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Geometric restrictions for the existence of viscosity solutions

by

P. CARDALIAGUET

Département de Mathématiques, Paris, France.
E-mail: cardaliaguet@ceremade.dauphine.fr

B. DACOROGNA

Département de Mathématiques,
Ecole Polytechnique Fédérale de Lausanne,
CH 1015 Lausanne, Suisse.
E-mail: Bernard.Dacorogna@epfl.ch

W. GANGBO*

School of Mathematics, Georgia Institute of Technology,
Atlanta, GA 30332, USA.
E-mail: gangbo@math.gtech.edu

and

N. GEORGY

Département de Mathématiques,
Ecole Polytechnique Fédérale de Lausanne,
CH 1015 Lausanne, Suisse.
E-mail: Nicolas.Georgy@epfl.ch

ABSTRACT. – We study the Hamilton-Jacobi equation

$$\begin{cases} F(Du) = 0 & \text{a.e. in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases} \tag{0.1}$$

where $F : \mathbb{R}^N \longrightarrow \mathbb{R}$ is not necessarily convex. When Ω is a convex set, under technical assumptions our first main result gives a necessary and

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sufficient condition on the geometry of Ω and on $D\varphi$ for (0.1) to admit a Lipschitz *viscosity solution*. When we drop the convexity assumption on Ω, and relax technical assumptions our second main result uses the viability theory to give a necessary condition on the geometry of Ω and on $D\varphi$ for (0.1) to admit a Lipschitz *viscosity solution*.

RÉSUMÉ. – Nous étudions l'équation de Hamilton-Jacobi suivante

$$\begin{cases} F(Du) & = & 0 & \text{p.p. dans} & \Omega \\ u & = & \varphi & \text{sur} & \partial\Omega \end{cases} \tag{0.2}$$

où $F : \mathbb{R}^N \longrightarrow \mathbb{R}$ n'est pas nécessairement convexe. Lorsque Ω est un ensemble convexe, notre premier résultat donne une condition nécessaire et suffisante sur la géométrie du domaine Ω et sur $D\varphi$ afin que (0.2) admette une *solution de viscosité* lipschitzienne. Si on enlève la condition de convexité du domaine Ω, notre second résultat permet, à l'aide du théorème de viabilité, de donner une condition nécessaire sur la géométrie du domaine Ω et sur $D\varphi$ afin que (0.2) admette une *solution de viscosité* lipschitzienne.

1. INTRODUCTION

In this article we give a necessary and sufficient geometric condition for the following Hamilton-Jacobi equation

$$\begin{cases} F(Du) & = & 0 & \text{a.e. in} & \Omega \\ u & = & \varphi & \text{on} & \partial\Omega \end{cases} \tag{1.1}$$

to admit a $W^{1,\infty}(\Omega)$ *viscosity* solution. Here, $\Omega \subset \mathbb{R}^N$ is a bounded, open set, $F : \mathbb{R}^N \longrightarrow \mathbb{R}$ is continuous and $\varphi \in C^1(\overline{\Omega})$. We prove that existence of *viscosity* solutions [1] depends strongly on geometric compatibilities of the set of zeroes of F, of φ and of Ω, however it does not depend on the smoothness of the data.

The Hamilton-Jacobi equations are classically derived from the calculus of variations, and the interest of finding *viscosity* solutions (notion introduced by M.G. Crandall-P.L. Lions [8]) of problem (1.1) is well-known

[1] Equation (1.1) may admit only continuous or even discontinuous *viscosity* solutions (see [4]). We are here interested only in $W^{1,\infty}$ solutions.

in optimal control and differential games theory (c.f. M. Bardi-I. Capuzzo Dolcetta [3], G. Barles [4], W.H. Fleming-H.M. Soner [13] and P.L. Lions [17]).

It has recently been shown by B. Dacorogna-P. Marcellini in [9], [10] and [11] (cf. also A. Bressan and F. Flores [6]) that (1.1) has infinitely (even G_δ dense) many solutions $u \in W^{1,\infty}(\Omega)$ provided the compatibility condition

$$D\varphi(x) \in int(conv(Z_F)) \cup Z_F, \quad \text{for every } x \in \Omega \tag{1.2}$$

holds, where

$$Z_F = \{\xi \in \mathbb{R}^N \ : \ F(\xi) = 0\}, \tag{1.3}$$

and $conv(Z_F)$ denotes the convex hull of Z_F and $int(conv(Z_F))$ its interior. In fact (1.2) is, in some sense, almost a necessary condition for the existence of $W^{1,\infty}(\Omega)$ solution of (1.1). The classical existence results on $W^{1,\infty}(\Omega)$ *viscosity* solution of (1.1) require stronger assumptions than (1.2) (see M. Bardi-I. Capuzzo Dolcetta, [3], G. Barles [4], W.H. Fleming-H.M. Soner [13] and P.L. Lions [17]).

Here we wish to investigate the question of existence of $W^{1,\infty}(\Omega)$ *viscosity* solution under the sole assumption (1.2). As mentioned above, the answer will be, in general, that such solutions do not exist unless strong geometric restrictions on the set Z_F, on Ω and on φ are assumed.

To understand better our results one should keep in mind the following example.

EXAMPLE 1.1. – *Let*

$$F(\xi_1, \xi_2) = -(\xi_1^2 - 1)^2 - (\xi_2^2 - 1)^2 \tag{1.4}$$

(Note that F is a polynomial of degree 4). Clearly,

$$\begin{cases} Z_F = \{\xi \in \mathbb{R}^2 \ : \ \xi_1^2 = \xi_2^2 = 1\} \\ \\ conv(Z_F) = \{\xi \in \mathbb{R}^2 \ : \ |\xi_1| \le 1 \ , \ |\xi_2| \le 1\} \\ \quad = \{\xi \in \mathbb{R}^2 \ : \ |\xi|_\infty = max\{|\xi_1| \ , \ |\xi_2|\} \le 1\} \\ \\ Z_F \subset \partial(conv(Z_F)) \text{ and } Z_F \neq \partial(conv(Z_F)). \end{cases} \tag{1.5}$$

Our article will be divided into two parts, obtaining essentially the same results. The first one (c.f. Section 2) will compare the Dirichlet problem (1.1) with an appropriate problem involving a certain gauge. The second one (c.f. Section 3) will use the viability approach.

We start by describing the first approach. We will assume there that Ω is convex. To the set $conv(Z_F)$ we associate its gauge, i.e.

$$\rho(\xi) = \inf\{\lambda > 0 \ : \ \xi \in \lambda conv(Z_F)\}. \tag{1.6}$$

(In the example $\rho(\xi) = |\xi|_\infty$).

The $W^{1,\infty}(\Omega)$ *viscosity* solutions of (1.1) will then be compared to those of

$$\begin{cases} \rho(Du) & = & 1 & \text{a.e. in} & \Omega \\ u & = & \varphi & \text{on} & \partial\Omega. \end{cases} \tag{1.7}$$

The compatibility condition on φ will then be

$$D\varphi(x) \in int(conv(Z_F)) \ , \ \forall x \in \overline{\Omega} \Leftrightarrow \rho(D\varphi(x)) < 1 \ , \ \forall x \in \overline{\Omega}.$$

We will first show (c.f. Theorem 2.2) that if $Z_F \subset \partial(conv(Z_F))$ and Z_F is bounded, then any $W^{1,\infty}(\Omega)$ *viscosity* solution of (1.1) is a *viscosity* solution of (1.7). However by classical results (c.f. S.H. Benton [5], A. Douglis [12], S.N. Kruzkov [16], P.L. Lions [17] and the bibliography there) we know that the *viscosity* solution of (1.7) is given by

$$u(x) = \inf_{y\in\partial\Omega}\{\varphi(y) + \rho^o(x-y)\}, \tag{1.8}$$

where ρ^o is the polar of ρ, i.e.

$$\rho^o(\xi^*) = \sup_{\rho(\xi)\neq 0}\left\{\frac{<\xi^*, \xi>}{\rho(\xi)}\right\}. \tag{1.9}$$

(In the example $\rho^o(\xi^*) = |\xi^*|_1 = |\xi_1^*| + |\xi_2^*|$.)

The main result of Section 2 (c.f. Theorem 2.6, c.f. also Theorem 3.2) uses the above representation formula to give a *necessary and sufficient condition* for existence of $W^{1,\infty}(\Omega)$ *viscosity* solutions of (1.1). This geometrical condition can be roughly stated as $\forall y \in \partial\Omega$ where the inward unit normal, $\nu(y)$, is uniquely defined (recall that here Ω is convex and therefore this is the case for almost every $y \in \partial\Omega$) there exists $\lambda(y) > 0$ such that

$$D\varphi(y) + \lambda(y)\nu(y) \in Z_F. \tag{1.10}$$

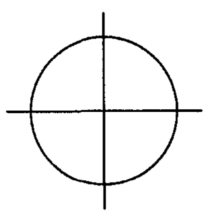
In particular if $\varphi \equiv 0$, we find that $\lambda(y) = \frac{1}{\rho(\nu(y))}$ and therefore the necessary and sufficient condition reads as

$$\frac{\nu(y)}{\rho(\nu(y))} \in Z_F. \tag{1.11}$$

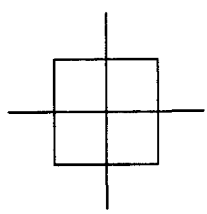
In the above example $Z_F = \{(-1,-1),(-1,1),(1,-1),(1,1)\}$, therefore the only convex Ω, which allows for $W^{1,\infty}(\Omega)$ *viscosity* solution of

$$\begin{cases} F(Du) &= 0 \quad \text{a.e. in} \quad \Omega \\ u &= 0 \quad \text{on} \quad \partial\Omega \end{cases}$$

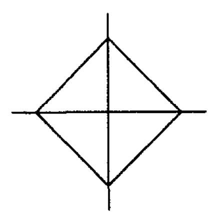
are rectangles whose normals are in Z_F. In particular for any smooth domain (such as the unit disk), (1.1) has no $W^{1,\infty}(\Omega)$ *viscosity* solution, while by the result of B. Dacorogna-P. Marcellini in [9], [10] and [11], (since $0 \in int(conv(Z_F))$) the existence of general $W^{1,\infty}(\Omega)$ solutions is guaranteed. Note that in the above example with Ω the unit disk, F and φ are analytic and therefore existence of $W^{1,\infty}(\Omega)$ *viscosity* solutions do not depend on the smoothness of the data.



