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A remark on multiplicity of solutions for the Ginzburg-Landau equation

by

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ABSTRACT. – In this paper we study the structure of certain level set of the Ginzburg-Landau functional which has similar topology with the configuration space. As an application, we generalize Almeida-Bethuel's result on multiplicity of solutions for the Ginzburg-Landau equation.

Key words: Ginzburg-Landau equation, Ljusternik-Schnirelman theory, renormalized energy

RÉSUMÉ. – On étudie la structure de certains ensembles de niveau de la fonctionnelle du type Ginzburg-Landau qui ont des topologies similaires à celles de l'espace de configuration. Comme application, on généralise le résultat d'Almeida-Bethuel sur la multiplicité des solutions des équations de G-L.
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1. INTRODUCTION

Let $\Omega \subset \mathbb{C}$ be a smooth, bounded and simply connected domain. Let $g : \partial\Omega \to \mathbb{C}$ be a prescribed smooth map with $|g(x)| = 1$, for all $x \in \partial\Omega$.

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The Ginzburg-Landau functional, for any $\varepsilon > 0$, is given by

$$E_\varepsilon(u) = \frac{1}{2}\int_\Omega \left\{ |\nabla u|^2 + \frac{1}{2\varepsilon^2}(1 - |u|^2)^2 \right\} \tag{1.1}$$

which is defined on the Hilbert space

$$H_g^1(\Omega, \mathbb{C}) = \{u \in H^1(\Omega, \mathbb{C}); u = g \text{ on } \partial\Omega\}.$$

It is easy to verify that E_ε is a positive, C^2-functional satisfying the Palais-Smale condition. So

$$\mu_\varepsilon = \min_{u \in H_g^1(\Omega, \mathbb{C})} E_\varepsilon(u)$$

is achieved by some $u_\varepsilon \in H_g^1(\Omega, \mathbb{C})$ and these minimizers satisfy the following Ginzburg-Landau equation:

$$\begin{cases} -\Delta u = \frac{1}{\varepsilon^2} u(1 - |u|^2) & \text{in } \Omega \\ u = g & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

The Ginzburg-Landau equation (1.2) has been extensively studied by F. Bethuel, H. Brezis and F. Hélein [BBH1, 2] and many others. A complete characterization of asymptotic behavior (as $\varepsilon \to 0^+$) for minimizing solutions of (1.2) is given. It has been shown that the degree of g, denoted by $k = \deg(g, \partial\Omega)$, plays a crucial role in the asymptotic analysis of the minimizers. Without loss of generality, we will always assume $k \geqslant 0$ throughout this paper.

In this paper, we will study the multiplicity of the solutions for the Ginzburg-Landau equation (1.2), many such results have been given for special domains and/or boundary values (see for instance Almeida and Bethuel [AB1], Felmer and Del Pino [FP], F.H. Lin [Li]). The motivation of our paper comes from the recent work of Almeida-Bethuel [AB2, 3] concerning the existence of non-minimizing solutions of (1.2). They showed that if $k \geqslant 2$, the Ginzburg-Landau equation (1.2) has at least three distinct solutions, among which at least one is not minimizing. Based on topological arguments directly inspired by Almeida-Bethuel's work, we obtain our main result as follows

THEOREM 1. – *Assume that $k \geqslant 2$, there is some $\varepsilon_0 > 0$ (depending on Ω and g only) such that if $\varepsilon < \varepsilon_0$, the equation (1.2) has at least $k + 1$ distinct solutions.*

To prove Theorem 1, we will apply the standard Ljusternik-Schnirelman theory to a suitable covering space of a level set

$$E_\varepsilon^a = \{u \in H_g^1(\Omega, \mathbb{C}); E_\varepsilon(u) < a\},$$

for an a of the form

$$a = \mu_\varepsilon + \lambda \tag{1.3}$$

where λ is a fixed positive constant to be determined later. The proof is strongly related to the topological similarities between E_ε^a and the configuration space $\Sigma_k(\Omega)$ of k distinct points in Ω. As in [AB3], we need to use a map $\tilde{\Phi}$ from E_ε^a into $\Sigma_k(\Omega)$. More precisely, We may assign to each function u in E_ε^a, a set of k distinct points $\{a_1, \ldots, a_k\}$, called the vortices of u, where each vortex has the topological degree +1. The map $\tilde{\Phi} : E_\varepsilon^a \to \Sigma_k(\Omega)$ is not continuous. However this difficulty can been overcome by applying the notion of η-almost continuity given in [AB3]. The topological similarity between E_ε^a and $\Sigma_k(\Omega)$ allows us to define a covering space $\tilde{E}_\varepsilon^a$ of E_ε^a corresponding to the covering $F_k(\Omega) \to \Sigma_k(\Omega)$, where $F_k(\Omega)$ is the configuration space of ordered k distinct points in Ω. Again we have topological similarity between these two spaces, and we than can prove that the category of $\tilde{E}_\varepsilon^a$ is at least k. The Ljusternik-Schnirelman minimax theorem concludes that the functional $\tilde{E}_\varepsilon$ on $\tilde{E}_\varepsilon^a$, which is the composition of E_ε and the covering projection either has at least k distinct critical values or the dimension of the critical set is at least 1. These imply that E_ε has at least k critical points on E_ε^a. Finally, the fact that $E_\varepsilon^\infty = H_g^1(\Omega, \mathbb{C})$ is an affine space guarantee that E_ε^a has at least another critical point outside of E_ε^a, if $k \geqslant 2$.

