \partial_y \varphi_n \in L^2(\mathbb{R}^2)$. Multiplying φ_n by a function $\psi \in C^\infty_0(\mathbb{R}^2)$ with $0 \leq \psi \leq 1$, $\psi \equiv 1$ on B_R and $\text{supp } \psi \subset B_{2R}$, we may assume that $\text{supp } \varphi_n \subset B_{2R}$. Now since u_n is bounded in Y , we may assume that $u_n \rightharpoonup u = \partial_x \varphi$ weakly in Y , and

replacing if necessary φ_n by $\varphi_n - \varphi$, we may also assume that $\varphi = 0$. Then we have

$$\begin{aligned} \int_{B_{2R}} |u_n|^2 &= \int_{\mathbb{R}^2} |\hat{u}_n|^2 = \int_{\{|\xi_1| \leq R_1, |\xi_2| \leq R_1^2\}} |\hat{u}_n|^2 \\ &\quad + \int_{\{|\xi_1| \geq R_1\}} |\hat{u}_n|^2 + \int_{\{|\xi_1| \leq R_1, |\xi_2| \geq R_1^2\}} |\hat{u}_n|^2 \end{aligned}$$

where $\hat{f}(\xi_1, \xi_2)$ is the Fourier transform of $f(x, y)$. The third term satisfies

$$\int_{\{|\xi_1| \leq R_1, |\xi_2| \geq R_1^2\}} |\hat{u}_n|^2 = \int_{\{|\xi_1| \leq R_1, |\xi_2| \geq R_1^2\}} \frac{|\xi_1|^2}{|\xi_2|^2} |\hat{v}_n|^2 \leq \frac{1}{R_1^2} \|v_n\|_{L^2}^2.$$

The second term is bounded in the following way

$$\int_{\{|\xi_1| \geq R_1\}} |\hat{u}_n|^2 \leq \frac{1}{R_1^2} \|\partial_x u_n\|_{L^2}^2.$$

Fix $\varepsilon > 0$; then choosing R_1 sufficiently large leads to

$$\int_{\{|\xi_1| \geq R_1\}} |\hat{u}_n|^2 + \int_{\{|\xi_1| \leq R_1, |\xi_2| \geq R_1^2\}} |\hat{u}_n|^2 \leq \frac{\varepsilon}{2}.$$

We then use Lebesgue's dominated convergence theorem for the first term, having noted that since u_n tends to 0 weakly in $L^2(\mathbb{R}^2)$,

$$\hat{u}_n(\xi_1, \xi_2) = \int_{B_{2R}} e^{-ix\xi_1 - iy\xi_2} u_n(x, y) dx dy$$

tends to zero as $n \rightarrow +\infty$, for a.e. $(\xi_1, \xi_2) \in \mathbb{R}^2$, and that $|\hat{u}_n(\xi)| \leq |u_n|_{L^1(B_{2R})}$. ■

We now turn to the 3-dimensional case.

Proof of Theorem 3.2. – Again, we prove the existence of a minimum for I_λ , by using the concentration compactness principle. Many details are very similar to the two-dimensional case, so that we will omit them. Moreover, to avoid technicalities, we restrict the proof to the case $p = 1$.

First, we also have $I_\lambda > 0$ for any $\lambda > 0$, since from [4, p. 323], we have

$$\|u\|_{L^q} \leq C \|u\|_Y \quad \text{for any } q, \quad 2 \leq q \leq \frac{10}{3}.$$

Let $\lambda > 0$, and let u_n be a minimizing sequence for I_λ . Then there exists $\varphi_n \in L^6(\mathbb{R}^3)$, with $\partial_x \varphi_n = u_n$; let $v_n = \partial_y \varphi_n$ and $w_n = \partial_z \varphi_n$. We apply

the concentration-compactness lemma of [14] to $\rho_n = |\varphi_n|^6$. Since φ_n is bounded in $L^6(\mathbb{R}^3)$ by Sobolev's inequality, there exists a subsequence still denoted by ρ_n such that $\int \rho_n dx dy dz \xrightarrow{n \rightarrow +\infty} \beta \geq 0$. Applying the next lemma with $r = 6$ shows that $\beta > 0$.

LEMMA 3.4.

