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A generalization of the Weinstein-Moser theorems on periodic orbits of a Hamiltonian system near an equilibrium

by

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ABSTRACT. – We study the Hamiltonian system (HS) $\dot{x} = JH'(x)$ where $H \in C^2(\mathbb{R}^{2N}, \mathbb{R})$ satisfies $H(0) = 0$, $H'(0) = 0$ and the quadratic form $Q(x) = \frac{1}{2}\langle H''(0)x, x\rangle$ is non-degenerate. We fix $\tau_0 > 0$ and assume that $\mathbb{R}^{2N} \cong E \oplus F$ decomposes into linear subspaces E and F which are invariant under the flow associated to the linearized system (LHS) $\dot{x} = JH''(0)x$ and such that each solution of (LHS) in E is τ_0-periodic whereas no solution of (LHS) in $F - 0$ is τ_0-periodic. We write $\sigma(\tau_0) = \sigma_Q(\tau_0)$ for the signature of the quadratic form Q restricted to E. If $\sigma(\tau_0) \neq 0$ then there exist periodic solutions of (HS) arbitrarily close to 0. More precisely we show, either there exists a sequence $x_k \to 0$ of τ_k-periodic orbits on the energy level $H^{-1}(0)$ with $\tau_k \to \tau_0$; or for each λ close to 0 with $\lambda\sigma(\tau_0) > 0$ the energy level $H^{-1}(\lambda)$ contains at least $\frac{1}{2}|\sigma(\tau_0)|$ distinct periodic orbits of (HS) near 0 with periods near τ_0. This generalizes a result of Weinstein and Moser who assumed $Q|E$ to be positive definite.

RÉSUMÉ. – Nous considérons le système hamiltonien (HS) $\dot{x} = JH'(x)$ ou $H \in C^2(\mathbb{R}^{2N}, \mathbb{R})$ satisfait $H(0) = 0$, $H'(0) = 0$ et la forme quadratique $Q(x) = \frac{1}{2}\langle H''(0)x, x\rangle$ est non-dégénérée. Nous fixons $\tau_0 > 0$ et supposons $\mathbb{R}^{2N} \cong E \oplus F$ est la somme des sous-espaces linéaires E, F qui sont invariants sous le flot associé au système linéaire (LHS) $\dot{x} = JH''(0)x$. En plus chaque solution de (LHS) dans E est τ_0-périodique lorsqu'aucune des solutions de (LHS) dans $F - 0$ soit τ_0-périodique. Soit

$\sigma(\tau_0) = \sigma_Q(\tau_0)$ la signature de la forme quadratique Q restreint à E. Si $\sigma(\tau_0) \neq 0$ il existe des solutions périodiques de (HS) arbitrairement près de 0. Plus précisément nous démontrons que ou bien il existe une suite $x_k \to 0$ des solutions τ_k-périodique au niveau $H^{-1}(0)$ avec $\tau_k \to \tau_0$ ou bien pour chaque λ près de 0 tel que $\lambda\sigma(\tau_0) > 0$ il existe au moins $\frac{1}{2}|\sigma(\tau_0)|$ solutions périodiques au niveau $H^{-1}(\lambda)$ près de 0 avec des périodes près de τ_0. Ce résultat généralise un théorème de Weinstein et Moser qui supposent que $Q|E$ est positif défini.

1. INTRODUCTION

We consider the Hamiltonian system

$$\dot{x}(t) = JH'(x(t)) \tag{HS}$$

where

$H : \mathbb{R}^{2N} \to \mathbb{R}$ is of class C^2, $H(0) = 0$, $H'(0) = 0$; $H''(0)$ is non-singular and $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ is the usual symplectic matrix. Thus the origin is an equilibrium and we are interested in periodic solutions of (HS) in the neighborhood of the equilibrium. This is an old problem. It is well known that periodic orbits near 0 can only exist if the linearized Hamiltonian system

$$\dot{x}(t) = JH''(0)\,x(t) \tag{LHS}$$

has non-trivial periodic solutions, that is, if $JH''(0)$ has a pair of purely imaginary eigenvalues $\pm i\alpha$, $\alpha > 0$. Simple examples show that this necessary condition is not sufficient. We fix a period $\tau_0 = 2k\pi/\alpha$, $k \in \mathbb{N}$, of the linear system and let $E = E(\tau_0)$ be the space of periodic solutions of (LHS) which have (not necessarily minimal) period τ_0. Assume that E has a complement F which is invariant under the flow associated to (LHS). This means that all eigenvalues of $JH''(0)$ which are integer multiples of $2\pi i/\tau_0 = i\alpha/k$ are semisimple. The Lyapunov center theorem [L] guarantees the existence of a two-dimensional surface containing the origin which is foliated by periodic orbits of (HS) provided $\dim E = 2$. This non-resonance condition has been removed by Weinstein [W1] who proved the following.

THEOREM (Weinstein 1973). – *If the quadratic form* $Q(x) = \frac{1}{2}\langle H''(0)\,x,\, x\rangle$ *is positive definite then for each* $\lambda > 0$ *small there exists*

at least N geometrically different periodic orbits of (HS) *on the energy surface* $H^{-1}(\lambda)$.

