

Convex symmetrization and applications

by

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ABSTRACT. – We give a “generalized” version of the isoperimetric inequality when the perimeter is defined with respect to a convex, positively homogeneous function on \mathbb{R}^n . We use it to prove that, for any function u compactly supported in \mathbb{R}^n , the integral of a convex function of Du decreases when u is rearranged in the corresponding “convex” way. Similar arguments allow us, for example, to prove comparison results for solutions of the Dirichlet problem for elliptic equations when the differential operator satisfies suitable structure assumptions.

RÉSUMÉ. – Nous donnons une version « généralisée » de l’inégalité isopérimétrique lorsque la définition du périmètre dépend d’une fonction convexe et positivement homogène sur \mathbb{R}^n . Cette inégalité est employée pour démontrer que, pour toutes les fonctions u avec support compact dans \mathbb{R}^n , l’intégrale d’une fonction convexe de Du décroît quand u est réarrangée à une façon « convexe ». Avec des arguments du même type nous démontrons, par exemple, les résultats de comparaison pour les solutions du problème de Dirichlet pour des équations elliptiques quand l’opérateur différentiel satisfait des hypothèses de structure convenables.

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1. INTRODUCTION

It is well known that under Schwarz symmetrization certain quantities increase or decrease. As an example let us consider a nonnegative smooth function u defined on \mathbb{R}^n , with compact support. If $u^\#$ is the Schwarz symmetrization of u , the Pólya-Szegő principle states (*see*, e.g., [PS], [T1]) that:

$$(1.1) \quad \int_{\mathbb{R}^n} |Du|^2 dx \geq \int_{\mathbb{R}^n} |Du^\#|^2 dx.$$

We recall (*see* also Section 2) that the Schwarz symmetrization or spherically decreasing rearrangement of u is defined as

$$u^\#(x) = \sup\{t \geq 0 : |\{x : |u(x)| > t\}| > \omega_n |x|^n\},$$

where ω_n is the measure of the unit sphere in \mathbb{R}^n and $|E|$ denotes the measure of the set $E \subset \mathbb{R}^n$. In other words, $u^\#$ is a spherically symmetric and radially decreasing function such that its level sets (*i.e.* $\{x : u^\#(x) > t\}$) are balls which have the same measure as the level sets of u (*i.e.* $\{x : u(x) > t\}$). The proof of (1.1) is essentially based on the following inequality

$$(1.2) \quad -\frac{d}{dt} \int_{u>t} |Du| dx \geq n\omega_n^{1/n} |\{x : u(x) > t\}|,$$

which is a consequence of the isoperimetric inequality and of Fleming-Rishel formula.

Using similar methods, it is possible to prove that Schwarz symmetrization increases the L^p -norm of solutions of linear and nonlinear elliptic equations. Let u be a weak solution of the Dirichlet problem

$$\begin{cases} -(a_{ij}(x)u_{x_j})_{x_i} = f \\ u \in W_0^{1,2}(\Omega), \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is an open bounded set and a_{ij} are bounded functions satisfying

$$a_{ij}\xi_j\xi_i \geq |\xi|^2.$$

If v is the solution of the “symmetrized” problem

$$\begin{cases} -\Delta v = f^\# \\ v \in W_0^{1,2}(\Omega^\#), \end{cases}$$

where $\Omega^\#$ is the ball centered at the origin such that $|\Omega^\#| = |\Omega|$, then the inequality

$$(1.3) \quad u^\#(x) \leq v(x), \quad x \in \Omega^\#,$$

holds (see [T2] and, for various generalizations, [T3], [ALT], [FP], [BFM]). This means, for example that any norm of u increases under Schwarz symmetrization. Once again, one of the main tools to prove (1.3) is inequality (1.2).

At this point, one could ask if it is possible to find the way to “symmetrize” a function u in order to minimize the functional

$$(1.4) \quad \int_{\mathbb{R}^n} (H(Du))^2 dx,$$

where $H(\xi)$ is a nonnegative convex function, positively homogeneous of degree 1.

In order to solve the problem of minimizing (1.4) we will prove that for any bounded set E of finite perimeter, an inequality like

$$(1.5) \quad P_H(E) \geq \alpha_n |E|^{1-1/n}$$

holds (see Section 2), where α_n is a suitable “isoperimetric” constant and $P_H(E)$ is a “generalized” perimeter. Using (1.5) it is possible to prove the following version of the Pólya-Szegö principle (see Section 3):

$$\int_{\mathbb{R}^n} (H(Du))^2 dx \geq \int_{\mathbb{R}^n} (H(Du^*))^2 dx,$$

where u^* is the “convex” rearrangement of u with respect to H , that is, the level sets of u^* have the same measure as the level sets of u and are homothetic to the set K° which is polar to the set $\{x : H(x) \leq 1\}$.