This paper is organized as follows: In the next section we will recall some preliminary results about the configuration space and the construction of the map $\tilde{\Phi}$ in [AB3] and Theorem 1 will been proved in Section 3.

2. PRELIMINARIES

Our proof of Theorem 1 relies essentially on the properties of the map $\tilde{\Phi} : E_\varepsilon^a \to \Sigma_k(\Omega)$ described by Almeida and Bethuel [AB3]. With a such map, they showed that the fundamental group $\pi_1(E_\varepsilon^a)$ is non trivial for some suitable value a of the form (1.3) when ε is sufficiently small. We review here some basic facts about the configuration space and the construction of the map $\tilde{\Phi}$.

We study the configuration space and renormalized energy first. Let the metric on $\mathbb{C}^k$ be defined by the following norm

$$\|(z_1, \ldots, z_k)\| = \sum_{i=1}^{k} |z_i|. \tag{2.1}$$

The configuration space of the ordered k distinct points in Ω

$$F_k(\Omega) = \{(a_1, \ldots, a_k) \in \Omega^k; a_i \neq a_j \text{ for all } i \neq j\} \subset \mathbb{C}^k$$

with the inherited metric (2.1) on $\mathbb{C}^k$ is a smooth manifold. The cohomology ring $H^*(F_k(\Omega)) = H^*(F_k(\Omega), \mathbb{R})$ of the space $F_k(\Omega)$ has been determined by Arnol'd in 1969 (see [Ar]), which is generated by elements $\omega_{ij} \in H^1(F_k(\Omega)), 1 \leqslant i < j \leqslant k$ and subject to the following defining relations

$$\omega_{ij}\omega_{jl} + \omega_{jl}\omega_{il} + \omega_{il}\omega_{ij} = 0.$$

Arnol'd also showed that the pth Betti number B_p of $F_k(\Omega)$ is the coefficent of t^p in the polynomial

$$(1 + t)(1 + 2t) \cdots (1 + (k - 1)t).$$

In particular, $B_{k-1} = (k - 1)! \neq 0$, and this concludes that

LEMMA 2. – *The cuplength of $F_k(\Omega)$ is $k - 1$.*

The cuplength of a space X is the largest integer n such that there are n elements $\varphi_j \in H^{p_j}(X), p_j > 0, 1 \leqslant j \leqslant n$ and $\varphi_1 \cup \cdots \cup \varphi_n \neq 0$.

The symmetric group S_k on $\{1, \ldots, k\}$ acts isometrically on $F_k(\Omega)$ by permuting coordinates, i.e., for all $\sigma \in S_k$,

$$\sigma(a_1, \ldots, a_k) = (a_{\sigma(1)}, \ldots, a_{\sigma(k)}).$$

This action is free, and the quotient space $F_k(\Omega)/S_k$ is called the configuration space of k distinct point in Ω and it will be denoted by $\Sigma_k(\Omega)$.

On $\Sigma_k(\Omega)$, we have a natural metric such that the quotient map $\pi : F_k(\Omega) \to \Sigma_k(\Omega)$ is a Riemannian regular covering. This metric on $\Sigma_k(\Omega)$ is the same as the length of minimal connection introduced by Brezis, Coron and Lieb in [BCL], i.e., for $a = \{a_1, \ldots, a_k\}, a' = \{a'_1, \ldots, a'_k\} \in \Sigma_k(\Omega)$,

$$\|a - a'\| = L(a, a') = \inf_{\sigma \in S_k} \sum_{i=1}^{k} |a_i - a'_{\sigma(i)}|.$$

We now define the renormalized energy W_g on $\Sigma_k(\Omega)$ which is introduced by Bethuel-Brezis-Hélein in [BBH2] as follows, for $a = \{a_1, \dots, a_k\} \in \Sigma_k(\Omega)$,

$$W_g(a_1, \dots, a_k) = -\pi \sum_{i \neq j} \log |a_i - a_j| + \frac{1}{2} \int_{\partial\Omega} \phi \cdot (g \times g_\tau) - \pi \sum_{j=1}^{k} R(a_j)$$

where ϕ is the solution of

$$\begin{cases} \Delta \phi = 2\pi \sum_{i=1}^{k} \delta_{a_i} & \text{in } \Omega \\ \dfrac{\partial \phi}{\partial \nu} = g \times g_\tau & \text{on } \partial\Omega \\ \displaystyle\int_{\partial\Omega} \phi = 0. \end{cases}$$