No *viscosity* solution



No *viscosity* solution



One *viscosity* solution

It is interesting to note that if $F : \mathbb{R}^N \longrightarrow \mathbb{R}$ is convex and coercive (such as the eikonal equation), as in the classical literature, then $\partial(conv(Z_F)) \subset Z_F$. Therefore the above necessary and sufficient condition does not impose any restriction on the set Ω. However as soon as non convex F are considered, such as in the example, (1.10) drastically restricts the geometry of the set Ω, if existence of $W^{1,\infty}(\Omega)$ *viscosity* solution is to be ensured.

In Section 3 the basic ingredient for proving such a result is the viability Theorem (Theorem 3.3.2 of [2]). This Theorem gives an equivalence between the geometry of a closed set and the existence of solutions of some differential inclusion remaining in this set. The idea of putting together *viscosity* solutions and the viability Theorem is due to H. Frankowska in [15].

The main result of this section (c.f. Theorem 3.1, c.f. also Corollary 2.8) will show that if

$$\partial(conv(Z_F))\backslash Z_F \neq \emptyset \tag{1.12}$$

then we can always find an affine function φ with $D\varphi \in int(conv(Z_F))$ so that (1.1) has no $W^{1,\infty}(\Omega)$ *viscosity* solution.

The advantage of the second approach is that it will require weaker assumptions on F and on Ω than the first one. However the first approach will give more precise information since we will use the explicit formula for the *viscosity* solution of (1.7).

Some technical results are gathered in two appendixes.

2. COMPARISON WITH THE SOLUTION ASSOCIATED TO THE GAUGE

Throughout this section we assume that $F : \mathbb{R}^N \longrightarrow \mathbb{R}$ is continuous and that

- (H1) $Z_F \subset \partial(conv(Z_F))$.
  We recall that $Z_F = \{\xi \in \mathbb{R}^N : F(\xi) = 0\}$.
- (H2) Z_F is bounded.
- (H3) $D\varphi(x) \in int(conv(Z_F))$, $\forall x \in \overline{\Omega}$.

In addition we assume that the interior of the convex hull of Z_F is nonempty, i.e.

$$int(conv(Z_F)) \neq \emptyset \tag{2.1}$$

REMARKS 2.1.

(i) *In light of* (2.1) *we may assume without loss of generality that* $0 \in int(conv(Z_F))$, *since up to a translation this always holds.*

(ii) *Observe that* $int(conv(Z_F)) \neq \emptyset$ *is necessary for (H3) to make sense.*

(iii) *Recall that (H3) (without the interior) is, in some sense, necessary for existence of* $W^{1,\infty}(\Omega)$ *solutions (c.f. P.L. Lions [17]).*

(iv) *It is well-known (c.f. [18]) that the following properties hold:*

- ρ *is convex, homogeneous of degree one and* $\rho^{oo} = \rho$.
- $conv(Z_F) = \{z \in \mathbb{R}^N : \rho(z) \leq 1\}$.
- $\partial(conv(Z_F)) = \{z \in \mathbb{R}^N : \rho(z) = 1\}$.

- $\rho(z) > 0$ *for every* $z \neq 0$.

(v) *Since* $Z_F \subset \partial(conv(Z_F))$, *the function* F *has a definite sign in* $int(conv(Z_F))$. *We will assume, without loss of generality, that*

$$F(\xi) < 0, \tag{2.2}$$

for every $\xi \in int(conv(Z_F))$. *Otherwise in the following analysis we should replace* F *by* $-F$.

Our first result compares *viscosity* solutions of (1.1) and those of (1.7).

THEOREM 2.2. – *Let* $\Omega \subset \mathbb{R}^N$ *be a bounded open set, let* F *and* φ *satisfy (H1), (H2), (H3) and* (2.2). *Then any* $W^{1,\infty}(\Omega)$ *viscosity solution of* (1.1) *is also a* $W^{1,\infty}(\Omega)$ *viscosity solution of* (1.7). *Conversely if, in addition* $F > 0$ *outside* $conv(Z_F)$ *then a* $W^{1,\infty}(\Omega)$ viscosity *solution of* (1.7) *is also a* $W^{1,\infty}(\Omega)$ viscosity *solution of* (1.1).

REMARK 2.3. – *In the converse part of the above theorem the facts that* F *is continuous,* $F < 0$ *in* $int(conv(Z_F))$, *and* $F > 0$ *outside* $conv(Z_F)$ *implies that*

$$\partial(conv(Z_F)) = Z_F.$$

We recall the definition of subdifferential and superdifferential of functions (c.f. M. Bardi-I. Capuzzo Dolcetta [3], G. Barles [4] or W.H. Fleming-H.M. Soner [13]).

DEFINITION 2.4. – *Let* $u \in C(\Omega)$, *we define for* $x \in \Omega$ *the following sets,*

$$D^+u(x) = \left\{p \in \mathbb{R}^N : \limsup_{y \to x,\ y \in \Omega} \frac{u(y) - u(x) - < p, y - x >}{|x - y|} \leq 0\right\},$$

$$D^-u(x) = \left\{p \in \mathbb{R}^N : \liminf_{y \to x,\ y \in \Omega} \frac{u(y) - u(x) - < p, y - x >}{|x - y|} \geq 0\right\}.$$

$D^+u(x)$ $(D^-u(x))$ *is called superdifferential (subdifferential) of* u *at* x.

We recall a useful lemma stated in G. Barles [4].

LEMMA 2.5.

(i) $u \in C(\Omega)$ *is a viscosity subsolution of* $F(D(u(x))) = 0$ *in* Ω *if and only if,* $F(p) \leq 0$ *for every* $x \in \Omega$, $\forall p \in D^+u(x)$.

(ii) $u \in C(\Omega)$ *is a viscosity supersolution of* $F(D(u(x))) = 0$ *in* Ω *if and only if,* $F(p) \geq 0$ *for every* $x \in \Omega$, $\forall p \in D^-u(x)$.

We now give the proof of our first theorem.

Proof of Theorem 2.2.

1. Let $u \in W^{1,\infty}(\Omega)$ be a *viscosity* solution of (1.1).

(i) We first show that u is a *viscosity* supersolution of (1.7). Since u is a *viscosity* supersolution of (1.1), then in light of Lemma 4.2 and 2.5 we have for every $x \in \Omega$, and every $p \in D^-u(x)$,

$$p \in conv(Z_F) \text{ and } F(p) \geq 0. \tag{2.3}$$

Combining (2.2), (2.3) and (H1), we obtain that $p \in \partial(conv(Z_F))$, and so, $\rho(p) - 1 = 0$. Hence, by Lemma 2.5, u is a *viscosity* supersolution of (1.7).

(ii) We next show that u is a *viscosity* subsolution of (1.7). Since u is a *viscosity* subsolution of (1.1), then for every $x \in \Omega$, and $p \in D^+u(x)$, we have by Lemma 4.2, $p \in conv(Z_F)$ and so, $\rho(p) - 1 \leq 0$. We therefore deduce that u is a *viscosity* subsolution of (1.7).

Combining (i) and (ii) we have that $u \in W^{1,\infty}(\Omega)$, is a *viscosity* solution of (1.7).

2. We show that $u \in W^{1,\infty}(\Omega)$, the *viscosity* solution of (1.7) defined by (1.8), is also a *viscosity* solution of (1.1).

(iii) We recall that

$$F(\xi) > 0, \tag{2.4}$$

for all $\xi \in \mathbb{R}^N \setminus conv(Z_F)$. Since u is a *viscosity* supersolution of (1.7), then for every $x \in \Omega$, and $p \in D^-u(x)$, we have that $\rho(p) - 1 \geq 0$, i.e. $p \in \mathbb{R}^N \setminus int(conv(Z_F))$. From (2.4), it follows that $F(p) \geq 0$ and thus u is a *viscosity* supersolution of (1.1).

(iv) Since u is a *viscosity* subsolution of (1.7), we have for every $x \in \Omega$, and $p \in D^+u(x)$, that $\rho(p) - 1 \leq 0$, i.e. $p \in conv(Z_F)$ and then $F(p) \leq 0$. Thus u is a *viscosity* subsolution of (1.1).

Combining (iii) and (iv) we conclude that u is a *viscosity* solution of (1.1). □

We now state the main result of this section (see also Theorem 3.4).