In a natural way, using (1.5), we prove (see Section 4) comparison results for solutions of the Dirichlet problem:

$$(1.6) \quad \begin{cases} -(a_i(Du))_{x_i} = f \\ u \in W_0^{1,2}(\Omega), \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is an open bounded set and a_i satisfy

$$a_i(\xi)\xi_i \geq (H(\xi))^2,$$

and for solutions of the Cauchy problem:

$$\begin{cases} \frac{\partial u}{\partial t} + A(t, Du) = 0 & \text{in } Q = \Omega \times [0, T] \\ u(x, t) = 0 & \text{on } \partial\Omega \times [0, T] \\ u(x, 0) = \varphi(x) & \text{in } \Omega, \end{cases}$$

where

$$A(t, \xi) \geq H(\xi).$$

For example, it is possible to prove that if $H(\xi)$ is sufficiently regular (off the origin) a solution of (1.6) can be estimated in terms of the solution of the problem:

$$\begin{cases} -(H(Dw)H_{\xi_i}(Dw))_{x_i} = f^* & \text{in } \Omega^* \\ w = 0 & \text{on } \partial\Omega^*, \end{cases}$$

where f^* is the convex rearrangement of f with respect to H and Ω^* is the set homothetic to K^o such that $|\Omega^*| = |\Omega|$.

2. PRELIMINARIES

Let $H : \mathbb{R}^n \rightarrow [0, +\infty[$ be a convex function satisfying the homogeneity property:

$$(2.1) \quad H(tx) = |t|H(x), \quad \forall x \in \mathbb{R}^n, \quad \forall t \in \mathbb{R}.$$

Furthermore, assume that H satisfies

$$(2.2) \quad \alpha|\xi| \leq H(\xi) \leq \beta|\xi|, \quad \forall \xi \in \mathbb{R}^n,$$

for some positive constants $\alpha \leq \beta$. Because of (2.1) we can assume, without loss of generality, that the convex closed set

$$K = \{x \in \mathbb{R}^n : H(x) \leq 1\}$$

has measure $|K|$ equal to the measure ω_n of the unit sphere in \mathbb{R}^n . Sometimes, we will say that H is the gauge of K . If one defines (*see* [R]) the support function of K as:

$$H^o(x) = \sup_{\xi \in K} \langle x, \xi \rangle,$$

it is easy to verify that $H^\circ : \mathbb{R}^n \rightarrow [0, +\infty[$ is a convex, homogeneous function, and that H, H° are polar to each other in the sense that:

$$H^\circ(x) = \sup_{\xi \neq 0} \frac{\langle x, \xi \rangle}{H(\xi)},$$

and

$$H(x) = \sup_{\xi \neq 0} \frac{\langle x, \xi \rangle}{H^\circ(\xi)}.$$

For example it follows:

$$|\langle x, \xi \rangle| \leq H(x)H^\circ(\xi).$$

Clearly $H^\circ(x)$ itself is the gauge of the set:

$$K^\circ = \{x \in \mathbb{R}^n : H^\circ(x) \leq 1\}.$$

We say that K and K° are polar to each other. Finally we denote by κ_n the measure of K° . Further details can be found, e.g., in [La], [R].

Let Ω be an open subset of \mathbb{R}^n . It is possible to give the following definition of the total variation of a function $u \in BV(\Omega)$ with respect to a gauge function H (see [AB]):

$$\int_{\Omega} |Du|_H = \sup \left\{ \int_{\Omega} u \operatorname{div} \sigma \, dx : \sigma \in C_0^1(\Omega; \mathbb{R}^n), H^0(\sigma) \leq 1 \right\}.$$

This yields the following “generalized” definition of perimeter of a set E with respect to H :

$$P_H(E; \Omega) = \int_{\Omega} |D\chi_E|_H = \sup \left\{ \int_E \operatorname{div} \sigma \, dx : \sigma \in C_0^1(\Omega; \mathbb{R}^n), H^0(\sigma) \leq 1 \right\}.$$

The following co-area formula

$$\int_{\Omega} |Du|_H = \int_0^{\infty} P_H(\{u > s\}; \Omega) \, ds, \quad \forall u \in BV(\Omega),$$

and the equality

$$P_H(E; \Omega) = \int_{\Omega \cap \partial^* E} H(\nu^E) \, d\mathcal{H}^{n-1}(x)$$

hold, where $\partial^* E$ is the reduced boundary of E and ν^E is the outer normal to E (see [AB]).

One obtains readily that by definition $P_H(E; \Omega)$ is finite if and only if the usual perimeter:

$$P(E; \Omega) = \sup \left\{ \int_E \operatorname{div} \sigma \, dx : \sigma \in C_0^1(\Omega; \mathbb{R}^n), |\sigma| \leq 1 \right\}$$

is finite. In fact, (2.1) and (2.2) give:

$$\frac{1}{\beta} |\xi| \leq H^o(\xi) \leq \frac{1}{\alpha} |\xi|, \quad \forall \xi \in \mathbb{R}^n,$$

and then:

$$\alpha P(E; \Omega) \leq P_H(E; \Omega) \leq \beta P(E; \Omega).$$

Our aim is now to state an isoperimetric inequality which allows us to estimate from below the perimeter with respect to a generic gauge function H of a set E in terms of the measure of the set itself.