Two solutions x_1, x_2 are said to be geometrically different if their trajectories $x_1(\mathbb{R})$, $x_2(\mathbb{R})$ are disjoint, that is, if they are not obtained from each other by time translation. Clearly H is a first integral of (HS). Therefore the solutions can be parametrized by the energy but they do not form smooth surfaces in general. In [M] Moser weakened the assumptions of both the Lyapunov center theorem and Weinstein's theorem.

THEOREM (Moser 1976). – *If $Q|E$ is positive definite then for each $\lambda > 0$ small there exist at least $\frac{1}{2}\dim E$ geometrically different periodic orbits of* (HS) *on $H^{-1}(\lambda)$ with periods near τ_0.*

In the case $\dim E = 2$ the energy provides a smooth parametrization of the periodic orbits. Thus, Moser recovers the Lyapunov center theorem. If Q is positive definite then one can split $\mathbb{R}^{2N} \cong E(\tau_1) \oplus \ldots \oplus E(\tau_r)$ into subspaces $E(\tau_i)$ such that each $E(\tau_i)$ consists of τ_i-periodic orbits of (LHS), and $\tau_1, \ldots, \tau_r$ are rationally independent. Here we identify a periodic orbit $x \in E(\tau_i)$ with $x(0) \in \mathbb{R}^{2N}$. Weinstein's theorem follows by applying Moser's result to each of the $E(\tau_i)$.

The goal of this paper is to prove the following result.

THEOREM 1.1. – *Let $\sigma(\tau_0) = \sigma_Q(\tau_0)$ be the signature of the quadratic form Q restricted to $E = E(\tau_0)$. If $\sigma(\tau_0) \neq 0$ then one of the following statements hold.*

(i) *There exists a sequence of τ_k-periodic orbits x_k of* (HS) *which lie on the energy surface $H^{-1}(0) - \{0\}$ with $x_k \to 0$ and $\tau_k \to \tau_0$ as $k \to \infty$.*

(ii) *There exists $\lambda_0 > 0$ such that there are at least $\frac{1}{2}|\sigma(\tau_0)|$ geometrically different periodic solutions of* (HS) *on $H^{-1}(\lambda)$ with periods near τ_0 for $0 < |\lambda| \leq \lambda_0$ and $\lambda \cdot \sigma(\tau_0) > 0$. These solutions converge towards 0 as $\lambda \to 0$.*

The lower bound $\frac{1}{2}|\sigma(\tau_0)|$ in (ii) is optimal. (Observe that $\sigma(\tau_0)$ is an even integer.) Moser's theorem corresponds to the case $\sigma(\tau_0) = \dim E(\tau_0)$. It is not difficult to see that (i) cannot occur in this case. Whereas the energy surface $H^{-1}(\lambda)$ will in general not be compact any more if Q is not positive (or negative) definite, the intersection $H^{-1}(\lambda) \cap E$ is compact if $Q|E$ is positive definite. This compactness plays an important role in Moser's proof of his theorem. In order to prove Moser's theorem one can also apply Weinstein's method for bifurcation of non-degenerate periodic manifolds but again it is important that the manifold $H^{-1}(\lambda) \cap E$ is compact; *see* [W1, 2].

It is not difficult to see that $\sigma(\tau_0) \neq 0$ implies the existence of periodic orbits near the origin. In [CMY] Chow, Mallet-Paret and Yorke use the Fuller index in order to prove the existence of a connected branch of periodic solutions bifurcating from the origin if $\sigma(\tau_0) \neq 0$. In the case $|\sigma(\tau_0)| = 2$. Theorem 1.1 follows easily from their result. But they do not obtain a multiplicity result as in 1.1 (ii) if $|\sigma(\tau_0)| > 2$. Neither do they obtain the direction of the bifurcating solutions, that is, whether the solutions lie on $H^{-1}(\lambda)$ for $\lambda > 0$ or $\lambda < 0$. On the other hand the result of Chow *et al.* generalizes to ordinary differential equations with a first integral. It is not needed that these are Hamiltonian. In general it cannot be expected that the solutions obtained in 1.1 lie on connected branches which bifurcate from the origin. A detailed count of the number of bifurcating orbits parametrized by the period can be found in the paper [FR] by Fadell and Rabinowitz. However, their result does not even imply the existence of one periodic solution on every energy surface $H^{-1}(\lambda)$ with $|\lambda|$ small and $\lambda \cdot \sigma(\tau_0) > 0$. In addition they do not obtain the direction of the bifurcating solutions. It is interesting to observe that the periods of the bifurcating solutions may be less than τ_0 or bigger than τ_0. In other words, the direction of the bifurcating solutions with the period as parameter is not determined by the signature $\sigma(\tau_0)$. Our paper can be considered as a fixed energy analogue of the fixed period result of [FR]. As in [FR] we shall apply variational methods and Borel cohomology. In addition we use ideas from equivariant Conley index theory. It should be mentioned that one can also prove the result of Fadell and Rabinowitz in a similar spirit. Such an approach can be found in a paper by Floer and Zehnder [FZ] and, for more general bifurcation problems, in [B2].