THEOREM 2.6. – *Let F and φ satisfy (H1), (H2), (H3) and* (2.2). *If Ω is bounded, open and convex and $\varphi \in C^1(\overline{\Omega})$, then the two following conditions are equivalent*

1. *There exists $u \in W^{1,\infty}(\Omega)$* viscosity *solution of* (1.1).

2. *For every $y \in \partial\Omega$, where the unit inward normal in y (denoted $\nu(y)$) exists, there exists a unique $\lambda_0(y) > 0$ such that*

$$\begin{cases} D\varphi(y) + \lambda_0(y)\nu(y) \in Z_F \\ \rho(D\varphi(y) + \lambda_0(y)\nu(y)) = 1. \end{cases} \tag{2.5}$$

Before proving Theorem 2.6, we make few remarks, mention an immediate corollary and prove a lemma.

REMARKS 2.7. – (i) *By $\nu(y)$, the unit inward normal at y, exists we mean that it is uniquely defined there. Since Ω is convex, then this is the case for almost every $y \in \partial\Omega$.*

(ii) *In particular if $\varphi \equiv 0$, then*

$$\lambda_0(y) = \frac{1}{\rho(\nu(y))}$$

and so, the necessary and sufficient condition becomes

$$\frac{\nu(y)}{\rho(\nu(y))} \in Z_F.$$

(iii) *If F is convex and coercive, then* (2.5) *is always satisfied and therefore no restriction on the geometry of Ω is imposed by our theorem (as in the classical theory of M.G. Crandall-P.L. Lions [8]).*

COROLLARY 2.8. – *Let $\Omega \subset \mathbb{R}^N$ be a bounded open convex set, let $F : \mathbb{R}^N \longrightarrow \mathbb{R}$ be continuous and such that*

$$Z_F \subset \partial(conv(Z_F)) \text{ and } Z_F \neq \partial(conv(Z_F)).$$

Then there exists φ affine with $D\varphi(x) \in int(conv(Z_F))$, $\forall x \in \overline{\Omega}$ such that (1.1) *has no $W^{1,\infty}(\Omega)$ viscosity solutions.*

In section 3 we will strengthen this corollary by assuming only that $\partial(conv(Z_F))\backslash Z_F \neq \emptyset$.

We next state a lemma which plays a crucial role in the proof of Theorem 2.6.

LEMMA 2.9. – *Let Ω be bounded open and convex and $\varphi \in C^1(\overline{\Omega})$ with $\rho(D\varphi(x)) < 1$, $\forall x \in \overline{\Omega}$. Let u be defined by*

$$u(x) = \inf_{y \in \partial\Omega} \{\varphi(y) + \rho^o(x-y)\}, \quad x \in \overline{\Omega}.$$

Let $y(x) \in \partial\Omega$ be such that $u(x) = \varphi(y(x)) + \rho^o(x - y(x))$. The two following properties then hold

(i) *If* $D^-u(x)$ *is nonempty then the inward unit normal* $\nu(y(x))$ *at* $y(x)$ *exists (i.e. is uniquely defined).*

(ii) *Furthermore if* $p \in D^-u(x)$ *then there exists* $\lambda_0(y(x)) > 0$ *such that,* $p = D\varphi(y(x)) + \lambda_0(y(x))\nu(y(x))$, *where* $\nu(y(x))$ *is the unit inward normal to* $\partial\Omega$ *at* y.

Proof.

1. Let

$$I(x) = \{z \in \partial\Omega \ : \ u(x) = \varphi(z) + \rho^o(x-z)\}.$$

If $p \in D^-u(x)$ then for every compact set $K \subset \mathbb{R}^N$ and $h > 0$, we have

$$u(x+h\omega) - u(x) \geq < p, h\omega > + \epsilon(h), \quad \omega \in K \tag{2.6}$$

where ϵ satisfies $\liminf\limits_{h\to 0} \frac{\epsilon(h)}{h} = 0$.

In the sequel we assume without loss of generality that

$$0 \in int(\Omega), \tag{2.7}$$

since, by a change of variables (2.7) holds. Let ρ_Ω be the gauge associated to Ω i.e.

$$\rho_\Omega(z) = \inf\{\lambda > 0 : z \in \lambda\Omega\}.$$

We recall that

$$\partial\Omega = \{z \in \mathbb{R}^N \ : \ \rho_\Omega(z) = 1\}, \tag{2.8}$$

and

$$\Omega = \{z \in \mathbb{R}^N \ : \ \rho_\Omega(z) < 1\}. \tag{2.9}$$

Now, let $x_0 \in \Omega$, let $y_0 \in I(x_0)$ and let $q_0 \in \partial\rho_\Omega(y_0)$ (the subdifferential of ρ_Ω at y_0, in the sense of convex analysis, see R.T. Rockafellar [18]). Since ρ_Ω is a convex function, we have $\partial\rho_\Omega(y_0) = D^-\rho_\Omega(y_0)$ (see [4]). We have

$$\rho_\Omega(z) \geq \rho_\Omega(y_0) + < q_0, z - y_0 > , \quad z \in \mathbb{R}^N. \tag{2.10}$$

Note that $q_0 \neq 0$ since otherwise we would have $0 \in \partial\rho_\Omega(y_0)$ and so, y_0 would be a minimizer for ρ_Ω whereas $\rho_\Omega(y_0) > \rho_\Omega(0) = 0$. Define the hyperplane touching $\partial\Omega$ at y_0 and normal to q_0,

$$P_0 = \{z \in \mathbb{R}^N \ : < q_0, z - y_0 >= 0\},$$

and the barrier function

$$v(z) = \inf_{y \in P_0} \{\varphi(y) + \rho^o(x - y)\}.$$

2. *Claim* 1. – We have $u \leq v$ on Ω and $u(x_0) = v(x_0)$.

Indeed, for $x \in \Omega$, let $y_1(x) \in P_0$ be such that

$$v(x) = \varphi(y_1(x)) + \rho^o(x - y_1(x)),$$

and let

$$z_t = (1-t)x + ty_1(x), \quad t \in [0,1].$$

In light of (2.8), (2.9), (2.10), and the fact that $y_1(x) \in P_0$, we have

$$\rho_\Omega(z_0) = \rho_\Omega(x) < 1 \tag{2.11}$$

and

$$\rho_\Omega(z_1) = \rho_\Omega(y_1(x)) \geq 1. \tag{2.12}$$

Using (2.8), (2.11), and (2.12) we conclude that there exists $\mu \in (0,1]$ such that

$$z_\mu \in \partial\Omega.$$

Using the homogenity of ρ^o we obtain that

$$\begin{aligned} \rho^o(x - y_1(x)) &= \mu\rho^o(x - y_1(x)) + (1-\mu)\rho^o(x - y_1(x)) \\ &= \rho^o(x - z_\mu) + \rho^o(z_\mu - y_1(x)). \end{aligned}$$

We therefore deduce that

$$\begin{aligned} v(x) &= \varphi(y_1(x)) + \rho^o(x - y_1(x)) \\ &= \varphi(y_1(x)) + \rho^o(x - z_\mu) + \rho^o(z_\mu - y_1(x)). \end{aligned} \tag{2.13}$$

As $\rho(D\varphi) \leq 1$ we have (see Lemma 4.1)

$$\varphi(z_\mu) - \varphi(y_1(x)) \leq \rho^o(z_\mu - y_1(x)). \tag{2.14}$$

From (2.14) and the definition of u, we obtain

$$v(x) \geq \varphi(z_\mu) + \rho^o(x - z_\mu) \geq u(x).$$

So we have $v(x) \geq u(x)$. Observe also that $v(x_0) \leq u(x_0)$ and so, $v(x_0) = u(x_0)$. This concludes the proof of Claim 1.

3. *Claim* 2. – We have $p \in D^{-}v(x_0)$.

Indeed, in light of Claim 1 and (2.6) we have

$$v(x_0+hd)-v(x_0)-<p,hd>\geq u(x_0+hd)-u(x_0)-<p,hd>\geq \epsilon(h), \tag{2.15}$$

for every d in a compact set, and so,

$$p \in D^{-}v(x_0).$$

4. *Claim* 3. – $p - D\varphi(y_0)$ is parallel to q_0 (recall that $q_0 \neq 0$).

Let $q_1, \cdots, q_{N-1}$ be such that $\{q_0, \cdots, q_{N-1}\}$ is a set of orthogonal vectors. Using the definition of v, Claim 1 and the fact that

$$y_0 + hq_i \in P_0 \ , \ i = 1, \cdots, N-1, \tag{2.16}$$

we obtain

$$\begin{aligned} v(x_0 + hq_i) &\leq \varphi(y_0 + hq_i) + \rho^o(x_0 + hq_i - y_0 - hq_i) \\ &= \varphi(y_0 + hq_i) + \rho^o(x_0 - y_0) \\ &= \varphi(y_0 + hq_i) - \varphi(y_0) + v(x_0). \end{aligned} \tag{2.17}$$

Combining (2.15) and (2.17) we deduce that

$$h < p, q_i > \leq h < D\varphi(y_0), q_i > +\epsilon(h). \tag{2.18}$$

When we divide both sides of (2.18) by $h > 0$ and let h tend to 0 we obtain

$$< p, q_i > \leq < D\varphi(y_0), q_i > . \tag{2.19}$$

Similarly, when we divide both sides of (2.18) by $h < 0$ and let h tend to 0 we obtain

$$< p, q_i > \geq < D\varphi(y_0), q_i > . \tag{2.20}$$

Using (2.19) and (2.20) we conclude that

$$< p - D\varphi(y_0), q_i >= 0, \quad i = 1, \cdots, N-1,$$

thus,

$$p - D\varphi(y_0) = \lambda q_0, \tag{2.21}$$

for some $\lambda \in \mathbb{R}$. It is clear that $\lambda \neq 0$, since $\rho(p) = 1$ (by the fact that u is a supersolution of (1.7) and by Lemma 4.2) and $\rho(D\varphi(y_0)) < 1$.