PROPOSITION 2.1. – *Let $f \in BV(\Omega)$. A sequence $\{f_h\}_h \subset C^\infty(\Omega)$ exists, such that:*

$$\lim_{h \rightarrow \infty} \int_{\Omega} |f_h - f| = 0,$$

and

$$\lim_{h \rightarrow \infty} \int_{\Omega} |Df_h|_H = \int_{\Omega} |Df|_H.$$

Proof. – Suitably mollifying f it is possible to define a sequence $\{f_h\}_h \subset C^\infty(\Omega)$ (see [G], [M]). Then, for example, one can follow the proof of Theorem 1.17 of [G].

PROPOSITION 2.2. – *Let E be a set of finite perimeter in Ω . A sequence of C^∞ sets $\{E_h\}_h$ exists, such that:*

$$\lim_{h \rightarrow \infty} \int_{\Omega} |\chi_{E_h} - \chi_E| = 0,$$

and

$$\lim_{h \rightarrow \infty} \int_{\Omega} |D\chi_{E_h}|_H = P_H(E; \Omega).$$

Proof. – Mollifying the function χ_E as in Proposition 2.1, it is possible to find a sequence $\{u_h\}_h \subset C^\infty(\Omega)$ such that:

$$\lim_{h \rightarrow \infty} \int_{\Omega} |u_h - \chi_E| = 0,$$

and

$$\lim_{h \rightarrow \infty} \int_{\Omega} |Du_h|_H = P_H(E; \Omega).$$

Furthermore, one has also (see [G], [M]) $0 \leq u_h \leq 1$. By the co-area formula we have:

$$\int_{\Omega} |Du_h|_H = \int_0^1 P_H(\{u_h > s\}; \Omega) ds.$$

Sard's theorem implies that the sets $E_s^{(h)} = \{u_h > s\}$ have C^∞ boundary for almost every $s \in (0, 1)$. We consider only such levels s . Let us fix $\varepsilon \in]0, 1/4[$ and $h = h(\varepsilon)$ such that:

$$\int_{\Omega} |u_h - \chi_E| < \varepsilon.$$

Following [M] (Lemma 2, p. 299) we have:

$$(2.3) \quad \int_{\Omega} |\chi_E - \chi_{E_s^{(h)}}| \leq \varepsilon^{1/2},$$

for every $s \in [\varepsilon^{1/2}, 1 - \varepsilon^{1/2}]$.

On the other hand, for every h there exists $s_h \in (\varepsilon^{1/2}, 1 - \varepsilon^{1/2})$ such that:

$$(2.4) \quad (1 - 2\varepsilon^{1/2})P_H(E_{s_h}^{(h)}; \Omega) \leq \int_0^1 P_H(E_t^{(h)}; \Omega) dt.$$

In view of the properties of u_h , we also get:

$$(2.5) \quad P_H(E; \Omega) = \lim_{h \rightarrow \infty} \int_{\Omega} |Du_h|_H = \lim_{h \rightarrow \infty} \int_0^1 P_H(E_t^{(h)}; \Omega) dt.$$

By (2.3), it follows that $\chi_{E_{s_h}^{(h)}} \rightarrow \chi_E$ in $L^1(\Omega)$ and, by (2.4), (2.5),

$$\limsup_{\varepsilon \rightarrow 0} P_H(E_{s_h}^{(h)}; \Omega) \leq P_H(E; \Omega).$$

Taking into account the fact that, by definition, P_H is lower semicontinuous, the proposition follows. ■

It is well known that, if E is a smooth set (for example Lipschitz), then the following inequality holds:

$$(2.6) \quad P_H(E; \mathbb{R}^n) = \int_{\mathbb{R}^n} |D\chi_E|_H = \int_E H(\nu^E) d\sigma \geq n\kappa_n^{1/n} |E|^{1-1/n}.$$

It is proven in [Bu] (see also [BZ]) making use of Brunn-Minkowski inequality. Then, using Proposition 2.2 and (2.6), we have:

PROPOSITION 2.3. – *If E is a set of finite perimeter in \mathbb{R}^n , then:*

$$(2.7) \quad P_H(E; \mathbb{R}^n) \geq n\kappa_n^{1/n} |E|^{1-1/n}.$$

Remark 2.1. – The inequality (2.7) obviously reduces to the classical isoperimetric inequality due to De Giorgi [DG] when $H(\xi) = |\xi|$. As in the classical one, (2.7) holds as an equality when the set E is homothetic to K° .