• Let $\varphi \in L^6(\mathbb{R}^3)$ with $\partial_x \varphi \in Y$; then $\varphi \in L^{10}(\mathbb{R}^3)$ and there is a constant $C > 0$ such that

$$\|\varphi\|_{L^{10}} \leq C \|\partial_x^2 \varphi\|_{L^2}^{1/5} \|\partial_y \varphi\|_{L^2}^{2/5} \|\partial_z \varphi\|_{L^2}^{2/5}.$$

• For any r with $6 \leq r < 10$, there exist α_j with $0 < \alpha_j < 1$ for $j = 0, 1, 2, 3$ and a constant $C > 0$ such that if $\varphi \in L^6(\mathbb{R}^3)$ and $\partial_x \varphi \in Y$,

$$\|\partial_x \varphi\|_{L^r} \leq C \|\varphi\|_{L^6}^{\alpha_0} \|\partial_x \varphi\|_{L^2}^{\alpha_1} \|\partial_x^2 \varphi\|_{L^2}^{\alpha_2} \|\partial_y \varphi\|_{L^2}^{\alpha_3} \|\partial_z \varphi\|_{L^2}^{\alpha_3}.$$

Proof of Lemma 3.4.

• The first inequality and the case $r = 6$ in the second inequality follow directly from the generalized Sobolev inequality (see [4], p. 323).

• If $6 < r < 10$, then we cannot take directly $q = 3$ in the generalized Sobolev inequality but we first consider q such that $3 < \frac{4r}{2+r} < q < \frac{10}{3}$; then the generalized Sobolev inequality applies with $\mu_1 = \frac{3-(r/q)}{5-(r/2)}$, $\mu_2 = \mu_3 = 2\mu_1 - 1$ and $\mu_0 = r(\frac{1}{q} - \frac{\mu_1}{2})$, i.e. we have

$$\|\partial_x \varphi\|_{L^q} \leq C \|\varphi\|_{L^r}^{\mu_0} \|\partial_x^2 \varphi\|_{L^2}^{\mu_1} \|\partial_y \varphi\|_{L^2}^{\mu_2} \|\partial_z \varphi\|_{L^2}^{\mu_3}.$$

We obtain the desired inequality by interpolation, writing

$$\frac{1}{3} = \frac{\theta}{q} + \frac{1-\theta}{2}, \quad \text{with } \theta \in]0, 1[\quad \blacksquare$$

(i) We first show that “vanishing” cannot occur. If it occurred then an easy adaptation of Lemma I.1 of [14] part II would show that φ_n tends to zero in $L^r(\mathbb{R}^3)$ for any r such that $6 < r < 10$ (Note that Lemma I.1 of [14] does not apply directly because we are in the “limit case” $q = \frac{Np}{N-p}$ with the notations of [14], but using the fact that φ_n is however bounded in $L^s(\mathbb{R}^3)$ for $r \leq s \leq 10$, one easily checks that the proof of [14] adapts). But then, the use of Lemma 3.4 contradicts the constraint.

(ii) Next, assume that “dichotomy” occurs, i.e. that

$$\lim_{t \rightarrow +\infty} Q(t) = \alpha \in]0, \beta[, \quad \text{where for } t \geq 0,$$

$$Q(t) = \lim_{n \rightarrow +\infty} \sup_{(x_0, y_0, z_0) \in \mathbb{R}^3} \int_{(x_0, y_0, z_0) + B_t} |\varphi_n|^6 dx dy dz.$$

We define R_0, R_n, ξ_n, η_n in the same way as in the proof of Lemma 3.2, with $|u_n|^2 + |v_n|^2 + |\partial_x u_n|^2$ replaced by $|\varphi_n|^6$; we then set $\varphi_n^1 = \xi_n \varphi_n$, $\varphi_n^2 = \eta_n \varphi_n$, and $(u_n^1, v_n^1, w_n^1) = \nabla \varphi_n^1$, $(u_n^2, v_n^2, w_n^2) = \nabla \varphi_n^2$. By doing so, we have for n sufficiently large,

$$(3.4) \quad \begin{cases} \left| \|\varphi_n^1\|_{L^6} - \alpha \right| \leq C \varepsilon^{1/6} \\ \left| \|\varphi_n^2\|_{L^6} - (\beta - \alpha) \right| \leq C \varepsilon^{1/6}. \end{cases}$$

Now, using the fact that

$$\int_{R_0 \leq |\mathbf{x} - \mathbf{x}_n| \leq 2R_n} |\varphi_n|^6 d\mathbf{x} \leq 2\varepsilon, \quad \text{where } \mathbf{x} = (x, y, z) \in \mathbb{R}^3,$$

it is not difficult, although quite technical, to show that

$$(3.5) \quad \left| \|u_n^1\|_Y^2 + \|u_n^2\|_Y^2 - \|u_n\|_Y^2 \right| \leq \delta(\varepsilon) \xrightarrow{\varepsilon \rightarrow 0} 0.$$