Here ν denotes the unit outer normal to $\partial\Omega$ and τ is unit tangent to $\partial\Omega$ oriented so that $\nu \times \tau = 1$. And the function R is the regular part of ϕ, i.e.,

$$R(z) = \phi(z) - \sum_{i=1}^{k} \log |z - a_i|.$$

It is clear that $W_g(a) \to +\infty$ if $\text{dist}(a_j, \partial\Omega) \to 0$ for some i or if $|a_i - a_j| \to 0$ for some $i \neq j$. It has been proved in [BBH2] that, as $\varepsilon \to 0$, we have

$$\mu_\varepsilon = k\pi |\log \varepsilon| + W_g(a_1^*, \dots, a_k^*) + k\nu_0 + o(1),$$

where $o(1) \to 0$ as $\varepsilon \to 0$, ν_0 is a universal constant, and $(a_1^*, \dots, a_k^*)$ is a global minimum of the function W_g.

Next we will turn to the construction of the map $\tilde{\Phi}$. We will use a regularization technique, that is, for any $u \in E_\varepsilon^a$, we can associate a map u^h, which is a minimizer (not necessarily to be unique) of the following minimization problem

$$\inf_{v \in H_g^1(\Omega, \mathbb{C})} \left\{ E_\varepsilon(v) + \int_\Omega \frac{|u - v|^2}{2h^2} \right\} \tag{2.2}$$

where $h = \varepsilon^{\frac{2}{4k+1}} > 0$. We denote $u^h = T(u)$ where $T : H_g^1(\Omega, \mathbb{C}) \to H_g^1(\Omega, \mathbb{C})$. Clearly we have $u^h \in E_\varepsilon^a$ and it satisfies an equation similar to the Ginzburg-Landau equation (1.2). One of the main observations in [AB3] is that we can describe the "vortex structure" not only for the solutions of the Ginzburg-Landau equation, but also for such maps u^h. To be more precise, let us collect some of results of [AB3].

THEOREM 3 [AB3]. – *Assume that a is of the form (1.3) for some constant $\lambda > 0$. Then there is a constant $0 < \varepsilon_0' < 1$ depending only on Ω, g and λ, such that if $\varepsilon < \varepsilon_0'$, then for $u \in E_\varepsilon^a$, $|u| \leqslant 1$ on Ω, there is a point $a = \{a_1, \dots, a_k\}$ in $\Sigma_k(\Omega)$ such that*

$$|u^h(x)| \geqslant \frac{1}{2}, \quad \forall x \in \Omega \setminus \bigcup_{i=1}^{k} B(a_i, \rho)$$

where ρ satisfies $\varepsilon^{\bar{\chi}} \leqslant \rho \leqslant \varepsilon^{\chi}$, for some constants $\chi, \bar{\chi} \in]0, 1[$ independent of ε.

$$\deg(u^h, a_i) = \deg\left(\frac{u^h}{|u^h|}, \partial B(a_i, \rho)\right) = +1, \textit{ for all } 1 \leqslant i \leqslant k.$$

Moreover, there exists some constant $\beta > 0$ depending only on Ω, g and λ such that $\operatorname{dist}(a_i, \partial\Omega) \geqslant \beta$, *for all $1 \leqslant i \leqslant k$ and $|a_i - a_j| \geqslant \beta$, for all $1 \leqslant i \neq j \leqslant k$.*

Thus we can see that the properties of maps u^h are very close to that of minimizers of (1.2) as in [BBH], and it allows us to define vorties $\{a_1, \dots, a_k\}$ for u^h and each of the vortices has topological degree +1. That defines a map Ψ from $\operatorname{Im}(T(P(E_\varepsilon^a)))$ to $\Sigma_k(\Omega)$, by $\Psi(u^h) = \{a_1, \dots, a_k\}$, where the map $P : H_g^1(\Omega, \mathbb{C}) \to H_g^1(\Omega, \mathbb{C})$ defined by

$$\begin{cases} Pu(x) = u(x) & \text{if } |u(x)| \leqslant 1 \\ Pu(x) = \dfrac{u(x)}{|u(x)|} & \text{if } |u(x)| \geqslant 1 \end{cases}$$

is continuous. Composing P, T and Ψ, we define $\tilde{\Phi} : E_\varepsilon^a \to \Sigma_k(\Omega)$;

$$\tilde{\Phi}(u) = \Psi(T(Pu)).$$

As already noticed in [AB3], the minimizer u^h to the problem (2.2) may not be unique and moving slightly the points a_i's, the new positions would still match the requirements of Theorem 2. Hence the assignment of u^h and the vortices for u^h require some choices, so we can not expect the map $\tilde{\Phi}$ to be continuous. However the freedom in these choices are not too wild, and we can say that $\tilde{\Phi}$ is "almost" a continuous map from E_ε^a to $\Sigma_k(\Omega)$. More precisely, we have