3. PÓLYA-SZEGÖ PRINCIPLE

We first observe that if $u \in W^{1,1}(\Omega)$ then (see [AB])

$$\int_{\Omega} |Du|_H = \int_{\Omega} H(Du) dx.$$

The co-area formula then gives, for any $u \in W^{1,1}(\Omega)$,

$$(3.1) \quad -\frac{d}{dt} \int_{u>t} H(Du) dx = P_H(\{u > t\}; \Omega),$$

for almost every t .

Let now $u : \mathbb{R}^n \rightarrow [0, +\infty[$ be a smooth function with compact support. It is well known that for a.e. $t \in [0, \sup u]$,

$$-\frac{d}{dt} \int_{u>t} H(Du) dx = \int_{u=t} \frac{H(Du)}{|Du|} d\mathcal{H}^{n-1}$$

For $p \geq 1$ we have:

$$\int_{\mathbb{R}^n} H^p(Du) dx = \int_0^\infty \left(\int_{u=t} \frac{H^p(Du)}{|Du|} d\mathcal{H}^{n-1} \right) dt.$$

On the other hand, by Hölder inequality we get:

$$(3.2) \quad \int_{u=t} H\left(\frac{Du}{|Du|}\right) d\mathcal{H}^{n-1} \leq \left(\int_{u=t} \frac{H^p(Du)}{|Du|} d\mathcal{H}^{n-1} \right)^{\frac{1}{p}} \left(\int_{u=t} \frac{1}{|Du|} d\mathcal{H}^{n-1} \right)^{1-\frac{1}{p}}.$$

It follows that for almost every $t \in [0, \sup u[$ the inequality

$$\int_{u=t} \frac{H^p(Du)}{|Du|} d\mathcal{H}^{n-1} \geq \left(\int_{u=t} H\left(\frac{Du}{|Du|}\right) d\mathcal{H}^{n-1} \right)^p (-\mu'(t))^{1-p}.$$

holds, where, as usual, we have denoted by $\mu(t) = |\{x : u(x) > t\}|$ the distribution function of u .

Taking into account (3.1) and the isoperimetric inequality (2.7), we obtain:

$$(3.3) \quad \int_{\mathbb{R}^n} H^p(Du) dx \geq \int_0^\infty (-\mu'(t))^{1-p} (n\kappa_n^{1/n} \mu(t)^{1-1/n})^p dt.$$

Let us verify that the right hand side of (3.3) coincides with $\int_{\mathbb{R}^n} H^p(Du^*) dx$, where u^* is a suitable ‘‘convex’’ symmetrization of u . We set

$$u^*(x) = u^*(|K^o(x)|),$$

where $K^o(x) = \{\xi \in \mathbb{R}^n : H^o(\xi) \leq H^o(x)\}$ and $|K^o(x)| = \kappa_n (H^o(x))^n$. By construction the level sets of u^* are homothetic to K . This means that the isoperimetric inequality holds as an equality for the sets $\{u^* > t\}$, i.e.,

$$\int_{u^*=t} H\left(\frac{Du^*}{|Du^*|}\right) d\mathcal{H}^{n-1} = n\kappa_n^{1/n} \mu(t)^{1-1/n}, \quad \text{for a.e. } t.$$

On the other hand, the Hölder inequality in (3.2) also holds as an equality when $u = u^*$. In fact, we claim that $H(Du^*)$ is constant on the set $\{u^* = t\}$. In order to prove this claim, one has to observe that $DH^o(x)$ is, for a.e. x , a vector normal to $\partial K^o(x)$. Then the definition of H gives (see [R])

$$H(DH^o(x)) = \frac{\langle x, DH^o(x) \rangle}{H^o(x)}.$$

The homogeneity assumption (2.1) implies:

$$H(DH^o(x)) = 1.$$

It follows that:

$$(3.4) \quad \begin{aligned} H(Du^*(x)) &= -u^{*'}(|K(x)|) n\kappa_n (H^o(x))^{n-1} \\ &= -u^{*'}(|K(x)|) n\kappa_n^{1/n} |K(x)|^{1-1/n}, \end{aligned}$$

and the claim is proven.

One also obtains:

$$\begin{aligned}
 (3.5) \quad \int_{\mathbb{R}^n} H^p(Du^*) \, dx &= \int_0^\infty \left(\int_{u^*=t} \frac{H^p(Du^*)}{|Du^*|} \, d\mathcal{H}^{n-1} \right) dt \\
 &= \int_0^\infty \left[-u^{*'}(\mu(t)) n \kappa_n^{1/n} \mu(t)^{1-1/n} \right]^p (-\mu'(t)) \, dt \\
 &= \int_0^\infty \left[n \kappa_n^{1/n} \mu(t)^{1-1/n} \right]^p (-\mu'(t))^{1-p} \, dt.
 \end{aligned}$$

Now, taking into account the fact that

$$\left(\int_{\mathbb{R}^n} H^p(Du) \right)^{1/p}$$

is a norm in $W_0^{1,p}(\mathbb{R}^n)$ equivalent to the usual one, a density argument gives the following

THEOREM 3.1. – *Let H be a gauge function and let $u \in W_0^{1,p}(\mathbb{R}^n)$, $p \geq 1$. Then $u^* \in W_0^{1,p}(\mathbb{R}^n)$ and*

$$(3.6) \quad \int_{\mathbb{R}^n} H^p(Du) \, dx \geq \int_{\mathbb{R}^n} H^p(Du^*) \, dx.$$

Remark 3.1. – Clearly, in the case $H(\xi) = |\xi|$, Theorem 3.1 gives the well known Pólya-Szegő principle. In such a case, u^* coincides with the spherically symmetric decreasing rearrangement of u , which is usually denoted by $u^\#$ (see, e.g., [T1]).