In a certain sense the non-triviality of the signature is a necesary and sufficient condition for the existence of periodic orbits of (HS) near 0 if one does not know anything about the higher order terms of H. Namely, if Q is a non-degenerate quadratic form on $\mathbb{R}^{2N} \cong E \oplus F$ as above with $\sigma_Q(\tau_0/n) = 0$ for all $n \in \mathbb{N}$ then there exists a polynomial function $H(x) = Q(x) + o(\|x\|^2)$ such that (HS) does not have any small periodic solutions with period near τ_0 except 0. We shall prove this in § 6.

We conclude this introduction with a sketch of the proof of Theorem 1.1. The τ-periodic solutions of (HS) correspond to 1-periodic solutions of

$$\dot{x} = \tau J H'(x).$$

These in turn correspond to critical points of the action functional

$$\mathcal{A}(x) = \frac{1}{2} \int_0^1 \langle -J\dot{x}(t),\, x(t) \rangle \, dt$$

restricted to the hypersurface $\{x \in X : \mathcal{H}(x) = \lambda\}$. Here $X = H^1(S^1, \mathbb{R}^{2N})$ consists of the absolutely continuous 1-periodic functions $x : \mathbb{R} \to \mathbb{R}^{2N}$ with $\dot{x} \in L^2$ and

$$\mathcal{H} : X \to \mathbb{R}, \qquad \mathcal{H}(x) = \int_0^1 H(x(t))\, dt.$$

The period τ appears as Lagrange multiplier in this approach. We perform a reduction of the equation

$$\mathcal{A}'(x) = \tau \mathcal{H}'(x)$$

near $(\tau_0, 0)$ to a finite-dimensional variational problem and are left with the problem of finding critical points of a function

$$\mathcal{A}_0(v) = \mathcal{A}(v + \bar{w}(v))$$

restricted to the level set $\{v \in V : \mathcal{H}_0(v) = \mathcal{H}(v + \bar{w}(v)) = \lambda\}$. Here V is the kernel of the linearization

$$X \ni x \mapsto \mathcal{A}''(0)x - \tau_0 \mathcal{H}''(0)x \in X^* \cong X$$

and $\bar{w} : V \supset U(0) \to V^\perp \subset X$ is defined on a neighborhood $U(0)$ of 0 in V. Thus $V \cong E$ and one checks that $\mathcal{A}_0$ and $\mathcal{H}_0$ are of class C^1 and that $\mathcal{H}_0''(0)$ exists. In fact:

$$\langle \mathcal{H}_0''(0)v, w\rangle = \int_0^1 \langle \mathcal{H}''(0)v(t), w(t)\rangle\, dt.$$

This suffices to apply the Morse Lemma to $\mathcal{H}_0$ near 0. After a change of coordinates $\mathcal{H}_0$ looks near 0 like the non-degenerate quadratic form

$$q := Q|V : V \ni v \mapsto \frac{1}{2}\langle \mathcal{H}''(0)v, v\rangle \in \mathbb{R}.$$

Therefore the level surfaces $\mathcal{H}_0^{-1}(\lambda)$ look locally like the level surfaces of q. If q is positive definite (which is just the situation of Moser's theorem) on can conclude the proof easily upon observing that the functionals $\mathcal{A}$, $\mathcal{H}$, hence $\mathcal{A}_0$, $\mathcal{H}_0$ are invariant under the action of $S^1 = \mathbb{R}/\mathbb{Z}$ on X induced by the time shifts:

$$S^1 \times X \ni (\theta, x) \mapsto (x_\theta : t \mapsto x(t + \theta)) \in X.$$

Moreover, $\mathcal{H}_0^{-1}(\lambda) \cong q^{-1}(\lambda)$ is diffeomorphic to the unit sphere SV of V. And any C^1-functional $SV \to \mathbb{R}$ which is invariant under the action of S^1 has at least $\frac{1}{2}\dim V = \frac{1}{2}\dim E$ S^1-orbits of critical points.

This elementary argument from S^1-equivariant critical point theory does not work if q is indefinite. Instead we look at the local flow φ_λ on $\Sigma_\lambda := \mathcal{H}_0^{-1}(\lambda)$ which is essentially induced by the negative gradient of $\mathcal{A}_0|\Sigma_\lambda$. Since $\mathcal{A}_0$ and $\mathcal{H}_0$ are only of class C^1 the gradient vector field is

of class C^0, so it may not be integrable and has to be replaced by a locally Lipschitz continuous gradient-like vector field which leaves Σ_λ invariant for all λ and whose zero set is close to the set of critical points of $\mathcal{A}_0|\Sigma_\lambda$. We are not able to apply standard minimax methods because the function $\mathcal{A}_0$ is only defined near 0. The level surface $\Sigma_\lambda \subset U(0)$ can be chosen to be open manifolds or manifolds with boundary. In both cases it does not seem possible to detect critical values by looking at a change in the topology of the sublevel sets $\mathcal{A}_0^c = \{v \in \Sigma_\lambda : \mathcal{A}_0(v) \leq c\}$. However, one observes that the hypersurfaces Σ_λ change their topology as λ passes 0. In fact, they undergo a surgery. If $2n^+$ (respectively $2n^-$) is the maximal dimension of a subspace of V on which q is positive (respectively negative) definite then $\Sigma_{+\lambda}$ is obtained from $\Sigma_{-\lambda}$ upon replacing a handle of type $B^{2n^+} \times S^{2n^- - 1}$ by $S^{2n^+ - 1} \times B^{2n^-}$. It its this change in the topology of Σ_λ near 0 which forces the existence of stationary orbits of φ near the origin. In order to analyze the influence of this change on the flow φ_λ we use methods from equivariant Conley index theory and Borel cohomology. If $n^+ > n^-$ then as S^1-spaces $\Sigma_{+\lambda}$ has a richer cohomological structure than $\Sigma_{-\lambda}$. The difference $n^+ - n^- = \frac{1}{2}|\sigma(\tau_0)|$ is a lower bound for the number of stationary S^1-orbits of φ_λ on Σ_λ if $\lambda > 0$ is small.