5. *Claim* 4. – ρ_Ω is differentiable at y_0 (so $\nu(y_0)$ exists and $\nu(y_0) = q_0$ by definition of q_0).

Suppose there exists $q \in \partial\rho_\Omega(y_0)$ with $q \neq q_0$. We obtain repeating the same development as before, that

$$p - D\varphi(y_0) = \mu q, \tag{2.22}$$

for some $\mu \neq 0$. So

$$q = \alpha q_0 \tag{2.23}$$

with $\alpha = \frac{\lambda}{\mu} \neq 0$. If $\alpha < 0$, then any convex combination of q and q_0 is in $\partial\rho_\Omega(y_0)$ and thus $0 \in \partial\rho_\Omega(y_0)$ which yields that y_0 is a minimizer for ρ_Ω which, as already seen, is absurd. So we have $\alpha > 0$.

We will next prove that

$$\rho^o_\Omega(q) = 1, \tag{2.24}$$

for every $q \in \partial\rho_\Omega(y_0)$.

Assume for the moment that (2.24) holds and assume that $q \in \partial\rho_\Omega(y_0)$ satisfies (2.23). Then,

$$1 = \rho^o_\Omega(\alpha q_0) = \alpha\rho^o_\Omega(q_0) = \alpha.$$

Consequently, $\alpha = 1$, $q = q_0$ and so,

$$\partial\rho_\Omega(y_0) = \{q_0\}. \tag{2.25}$$

By (2.25) we deduce that ρ_Ω is differentiable at y_0 (see [18] Theorem 25.1).

We now prove (2.24). Denoting by ρ^*_Ω the Legendre tranform of ρ_Ω, one can readily check that

$$\rho^*_\Omega(x^*) = \begin{cases} 0 & \text{if } \rho^o_\Omega(x^*) \leq 1 \\ +\infty & \text{if } \rho^o_\Omega(x^*) > 1. \end{cases} \tag{2.26}$$

We recall the following well known facts:

$$\rho_\Omega(y_0) + \rho^*_\Omega(q) = < y_0, q >, \tag{2.27}$$

for every $q \in \partial\rho_\Omega(y_0)$, (see [18] Theorem 23.5) and

$$< y_0, q > \leq \rho_\Omega(y_0)\rho^o_\Omega(q). \tag{2.28}$$

Since $y_0 \in \partial\Omega$, we have $\rho_\Omega(y_0) = 1$, which, together with (2.27) and (2.28) implies that

$$\rho^o_\Omega(q) \geq 1 + \rho^*_\Omega(q). \tag{2.29}$$

Hence, $\rho^o_\Omega(q)$ being finite, we deduce $\rho^*_\Omega(q) = 0$. Using (2.26) and (2.29) we obtain that

$$\rho^o_\Omega(q) = 1. \tag{2.30}$$

6. *Claim* 5. – We have $p = D\varphi(y_0) + \lambda_0\nu(y_0)$, where $\nu(y_0)$ is the unit inward normal at y_0.

By Claim 3 and Claim 4, there exists $\lambda_0 \in \mathbb{R}$ such that

$$p = D\varphi(y_0) + \lambda_0\nu(y_0). \tag{2.31}$$

The task will be to show that $\lambda_0 > 0$. Let

$$x_h = (1-h)x_0 + hy_0, \quad h \in (0,1).$$

We have

$$u(x_h) = \inf_{y\in\partial\Omega}\{\varphi(y) + \rho^o(x_h - y)\} \leq \varphi(y_0) + \rho^o(x_h - y_0). \tag{2.32}$$

Using the definition of x_h and the homogeneity of ρ^o we get

$$\rho^o(x_h - y_0) = \rho^o((1-h)(x_0 - y_0)) = (1-h)\rho^o(x_0 - y_0),$$

which, along with (2.32) implies

$$u(x_h) \leq \varphi(y_0) + \rho^o(x_0 - y_0) - h\rho^o(x_0 - y_0) = u(x_0) - h\rho^o(x_0 - y_0). \tag{2.33}$$

In light of (2.6) and (2.33), we have

$$h < p, y_0 - x_0 > + \epsilon(h) \leq -h\rho^o(x_0 - y_0),$$

which yields,

$$< p, y_0 - x_0 > \leq -\rho^o(x_0 - y_0). \tag{2.34}$$

Using the definition of ρ^o (see (1.9)) we have

$$- < D\varphi(y_0), y_0 - x_0 > = < D\varphi(y_0), x_0 - y_0 > \leq \rho(D\varphi(y_0))\rho^o(x_0 - y_0).$$

Also, by (H3), there exists $\delta > 0$ such that

$$\rho(D\varphi(z)) \leq 1 - \delta, \quad z \in \overline{\Omega}. \tag{2.35}$$

Combining (2.34) and (2.35) we obtain

$$< p - D\varphi(y_0), y_0 - x_0 > \leq -\delta\rho^o(x_0 - y_0). \tag{2.36}$$

Moreover, since we can express $y_0 - x_0$ as a linear combination of the normal $\nu(y_0)$ and the tangential vectors $\{q_i\}_{i=1}^{N-1}$ at $\partial\Omega$ in y_0, there exist α and μ_i with $i = 1, \cdots, N-1$ such that

$$y_0 - x_0 = \alpha\nu(y_0) + \sum_{i=1}^{N-1} \mu_i q_i.$$

As $x_0 \in \Omega$ and Ω is convex, $\alpha < 0$. Using (2.31), and (2.36) we obtain

$$\alpha\lambda_0 = \alpha < p - D\varphi(y_0), \nu(y_0) > \leq -\delta\rho^o(x_0 - y_0).$$

Thus, $\lambda_0 > 0$. □

We now give the proof of the main theorem

Proof of Theorem 2.6.

1. (1) $\Rightarrow$ (2) We assume that u is a *viscosity* solution of (1.1).

From Theorem 2.2, we have that every *viscosity* solution of (1.1) is a *viscosity* solution of (1.7) and therefore by (1.8) u can be written as

$$u(x) = \inf_{y \in \partial\Omega} \{\varphi(y) + \rho^o(x - y)\}. \tag{2.37}$$

Let $y_0 \in \partial\Omega$ be a point where $\partial\rho_\Omega(y_0) = \{\nu(y_0)\}$ (see the notations of the proof of Lemma 2.9). Let $x \in \Omega$ be such that u is differentiable at x and x sufficiently close from y_0. Moreover the minimum in (2.37) is attained, at some $y(x) \in \partial\Omega$ close to y_0. In light of Lemma 2.9 there exists $\lambda_0(y(x)) > 0$ such that

$$Du(x) = D\varphi(y(x)) + \lambda_0(y(x))\nu(y(x)), \tag{2.38}$$

(i.e $Du(x) - D\varphi(y(x))$ is perpendicular to the tangential hyperplane).

Note that $\lambda_0(y(x))$ is bounded by $2|Du|_\infty$. Indeed, using the homogeneity of ρ, assuming that $|\nu(y(x))| = 1$ we have

$$|\lambda_0(y(x))\nu(y(x))| \leq |Du(x)| + |D\varphi(y(x))| \leq 2|Du|_\infty. \tag{2.39}$$

As u is a solution of (1.1), i.e. $Du(x) \in Z_F$, we obtain that

$$D\varphi(y(x)) + \lambda_0(y(x))\nu(y(x)) \in Z_F. \tag{2.40}$$

Letting x tend to y_0, we obtain that $y(x)$ tends to y_0. Since $\partial\rho_\Omega(y_0) = \{\nu(y_0)\}$ we have from Theorem 25.1 in [18] that ρ_Ω is differentiable at y_0. By Lemma 2.9 we have that $\partial\rho_\Omega(y(x)) = \{\nu(y(x))\}$ and ρ_Ω is differentiable at $y(x)$. Using Theorem 25.5 in [18], we obtain that $\nu(y(x))$ tends to $\nu(y_0)$. Also, by (2.39) $\lambda_0(y(x))$ tends, up to a subsequence, to a limit, denoted λ_0 when x goes to y_0. Since Z_F is closed, and F is continuous, and so is $D\varphi$, (2.40) implies

$$D\varphi(y_0) + \lambda_0\nu(y_0) \in Z_F.$$

As $\lambda_0(y(x)) > 0$, we have that $\lambda_0 \geq 0$. Moreover u is solution of (1.7) and so λ_0 is uniquely determined by the equation

$$\rho(D\varphi(y_0) + \lambda_0\nu(y_0)) = 1.$$

As $\rho(D\varphi(y_0)) < 1$, we have that $\lambda_0 \neq 0$ and so $\lambda_0 > 0$. This establishes that (1) $\Rightarrow$ (2).

2. (2) $\Rightarrow$ (1) Conversely, assume that (2.5) holds.

Using (1.8) we obtain that u defined by

$$u(x) = \inf_{y\in\partial\Omega}\{\varphi(y) + \rho^o(x-y)\}$$

is the *viscosity* solution of (1.7). We have to show that u is a *viscosity* solution of (1.1).