Consider for example

$$\begin{aligned} & \left| \int_{\mathbb{R}^3} [\xi_n^2 |\nabla \varphi_n|^2 - |\nabla(\xi_n \varphi_n)|^2] \right| \\ & \leq \int_{R_1 \leq |\mathbf{x} - \mathbf{x}_n| \leq 2R_1} (|\nabla \xi_n|^2 \varphi_n^2 + 2|\xi_n \nabla \xi_n| |\varphi_n \nabla \varphi_n|) \\ & \leq C \|\nabla \xi_n\|_{L^3}^2 \varepsilon^{1/3} + \|\nabla \varphi_n\|_{L^2} \varepsilon^{1/6} \|\xi_n \nabla \xi_n\|_{L^3} \end{aligned}$$

and we conclude by using the fact that $\nabla \varphi_n$ is bounded in L^2 , and that $\nabla \xi_n$ is bounded in L^3 , independently of n . Using this and the second inequality in Lemma 3.4, we also have

$$(3.6) \quad \|u_n^1 + u_n^2 - u_n\|_{L^3} \leq \delta(\varepsilon) \xrightarrow{\varepsilon \rightarrow 0} 0.$$

Finally, (3.4), (3.5) and (3.6) lead to a contradiction with the subadditivity condition implied by the relation $I_\lambda = \lambda^{2/3} I_1$, exactly as in the 2-dimensional case.

(iii) The only remaining possibility is that $\exists \mathbf{x}_n \in \mathbb{R}^3, \forall \varepsilon > 0, \exists R < +\infty$ such that for n sufficiently large

$$(3.7) \quad \int_{\mathbf{x}_n + B_R} |\varphi_n|^6 \geq \beta - \varepsilon.$$

Now, it is easily checked that Lemma 3.3 is also true in dimension 3, hence the sequence (φ_n) is relatively compact in $L^6_{loc}(\mathbb{R}^3)$ by Sobolev inequality. This together with (3.7) shows that, modulo a subsequence, $\varphi_n(\cdot - \mathbf{x}_n) \rightarrow \varphi$ strongly in $L^6(\mathbb{R}^3)$ and $u_n(\cdot - \mathbf{x}_n) \rightharpoonup u = \partial_x \varphi \in Y$ weakly in Y . Lastly, by the use of Lemma 3.4 with $r = 6$, $u_n \rightarrow u$ strongly in $L^3(\mathbb{R}^3)$ and u is a solution of I_λ .

4. REGULARITY PROPERTIES OF THE SOLITARY WAVES

In this Section, we prove that any solitary wave of (1.1) (resp. (1.2)) is a C^∞ function, provided p is an integer. More precisely we have

THEOREM 4.1. – *Any solitary wave solution of (1.1) (resp. (1.2)) belongs to $H^\infty(\mathbb{R}^d)$ provided $\varepsilon = -1$ and $p = 1, 2, 3$ (resp. $a = b = -1$ and $p = 1$). Moreover, $v = D_x^{-1}u_y$ (resp. $v = D_x^{-1}u_y$ and $w = D_x^{-1}u_z$) belong to $H^\infty(\mathbb{R}^d)$.*

Proof. – We are reduced to prove regularity results for the nonlinear elliptic equation

$$(4.1) \quad -\Delta u + \partial_x^4 u = -\partial_x^2(u^{p+1}) \quad \text{in } \mathbb{R}^d,$$

where $p = 1, 2, 3$ if $d = 2$ and $p = 1$ if $d = 3$.

The difficulty arises from the non isotropy of the symbol of the linear elliptic operator $-\Delta + \partial_x^4$. We will proceed by bootstrapping, using the following variant due to Lizorkin [15] of the Hörmander-Mikhlin multipliers theorem.

PROPOSITION 4.1. – [15] *Let $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}$ be C^n for $|\xi_j| > 0, j = 1, \dots, n$. Assume that there exists $M > 0$ such that*

$$(4.2) \quad \left| \xi_1^{k_1} \dots \xi_n^{k_n} \frac{\partial^k \Phi}{\partial \xi_1^{k_1} \dots \partial \xi_n^{k_n}}(\xi) \right| \leq M,$$

with $k_i = 0$ or $1, k = k_1 + k_2 + \dots + k_n = 0, 1, \dots, n$. Then $\Phi \in M_q(\mathbb{R}^n), 1 < q < +\infty$, i.e. Φ is a Fourier multiplier on $L^q(\mathbb{R}^n)$.