The paper is organized as follows. In § 2 we present a variational formulation of the problem and perform the finite-dimensional reduction. Then in § 3 we collect a number of more or less known results on the equivariant Conley index and how Borel cohomology can be used to analyze equivariant flows. In § 4 we construct the locally Lipschitz continuous vector field and begin to study the induced local flow. Finally in § 5 we put the pieces together and prove Theorem 1.1. The paper concludes with a number of remarks and related results in § 6.

2. VARIATIONAL FORMULATION AND FINITE-DIMENSIONAL REDUCTION

The treatment of (HS) which we describe in this section is a generalization of the one in [MW], Chapter 6, where a proof of Moser's theorem is given. We give a sketch which contains details whenever we deviate from [MW]. This is necessary in order to make the paper readable because our indefinite case is not treated in the literature the way we need it and requires a number of changes and additions.

We first recall that the subspaces $E, F \subset \mathbb{R}^{2N}$ are symplectic subspaces, that is, the symplectic form $\omega(x, y) = \langle Jx, y \rangle$ is nondegenerate if restricted

to E or F. After making a linear symplectic change of coordinates in $\mathbb{R}^{2N}$ we may therefore assume that the quadratic part Q of H has the form

$$Q(x) = Q(x_E) + Q(x_F) \qquad \text{where} \quad x = x_E + x_F \in E \oplus F.$$

Since each solution of (LHS) in E is periodic we may assume in addition that

$$Q(x_E) = \frac{1}{2} \sum_{k=1}^{n} \alpha_k (q_k^2 + p_k^2) \qquad \text{where} \quad x_E = (q_1, \ldots, q_n, p_1, \ldots, p_n)$$

Here the α_k are integer multiples of $2\pi/\tau_0$. Now we make a change of variables and look at

$$\dot{x} = \tau J H'(x). \tag{HS}_\tau$$

Clearly, 1-periodic solutions of $(\mathrm{HS})_\tau$ correspond to τ-periodic solutions of (HS). We want to find 1-periodic solution of $(\mathrm{HS})_\tau$ for τ near τ_0 and x near 0. Let $X = H^1(S^1, \mathbb{R}^{2N})$ be the Sobolev space of 1-periodic functions $x : \mathbb{R} \to \mathbb{R}^{2N}$ which are absolutely continuous with square integrable derivative. This is a Hilbert space with the usual scalar product

$$\langle x, y \rangle := \langle x, y \rangle_{L^2} + \langle \dot{x}, \dot{y} \rangle_{L^2}$$

and associated norm $\|x\|^2 = \|x\|_{L^2}^2 + \|\dot{x}\|_{L^2}^2$. In the sequel we shall also use $\langle \ , \ \rangle$ to denote the scalar product in $\mathbb{R}^{2N}$. We define the action functional

$$\mathcal{A} : X \to \mathbb{R}, \qquad \mathcal{A}(x) := \frac{1}{2} \int_0^1 \langle -J\dot{x}(t), x(t) \rangle \, dt$$

and

$$\mathcal{H} : X \to \mathbb{R}, \qquad \mathcal{H}(x) := \frac{1}{2} \int_0^1 H(x(t)) \, dt.$$

It is well known (and not difficult to see) that $\mathcal{A}$ is a quadratic form of class C^∞ and $\mathcal{H}$ is of class C^2 with derivatives

$$d\mathcal{A}(x)\, y = \langle \mathcal{A}'(x), y \rangle = \int_0^1 \langle -J\dot{x}(t), y(t) \rangle \, dt$$

$$d\mathcal{H}(x)\, y = \langle \mathcal{H}'(x), y \rangle = \int_0^1 \langle H'(x(t)), y(t) \rangle \, dt$$

and

$$d^2\mathcal{H}(0)(y, z) = \langle \mathcal{H}''(0)\, y, z \rangle = \int_0^1 \langle H''(0)\, y(t), z(t) \rangle \, dt.$$

Thus a critical point $x \in \mathcal{H}_0^{-1}(\lambda)$ of $\mathcal{A}|\mathcal{H}^{-1}(\lambda)$ satisfies $(\mathrm{HS})_\tau$ with

$$\tau = \langle \mathcal{A}'(x), \mathcal{H}'(x) \rangle / \|\mathcal{H}'(x)\|^2$$

appearing as Lagrange multiplier.