- Since u is a *viscosity* subsolution of (1.7), then for every $x \in \Omega$ and $\forall p \in D^+u(x)$, we have from Lemma 4.2, $p \in conv(Z_F)$ (i.e. $\rho(p) \leq 1$). As (H1) is satisfied (with the convention: $F(\xi) < 0$, $\forall \xi \in int(conv(Z_F))$) and as F is continuous, it follows that $F(p) \leq 0$. So u is a *viscosity* subsolution of (1.1).
- u is also a *viscosity* supersolution of (1.7), and so, for every $x \in \Omega$ and every $p \in D^-u(x)$ we have $\rho(p) \geq 1$ and, from Lemma 4.2, since $p \in conv(Z_F)$ (i.e. $\rho(p) \leq 1$), we obtain $\rho(p) = 1$. From Lemma 2.9, there exists $y(x) \in \partial\Omega$ where the inward unit normal is well defined such that

$$p = D\varphi(y(x)) + \lambda(y(x))\nu(y(x)).$$

Since $\rho(p) = 1$, then $\lambda(y(x)) > 0$ is uniquely determined by

$$\rho(D\varphi(y(x)) + \lambda(y(x))\nu(y(x))) = 1.$$

And so from (2.5), we deduce that $p \in Z_F$. Thus $F(p) = 0$, $\forall p \in D^- u(x)$. We have therefore obtained that u is a *viscosity* supersolution of (1.1).

The two above obsevations complete the proof of the sufficiency part of the theorem. □

We conclude this section with the proof of Corollary 2.8.

Proof of Corollary 2.8.

To prove that there exists $\varphi \in C^1(\overline{\Omega})$ such that the problem (1.1) has no *viscosity* solution, it is sufficient using Theorem 2.6 to find $y \in \partial\Omega$, where $\nu(y)$ the unit inward normal exists, such that

$$D\varphi(y) + \lambda\nu(y) \notin Z_F, \quad \forall\lambda > 0.$$

1. Without loss of generality, we suppose that $0 \in int(conv(Z_F))$. Let ρ be the gauge associated with the set $conv(Z_F)$. We have that ρ is differentiable for $H^{N-1}|_{\partial(conv(Z_F))}$ almost every $\alpha \in \partial(conv(Z_F))$. So, since $Z_F \neq \partial(conv(Z_F))$ and Z_F is closed, we can choose $\alpha \in \partial(conv(Z_F)) \setminus Z_F$ such that α is a point of differentiability of ρ.

2. We first prove that there exists $y_0 \in \partial\Omega$, where $\nu(y_0)$ exists, with

$$\alpha + \lambda\nu(y_0) \in int(conv(Z_F)), \tag{2.41}$$

for $\lambda < 0$ small enough. Let $a = D\rho(\alpha)$. By Lemma 4.3 the exists $y_0 \in \partial\Omega$ such that the normal $\nu(y_0)$ to $\partial\Omega$ at y_0 exists and

$$< a, \nu(y_0) > > 0. \tag{2.42}$$

Using (2.42) and the fact that ρ is differentiable at α we obtain (keeping in mind that $\rho(\alpha) = 1$)

$$\rho(\alpha + \lambda\nu(y_0)) = \rho(\alpha) + \lambda[< a, \nu(y_0) + O(\lambda)/\lambda] < 1$$

for $\lambda < 0$ small enough. This concludes the proof of (2.41).

3. Choose $y \in \partial\Omega$, where $\nu(y)$ exists, and $\bar{\lambda} < 0$, such that $\beta = \alpha + \bar{\lambda}\nu(y) \in int(conv(Z_F))$ (such $\bar{\lambda}$ exists by the previous step). Observe that by convexity of ρ we have since $\rho(\alpha) = 1$ and $\rho(\alpha + \bar{\lambda}\nu(y)) < 1$ that $\rho(\alpha + \lambda\nu(y)) > 1$ for every $\lambda > 0$. Let $\varphi(x) = < \beta, x >$. We therefore have

$$D\varphi(x) + \lambda\nu(y) = \beta + \lambda\nu(y) \notin Z_F$$

for every $\lambda > 0$. That is the claimed result. □

3. THE VIABILITY APPROACH

In the previous section, we have assumed that $Z_F \subset \partial(conv(Z_F))$ and Ω is convex. We have proved that a necessary and sufficient conditions for the Hamilton-Jacobi equation

$$\begin{cases} F(Du) = 0 & \text{a.e. in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases} \tag{3.1}$$

to admit a $W^{1,\infty}(\Omega)$ *viscosity* solution is that, for any $y \in \partial\Omega$ where there is an inward unit normal, $\nu(y)$, there exists $\lambda(y) > 0$ such that

$$D\varphi(y) + \lambda(y)\nu(y) \in Z_F.$$

In this section, we no longer assume that $Z_F \subset \partial(conv(Z_F))$ and Ω is convex. We investigate the existence of a $W^{1,\infty}(\Omega)$ *viscosity* solution for Hamilton-Jacobi equation (3.1) for any φ satisfying the compatibility condition $D\varphi(y) \in int(conv(Z_F))$.

The main result of this section is that, if

$$\partial(conv(Z_F))\backslash Z_F \neq \emptyset,$$

then there is some affine map φ satisfying the compatibility condition, and for which there is no $W^{1,\infty}(\Omega)$ *viscosity* solution to (3.1) (c.f. Corrolary 2.8).

THEOREM 3.1. – *Let $F : \mathbb{R}^N \to \mathbb{R}$ be continuous such that the set $Z_F = \{\xi \in \mathbb{R}^N \mid F(\xi) = 0)$ is compact and $\partial(conv(Z_F))\backslash Z_F \neq \emptyset$.*

Then for any bounded domain $\Omega \subset \mathbb{R}^N$, there is some affine function φ with $D\varphi \in int(conv(Z_F))$ such that the problem

$$\begin{cases} F(Du) = 0 & a.e.\ in\ \Omega \\ u = \varphi & on\ \partial\Omega, \end{cases}$$

has no $W^{1,\infty}(\Omega)$ viscosity solution.

The proof of Theorem 3.1 is a consequence of Theorem 3.4 below. For stating this result, we need the definition of generalized normals (see also [1]).

DEFINITION 3.2. – *Let K be a locally compact subset of $\mathbb{R}^P$, $x \in K$. A vector $v \in \mathbb{R}^P$ is tangent to K at x if there are $h_n \to 0^+$, $v_n \to v$ such that $x + h_n v_n$ belongs to K for any $n \in N$.*

A vector $\nu \in \mathbb{R}^P$ *is a generalized normal to* K *at* x *if for every tangent* v *to* K *at* x

$$< v, \nu > \leq 0.$$

We denote by $N_K(x)$ *the set of generalized normals to* K *at* x.

REMARK 3.3. – i) *If the boundary of* K *is piecewise* C^1, *then the generalized normals coincide with the usual outward normals at any point where these normals exist.*

ii) *If* Ω *is an open subset of* $\mathbb{R}^P$ *and* x *belongs to* $\partial\Omega$, *then a generalized normal* $\nu \in N_{\mathbb{R}^P\setminus\Omega}(x)$ *can be regarded as an interior normal to* Ω *at* x.

THEOREM 3.4. – *Let* $\Omega \subset \mathbb{R}^N$ *be a bounded domain and let* $F : \mathbb{R}^N \to \mathbb{R}$ *be continuous such that the set* $Z_F = \{\xi \in \mathbb{R}^N \mid F(\xi) = 0\}$ *is compact. Let* $\varphi(y) =< b, y >$ *with* $b \in int(conv(Z_F))$.

If $F(\xi) < 0$ *(resp.* $F(\xi) > 0$*) for every* $|\xi|$ *sufficiently large and if equation* (3.1) *has a* $W^{1,\infty}(\Omega)$ viscosity *supersolution (resp. subsolution), then for any* $y \in \partial\Omega$, *for any non zero generalized normal* $\nu_y \in N_{\mathbb{R}^N\setminus\Omega}(y)$ *to* Ω *at* y, *there is some* $\lambda \geq 0$ *such that*

$$b + \lambda\nu_y \in Z_F.$$

REMARK 3.5. – *In some sense, Theorem* 3.4 *improves the necessary part of Theorem* 2.6 *since we do not assume any more that* $Z_F \subset \partial(conv(Z_F))$ *and that* Ω *is convex. Moreover, this result gives a necessary condition of existence for sub or supersolution.*

For proving Theorem 3.4 and 3.1, we assume for a moment that the following lemma holds.

LEMMA 3.6. – *Let* $\Omega \subset \mathbb{R}^N$ *and* F *be as in Theorem* 3.4. *If there is some* $a \in \mathbb{R}^N\setminus\{0\}$ *such that*

1. $\forall\lambda \geq 0, \ F(\lambda a) < 0$,
2. $\exists x \in \partial\Omega$ *such that* $a \in N_{\mathbb{R}^N\setminus\Omega}(x)$,

then there is no $W^{1,\infty}(\Omega)$ viscosity *supersolution to*

$$\begin{cases} F(Du) = 0 & a.e.\ in\ \Omega \\ u = 0 & on\ \partial\Omega. \end{cases}$$

Proof of Theorem 3.4.