We now give sharp Sobolev-like inequalities as a simple application of the above results. Using Theorem 3.1 and the arguments in [T1], one gets:

COROLLARY 3.2. – *Under the assumptions of Theorem 3.1 we have:*

$$(3.7) \quad \|u\|_{p^*} \leq \frac{\omega_n^{1/n}}{\kappa_n^{1/n}} c_{n,p} \left(\int_{\mathbb{R}^n} H^p(Du) \, dx \right)^{1/p},$$

where $c_{n,p}$ is the best constant in Sobolev inequality given in [T1].

Proof. – The fact that one can bound the norm of u in L^{p^*} by the integral $(\int_{\mathbb{R}^n} H^p(Du) \, dx)^{1/p}$ times a suitable constant is a consequence of the Sobolev inequality and of the hypotheses on H . The best constant can be obtained observing that by Theorem 3.1 the ratio

$$\frac{\|u\|_{p^*}}{\left(\int_{\mathbb{R}^n} H^p(Du) \, dx \right)^{1/p}}$$

increases when u is replaced by u^* . But, once one supposes $u = u^*$, (3.5) gives:

$$(3.8) \quad \frac{\|u\|_{p^*}}{\left(\int_{\mathbb{R}^n} H^p(Du) dx\right)^{1/p}} = \frac{\omega_n^{1/n} \|u^\#\|_{p^*}}{\kappa_n^{1/n} \|Du^\#\|_p},$$

where $u^\#$ denotes the spherically symmetric rearrangement of u . So, using the estimate of the right hand side of (3.8) given in [T1], we get (3.7). ■

4. APPLICATIONS TO PDE's

In this section we will give some applications of the results in Section 2. In particular we will obtain sharp estimates for solutions of elliptic equations and of Hamilton-Jacobi equations.

4.1. Elliptic equations

Let $u \in W_0^{1,2}(\Omega)$ be a solution of the problem

$$(4.1) \quad \begin{cases} -\operatorname{div}(a(x, u, Du)) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where $f \in L^{\frac{2n}{n+2}}(\Omega)$ if $n \geq 3$, $f \in L^p(\Omega)$, $p > 1$, if $n = 2$, and $a(x, \eta, \xi) \equiv \{a_i(x, \eta, \xi)\}_{i=1, \dots, n}$ are Carathéodory functions satisfying

$$(4.2) \quad \langle a(x, \eta, \xi), \xi \rangle \geq (H(\xi))^2 \quad \text{a.e. } x \in \Omega, \quad \eta \in \mathbb{R}, \quad \xi \in \mathbb{R}^n,$$

and $H(\xi)$ is a gauge function as in Section 2.

A weak solution of (4.1) satisfies

$$\int_{\Omega} \langle a(x, u, Du), D\varphi \rangle = \int_{\Omega} f\varphi, \quad \forall \varphi \in W_0^{1,2}(\Omega).$$

Such a solution exists for example if the vector $a(x, \eta, \xi)$ satisfies:

$$|a(x, \eta, \xi)| \leq g(x) + c[|\eta| + |\xi|], \quad g \in L^2(\Omega),$$

$$\langle a(x, \eta, \xi) - a(x, \eta, \xi'), \xi - \xi' \rangle > 0,$$

for a.e. $x \in \Omega$, $\forall \eta \in \mathbb{R}$, $\xi, \xi' \in \mathbb{R}^n$.

Using, for $h > 0$, $t > 0$, the following test function

$$\varphi_h(x) = \begin{cases} h & \text{if } |u| > t + h \\ (|u| - t) \operatorname{sign} u & \text{if } t < |u| \leq t + h \\ 0 & \text{if } |u| \leq t \end{cases}$$

one gets, in a standard way,

$$-\frac{d}{dt} \int_{|u|>t} \langle a(x, u, Du), Du \rangle \leq \int_0^{\mu(t)} f^*(s) ds,$$

where $\mu(t)$ is the distribution function of u . Taking into account the assumption (4.2), it follows:

$$-\frac{d}{dt} \int_{|u|>t} (H(Du))^2 \leq \int_0^{\mu(t)} f^*(s) ds.$$

At this point, one has to use the isoperimetric inequality (2.7) in order to estimate from below the left-hand side of the above inequality. Proceeding for example as in [T2], one obtains:

$$1 \leq \frac{(-\mu'(t))}{n^2 \kappa_n^{2/n} \mu(t)^{2-2/n}} \int_0^{\mu(t)} f^*(s) ds \quad \text{for a.e. } t \in [0, \sup u].$$