PROPOSITION 4 [AB3]. – *Assume that* a, ε_0', $\bar{\chi}$ *are as in Theorem 3. Then for all* $\varepsilon < \varepsilon_0'$, $u, v \in E_\varepsilon^a$ *we have*

$$\|\tilde{\Phi}(u) - \tilde{\Phi}(v)\| \leqslant C_1\Big(|\log \varepsilon| \varepsilon^{\frac{2}{4k+1}} + \varepsilon^{\bar{\chi}} + \|u - v\|_{H_0^1(\Omega,\mathbb{C})}\Big)$$

where C_1 *is a constant depending only on* Ω *and* g.

Remark. – In [AB3], Almeida-Bethuel studied the more general configuration space corresponding to the "vortices" of the map u^h for $u \in E_\varepsilon^a$, where a is of the form

$$\mu_\varepsilon \leqslant a \leqslant K_1(|\log \varepsilon| + 1),$$

and the map $\tilde{\Phi}$ from E_ε^a to the configuration space. We refer reader to [AB3] for the details.

Here is the notion of η-almost continuity introduced in [AB3]: A map $\Phi : X \to Y$ from a metric space X to a metric space Y is said to be η-almost continuous, if for all $x \in X$ and $\varepsilon > 0$, there is a δ, such that for all x' with $d_X(x, x') < \delta$, we have $d_Y(\Phi(x), \Phi(x')) \leq \eta + \varepsilon$. Proposition 4 says that the map $\tilde{\Phi}$ is actually η-almost equi-continuous for $\eta = C_1(|\log \varepsilon| \varepsilon^{\frac{2}{4k+1}} + \varepsilon^{\bar{\chi}})$.

By Theorem 3, the image of $\tilde{\Phi}$ lies in the set

$$\begin{aligned}&\Sigma_{k,\beta}(\Omega)\\ &= \{\{a_1, \ldots, a_k\} \in \Sigma_k(\Omega);\ \text{dist}(a_i, \partial\Omega) \geqslant \beta, \text{ and } |a_i - a_j| \geqslant \beta \text{ for } i \neq j\}\end{aligned}$$

which is compact in Σ_k. So we have

PROPOSITION 5 [AB3]. – *We have an* η_0 *which only depends on* β, *such that for any* $\eta \leqslant \eta_0$ *and compact set* $W \in H_g^1(\Omega, \mathbb{C})$, *if* $\tilde{\Phi}$ *is* η*-almost continuous and* $\tilde{\Phi}(W) \subset \Sigma_{k,\beta}(\Omega)$, *then there exists a continuous map* $\Phi : W \to \Sigma_k(\Omega)$ *such that*

$$\|\tilde{\Phi}(u) - \Phi(u)\| \leqslant 3\eta \quad \textit{for all } u \in W.$$

3. PROOF OF THEOREM 1

In this section, we are going to prove Theorem 1 which is stated in §1.

Let $K \subset \Sigma_k(\Omega)$ be a compact core, i.e., K is compact and the natural inclusion $i : K \to \Sigma_k(\Omega)$ is a homotopy equivalence. Actually, $\Sigma_{k,\beta}(\Omega)$

is a compact core for sufficently small β. We start with a construction of maps $f_\varepsilon : K \to E_\varepsilon^a$.

LEMMA 6. – *There are constants* $\varepsilon_0'' > 0$, λ *and* C_2 *such that for all* $\varepsilon \leqslant \varepsilon_0''$, *we can define* $f_\varepsilon : K \to E_\varepsilon^a$, *where* $a = k\pi|\log\varepsilon| + \lambda$ *such that*

$$\|\tilde{\Phi} \cdot f_\varepsilon - \mathrm{id}\| \leqslant \eta$$

on K, *where* η *is given by*

$$\eta = C_2\Big(|\log\varepsilon|\varepsilon^{\frac{2}{4k+1}} + \varepsilon^{\bar{\chi}}\Big).$$