Assume for instance that $F(\xi) < 0$ for $|\xi|$ sufficiently large. Fix $b \in int(conv(Z_F))$ and $a \neq 0$ for which there is some $x \in \partial\Omega$ such that $a \in N_{\mathbb{R}^N\setminus\Omega}(x)$.

If $F(b) \geq 0$, then the result is clear because F is continuous and $F(b + \lambda a)$ is negative for λ sufficiently large.

Let us now assume that $F(b) < 0$. Let u be a $W^{1,\infty}(\Omega)$ supersolution to

$$\begin{cases} F(Du) = 0 & \text{a.e. in } \Omega \\ u(y) = < b, y > & \text{on } \partial\Omega. \end{cases}$$

Set $\tilde{F}(\xi) := F(\xi + b)$ and $\tilde{u}(y) := u(y) - < b, y >$. It is easy to check that $\tilde{u}$ is a supersolution to

$$\begin{cases} \tilde{F}(D\tilde{u}) = 0 & \text{a.e. in } \Omega \\ \tilde{u}(y) = 0 & \text{on } \partial\Omega. \end{cases}$$

So, from Lemma 3.6 there is some $\lambda_0 \geq 0$ such that $\tilde{F}(\lambda_0 a) \geq 0$, i.e., $F(b + \lambda_0 a) \geq 0$. Since $F(b + \lambda a)$ is negative for λ sufficiently large, there is $\lambda \geq \lambda_0$ such that $F(b + \lambda a) = 0$.

We have therefore proved that there is $\lambda \geq 0$ such that $b + \lambda a \in Z_F$. □

Proof of Theorem 3.1.

Since F is continuous and Z_F is bounded, $F(\xi)$ has a constant sign for $|\xi|$ sufficiently large. Say it is negative.

Let $b \in \partial(conv(Z_F)) \backslash Z_F$ and $r > 0$ be such that $B_r(b) \cap Z_F = \emptyset$. From the Separation Theorem, there is some $a \in \mathbb{R}^N$, $|a| = 1$, such that

$$< b, a > = \sup_{\xi \in Z_F} < \xi, a > .$$

Note that $F(b) < 0$. Indeed, F is continuous and $F(b + \lambda a) < 0$ for large λ. Moreover, $b + \lambda a$ never belongs to Z_F for positive λ because

$$< (b + \lambda a), a > > \sup_{\xi \in Z_F} < \xi, a > .$$

From Lemma 5.3 in Appendix 2, there is some $x \in \partial\Omega$ and a generalized normal $\nu_x \in N_{\mathbb{R}^N \backslash \Omega}(x)$ such that

$$< \nu_x, a > > 0$$

Set $0 < \epsilon = < \nu_x, a >$, $\sigma = r\epsilon/(|\nu_x| + \epsilon)$, $b_\sigma = b - \sigma a$. Let $\lambda \geq 0$. We are going to prove that $b_\sigma + \lambda\nu_x \notin Z_F$. If $\lambda \leq \sigma/\epsilon$, then

$$|b_\sigma + \lambda\nu_x - b| = |\lambda\nu_x - \sigma a| \leq \lambda|\nu_x| + \sigma \leq r,$$

so that $F(b_\sigma + \lambda\nu_x) < 0$ because $B_r(b) \cap Z_F = \emptyset$ and $F(b) < 0$.

If $\lambda > \sigma/\epsilon$, then

$$< (b_\sigma + \lambda\nu_x), a > \geq < b, a > -\sigma + \lambda\epsilon \ > < b, a > = \sup_{\xi \in Z_F} < \xi, a >,$$

so that $b_\sigma + \lambda\nu_x \notin Z_F$.

Since ν_x is a generalized normal to $\mathbb{R}^N \backslash \Omega$ at x and since $b_\sigma + \lambda\nu_x \notin Z_F$ for any $\lambda \geq 0$, Theorem 3.4 states that there is no *viscosity* supersolution $W^{1,\infty}(\Omega)$ to the problem (3.1) with $\varphi(y) =< b_\sigma, y >$. □

Proof of Lemma 3.6.

The main tool for proving Lemma 3.6 is the viability theorem. The viability theorem (c.f. Theorem 3.3.2 and 3.2.4 in [2]) states that, if G is a compact convex subset of $\mathbb{R}^P$ and K is a locally compact subset of $\mathbb{R}^P$, then there is an equivalence between

i) $\forall x \in K$, there exists $\tau > 0$ and a solution to

$$\begin{cases} x'(t) \in G & \text{a.e. } t \in [0,\tau), \\ x(t) \in K & \forall t \in [0,\tau), \\ x(0) = x \end{cases} \tag{3.2}$$

ii) $\forall x \in K$, $\forall \nu \in N_K(x)$, $\inf_{g \in G} < g, \nu > \leq 0$.

As usual, the solution of the constrained differential inclusion (3.2) can be extended on a maximal interval of the form $[0, \tau)$ such that either $\tau = +\infty$, or $x(\tau)$ belongs to $\partial K \backslash K$.

Assume now that, contrary to our claim, there is some $W^{1,\infty}(\Omega)$ *viscosity* supersolution u to the problem. We will proceed by contradiction.

First step: We claim that

$$\forall x \in \Omega, \ u(x) > 0. \tag{3.3}$$

Indeed, otherwise, there is some $x \in \Omega$ minimum of u. Note that $0 \in D^- u(x)$, so that $F(0) \geq 0$ because u is a *viscosity* supersolution. This is in contradiction with $F(\lambda a) < 0$ for all $\lambda \geq 0$.

The proof of the lemma consists in showing that inequality (3.3) does not hold.

Second step: Without loss of generality we set $|a| = 1$. Since Z_F is compact and $F(\lambda a) < 0$ for $\lambda \geq 0$, there is some positive ϵ such that

$$\forall \lambda \geq 0, \forall \xi \in \mathbb{R}^N, \text{ if } |\xi - \lambda a| \leq \lambda \epsilon, \text{ then } F(\xi) < 0. \tag{3.4}$$

Since u is a $W^{1,\infty}(\Omega)$ supersolution, we know, from a result due to H. Frankowska [15] (see also Lemma 5.1 in Appendix 2), that

$$\forall x \in \Omega,\ \forall (\nu_x, \nu_\rho) \in N_{Epi(u)}(x, u(x)),\ \nu_\rho < 0 \text{ and } F\left(\frac{\nu_x}{|\nu_\rho|}\right) \geq 0.$$

Let $x \in \Omega$ and $(\nu_x, \nu_\rho) \in N_{Epi(u)}(x, u(x))$. Since $F(\frac{\nu_x}{|\nu_\rho|}) \geq 0$, we have thanks to (3.4),

$$\forall \lambda \geq 0,\ |\frac{\nu_x}{|\nu_\rho|} - \lambda a| > \lambda \epsilon.$$

An easy computation shows that this inequality implies

$$< a, \nu_x > -(1 - \epsilon^2)^{1/2}\ |\nu_x| \leq 0.$$

Let $G = \{a + (1 - \epsilon^2)^{1/2}\ B\} \times \{0\}$ where B is the closed unit ball of $\mathbb{R}^N$. Then the previous inequality is equivalent with the following

$$\inf_{g \in G} < g, (\nu_x, \nu_\rho) > \leq 0,$$

so that $K = Epi(u) \cap (\Omega \times \mathbb{R})$ is a locally compact subset such that

$$\forall x \in \Omega,\ \forall (\nu_x, \nu_\rho) \in N_K(x, u(x)),\ \inf_{g \in G} < g, (\nu_x, \nu_\rho) > \leq 0.$$

In particular, it satisfies the condition (ii) of the viability theorem.

Thus, from the viability theorem, $\forall (x, u(x)) \in K$, there is a maximal solution to

$$\begin{cases} (x'(t), \rho'(t)) \in G, & \text{a.e. } t \in [0, \tau) \\ (x(t), \rho(t)) \in K, & \forall t \in [0, \tau) \\ x(0) = x,\ \rho(0) = u(x) \end{cases} \tag{3.5}$$

where either $\tau = \infty$ or $x(\tau) \in \partial\Omega$.

Let us point out that $\rho'(t) = 0$, so that $\rho(t) = u(x)$ on $[0, \tau)$.

Third step: Let $x \in \partial\Omega$ be such that $a \in N_{\mathbb{R}^N\setminus\Omega}(x)\setminus\{0\}$. We claim that there is a solution to (3.5) starting from $(x, u(x)) = (x, 0)$ defined on $(0, \tau)$.

Since a belongs to $N_{\mathbb{R}^N\setminus\Omega}(x)\setminus\{0\}$, from Lemma 5.2 of the Appendix 2, applied to $C = \{a + (1-\epsilon^2)^{1/2}\ B\}$, there is some $\alpha > 0$ such that

$$\forall c \in C,\ \forall b \in \mathbb{R}^N \text{ with } |b| \leq 1,\ \forall \theta \in (0, \alpha),\ \ x + \theta(c + \alpha b)\ \in\ \Omega.$$

Since $0 \notin C$, we can choose also $\alpha > 0$ sufficiently small such that $0 \notin C + \alpha B$, where B is the closed unit ball.

We denote by S the set

$$S = \{x + \theta(c + \alpha b),\ c \in C,\ b \in \mathbb{R}^N \text{ with } |b| \leq 1,\ \theta \in (0, \alpha)\}.$$

It is a subset of Ω and $x \in \partial S$.