We first consider the case $d = 2$. Setting $g \equiv -u^{p+1}$, (4.1) yields

$$(4.3) \quad \hat{u}_{xx} = -\xi_1^2 \hat{u} = \frac{\xi_1^4 \hat{g}}{|\xi|^2 + \xi_1^4}.$$

LEMMA 4.1. – *Let $u \in Y$ be a solution of (4.1). Then*

$$u \in \{f \in L^6(\mathbb{R}^2) \cap L^{6/(p+1)}(\mathbb{R}^2), \partial_x f \in L^{12/(p+2)}(\mathbb{R}^2), \\ \partial_y f \in L^{6/(p+1)}(\mathbb{R}^2); \partial_x^2 f \in L^{6/(p+1)}(\mathbb{R}^2)\}.$$

Proof of Lemma 4.1. – By [4], Theorem 15.7, p. 323, one has $Y \subset L^6(\mathbb{R}^2)$ and therefore $u^{p+1} \in L^{6/(p+1)}(\mathbb{R}^2)$. It is easily checked out that $\Phi_1(\xi) = \frac{\xi_1^2}{|\xi|^2 + \xi_1^4}, \Phi_2(\xi) = \frac{\xi_1^4}{|\xi|^2 + \xi_1^4}$ and $\Phi_3(\xi) = \frac{\xi_2 \xi_1^2}{|\xi|^2 + \xi_1^4}$ satisfy

the assumption of Proposition 4.1, yielding $u, \partial_x^2 u, \partial_y u, \in L^{6/(p+1)}(\mathbb{R}^2)$. The claim that $\partial_x u \in L^{12/(p+2)}(\mathbb{R}^2)$ follows by interpolation between $u \in L^6(\mathbb{R}^2)$ and $\partial_x^2 u \in L^{6/(p+1)}(\mathbb{R}^2)$. ■

Lemma 4.1 implies that $u, \partial_y u, \partial_x^2 u \in L^{6/(p+1)}(\mathbb{R}^2)$. By [4], Theorem 10.2, one has $u \in L^q(\mathbb{R}^2)$, where $q = +\infty$ if $1 \leq p < 3$, $3/2 < q < +\infty$ if $p = 3$. In any case, one has, for $f = -(u^{p+1})_{,xx} = -p(p+1)u^{p-1}u_x^2 - (p+1)u^p u_{,xx}$,

$$\begin{aligned} f &\in L^{6/(p+1)}(\mathbb{R}^2), & 1 \leq p < 3, \\ f &\in L^q(\mathbb{R}^2), & \forall q, \quad 1 \leq q < 3/2, \quad p = 3. \end{aligned}$$

Another application of Lizorkin’s theorem leads to

$$\partial_x^2 u, \partial_x^4 u, \partial_x^2 \partial_y u, \partial_y^2 u \in L^q(\mathbb{R}^2) \begin{cases} q = 6/(p+1), & \text{if } 1 \leq p < 3 \\ 1 < q < 3/2, & \text{if } p = 3. \end{cases}$$

Let $v = \partial_x^2 u$. Then $v, \partial_x^2 v, \partial_y v \in L^q(\mathbb{R}^2)$, and by the aforementioned result of [4],

$$\begin{aligned} v &\in L^\infty(\mathbb{R}^2), & 1 \leq p < 3 \\ v &\in L^r(\mathbb{R}^2), & \forall r, \quad 3/2 \leq r < +\infty, \quad p = 3. \end{aligned}$$

In both cases, we obtain that $f \in L^r(\mathbb{R}^2), \forall r, 2 \leq r < +\infty$, and Lizorkin’s Theorem implies that $\partial_x^4 u, \partial_x^2 \partial_y u, \partial_y^2 u \in L^q(\mathbb{R}^2), \forall q, 2 \leq q < +\infty$, which implies that $\partial_x f, \partial_y f \in L^q(\mathbb{R}^2), \forall q, 2 \leq q < +\infty$.

Reiteration of the process leads to the proof of Theorem 4.1 for $d = 2$ (the regularity of $D_x^{-1}u_y$ is obtained by using equation (1.3) and the regularity of u).

In the case $d = 3$, (4.1) reads

$$(4.4) \quad \begin{cases} -\Delta u + \partial_x^4 u = \partial_x^2 g, \\ \Delta = \partial_x^2 + \partial_y^2 + \partial_z^2, \end{cases}$$

where $g = -u^2 \in L^{5/3}(\mathbb{R}^3)$, because $Y \subset L^{10/3}(\mathbb{R}^3)$ (see [4], Theorem 15.7). Lizorkin’s Theorem still applies to (4.4) and leads to

LEMMA 4.2. – *Let $u \in Y$ be a solution of (4.4) (with $p = 1$). Then*

$$u \in \{f \in L^{10/3} \cap L^{5/3}(\mathbb{R}^3), \partial_x f \in L^{20/9}(\mathbb{R}^3), \partial_y f, \partial_z f, \partial_x^2 f \in L^{5/3}(\mathbb{R}^3)\}.$$