Proof. – Since K is compact, we can pick $\eta_K > 0$ such that for any $\{a_1, \ldots, a_k\} \in K$, the balls $B(a_i, 4\eta_K) \subset \Omega$ and are pairwise disjoint. Now once $\varepsilon \leqslant 4\eta_K$, we can construct a map $f_\varepsilon : \Sigma_k(\Omega) \to H_g^1(\Omega, \mathbb{C})$ as follows: for any $a = \{a_1, \ldots, a_k\} \in \Sigma_k(\Omega)$, let

$$\Omega_{\varepsilon,a} = \Omega \setminus \bigcup_{i=1}^{k} B(a_i, \varepsilon),$$

then on $\Omega_{\varepsilon,a}$, $f_\varepsilon(a)$ is defined by

$$f_\varepsilon(a)(z) = e^{i\varphi_{\varepsilon,a}(z)} \prod_{j=1}^{k} \frac{z - a_j}{|z - a_j|}$$

where the function $\varphi_{\varepsilon,a}$ is defined on Ω by the following equation

$$\begin{cases} \Delta\varphi_{\varepsilon,a}(z) = 0 & \text{in } \Omega \\ e^{i\varphi_{\varepsilon,a}(z)} \displaystyle\prod_{j=1}^{k} \frac{z - a_j}{|z - a_j|} = g & \text{on } \partial\Omega. \end{cases}$$

Notice that for a given a the map $\varphi_{\varepsilon,a}$ is uniquely defined, up to an integer multiple of 2π. In fact, we can choose this constant such that the map $a \to e^{i\varphi_{\varepsilon,a}}$ is continuous by the standard lifting argument. On each $B(a_i, \varepsilon)$, $f_\varepsilon(a)$ is defined by

$$\begin{cases} \Delta f_\varepsilon(a) = 0 & \text{in } B(a_i, \varepsilon) \\ f_\varepsilon(a)(z) = e^{i\varphi_{\varepsilon,a}(z)} \displaystyle\prod_{j=1}^{k} \frac{z - a_j}{|z - a_j|} & \text{on } \partial B(a_i, \varepsilon). \end{cases}$$

It is then easy to check that f_ε is a continuous map from $\Sigma_k(\Omega)$ to $H_g^1(\Omega, \mathbb{C})$.

Moreover we can estimate the energy $E_\varepsilon(f_\varepsilon(a))$. Using the same analysis as in [Section I, BBH2], we have a constant C which depends on Ω and g only, such that

$$E_\varepsilon(f_\varepsilon(a)) \leqslant W_g(a_1, \ldots, a_k) + k\pi|\log\varepsilon| + C.$$

Let

$$\lambda' = \sup_{a\in K} W_g(a_1, \ldots, a_k),$$

it is finite by the compactness of K. So

$$E_\varepsilon(f_\varepsilon(a)) \leq k\pi|\log\varepsilon| + \lambda' + C.$$

Hence there is an $\varepsilon_1 > 0$ such that for all $\varepsilon \leqslant \varepsilon_1$, $f_\varepsilon(a) \in E_\varepsilon^a$, for $a = \mu_\varepsilon + \lambda$ provided λ is chosen large enough (but independent of ε).

Now suppose that $\varepsilon \leqslant \varepsilon_0'' = \min\{\varepsilon_0', \varepsilon_1, 4\eta_K\}$, and denote $f_\varepsilon(a)$ by $f_{\varepsilon,a}$ for simplicity. Let $a = \{a_1, \ldots, a_k\}$ be given in K, and $a' = \{a_1', \ldots, a_k'\}$ be the vortices for $(f_{\varepsilon,a})^h$, i.e., $\tilde\Phi(f_{\varepsilon,a}) = \{a_1', \ldots, a_k'\}$. According to Theorem 3, on $\Omega_{\rho,a'} = \Omega \setminus \bigcup_{i=1}^k B(a_i', \rho)$, we have

$$|f_{\varepsilon,a}^h(x)| \geqslant \frac{1}{2}, \text{ for all } x \in \Omega_{\rho,a'},$$

where $\varepsilon^\chi \leqslant \rho \leqslant \varepsilon^{\bar\chi}$. We may therefore consider on $\tilde\Omega = \Omega_{\rho,a'} \setminus \bigcup_{i=1}^k B(a_i, \varepsilon)$, the map $\xi = \frac{f_{\varepsilon,a}^h}{|f_{\varepsilon,a}^h|} f_{\varepsilon,a}^{-1}$. ξ takes its values in S^1 and satisfies $\xi \equiv 1$ on $\partial\Omega$. Moreover we have

$$|\xi - 1| \leq 4|f_{\varepsilon,a}^h - f_{\varepsilon,a}|.$$

This yields

$$\begin{aligned}\int_{\tilde\Omega} |\xi - 1|^2 \leqslant 16 \int_{\tilde\Omega} |f_{\varepsilon,a}^h - f_{\varepsilon,a}|^2 &\leqslant 32\varepsilon^{\frac{4}{4k+1}}(E_\varepsilon(f_{\varepsilon,a}) - E_\varepsilon(f_{\varepsilon,a}^h)) \\ &\leqslant C|\log\varepsilon|\varepsilon^{\frac{4}{4k+1}} \end{aligned} \tag{3.1}$$

for some constant C depending only on g, K and Ω.