Let $x_n \in S$ converge to x, $(x_n(\cdot), \rho_n(\cdot))$ be maximal solutions to (3.5) with initial data $(x_n, u(x_n))$ defined on $[0, \tau_n)$. Let us first prove by contradiction that the sequence τ_n is bounded from below by some positive τ. Assume on the contrary that $\tau_n \to 0^+$. Note that

$$\forall n \in N,\ \ x_n(\tau_n) \in x_n + \tau_n C,$$

because $x'(t) \in C$ which is convex compact. Thus, for any n, there is $c_n \in C$ such that $x_n(\tau_n) = x_n + \tau_n c_n$.

Since $x_n \in S$, for any $n \geq N$ there are $\theta_n \in (0, \alpha)$, $b_n \in B$ and $c'_n \in C$ such that $x_n = x + \theta_n(c'_n + \alpha b_n)$. Since x_n converges to x and $0 \notin C + \alpha B$, we have $\theta_n \to 0^+$. Let N_0 be such that $\forall n \geq N_0$, $\theta_n + \tau_n < \alpha$.

Then

$$x_n(\tau_n)\ =\ x + (\theta_n + \tau_n)\left[\frac{\theta_n}{\theta_n + \tau_n} c'_n + \frac{\tau_n}{\theta_n + \tau_n} c_n + \alpha \frac{\theta_n}{\theta_n + \tau_n} b_n\right].$$

Since C is convex,

$$\frac{\theta_n}{\theta_n + \tau_n} c'_n + \frac{\tau_n}{\theta_n + \tau_n} c_n \tag{3.6}$$

belongs to C. Moreover,

$$\left|\frac{\theta_n}{\theta_n + \tau_n} b_n\right| \leq 1. \tag{3.7}$$

Thus, for any $n \geq N_0$, $x_n(\tau_n)$ belongs to S which is a subset of Ω and we have a contradiction with $x_n(\tau_n) \in \partial\Omega$.

So we have proved that the sequence τ_n is bounded from below by some positive τ.

Since G is convex compact and since the solutions $(x_n(\cdot), \rho_n(\cdot))$ are defined on $[0, \tau]$, the solutions $(x_n(\cdot), \rho_n(\cdot))$ converge up to a subsequence to some $(x(\cdot), \rho(\cdot))$ solution to

$$\begin{cases} (x'(t), \rho'(t)) \in G, & \text{a.e. } t \in [0, \tau) \\ (x(t), \rho(t)) \in \overline{K}, & \forall t \in [0, \tau) \\ x(0) = x, \ \rho(0) = u(x) = 0 \end{cases}$$

(see Theorem 3.5.2 of [2] for instance).

Since, $x'(t) \in C$, for any $t \in [0, \tau]$ there is some $c(t) \in C$ such that $x(t) = x + tc(t)$. Thus, for $t \in (0, \inf\{\tau, \alpha\})$, $x(t)$ belongs to S and so to Ω.

In particular, $(x(t), \rho(t)) = (x(t), 0)$ belongs to the epigraph of u for $t \in (0, \tau')$ (with $\tau' = \inf\{\tau, \alpha\}$), i.e.,

$$\forall t \in (0, \tau'), \quad u(x(t)) \leq 0.$$

This is in contradiction with inequality (3.3). □

4. APPENDIX 1

We now state two lemmas which are well-known in the literature. The first one is a Mac Shane type extension lemma for Lipschitz functions. The second one can be found in F.H. Clarke [7] and H. Frankowska [14]. However for the sake of completeness we prove them again.

LEMMA 4.1. – *Let Ω be a convex set of $\mathbb{R}^N$ and $u \in W^{1,\infty}(\Omega)$ with $\rho(Du(x)) \leq 1$ a.e. in Ω, then there exists an extension $\tilde{u} \in W^{1,\infty}(\mathbb{R}^N)$ of u with $\rho(D\tilde{u}(x)) \leq 1$ a.e. in $\mathbb{R}^N$.*

Proof.

The task here is to check that $\tilde{u}$ given by

$$\tilde{u}(x) = \sup_{y \in \Omega}\{u(y) - \rho^o(y - x)\}, \quad \forall x \in \mathbb{R}^N.$$

satisfies the requirements of Lemma 4.1. (Note the similarity with the *viscosity* solution (1.8).)

1. We first show that $\tilde{u}$ is an extension of u.

For this, it will be sufficient to show

$$\rho(Du(x)) \le 1 \text{ a.e. } \Longrightarrow u(y) - u(x) \le \rho^o(y-x). \tag{4.1}$$

To prove (4.1) we proceed by regularization. We introduce the mollifier function

$$f(x) = \begin{cases} Ce^{\frac{1}{|x|^2-1}} & \text{if} \quad |x| < 1 \\ 0 & \text{if} \quad |x| \ge 1. \end{cases}$$

and the sequence $f_n(x) = n^N f(nx)$ where C is chosen so that $\int f = 1$. First, we extend u, as a Lipschitz function, to the whole of $\mathbb{R}^N$ and we still denote this extension by u (this can be done by Mac Shane lemma). We then set

$$u_n(x) = \int_{\mathbb{R}^N} f_n(x-y)u(y)\, \mathrm{d}y.$$

It is well known that $u_n \to u$ uniformly on every compact set. Let Ω_δ be the compact subset of Ω defined by

$$\Omega_\delta = \{x \in \Omega : dist(x, \partial\Omega) \ge \delta\}.$$

for $\delta > 0$ and $n > \frac{1}{\delta}$. As ρ is convex and homogeneous of degree one, using Jensen inequality, we obtain that

$$\rho(D(u_n(x))) \le \int_{\mathbb{R}^N} f_n(x-y)\rho(D(u(y))\, \mathrm{d}y \le 1, \quad \forall x \in \Omega_\delta. \tag{4.2}$$

Since u_n is of class C^1, (4.2) implies that for $x, y \in \Omega_\delta$, there exists $\tilde{x} \in \mathbb{R}^N$ such that

$$\begin{aligned} u_n(y) - u_n(x) &= < Du_n(\tilde{x}), y - x > \\ &\le \rho(Du_n(\tilde{x}))\rho^o(y-x) \\ &\le \rho^o(y-x), \end{aligned}$$

and so, letting n tend to infinity, we obtain

$$u(y) - u(x) \le \rho^o(y-x).$$

Letting then δ tend to 0, we have deduced (4.1) and so, $\tilde{u}$ is an extension of u.

2. We next show that

$$\tilde{u}(z) - \tilde{u}(x) \le \rho^o(z-x), \quad x, z \in \mathbb{R}^N. \tag{4.3}$$

Indeed we have

$$\begin{aligned}\tilde{u}(z) - \tilde{u}(x) &= \sup_{y\in\Omega}\{u(y) - \rho^o(y-z)\} - \sup_{y\in\Omega}\{u(y) - \rho^o(y-x)\}\\ &\le \sup_{y\in\Omega}\{-\rho^o(y-z) + \rho^o(y-x)\}\\ &\le \rho^o(z-x).\end{aligned}$$

3. We then show that (4.3) implies that $\rho(D\tilde{u}(x)) \le 1$ a.e.

As $\tilde{u}$ is a Lipschitz function we can use Rademacher theorem and obtain that for almost every $x \in \mathbb{R}^N$

$$\lim_{h\to 0} \frac{\tilde{u}(x+h) - \tilde{u}(x) - < D\tilde{u}(x), h >}{|h|} = 0.$$

This means that for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$\frac{\tilde{u}(x+h) - \tilde{u}(x) - < D\tilde{u}(x), h >}{|h|} \le \epsilon.$$

for every $|h| \le \delta$, and so,

$$\frac{\tilde{u}(x+h) - \tilde{u}(x) - < D\tilde{u}(x), h >}{\rho^o(-h)} \le \epsilon \frac{|h|}{\rho^o(-h)}.$$

From (4.3), we get that

$$-1 - \frac{< D\tilde{u}(x), h >}{\rho^o(-h)} \le \epsilon \frac{|h|}{\rho^o(-h)}. \tag{4.4}$$

As ρ is convex and homogeneous of degree one, we have

$$\rho(D\tilde{u}(x)) = \rho^{oo}(D\tilde{u}(x)) = \sup_{|\lambda|\le\delta} \frac{< D\tilde{u}(x), \lambda >}{\rho^o(\lambda)}. \tag{4.5}$$

Taking the supremum over every $|h| \le \delta$ in (4.4) we obtain

$$-1 + \sup_{|h|\le\delta} \frac{< D\tilde{u}(x), -h >}{\rho^o(-h)} \le \sup_{|h|\le\delta} \epsilon \frac{|h|}{\rho^o(-h)} = \epsilon D$$

where,

$$0 < \sup_{|h|\le\delta} \frac{|h|}{\rho^o(-h)} = D < \infty.$$

Letting now ϵ tend to 0, and using (4.5) we obtain

$$\rho(D\tilde{u}(x)) \leq 1. \qquad \square$$

LEMMA 4.2. – *Let $u \in W^{1,\infty}(\Omega)$ with $Du(y) \in conv(Z_F)$ a.e. (i.e $\rho(Du) \leq 1$ a.e.), then*

$$D^+u(x) \cup D^-u(x) \subset conv(Z_F),$$

for every $x \in \Omega$.

Proof.