Integrating both sides and using the definition of decreasing rearrangement of u , we have:

$$(4.3) \quad u^*(s) \leq \frac{1}{n^2 \kappa_n^{2/n}} \int_s^{|\Omega|} \frac{dr}{r^{2-2/n}} \int_0^r f^*(t) dt, \quad \forall s \in]0, |\Omega|].$$

It is easy to recognize that the function on the right hand side of (4.3) is proportional to the decreasing rearrangement of the solution of a suitable "symmetrized" problem. More precisely, (4.3) can be written as:

$$(4.4) \quad u^*(s) \leq \frac{\omega_n^{2/n}}{\kappa_n^{2/n}} v^*(s),$$

where $v^*(s)$ is the decreasing rearrangement of the solution v of the Dirichlet problem:

$$(4.5) \quad \begin{cases} -\Delta v = f^\# & \text{in } \Omega^\# \\ v = 0 & \text{on } \partial\Omega^\# \end{cases}$$

and $\Omega^\#$ denotes the sphere centered at the origin such that $|\Omega^\#| = |\Omega|$. Then we can state:

THEOREM 4.1. – *Let $u \in W_0^{1,2}(\Omega)$ be a solution of Problem (4.1) under the Hypothesis (4.2). If $v \in W_0^{1,2}(\Omega^\#)$ is the solution of (4.5), then (4.4) holds.*

Remark 4.1. – A result similar to Theorem 4.1 can be easily obtained using Hypothesis (2.2) and Talenti’s result (see [T2]), but our result gives a sharper estimate. In fact, by (4.2) and (2.2) it follows that the vector $a(x, \eta, \xi)$ satisfies the hypothesis

$$\langle a(x, \eta, \xi), \xi \rangle \geq \alpha^2 |\xi|^2.$$

Talenti’s result then implies

$$(4.6) \quad u^*(s) \leq \frac{1}{\alpha^2} v^*(s)$$

where v is the solution of (4.5). It is easy to show that

$$\frac{1}{\alpha^2} \geq \left(\frac{\omega_n}{\kappa_n} \right)^{2/n}.$$

In fact, the hypotheses $|K| = \omega_n$ and $H(\xi) \geq \alpha|\xi|$ imply that $K^o \supseteq \{\xi \in \mathbb{R}^n : |\xi| \leq \alpha\}$ and

$$\kappa_n = |K^o| \geq \alpha^n \omega_n$$

and then (4.4) is sharper than (4.6). We also observe that $\frac{\omega_n^{2/n}}{\kappa_n^{2/n}} = \frac{1}{\alpha^2}$ if and only if $H(\xi) = |\xi|$.

Example. – In order to give an idea of the improvement obtained, we give an example where κ_n can be explicitly calculated. Let us consider

$$H(\xi) = \frac{\Gamma(\frac{1}{4})}{\sqrt{2}\pi^{3/4}} (\xi_1^4 + \xi_2^4)^{1/4}, \quad \xi \in \mathbb{R}^2.$$

It is easy to show that the hypotheses required on H are satisfied and that

$$H^o(\xi) = \frac{\sqrt{2}\pi^{3/4}}{\Gamma(\frac{1}{4})} (\xi_1^{4/3} + \xi_2^{4/3})^{3/4}.$$

It follows that $\kappa_2 = 3$ and $\frac{\omega_2}{\kappa_2} = \frac{\pi}{3}$.

On the other hand, the best choice of α in (2.2) is $\alpha = \frac{\Gamma(\frac{1}{4})}{(2\pi)^{3/4}}$ and then

$$\frac{\omega_2}{\kappa_2} = \frac{\pi}{3} < \frac{1}{\alpha^2} \simeq 1.1981. \quad \blacksquare$$

In the case when $H(\xi) = |\xi|$, the estimate (4.4) can be written as

$$u^\# \leq v(x), \quad x \in \Omega^\#.$$

One could ask if, also in the general case, (4.4) gives an estimate of u in terms of a suitably symmetrized Dirichlet problem. We readily observe that (4.4) can be written as

$$(4.7) \quad u^\#(x) \leq \frac{\omega_n^{2/n}}{\kappa_n^{2/n}} v(x), \quad x \in \Omega^*,$$

where $v(x)$ minimizes the functional

$$(4.8) \quad \int_{\Omega^\#} \left[\frac{1}{2} |Dw|^2 - f^\# w \right] dx, \quad w \in W_0^{1,2}(\Omega^\#).$$

Now we show that

$$(4.9) \quad \frac{\omega_n^{2/n}}{\kappa_n^{2/n}} v(x) = z^\#(x),$$

where $z = z^*$, $z \in W_0^{1,2}(\Omega^*)$, minimizes the functional:

$$(4.10) \quad F^o(w) = \int_{\Omega^*} \left[\frac{1}{2} H^2(Dw) - f^* w \right] dx, \quad w \in W_0^{1,2}(\Omega^*),$$

and Ω^* is the set homothetic to K^o such that $|\Omega^*| = |\Omega|$.