The previous lemma implies in particular that $u, \partial_y u, \partial_z u, \partial_x^2 u \in L^{5/3}(\mathbb{R}^3)$ which implies [4], $u \in L^5(\mathbb{R}^3)$. We apply Lizorkin’s Theorem from $\partial_x^2(u^2) = 2u_x^2 + 2u\partial_x^2 u \in L^{5/4}(\mathbb{R}^3)$ to obtain

$$\partial_x^2 u, \partial_x^4 u, \partial_x^2 \partial_y u, \partial_x^2 \partial_z u, \partial_y^2 u, \partial_z^2 u \in L^{5/4}(\mathbb{R}^3).$$

Another application of [4] to $v = \partial_x^2 u$ (note that $\partial_y v, \partial_z v, \partial_x^2 v \in L^{5/4}(\mathbb{R}^3)$) yields $\partial_x^2 u \in L^{5/2}(\mathbb{R}^3)$. Similarly [4] applied to u (noticing that $\partial_x^4 u, \partial_y^2 u, \partial_z^2 u \in L^{5/4}(\mathbb{R}^3)$) implies that $u \in L^q(\mathbb{R}^3), \forall q, 5/4 \leq q < +\infty$ and by interpolation, $u_x \in L^q(\mathbb{R}^3), \forall q, 5/4 \leq q < 4$. This leads to $u_x^2 + uu_{xx} \in L^q(\mathbb{R}^3), \forall q, 5/8 \leq q < 2$ which by Lizorkin's Theorem implies that

$$\partial_x^4 u, \partial_x^2 \partial_y u, \partial_x^2 \partial_z u, \partial_y^2 u, \partial_z^2 u \in L^q(\mathbb{R}^3), \quad \forall q, \quad 1 < q < 2.$$

Thanks to [4] applied to $v = \partial_x^2 u$, we thus have $\partial_x^2 u \in L^q(\mathbb{R}^3), \forall q, 1 < q < 10, u \in L^\infty(\mathbb{R}^3)$ and by interpolation $\partial_x^3 u \in L^q(\mathbb{R}^3), \forall q, 1 < q < 10/3$.

We also obtain by [4] that $\partial_x^{\alpha_1} \partial_y^{\alpha_2} \partial_z^{\alpha_3} u \in L^q(\mathbb{R}^3)$, where

$$\frac{\alpha_1}{4} + \frac{\alpha_2}{2} + \frac{\alpha_3}{2} < \frac{3}{8} + \frac{5}{4q}, \quad i.e.$$

$$\alpha_1 + 2\alpha_2 + 2\alpha_3 < \frac{3}{2} + \frac{5}{q}, \quad \text{and} \quad 1 < q \leq +\infty.$$

In particular, $\partial_y u, \partial_z u \in L^q(\mathbb{R}^3), \forall q < 10, \partial_{xy}^2 u, \partial_{xz}^2 u \in L^q(\mathbb{R}^3), \forall q < 10/3, \partial_x u \in L^\infty(\mathbb{R}^3)$.

Lizorkin's Theorem implies that $\partial_x^4 u, \partial_x^2 \partial_y u, \partial_x^2 \partial_z u, \partial_y^2 u, \partial_z^2 u \in L^q(\mathbb{R}^3), \forall q < 10$ and also (because $3u_x u_{xx} + uu_{xxx} \in L^q(\mathbb{R}^3), \forall q < 10/3$), that

$$\partial_x^5 u, \partial_x^3 \partial_y u, \partial_x^3 \partial_z u, \partial_x \partial_y^2 u, \partial_x \partial_z^2 u \in L^q(\mathbb{R}^3), \quad \forall q < 10/3.$$

By [4] again,

$$\partial_x^{\alpha_1} \partial_y^{\alpha_2} \partial_z^{\alpha_3} u \in L^q(\mathbb{R}^3), \quad \alpha_1 + 2\alpha_2 + 2\alpha_3 < \frac{7}{2} + \frac{5}{q}.$$

Thus, $u, \partial_x u, \partial_x^2 u, \partial_y u, \partial_z u, \partial_x^3 u, \partial_x \partial_z u, \partial_x \partial_y u \in L^\infty(\mathbb{R}^3)$. This implies, setting $f = u_x^2 + uu_{xx}$, that $f, \partial_x f \in L^\infty(\mathbb{R}^3)$, and $\partial_y f, \partial_z f \in L^q(\mathbb{R}^3), \forall q < 10$. Theorem 4.1 is now obtained by reiteration (again, the regularity of $D_x^{-1} u_y$ and $D_x^{-1} u_z$ follows from the regularity of u and equation (1.4) which gives $\Delta D_x^{-1} u_y$ and $\Delta D_x^{-1} u_z \in H^\infty$). ■

Remark 4.1. – In the case where p is not an integer ($1 \leq p < 4$ if $d = 2$ and $1 \leq p < 4/3$ if $d = 3$), the previous method only gives a finite order regularity for the solitary waves, since in this case, $f(u)$ is not a C^∞ function of u .