We first show that $D^+u(x) \subset conv(Z_F)$. Observe that from (4.1) we have:

$$\frac{u(x+h)-u(x)}{\rho^o(-h)} \geq -1.$$

Using the definition of D^+u we have for every $x \in \Omega$ and $p \in D^+u(x)$

$$\limsup_{h\to 0} \frac{u(x+h)-u(x)-<p,h>}{|h|} \leq 0.$$

Proceeding as in Lemma 4.1, we observe that for every $p \in D^+u(x)$, and every $\epsilon > 0$, there exists $\delta > 0$

$$\frac{u(x+h)-u(x)-<p,h>}{|h|} \leq \epsilon,$$

for every $|h| \leq \delta$. We therefore get

$$-1+\frac{<p,-h>}{\rho^o(-h)} \leq \epsilon\frac{|h|}{\rho^o(-h)}$$

since ρ is convex and homogeneous of degree one. Taking the supremum over every $|h| \leq \delta$, we obtain

$$-1+\sup_{|h|\leq\delta}\frac{<p,-h>}{\rho^o(-h)} \leq \epsilon\sup_{|h|\leq\delta}\frac{|h|}{\rho^o(-h)}. \tag{4.6}$$

Defining

$$0 < D = \sup_{|h|\leq\delta}\frac{|h|}{\rho^o(-h)} < \infty,$$

and using (4.6), we get

$$-1+\rho(p) \leq \epsilon D.$$

Letting ϵ tend to 0, we obtain $\rho(p) \leq 1$. Using the same argument for $D^-u(x)$ we conclude that

$$D^+u(x) \cup D^-u(x) \subset conv(Z_F). \qquad \square$$

In the proof of Corollary 2.8, we used the following result (see also Lemma 5.3).

LEMMA 4.3. – *Let Ω be a bounded, open and convex set. For every $a \in \mathbb{R}^N \setminus \{0\}$ there exists $y \in \partial\Omega$, where $\nu(y)$ the unit inward normal exists, such that*

$$< a, \nu(y) > > 0.$$

Proof.

1. By the divergence theorem, we have

$$\int_{\partial\Omega} < a, \nu(y) > \, d\sigma(y) = 0.$$

It is then clear from the above identity that the claim of this lemma will follow if we can prove that $< a, \nu(y) > \neq 0$ on a set of positive (relative to $\partial\Omega$) measure. This will be achieved in the next step.

2. Suppose on the contrary that there exists $a \neq 0$ for which the conclusion of the Lemma fails. We may assume without loss of generality that $0 \in \Omega$. Let ρ_Ω be the gauge of Ω. Then for each $\mu \in \mathbb{R}$ and each $y \in \partial\Omega$ such that $\nu(y)$ exists we have (keeping in mind that $\rho_\Omega(y) = 1$)

$$\rho_\Omega(y + \mu a) \geq \rho_\Omega(y) + \mu < a, \nu(y) > = 1,$$

and so, using that the set where $\nu(y)$ exists is dense in $\partial\Omega$ we deduce that

$$\rho_\Omega(y + \mu a) \geq 1, \qquad (4.7)$$

for every $y \in \partial\Omega$. Next, take $x \in \Omega$ and $\bar{\mu} \in \mathbb{R}$ such that $x + \bar{\mu}a \in \partial\Omega$. We have, for $y = x + \bar{\mu}a$,

$$1 > \rho_\Omega(x) = \rho_\Omega(y - \bar{\mu}a),$$

which is at variance with (4.7). $\square$

5. APPENDIX 2

We collect here some lemmas needed throughout the proofs of Theorem 3.1 and 3.4 and Lemma 3.6. Lemma 5.1 appeared in [15], but we will give a proof for sake of completeness. Lemma 5.2 and 5.3 are well known results of non smooth analysis, although it is not easy to find a proof in the literature. We think that the proof of Lemma 5.3 is new and interesting.

LEMMA 5.1. – *If Ω is an open subset of $\mathbb{R}^N$ and u is a $W^{1,\infty}(\Omega)$ supersolution of*

$$F(Du) = 0 \text{ a.e. in } \Omega,$$

then

$$\forall x \in \Omega,\ \forall (\nu_x, \nu_\rho) \in N_{Epi(u)}(x, u(x)) \backslash \{(0,0)\},\ \nu_\rho < 0 \text{ and } F\left(\frac{\nu_x}{|\nu_\rho|}\right) \geq 0.$$

Let us point out that the converse of this result holds also true (see [15]).

LEMMA 5.2. – *Let Ω be an open subset of $\mathbb{R}^N$, $x \in \partial\Omega$ and $a \in N_{\mathbb{R}^N \backslash \Omega}(x)$ with $a \neq 0$. Let C be a compact subset of $\mathbb{R}^N$ be such that*

$$\inf_{c \in C} < c, a > \ > \ 0.$$

Then there is some $\alpha > 0$ such that

$$\forall c \in C,\ \forall b \in \mathbb{R}^N \text{ with } |b| \leq 1,\ \forall \theta \in (0, \alpha),\ \ x + \theta(c + \alpha b) \ \in \ \Omega.$$

LEMMA 5.3. – *If $\Omega \subset \mathbb{R}^N$ is open and bounded, then, for any $a \in \mathbb{R}^N \backslash \{0\}$, there is some $x \in \partial\Omega$ and a generalized normal $\nu_x \in N_{\mathbb{R}^N \backslash \Omega}(x)$ such that*

$$< \nu_x, a > \ > \ 0.$$

Proof of Lemma 5.1.

Let $(\nu_x, \nu_\rho) \neq (0,0)$ be a generalized normal to $Epi(u)$ at $(x, u(x))$. We have to prove that $\nu_\rho < 0$ and $\nu_x / |\nu_\rho|$ belongs to $D^- u(x)$.

Since $(x, u(x)) + t(0,1)$ belongs to $Epi(u)$ for $t > 0$, $(0,1)$ is tangent to $Epi(u)$ at $(x, u(x))$, and so $< (0,1), (\nu_x, \nu_\rho) > \ \leq 0$. In particular, $\nu_\rho \leq 0$.

Assume for a while that $\nu_\rho = 0$. Then, $\nu_x \neq 0$. Set $h_n := 1/n$. Since u is Lipschitz, the sequence

$$\frac{(x + h_n\nu_x, u(x + h_n\nu_x) - (x, u(x))}{h_n} \tag{5.1}$$

is bounded and it converges, up to a subsequence, to some (ν_x, θ) which is tangent to $Epi(u)$ at $(x, u(x))$.

Thus $< (\nu_x, 0), (\nu_x, \theta) > \leq 0$ which is impossible since $\nu_x \neq 0$. So $\nu_\rho < 0$.

Set $p := \nu_x/|\nu_\rho|$. We now have to check that, $\forall v \in \mathbb{R}^N$,

$$\liminf_{h\to 0^+} \frac{u(x + hv) - u(x) - h < p, v >}{h} \geq 0.$$

Fix $v \in \mathbb{R}^N\backslash\{0\}$ and denote by θ the lower limit as above. Since u is Lipschitz, θ is finite. We have to prove that $\theta \geq 0$.

Let $\{h_n\}$ be a sequence converging to 0 such that

$$\frac{u(x + h_n v) - u(x) - h_n < p, v >}{h_n} \tag{5.2}$$

converge to θ.

Note that

$$\frac{(x + h_n v, u(x + h_n v)) - (x, u(x))}{h_n} \tag{5.3}$$

converges to $(v, < p, v > +\theta)$. Thus $(v, < p, v > +\theta)$ is tangent to $Epi(u)$ at $(x, u(x))$ and

$$< (v, < p, v > +\theta), (\nu_x, \nu_\rho) > \leq 0.$$

This implies that

$$< v, \nu_x > + < \left(\frac{\nu_x}{-\nu_\rho}\right), v > \nu_\rho + \theta\nu_\rho \leq 0.$$

So $\theta \geq 0$ because $\nu_\rho < 0$.

Since u is a supersolution and $\nu_x/|\nu_\rho| \in D^- u(x)$, we deduce from Lemma 2.5, $F(\nu_x/|\nu_\rho|) \geq 0$. □

Proof of Lemma 5.2.

Assume that, contrary to our claim, for any $n > 0$ there are $0 < \theta_n \leq \frac{1}{n}$, $c_n \in C$, $b_n \in B$ with $x + \theta_n(c_n + \frac{1}{n}b_n) \notin \Omega$.

Then c_n converges, up to a subsequence, to some $c \in C$. Clearly c is tangent to $\mathbb{R}^N \backslash \Omega$ at x.

Since $a \in N_{\mathbb{R}^N \backslash \Omega}(x)$, this implies that $< a, c > \leq 0$, which is in contradiction with the assumption. □

Proof of Lemma 5.3.

Assume that the conclusion of the lemma is false. Then

$$\forall x \in \partial\Omega,\ \forall \nu_x \in N_{\mathbb{R}^N \backslash \Omega}(x),\ < \nu_x, a > \leq 0.$$

This means (from the viability Theorem (again !) applied to the closed set $K := \mathbb{R}^N \backslash \Omega$ and $G := a$) that for any $x \in \partial\Omega$, the solution to $x'(t) = a$, $x(0) = x$ remains in K forever.

Let now y belong to Ω. Since Ω is bounded, there is some τ sufficiently large such that $x - \tau a \notin \Omega$. The previous remark applied to $x - \tau a$ yields that $x(t) = x - \tau a + ta$ belongs to $\mathbb{R}^N \backslash \Omega$ for any $t \geq 0$, which, for $t = \tau$, is in contradiction with $x \in \Omega$. □

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