On the other hand, for any $1 \leqslant i \leqslant k$, we have

$$\deg(\xi, \partial B(a_i', \rho)) = -\deg(\xi, \partial B(a_i, \varepsilon)) = 1.$$

So for any regular value $y \in S^1$ of ξ and $y \neq 1$, $\xi^{-1}(y)$ is a connection between balls $B(a_i, \varepsilon)$ and $B(a_i', \rho)$. By the definition of length of minimal connection L given in (2.2), we get

$$L(a', a) - k(\rho + \varepsilon) \leqslant \mathcal{H}^1(\xi^{-1}(y)) \text{ for almost every } y \in S^1.$$

Let

$$N = \left\{ y \in S^1, \frac{1}{8} \leq |y-1| \leq \frac{1}{4} \right\}$$

and take $A = \xi^{-1}(N)$, using the coarea formula of Federer-Fleming, we obtain

$$\begin{aligned} \int_N \mathcal{H}^1(\xi^{-1}(y))dy &= \int_A |\nabla \xi| \\ &\leqslant \left(\int_A |\nabla \xi|^2 \right)^{1/2} (\text{meas } A)^{1/2}. \end{aligned} \tag{3.2}$$

By (3.1), we have

$$(\text{meas } A) \leq 64 \int_{\tilde{\Omega}} |\xi - 1|^2 \leq C |\log \varepsilon| \varepsilon^{\frac{4}{4k+1}};$$

On the other hand

$$\int_{\tilde{\Omega}} |\nabla \xi|^2 \leq 8 \left(\int_{\Omega} |\nabla f^h_{\varepsilon,a}|^2 + |\nabla f_{\varepsilon,a}|^2 \right) \leq C |\log \varepsilon|.$$

Together with (3.2) we get that

$$L(a, a') - k(\rho + \varepsilon) \leqslant \frac{1}{(\text{meas } N)} \int_N \mathcal{H}^1(\xi^{-1}(y))dy \leqslant C |\log \varepsilon| \varepsilon^{\frac{2}{4k+1}},$$

that is the conclusion we required. □

For any $a \in K$, the ball $B(a, 4\eta_K) \subset \Sigma_k(\Omega)$ with radius $4\eta_K$, where η_K is the constant in the proof of Lemma 6, is in fact isometric to a standard ball in $\mathbb{C}^k$. To see this, let $\tilde{K} = \pi^{-1}(K) \subset F_k(\Omega)$, which is also a compact core of $F_k(\Omega)$, and for any $\tilde{a} \in \pi^{-1}(a)$, the condition that $B(a_i, 4\eta_K)$'s are pairwise disjoint implies that the ball $B(\tilde{a}, 4\eta_K) \subset \mathbb{C}^k$ is contained in $F_k(\Omega)$ entirely, and $B(a, 4\eta_K)$ is isometric to $B(\tilde{a}, 4\eta_K)$.

LEMMA 7. – *There is an ε_o, such that for any $\varepsilon \leqslant \varepsilon_o$, the map f_ε induces an injection*

$$f_{\varepsilon *} : \pi_1(K) \to \pi_1(E^a_\varepsilon),$$

where a is chosen by Lemma 6.

Proof. – The constant $\varepsilon_0 \leqslant \varepsilon''_0$ is chosen such that

$$\max(C_1, C_2)\left(|\log \varepsilon_0| \varepsilon_0^{\frac{2}{4k+1}} + \varepsilon_0^{\bar{\chi}} \right) \leqslant \min\{\eta_0, \eta_K\},$$

where C_1 is as in Proposition 4, η_0 as in Proposition 5 and ε_0'', C_2 and η_K as in Lemma 6.

For each element $\alpha \in \pi_1(K)$, we can choose a closed path $c : S^1 \to K$ which representing α. Now for $\varepsilon \leqslant \varepsilon_0$, if $f_\varepsilon \cdot c : S^1 \to E_\varepsilon^a$ is null homotopic, we get a map $\tilde{f} : D^2 \to E_\varepsilon^a$, such that $\tilde{f}|_{\partial D^2} = f_\varepsilon \cdot c$. By Proposition 5, on the compact set $\tilde{f}(D^2) \subset E_\varepsilon^a$, we can define a continuous map $\Phi : \tilde{f}(D^2) \to \Sigma_k(\Omega)$, such that for any $u \in \tilde{f}(D^2)$, $\|\Phi(u) - \tilde{\Phi}(u)\| < 3\eta_K$. The map $\Phi \cdot \tilde{f}|_{\partial D^2} = \Phi \cdot f_\varepsilon \cdot c : S^1 \to \Sigma_k(\Omega)$ is null homotopic. On the other hand, by Lemma 6,

$$\|\Phi \cdot f_\varepsilon \cdot c(t) - c(t)\| \leqslant \|\Phi \cdot f_\varepsilon \cdot c(t) - \tilde{\Phi} \cdot f_\varepsilon \cdot c(t)\| + \|\tilde{\Phi} \cdot f_\varepsilon \cdot c(t) - c(t)\| < 4\eta_K.$$