Remark 4.2. – It is worth noticing that, contrarily to the solitary waves of Korteweg-de Vries or nonlinear Schrödinger equations, the solitary waves

of Kadomtsev-Petviashvili equations are neither radial, nor positive, and in general do not decay exponentially to 0 at infinity, as shows the example of “lumps” solutions of the KPI equation :

$$u(x - ct, y) = \frac{8c(1 - \frac{c}{3}(x - ct)^2 + \frac{c^2}{3}y^2)}{(1 + \frac{c}{3}(x - ct)^2 + \frac{c^2}{3}y^2)^2}$$

We do not know whether the solitary waves obtained in Theorems 2.1 and 2.2 decay with the same algebraic rate or not. (Added in proofs: a positive answer to this question has been given in [23].)

5. AN EXTENSION AND FINAL REMARKS

The results of the previous sections can be extended in various ways. We consider here the 2D and 3D versions of a fifth order KdV equation which have been investigated numerically by Abramyan and Stepanyants [3] and by Karpman and Belashov ([11], [12]). They read

$$(5.1) \quad \begin{cases} u_t + u^p u_x + u_{xxx} + \delta u_{xxxxx} - v_y = 0, \\ v_x = u_y \end{cases}$$

in the 2-dimensional case and

$$(5.2) \quad \begin{cases} u_t + u^p u_x + u_{xxx} + \delta u_{xxxxx} - v_y - w_z = 0, \\ v_x = u_y, \\ w_x = u_z \end{cases}$$

in the 3-dimensional case. In both cases, $\delta = \pm 1$. Let

$$Z = \{u \in Y, \partial_x^2 u \in L^2(\mathbb{R}^d)\}$$

endowed with the norm

$$\|\partial_x \varphi\|_Z = (c\|\partial_x \varphi\|_{L^2}^2 + \|\partial_y \varphi\|_{L^2}^2 + \|\partial_z \varphi\|_{L^2}^2 + \|\partial_x^2 \varphi\|_{L^2}^2 + \|\partial_x^3 \varphi\|_{L^2}^2)^{1/2}$$

where $u = \partial_x \varphi$. Here is our result concerning (5.1) and (5.2).

THEOREM 5.1. - (i) *The equation (5.1) has no nontrivial solitary wave $u = \partial_x \varphi \in Z$ satisfying $u \in H^1(\mathbb{R}^2) \cap L^\infty_{loc}(\mathbb{R}^2)$, $\partial_x^2 u$, $\partial_x^4 u$ and $\partial_y^2 \varphi \in L^2_{loc}(\mathbb{R}^2)$, if*

$$\delta = 1 \quad \text{and} \quad p \geq 4$$

or

$$\delta = 1, \quad 1 \leq p < 4,$$

and c is sufficiently large.

For $\delta = -1$ and p arbitrary, it admits a non trivial solitary wave $u \in Z$ which is a $H^\infty(\mathbb{R}^2)$ function when p is an integer.

(ii) The equation (5.2) has no nontrivial solitary wave $u = \partial_x \varphi \in Z$ satisfying $u \in H^1(\mathbb{R}^3) \cap L^{2(p+1)}(\mathbb{R}^3) \cap L^\infty_{loc}(\mathbb{R}^3)$, $\partial_x^2 u$, $\partial_x^4 u$, $\partial_y^2 \varphi$ and $\partial_z^2 \varphi \in L^2_{loc}(\mathbb{R}^3)$, if

$$p \geq 8/3 \quad \delta = -1$$

or

$$p \geq 2 \quad \delta = +1$$

or

$$1 \leq p < 2, \quad \delta = +1 \text{ and } c \text{ sufficiently large.}$$

For $\delta = -1$ and $1 \leq p < 8/3$, it admits a non trivial solitary wave $u \in Z$ which is a H^∞ function if $p = 1, 2$.

Remark 5.1. – Solitary waves for (5.1) (5.2) have been observed numerically for $\delta = -1$, $p = 1$ in [3], [11], [12].