Since the action functional is strongly indefinite it is easier to make a reduction to a finite-dimensional constrained variational problem first. Let $\mathcal{L} := \mathcal{A}''(0) - \tau_0 \mathcal{H}''(0)$ and $V := \ker \mathcal{L} \subset X$. Then $\mathcal{L} : X \to X$ is a Fredholm operator of index 0. The kernel consists of the 1-periodic solutions of

$$\dot{x}(t) = \tau_0 J H''(0) x(t) \qquad (\mathrm{LHS})_{\tau_0}$$

so $V \cong E$. The Hilbert space X decomposes into the orthogonal direct sum of V and the image W of $\mathcal{L}$ because $\mathcal{L}$ is self adjoint. At this point we also recall the action of $S^1 = \mathbb{R}/\mathbb{Z}$ on X given by translation. For $x \in X$ and $\theta \in S^1$ we define $x_\theta \in X$ by setting

$$x_\theta(t) := x(t + \theta).$$

Clearly the scalar product in X is invariant under this action, *i.e.* $\langle x_\theta, y_\theta \rangle = \langle x, y \rangle$, and so are $\mathcal{A}$ and $\mathcal{H}$: $\mathcal{A}(x_\theta) = \mathcal{A}(x)$ and $\mathcal{H}(x_\theta) = \mathcal{H}(x)$. Therefore $\mathcal{L}$ is equivariant, *i.e.* $\mathcal{L}(x_\theta) = (\mathcal{L}x)_\theta$, and V, W are invariant subspaces of X. Let $P : X \to X$ be the orthogonal projection onto V. We write $x = v + w$ with $v = Px$, $w = (I - P)x$. Now $(\mathrm{HS})_\tau$ is equivalent to the system

$$P(\mathcal{A}'(v + w) - \tau \mathcal{H}'(v + w)) = 0, \tag{2.1}$$

$$(I - P)(\mathcal{A}'(v + w) - \tau \mathcal{H}'(v + w)) = 0. \tag{2.2}$$

By the implicit function theorem (2.2) can be solved for w in terms of τ, v near $v = 0$, $w = 0$, $\tau = \tau_0$ because $\mathcal{L} : W \to W$ is an isomorphism. In a neighborhood $U(\tau_0, 0)$ in $\mathbb{R} \times V$ (2.2) defines a C^1-map

$$w^* : U(\tau_0, 0) \to W$$

such that (2.2) is satisfied near $(\tau_0, 0, 0) \in \mathbb{R} \times V \times W$ iff $w = w^*(\tau, v)$. One easily checks that

$$w^*(\tau, 0) = 0 \quad \text{for all } \tau \qquad \text{and} \qquad \frac{\partial w^*}{\partial v}(\tau_0, 0) = 0. \tag{2.3}$$

Moreover, w^* is equivariant: $w^*(\tau, v_\theta) = (w^*(\tau, v))_\theta$. It remains to solve the bifurcation equation

$$P(\mathcal{A}'(v + w^*(\tau, v)) - \tau(\mathcal{H}'(v + w^*(\tau, v))) = 0. \tag{2.4}$$

In order to do this we look at the quadratic form q induced by $\mathcal{H}''(0)$ on V:

$$q(v) = \int_0^1 \langle H''(0) v(t), v(t) \rangle \, dt = \langle H''(0) v(0), v(0) \rangle.$$

We decompose V as $V^+ \oplus V^-$ such that q is positive definite on V^+ and negative definite on V^-. We write $v = v^+ + v^-$ according to this decomposition. Both subspaces can be chosen to be invariant under the action of S^1 on V. This implies that $\dim V^+ = 2n^+$ and $\dim V^- = 2n^-$ are even integers because there are no fixed points of the action except 0. The signature of q on V is $\sigma(\tau_0) = 2(n^+ - n^-)$.

Now we consider the inner product of equation (2.4) with $v^+ - v^-$. Observing that the image of $\mathcal{L}$ is orthogonal to V this yields the equation

$$\langle (\mathcal{A}' - \tau \mathcal{H}' - \mathcal{L})(v + w^*(\tau, v)),\ v^+ - v^- \rangle = 0 \tag{2.5}$$

which is defined for $(\tau, v) \in U(\tau_0, 0) \subset \mathbb{R} \times V$. We want to solve this for τ in terms of v near $(\tau_0, 0)$. To this end we set

$$g(\tau, v) := \tau - \tau_0 - \frac{\langle (\mathcal{A}' - \tau_0 \mathcal{H}' - \mathcal{L})(v + w^*(\tau, v)),\ v^+ - v^- \rangle}{\langle \mathcal{H}'(v + w^*(\tau, v)),\ v^+ - v^- \rangle}$$

for $v \neq 0$ and

$$g(\tau, v) := \tau - \tau_0 \quad \text{for } v = 0.$$

As in the proof of Lemma 6.11 of [MW] one checks that $g : U(\tau_0, v) \to \mathbb{R}$ is well defined and continuous. Moreover, $\frac{\partial g}{\partial \tau}(\tau, v)$ exists for all $(\tau, v) \in U(\tau_0, 0)$ and is continuous with $\frac{\partial g}{\partial \tau}(\tau_0, v) = 1$. In addition $\frac{\partial g}{\partial v}(\tau, v)$ exists for $v \neq 0$ and is continuous. The implicit function theorem yields a continuous map

$$\tau^* : U(0) \to \mathbb{R}$$

where $U(0) \subset V$ is a neighborhood of 0 in V, with the following properties. The equation $g(\tau, v) = 0$ is satisfied near $(\tau_0, 0)$ iff $\tau = \tau^*(v)$. In particular, $\tau^*(0) = \tau_0$. The map τ^* is of class C^1 in $U(0) - \{0\}$.