First of all we observe that a minimizer z of F^o such that $z = z^*$ exists because we have, for any $w \in W_0^{1,2}(\Omega^*)$,

$$F^o(w^*) \leq F^o(w).$$

This inequality is a consequence of Pólya-Szegő inequality (3.6) and of the following Hardy-Littlewood inequality

$$\int_{\Omega} f w \, dx \leq \int_{\Omega^*} f^* w^* \, dx = \int_{\Omega^\#} f^\# w^\#.$$

This means that, in order to minimize $F^o(w)$ one can consider $w = w^*$. On the other hand, because of equality (3.5), we have:

$$F^o(w) = \frac{1}{2} \int_{\Omega^\#} \frac{\kappa_n^{2/n}}{\omega_n^{2/n}} |Dw^\#|^2 dx - \int_{\Omega^\#} f^\# w^\# dx \equiv F^\#(w^\#),$$

for any $w = w^*$. This means that the minimizer $z = z^*$ of the functional F° is such that $z^\#$ minimizes the functional $F^\#$. In other words $z^\#$ minimizes the functional

$$\int_{\Omega^\#} \frac{1}{2} |Dw|^2 dx - \int_{\Omega^\#} \frac{\omega_n^{2/n}}{\kappa_n^{2/n}} f^\# w dx,$$

that is the functional (4.8) with $\frac{\omega_n^{2/n}}{\kappa_n^{2/n}} f^\#$ in place of $f^\#$. Equality (4.9) immediately follows. Taking into account (4.7) we have that (4.4) can be written as

$$(4.11) \quad u^*(x) \leq z(x), \quad x \in \Omega^*,$$

where $z = z^*$, $z \in W_0^{1,2}(\Omega^*)$, is a minimizer of functional (4.10).

If H is enough regular one has that z is the solution of the problem

$$(4.12) \quad \begin{cases} -(H(Dw)H_{\xi_i}(Dw))_{x_i} = f^* & \text{in } \Omega^* \\ w = 0 & \text{on } \partial\Omega^* \end{cases}$$

In other words, as in the case considered in [T2], one could say that *among the Dirichlet problems (4.1) where Ω has a fixed measure, the right hand side of the equation has a fixed distribution function and the differential operator satisfies condition (4.2), problem (4.12) has the biggest solution.*

Taking into account the above considerations it appears natural to deal with minimizers of functionals of the following type:

$$(4.13) \quad F(u) = \int_{\Omega} [A(Du) - fu] dx, \quad u \in W_0^{1,2}(\Omega),$$

where $A(\xi)$ is convex, continuous and coercive. Under the hypotheses made on $A(\xi)$ it is well known that $u \in W_0^{1,2}(\Omega)$ minimizes $F(u)$ if and only if

$$(4.14) \quad \int_{\Omega} [\tilde{A}(Du; Dv) - fv] dx \leq 0, \quad \forall v \in W_0^{1,2}(\Omega),$$

where

$$(4.15) \quad \tilde{A}(\xi'; \xi'') = \lim_{\varepsilon \rightarrow 0^+} \frac{A(\xi' + \varepsilon \xi'') - A(\xi')}{\varepsilon}.$$

Using (4.14) and the arguments of the proof of Theorem 4.1, it is possible to prove the following:

THEOREM 4.2. – *Let $u \in W_0^{1,2}(\Omega)$ be a minimizer of functional F in (4.13) and let us assume that $\hat{A}(\xi'; \xi'')$ in (4.15) satisfies*

$$\hat{A}(\xi; \xi) \geq H^2(\xi), \quad \forall \xi \in \mathbb{R}^n,$$

where $H(\xi)$ is a gauge function as in Section 2. Then (4.11) holds.

4.2. Hamilton-Jacobi equations

Let $T > 0$ and let $u \in W^{1,2}(0, T; H_0^1(\Omega))$ be a generalized solution of the following Cauchy problem:

$$(4.16) \quad \begin{cases} \frac{\partial u}{\partial t} + A(t, Du) = 0 & \text{in } Q = \Omega \times [0, T] \\ u(x, t) = 0 & \text{on } \partial\Omega \times [0, T] \\ u(x, 0) = \varphi(x) & \text{in } \Omega, \end{cases}$$

where $\varphi \geq 0$, $\varphi \in C(\bar{\Omega})$, and

$$(4.17) \quad A(t, \xi) \geq H(\xi), \quad \text{for a.e. } \xi \in \mathbb{R}^n, \quad t \in [0, T],$$

being $H(\xi)$ a gauge function. We will obtain a comparison result between u and the viscosity solution of the “symmetrized” problem

$$(4.18) \quad \begin{cases} \frac{\partial v}{\partial t} + H(Dv) = 0 & \text{in } \Omega^* \times [0, T] \\ v(x, t) = 0 & \text{on } \partial\Omega^* \times [0, T] \\ v(x, 0) = \varphi^*(x) & \text{in } \Omega^*. \end{cases}$$