Proof of Theorem. 5.1. – We first prove the non existence part. The proof is similar to that of Theorem 1.1 though the algebra is different. We just give the Pohojaev type identities. For equation (5.1) we get successively:

$$(5.4) \quad \int \left[-\frac{c}{2}u^2 + \frac{1}{p+2}u^{p+2} - \frac{3}{2}u_x^2 + \frac{5}{2}\delta u_{xx}^2 + \frac{v^2}{2} \right] dx dy = 0$$

$$(5.5) \quad \int \left[\frac{c}{2}u^2 - \frac{1}{(p+1)(p+2)}u^{p+2} + \frac{u_x^2}{2} - \frac{\delta}{2}u_{xx}^2 - \frac{v^2}{2} \right] dx dy = 0$$

$$(5.6) \quad \int \left[-cu^2 + \frac{u^{p+2}}{p+1} - u_x^2 + \delta u_{xx}^2 - v^2 \right] dx dy = 0$$

Substracting (5.4) from (5.5) we obtain

$$(5.7) \quad \int \left[cu^2 - \frac{1}{p+1}u^{p+2} + 2u_x^2 - 3\delta u_{xx}^2 - v^2 \right] dx dy = 0$$

and adding (5.6) (5.7) yields

$$(5.8) \quad \int [u_x^2 - 2\delta u_{xx}^2 - 2v^2] dx dy = 0,$$

while adding (5.4) and (5.5) implies

$$(5.9) \quad \frac{p}{(p+1)(p+2)} \int u^{p+2} dx dy = \int [u_x^2 - 2\delta u_{xx}^2] dx dy.$$

The next steps consists in plugging (5.8) and (5.9) into (5.7) to get

$$(5.10) \quad \int \left[2cu^2 + \left(\frac{p-4}{p} \right) u_x^2 + \frac{8\delta}{p} u_{xx}^2 \right] dx dy = 0$$

which proves that no solitary wave can exist for $\delta = +1$ and $p \geq 4$, or for $\delta = 1$, $1 \leq p < 4$, when c is sufficiently large, namely $c > \frac{(p-4)^2}{64p}$ (we have used the inequality $\|u_x\|_{L^2}^2 \leq \|u\|_{L^2} \|u_{xx}\|_{L^2}$).

As for equation (5.2), we find the Pohojaev identities

$$(5.11) \quad \int \left[-\frac{c}{2} u^2 + \frac{u^{p+2}}{p+2} - \frac{3}{2} u_x^2 + \frac{5}{2} \delta u_{xx}^2 + \frac{v^2}{2} + \frac{w^2}{2} \right] dx dy dz = 0$$

$$(5.12) \quad \int \left[\frac{c}{2} u^2 - \frac{1}{(p+1)(p+2)} u^{p+2} + \frac{1}{2} u_x^2 - \frac{\delta}{2} u_{xx}^2 - \frac{v^2}{2} + \frac{w^2}{2} \right] dx dy dz = 0$$

$$(5.13) \quad \int \left[\frac{c}{2} u^2 - \frac{1}{(p+1)(p+2)} u^{p+2} + \frac{1}{2} u_x^2 - \frac{\delta}{2} u_{xx}^2 + \frac{v^2}{2} - \frac{w^2}{2} \right] dx dy dz = 0$$

$$(5.14) \quad \int \left[-cu^2 + \frac{u^{p+2}}{p+1} - u_x^2 + \delta u_{xx}^2 - v^2 - w^2 \right] dx dy dz = 0.$$

We add (5.12) and (5.13) to obtain

$$(5.15) \quad \int \left[\frac{c}{2} u^2 - \frac{1}{(p+1)(p+2)} u^{p+2} + \frac{1}{2} u_x^2 - \frac{\delta}{2} u_{xx}^2 \right] dx dy dz = 0.$$

Now we add (5.11) and $(p+1)$ times (5.15) and get

$$(5.16) \quad \int [cpu^2 + u_x^2(p-2) - \delta u_{xx}^2(p-4) + v^2 + w^2] dx dy dz = 0.$$

Our claim for $\delta = 1$, $1 \leq p \leq 4$ results from (5.16) (In the case $1 \leq p < 2$ we have non existence for $c \geq \frac{(2-p)^2}{4p(4-p)}$).

To solve the remaining cases we need a few more identities. First, subtracting (5.13) from (5.12) yields (independently of the value of δ) :

$$(5.17) \quad \int v^2 dx dy dz = \int w^2 dx dy dz.$$

Then adding 2 times (5.13) to (5.14), and using (5.17), we obtain

$$\frac{1}{(p+1)(p+2)} \int w^{p+2} dx dy dz = \frac{2}{p} \int v^2 dx dy dz.$$

This identity plugged into (5.15) yields

$$(5.18) \quad \int [cpu^2 + pu_x^2 - p\delta u_{xx}^2 - 4v^2] dx dy dz = 0,$$

which with (5.16) and (5.17) imply

$$(5.19) \quad \int [3cpu^2 + (3p-4)u_x^2 - \delta(3p-8)u_{xx}^2] dx dy dz = 0.$$

This rules out the case $\delta = -1, p \geq 8/3$. Note that the cut-off value $p = 8/3$ for p corresponds to the imbedding $Z \subset L^{14/3}(\mathbb{R}^3)$ (see below).