Observe that equation (2.5) is equivalent to $g(\tau, v) = 0$ for $v \neq 0$. Therefore, setting $\bar{w}(v) := w^*(\tau(v), v)$ it remains to solve

$$P(\mathcal{A}'(v + \bar{w}(v)) - \tau^*(v) \mathcal{H}'(v + \bar{w}(v))) = 0. \tag{2.6}$$

This is defined for $v \in U(0) \subset V$. The equivariance of w^* implies that g is invariant, hence τ^* is invariant and $\bar{w}$ is equivariant: $\bar{w}(v_\theta) = (\bar{w}(v))_\theta$ for $v \in U(0)$ and $\theta \in S^1$. Since $\tau^* \in C^1(U(0) - \{0\})$ we have $\bar{w} \in C^1(U(0) - \{0\}, W)$. Using (2.3) it follows that

$$\bar{w}(v)/\|v\| \to 0 \quad \text{as } v \to 0,$$

Hence $d\bar{w}(0) = 0$. We claim that $\bar{w} \in C^1(U(0), W)$, that is $d\bar{w}(v) \to 0$ as $v \to 0$. By the definition of $\bar{w}$ and τ^* we obtain from (2.2) the equation

$$(I - P)(\mathcal{A}'(v + \bar{w}(v)) - \tau^*(v) \mathcal{H}'(v + \bar{w}(v))) = 0. \tag{2.7}$$

Differentiating (2.7) yields

$$\|d\bar{w}(v)\| \leq \|d\tau^*(v)\| \cdot o(\|v\|) + o(1) \quad \text{as } v \to 0.$$

In order to estimate $d\tau^*(v)$ we differentiate the equation $g(\tau^*(v), v) = 0$. This gives

$$\|d\tau^*(v)\| = \|d\bar{w}(v)\| \cdot o(1)/\|v\| + o(1)/\|v\| \quad \text{as } v \to 0. \tag{2.9}$$

Combining (2.8) and (2.9) we obtain

$$(1 - o(1)) \cdot \|d\bar{w}(v)\| \leq o(1) \quad \text{as } v \to 0,$$

hence $d\bar{w}(v) \to 0$ as $v \to 0$.

Equation (2.6) can be reformulated as a finite-dimensional variational problem. We define

$$\mathcal{A}_0 : U(0) \to \mathbb{R}, \qquad v \mapsto \mathcal{A}(v + \bar{w}(v)),$$

and

$$\mathcal{H}_0 : U(0) \to \mathbb{R}, \qquad v \mapsto \mathcal{H}(v + \bar{w}(v)).$$

These functionals are of class C^1. They are invariant under the action of S^1 on $U(0)$. Moreover, $\mathcal{H}_0(0) = 0$ and $\mathcal{H}'_0(0) = 0$. We claim that $\mathcal{H}''_0(0)$ exists and

$$\langle \mathcal{H}''_0(0)\, v, u\rangle = \int_0^1 \langle \mathcal{H}''(0)\, v(t), u(t)\rangle\, dt.$$

This can be seen as follows:

$$\begin{aligned}\mathcal{H}'_0(v)\, u &= \mathcal{H}'(v + \bar{w}(v))\,(u + d\bar{w}(v)\, u) \\ &= \langle \mathcal{H}''(0)\,(v + \bar{w}(v)),\ u + d\bar{w}(v)\, u\rangle + o(\|v\|) \\ &= \langle \mathcal{H}''(0)\, v, u\rangle + o(\|v\|)\end{aligned}$$

uniformly for bounded $u \in V$. Here we used that $\bar{w} \in C^1(U(0), W)$, $\bar{w}(0) = 0$ and $d\bar{w}(0) = 0$.

If $v \in U(0) - \{0\}$ is a critical point of $\mathcal{A}_0|\mathcal{H}_0^{-1}(\lambda)$ then there exists a Lagrange multiplier τ such that

$$\mathcal{A}'_0(v) = \tau\, \mathcal{H}'_0(v).$$

A simple computation shows that this implies $\tau = \tau^*(v)$ and that

$$x = v + \bar{w}(v) = v + w^*(\tau^*(v), v)$$

satisfies $\mathcal{A}'(x) = \tau\, \mathcal{H}'(x)$, so x solves $(\mathrm{HS})_\tau$ and $\bar{x} : t \mapsto x(t/\tau)$ solves (HS). The periodic orbit $\bar{x}$ satisfies

$$\frac{1}{\tau}\int_0^\tau H(\bar{x}(t))\, dt = \int_0^\tau H(x(t))\, dt = \mathcal{H}(x) = \mathcal{H}_0(v) = \lambda.$$

This implies $H(\bar{x}(t)) = \lambda$ for all t, so $\bar{x}$ lies on the energy surface $H^{-1}(\lambda)$.