Then we can find a unique minimum geodesic in $\Sigma_k(\Omega)$ connecting $\Phi \cdot f_\varepsilon \cdot c(t)$ and $c(t)$. This implies that $\Phi \cdot f_\varepsilon \cdot c$ is homotopic to c. So α is a trivial element in $\pi_1(K)$, and this means that $f_{\varepsilon *}$ is injective. □

Since $\pi_* : \pi_1(\tilde{K}) \to \pi_1(K)$ and $f_{\varepsilon *} : \pi_1(K) \to \pi_1(E_\varepsilon^a)$ are injective, so is $f_{\varepsilon *} \cdot \pi_* : \pi_1(\tilde{K}) \to \pi_1(E_\varepsilon^a)$. Consider a covering space $p : \tilde{E}_\varepsilon^a \to E_\varepsilon^a$ corresponding to the group $f_{\varepsilon *} \cdot \pi_*(\pi_1(\tilde{K})) \subset \pi_1(E_\varepsilon^a)$, the map $f_\varepsilon \cdot \pi : \tilde{K} \to E_\varepsilon^a$ can be lift to a map $\tilde{f} : \tilde{K} \to \tilde{E}_\varepsilon^a$ such that the following diagram commutes

$$\begin{array}{ccc} \tilde{K} & \longrightarrow & \tilde{E}_\varepsilon^a \\ \downarrow & & \downarrow \\ K & \longrightarrow & E_\varepsilon^a. \end{array}$$

LEMMA 8. – *The map $\tilde{f}$ induces maps $\tilde{f}_* : H_p(\tilde{K}) \to H_p(\tilde{E}_\varepsilon^a)$ on the homology groups which are injective for all p.*

Proof. – The argument here goes in the same fashion as the proof of Lemma 7. Consider a singular cycle $c \in Z_p(\tilde{K})$ such that $\tilde{f}_*([c]) = 0$ in $H_p(\tilde{E}_\varepsilon^a)$. This means that we have a $p+1$-chain $c' \in C_{p+1}(\tilde{E}_\varepsilon^a)$ and $\partial c' = \tilde{f}_*(c)$. The set $W = \tilde{f}(\tilde{K}) \bigcup \mathrm{support}(c')$ is compact in $\tilde{E}_\varepsilon^a$. Then we define a continues map $\Phi_1 : p(W) \to \Sigma_k(\Omega)$ such that for any $u \in p(W)$, $\|\Phi_1(u) - \tilde{\Phi}(u)\| < 3\eta_K$.

Notice that $\|\Phi_1 \cdot f_\varepsilon - \mathrm{id}\| < 4\eta_K$, as before, we have $\Phi_{1*} \cdot f_{\varepsilon *} = \mathrm{id}$. This implies that $\Phi_{1*} \cdot p_*(\pi_1(W)) \subset \Phi_{1*} \cdot f_{\varepsilon *} \cdot \pi_*(\pi_1(\tilde{K})) = \pi_*(\pi_1(\tilde{K}))$. So we can lift $\Phi_1 \cdot p : W \to \Sigma_k(\Omega)$ to $\tilde{\Phi}_1 : W \to F_k(\Omega)$.

In fact, we can make $\|\tilde{\Phi}_1 \cdot \tilde{f} - \mathrm{id}\| < 4\eta_K$. Since $\|\Phi_1 \cdot p \cdot \tilde{f} - \pi\| < 4\eta_K$, there is a homotopy H_t such that $H_0 = \pi$ and $H_1 = \Phi_1 \cdot p \cdot \tilde{f}$. Lift this homotopy to a homotopy $\tilde{H}_t$ with $\|\tilde{H}_0 - \tilde{H}_1\| < 4\eta_K$ and $\tilde{H}_0 = \mathrm{id}_{\tilde{K}}$. Define $\tilde{\Phi}_2 : \tilde{f}(\tilde{K}) \to F_k(\Omega)$ by $\tilde{\Phi}_2(\tilde{f}(a)) = \tilde{H}_1(a)$. Note that

$$\pi \cdot \tilde{\Phi}_2 = \Phi_1 \cdot p = \pi \cdot \tilde{\Phi}_1|_{\tilde{f}(\tilde{K})}.$$

$\tilde{\Phi}_2$ and $\tilde{\Phi}_1$ differ by a deck transformation, i.e., there is an elements $\sigma \in S^k$, such that

$$\tilde{\Phi}_2 = \sigma \cdot \tilde{\Phi}_1|_{\tilde{f}(\tilde{K})}.$$

Replace $\tilde{\Phi}_1$ by $\sigma \cdot \tilde{\Phi}_1$, which is also a lifting of $\Phi_1 \cdot p : W \to \Sigma_k(\Omega)$ and $\|\sigma \cdot \tilde{\Phi}_1 \cdot \tilde{f} - \text{id}\| < 4\eta_K$. The new lifting will still denoted by $\tilde{\Phi}_1$.