More precisely, we have:

THEOREM 4.3. – *Suppose $u \in W^{1,2}(0, T; W_0^{1,2}(\Omega))$ is a non-negative generalized solution of (4.16). Then we have:*

$$u^*(x, t) \leq v(x, t) \quad \text{in } \Omega^* \times [0, T]$$

where $v(x, t)$ is the viscosity solution of (4.18) and $u^*(\cdot, t)$ denotes the convex rearrangement of $u(\cdot, t)$ with respect to H for fixed t .

Proof. – Using hypothesis (4.17) and proceeding for example as in [FPV] (see also [GN]) it is possible to obtain:

$$(4.19) \quad -\frac{\partial}{\partial \theta} \int_{u>\theta} \frac{\partial u}{\partial t} dx - \frac{\partial}{\partial \theta} \int_{u>\theta} H(Du) dx \leq 0$$

for almost every $t \in [0, T]$ and $\theta \geq 0$, where the notation

$$\int_{u>\theta} (\dots) = \int_{\{x \in \Omega: u(x,t) > \theta\}} (\dots)$$

is used.

If, for a fixed t , we denote by $\mu(\theta, t)$ the distribution function of $u(x, t)$, using a result contained for example in [Ba], [MR], [ADLT] one obtains, for almost every $\theta \geq 0$,

$$(4.20) \quad \frac{\partial}{\partial \theta} \int_{u>\theta} \frac{\partial u}{\partial t} dx = \frac{\partial u^*}{\partial t}(\mu(\theta, t), t) \left(\frac{\partial \mu}{\partial \theta} \right),$$

where $u^*(\cdot, t)$ is the decreasing rearrangement of $u(\cdot, t)$. On the other hand the isoperimetric inequality (2.7) implies

$$(4.21) \quad -\frac{\partial}{\partial \theta} \int_{u>\theta} H(Du) dx \geq n\kappa_n^{1/n} \mu(\theta, t)^{1-1/n}.$$

Now, using standard techniques, (4.19), (4.20) and (4.21) give:

$$-\frac{\partial u^*}{\partial s} \leq -\frac{s^{\frac{1}{n}-1}}{n\kappa_n^{1/n}} \frac{\partial u^*}{\partial t}.$$

Putting $s = \kappa_n (H^\circ(x))^n$ and taking into account (3.4) we have:

$$H(Du^*(x, t)) \leq -\frac{\partial}{\partial t} u^*(x, t).$$

The assertion then follows from the well known properties of viscosity solutions (see, e.g. [Li] Theorem 11.2). ■

Remark 4.2. – As a final remark we would like to point out that, using methods similar to those indicated above, it is possible to obtain comparison results for solutions of equations in a more general form and also for solutions of equations of different type. For example one could consider solutions of the problem:

$$(4.22) \quad \begin{cases} -\operatorname{div}(a(x, u, Du)) + b(x, Du) + c(x, u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where the assumptions on $a(x, \eta, \xi)$ and f are the same as for problem (4.1), while on $b(x, \xi)$ and $c(x, u)$ we assume:

$$|b(x, \xi)| \leq B \left(\frac{\kappa_n}{\omega_n} \right)^{1/n} H(\xi), \quad c(x, \eta) \eta \geq 0.$$

An estimate like (4.4) can be found (*see* also [ALT]), where now $v(x)$ is solution of the problem:

$$\begin{cases} -\Delta v + B \frac{x_i}{|x|} v_{x_i} = f^\# & \text{in } \Omega^\# \\ v = 0 & \text{on } \partial\Omega^\#. \end{cases}$$

In a completely analogous way (*see* also [ALT]) one can give comparison results for parabolic equations of the type:

$$\begin{cases} u_t - \operatorname{div}(a(x, t, Du)) + b(x, t, Du) + c(x, t, u) = f(x, t), \\ u \in L^2(0, T; H_0^1(\Omega)) \cap C([0, T]; L^2(\Omega)), \quad u(x, 0) = u_0(x), \end{cases} \quad \text{in } Q = \Omega \times [0, T]$$

under the assumptions

$$\begin{aligned} \langle a(x, t, \xi), \xi \rangle &\geq \nu(t) (H(\xi))^2, \\ |b(x, t, \xi)| &\leq B(t) H(\xi), \\ c(x, t, \eta) \eta &\geq 0, \end{aligned}$$

where $\nu(t)$ and $B(t)$ suitably depend on t .

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