If 0 is an isolated periodic solution of (HS) on $H^{-1}(0)$ then the critical points of $\mathcal{A}_0|\mathcal{H}_0^{-1}(\lambda)$ which we find converge towards 0 as $\lambda \to 0$. hence, their Lagrange multipliers converge towards τ_0. Moreover, if v_1 and v_2 are critical points of $\mathcal{A}_0|\mathcal{H}_0^{-1}(\lambda)$ which lie on different S^1-orbits then the associated periodic solutions $\bar{x}_1$, $\bar{x}_2$ are geometrically different provided λ is close to 0. Namely, if $\bar{x}_1$ and $\bar{x}_2$ differ only by a time shift then either $\tau^*(v_1) = \tau^*(v_2)$, hence v_1 and v_2 lie on the same S^1-orbit; or $|\tau^*(v_1) - \tau^*(v_2)|$ is an integer multiple of the minimal period of $\bar{x}_1$ and $\bar{x}_2$. The minimal periods of periodic solutions of (HS) near 0 are bounded away from 0 (*cf.* [Yo]), so the last case cannot occur if λ is close to 0 because then also $|\tau^*(v_1) - \tau^*(v_2)|$ is close to 0.

In the following proposition we summarize what we have achieved in this section.

PROPOSITION 2.10. – *Periodic solutions of* (HS) *on the energy level* $H^{-1}(\lambda)$ *near the equilibrium with period near* τ_0 *correspond to critical points of the functional* $\mathcal{A}_0$ *constrained to the hypersurface* $H_0^{-1}(\lambda)$. *The functionals* $\mathcal{A}_0$, $\mathcal{H}_0 : U(0) \to \mathbb{R}$ *are of class* C^1 *defined on an open neighborhood* $U(0)$ *of* 0 *in* $V \cong E$. *Moreover,* $\mathcal{H}_0(0) = 0$, $\mathcal{H}_0'(0) = 0$, $\mathcal{H}_0''(0)$ *exists and*

$$\langle \mathcal{H}_0''(0)\, v,\ u\rangle = \int_0^1 \langle H''(0)\, v(t),\ u(t)\rangle\, dt.$$

Both functionals $\mathcal{A}_0$ *and* $\mathcal{H}_0$ *are invariant with respect to the action of* S^1 *on* V. *Different* S^1*-orbits of critical points of* $\mathcal{A}_0|\mathcal{H}_0^{-1}(\lambda)$ *correspond to geometrically different periodic solutions of* (HS) *on* $H^{-1}(\lambda)$.

3. CONLEY INDEX AND BOREL COHOMOLOGY

In Section 4 we shall construct an equivariant local flow in $U(0)$ whose stationary solutions are close to the critical points of $\mathcal{A}_0|\mathcal{H}_0^{-1}(\lambda)$ for λ close to 0. In order to analyse this flow we use a cohomological version of the equivariant Conley index. We assume the reader to be familiar with the standard version of Conley index theory as developed for instance in [Co], [CoZ] or [Sa]. We just introduce the basic notions without proofs.

Let M be a locally compact metric space on which the group S^1 acts continuously. Let φ be a continuous equivariant local flow on M, that is, $\varphi : \mathcal{O} \to M$, $(t, x) \mapsto \varphi^t(x)$, is a continuous map defined on an open S^1-invariant subset $\mathcal{O}$ of $\mathbb{R} \times M$ such that:

– $\{0\} \times M \subset \mathcal{O}$ and $\mathcal{O} \cap \mathbb{R} \times \{x\}$ is an interval for any $x \in M$;

– $\varphi^t : \mathcal{O}^t = \{x \in M : (t, x) \in \mathcal{O}\} \to M$ is equivariant;

– $\varphi^s \circ \varphi^t = \varphi^{s+t} : \mathcal{O}^{s+t} \to M$.

A compact subset S of M is said to be *isolated invariant* if S is invariant with respect to the action of S^1 and if there exists a neighborhood N of S in M with

$$S = \mathrm{inv}(N) := \{x \in M : \varphi^t(x) \in N \quad \text{for all } t \in \mathbb{R}\}.$$

In particular, S is also invariant with respect to the flow: $\varphi^t(x)$ is defined for all $x \in S$ and $t \in \mathbb{R}$ and lies in S. A neighborhood N as above is called an *isolating neighborhood* of S.

An *index pair* for an isolated invariant set S is a pair (N, A) of compact S^1-invariant subsets $A \subset N$ of M with:

– $N - A$ is an isolating neighborhood of S;

– A is positively invariant with respect to N, that is, if $x \in A$ and $\varphi^t(x) \in N$ for all $0 \leq t \leq t_0$ then $\varphi^t(x) \in A$ for $0 \leq t \leq t_0$;

– A is an exit set for$N$, that is, if $x \in N$ and $\varphi^{t_0}(x) \notin N$ for some $t_0 > 0$ then there exists $t \in [0, t_0]$ with $\varphi^t(x) \in A$.

The starting point of (S^1-equivariant) Conley index theory is the following result.