Now $\tilde{\Phi}_1$ maps the chain c' into a chain in $C_{p+1}(F_k(\Omega))$, and $\partial\tilde{\Phi}_1(c') = \tilde{\Phi}_1(\partial c') = \tilde{\Phi}_1 \cdot \tilde{f}_*(c)$. We get that $\tilde{\Phi}_1 \cdot \tilde{f}_*(c)$ is a boundary in $C_p(F_k(\Omega))$. On the other hand, $\tilde{\Phi}_1 \cdot \tilde{f}$ is homotopic to the natural inclusion $i : \tilde{K} \to F_k(\Omega)$. So c is homologous to $\tilde{\Phi}_1 \cdot \tilde{f}_*(c)$, and c is null homologous as well. This shows that $\tilde{f}_*$ is injective. □

The lemma allows us to estimate the category of $\tilde{E}^a_\varepsilon$.

COROLLARY 9. – *The category* $cat(\tilde{E}^a_\varepsilon)$ *of* $\tilde{E}^a_\varepsilon$ *is at least* k.

Proof. – By Lemma 8, the map $f^* : H^*(\tilde{E}^a_\varepsilon) \to H^*(\tilde{K})$ between cohomology rings are surjective, and this implies that the cuplength of E^a_ε is at least the cuplength of $\tilde{K}$, which is the same as the cuplength of $F_k(\Omega)$. By Lemma 2, the cuplength of $\tilde{E}^a_\varepsilon$ is at least $k - 1$. Finally, according to [BG], the category $\text{cat}(\tilde{E}^a_\varepsilon)$ of $\tilde{E}^a_\varepsilon$ is at least the cuplength of $\tilde{E}^a_\varepsilon$ plus one. This completes the proof. □

Now we are in the position to complete the proof of Theorem 1. The Lusternik-Schnirelman minimax theorem we will use is the following

THEOREM 10. – *Suppose* F *is a* C^2 *non-negative functional defined on a smooth Hilbert manifold* M *such that*

i) the backwards gradient flow is complete;

ii) F *satisfies the following weak Palais-Smale condition: if we have a sequence* $\{u_n\}$ *in* M *such that* $F(u_n) \to c$ *and* $\|\nabla F(u_n)\| \to 0$ *as* $n \to \infty$, *then* c *is a critical value;*

iii) $\text{cat}M = k$.

Then we have either F *has at least* k *distinct critical values in* $[0, a]$ *or the dimension of the critical set of* F *is at least* 1.

The proof is standard, we refer reader to [Pa].

Proof of Theorem 1. – Now we want to apply Theorem 10 to the positive functional $\tilde{E}_\varepsilon = E_\varepsilon \cdot p : \tilde{E}^a_\varepsilon \to \mathbb{R}$. Notice that $\tilde{E}_\varepsilon$ and E_ε have the same critical values and critical sets of the two functionals have the same dimension. If all three conditions in the theorem hold, both conclusions will imply that E_ε has at least k critical points on E^a_ε.

We now check the three conditions in Theorem 10. First, the backwards gradient flow of $\tilde{E}_\varepsilon$ is a lift of the backwards flow of E_ε, so it is complete. Second, let $\{u_n\}$ be a sequence in $\tilde{E}_\varepsilon^a$ such that $\tilde{E}_\varepsilon(u_n) \to c$ and $\|\nabla \tilde{E}_\varepsilon(u_n)\| \to 0$ as $n \to \infty$, then $E_\varepsilon(p(u_n)) \to c$ and $\|\nabla E_\varepsilon(p(u_n))\| \to 0$. We know that E_ε satisfies Palais-Smale condition, so $p(u_n)$ has a subsequence converges to a critical point. This shows that c is a critical value of E_ε and then it is a critical value of $\tilde{E}_\varepsilon$ as well. Finally, cat$\tilde{E}_\varepsilon^a \geqslant k$ is the conculsion of Corollary 9. So we now can conclude that E_ε has at least k critical points on E_ε^a.

Outside of E_ε^a, E_ε has at least another critical point, since $H_g^1(\Omega, \mathbb{C})$ is contractible, but E_ε^a is not (if $k \geqslant 2$). So totally E_ε will have at least $k+1$ critical points on $H_g^1(\Omega, \mathbb{C})$. □

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