In order to treat the last case $\delta = 1, p \geq 4$, we subtract $\frac{1}{3}$ times (5.19) from (5.16) to get

$$(5.20) \quad \int [-u_x^2 + 2\delta u_{xx}^2 + 3v^2] dx dy dz = 0.$$

Eliminating the u_{xx}^2 term thanks to (5.18) leads to

$$\int \left[2cu^2 + u_x^2 + \left(3 - \frac{8}{p} \right) v^2 \right] dx dy dz = 0,$$

which rules out the case $p \geq 8/3$ (for $\delta = \pm 1$). This concludes the non existence proof for $d = 3$.

We now turn to the existence proof. Let $d = 2$ or $3, \delta = -1, c > 0$ and $1 \leq p < 8/3$ if $d = 3$ (p arbitrary if $d = 2$). The proof of existence of a solitary wave for (5.1) or (5.2) is similar to the proof of Theorem 2.1. Consider for $\lambda > 0$ the minimization problem

$$J_\lambda = \inf \left\{ \|u\|_Z^2, u \in Z, \int_{\mathbb{R}^d} w^{p+2} = \lambda \right\}.$$

Note that J_λ is well defined since we infer from [4] that

$$Z \subset L^q(\mathbb{R}^d), \quad \begin{cases} \forall q, \quad 2 \leq q < +\infty & \text{if } d = 2 \\ 2 \leq q \leq \frac{14}{3} & \text{if } d = 3. \end{cases}$$

We then proceed as in the proof of Theorem 2.1 and find a minimum $u \in Z$ of J_λ , which after a scale change satisfies

$$(5.21) \quad \begin{cases} -cu_x + u^p u_x + u_{xxx} - u_{xxxxx} - v_y = 0, & \text{if } d = 2 \\ v_x = u_y \end{cases}$$

and

$$(5.22) \quad \begin{cases} -cu_x + u^p u_x + u_{xxx} - u_{xxxxx} - v_y - w_z = 0, & \text{if } d = 3, \\ v_x = u_y, \\ w_x = u_z. \end{cases}$$

Note that here, no rescaling allow us to suppress c in equations (5.21) and (5.22). This explains why we have to introduce c in the minimization problem J_λ .

The regularity of the solitary wave is obtained by the same method as in Theorem 4.1. ■

We conclude this section by some remarks and open questions.

1. We postpone to a subsequent paper [23] the study of further properties of the solitary waves and the study of solitary waves of Kadomtsev-Petviashvili type equations with non pure power nonlinearities (*see* a physical example of such equations in [16]).
2. One could also consider solitary waves propagating along an arbitrary direction, *i.e.* of the form $u(x - c_1 t, y - c_2 t, z - c_3 t)$ (resp. $u(x - c_1 t, y - c_2 t, z - c_3 t)$), with $c_1 > 0$. In this case, the change of variables $x' = x, y' = y - \frac{1}{2}\varepsilon c_2 x$ (resp. $x' = x, y' = y - \frac{1}{2}ac_2 x, z' = z - \frac{1}{2}bc_3 x$) allows to get back to a solitary wave propagating along the x -direction with a velocity $c = c_1 + \varepsilon c_2^2/4$ (resp. $c = c_1 + ac_2^2/4 + bc_3^2/4$). The computations in Section 2 then show that no such solitary wave exist if $\varepsilon = +1$, or $\varepsilon = -1, c_1 > c_2^2/4$ and $p \geq 4$, or $\varepsilon = -1, c_1 < c_2^2/4$ and $p \leq 4$ (resp. $ab = -1$, or $a = b = 1$, or $a = b = -1, c_1 > (c_2^2 + c_3^2)/4$ and $p \geq 4/3$, or $a = b = -1, c_1 < (c_2^2 + c_3^2)/4$ and $p \leq 4/3$). On the other hand, the existence theorem in Section 3 shows that when $\varepsilon = -1, p < 4$ and $c_1 > c_2^2/4$ (resp. $a = b = -1, p < 4/3$ and $c_1 > (c_2^2 + c_3^2)/4$), there is a solitary wave solution $u \in Y$.

3. An interesting question is that of stability of solitary waves. To our knowledge no rigorous result is known so far concerning the orbital stability of solitary waves for Kadomtsev-Petviashvili type equations (even for the lumps of KPI). As to instability, it is claimed in [20] that the solitary waves of (4.1) with $\varepsilon = -1$, $f(u) = u^p$ are unstable for $p > 4/3$. (Added in proofs: see [24] for an answer to this question.)

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