PROPOSITION 3.1. – *For any neighborhood U of an isolated invariant subset S of M there exists an index pair (N, A) for S contained in U. If (N, A) and (N', A') are two index pairs for S then the quotient spaces N/A and N'/A' are homotopy equivalent as S^1-spaces with base points.*

The *Conley index* $\mathcal{C}(S)$ of an isolated invariant set S is the based S^1-homotopy type of N/A where (N, A) is an index pair for S. We use the convention $N/\emptyset := N \sqcup \mathrm{pt}$.

An *S^1-Morse decomposition* of a compact invariant set $S \subset M$ is a finite family $(M(\pi) : \pi \in P)$ of pairwise disjoint compact invariant sets $M(\pi) \subset S$ with the following property:

– There exists an ordering $\pi_1, \ldots, \pi_n$ of P such that for every $x \in S - \bigcup_{\pi \in P} M(\pi)$ there exist indices $i, j \in \{1, \ldots, n\}$ with $i < j$ and $\omega(x) \subset M(\pi_i)$ and $\alpha(x) \subset M(\pi_j)$. Here $\alpha(x)$ and $\omega(x)$ denote the alpha and omega limit set of x respectively.

Next we recall Borel cohomology; *see* [tD] for its basic properties. Let $H^*(-; \mathbb{Q})$ denote Alexander-Spanier or Čech cohomology (*cf.* [D] or [Sp]). Let ES^1 be a contractible space with a free action of S^1; for example we may take the unit sphere of an infinite-dimensional normed complex vector space where S^1 considered as the group of complex numbers of modulus

1 acts via scalar multiplication. ES^1 is determined up to equivariant homotopy. For S^1-spaces $B \subset A$ we write

$$h^*(A, B) := H^*((ES^1 \times A)/S^1, (ES^1 \times B)/S^1; \mathbb{Q})$$

for the *Borel cohomology* of (A, B). Here $(ES^1 \times A)/S^1$ denotes the orbit space of the diagonal actional of S^1 on $ES^1 \times A$. Observe that $ES^1/S^1 = BS^1$ is the classifying space of S^1. It is unique up to homotopy and homotopy equivalent to $\mathbb{C}P^\infty$. Therefore the coefficient ring is

$$R := h^*(\mathrm{pt}) \tilde{=} H^*(BS^1; \mathbb{Q}) \tilde{=} \mathbb{Q}[c]$$

with a generator $c \in H^2(BS^1; \mathbb{Q}) \tilde{=} \mathbb{Q}$. The cup product in cohomology turns $h^*(A)$ into a graded commutative ring with unit and $h^*(A, B)$ into a module over $h^*(A)$. The homomorphism $R \to h^*(A)$ induced by $A \to \mathrm{pt}$ induces an R-module structure on each $h^*(A, B)$.

DEFINITION 3.2. – For a pair (A, B) of S^1-spaces the length $\ell(A, B)$ of (A, B) is defined to be

$$\begin{aligned}\ell(A, B) &:= \min\{k \in \mathbb{N} : c^k \in R \text{ annihilates } h^*(A, B)\} \\ &= \min\{k \in \mathbb{N} : c^k \xi = 0 \text{ for all } \xi \in h^*(A, B)\} \\ &= 1 + \max\{k \in \mathbb{N} : c^k \xi \neq 0 \text{ for some } \xi \in h^*(A, B)\}.\end{aligned}$$

We use the convention $\min \emptyset = \infty$.

This notion is due to Fadell and Rabinowitz [FR], at least if $B = \emptyset$. They call it cohomological index for S^1 because it is defined analogously to the cohomological index for $\mathbb{Z}/2$ introduced by Yang [Ya].

PROPOSITION 3.3. – *The length ℓ has the following properties.*

(a) Monotonicity: *If there exists an equivariant map $A \to A'$ then $\ell(A) \leq \ell(A')$.*

(b) $\ell(A, B) \leq \ell(A)$ *for any invariant subspace B of A.*

(c) Subadditivity: *If A and A' are open subset of $A \cup A'$ then $\ell(A \cup A') \leq \ell(A) + \ell(A')$.*

(d) $\ell(\coprod_{i \in I} A_i) = \sup\{\ell(A_i) : i \in I\}$

(e) Continuity: *Any locally closed invariant subset B of a metrizable S^1-space A has an invariant neighborhood N with $\ell(N) = \ell(B)$. Here B is said to be locally closed if it is the intersection of a closed and an open subset of A*

(f) Triangle inequality: *For any triple $C \subset B \subset A$ of S^1-spaces*

$$\ell(A, B) + \ell(B, C) \geq \ell(A, C).$$

(g) *If $C \subset B \subset A$ are invariant subspaces and C is closed in A then*

$$\ell(A, B) = \ell(A/C, B/C).$$

(h) *If V is a representation of S^1 without nontrivial fixed points* (i.e. $\zeta v = v$ *for all* $\zeta \in S^1$ *implies* $v = 0$) *and SV is the unit sphere then*

$$\ell(SV) = \frac{1}{2}\dim V.$$

This also holds if $\dim V = \infty$.

Proof. – The statements follow easily from the properties of h^*. *See* [FR] for the proof of a), c), e), h) or [B2], § 4.4.

We conclude this section with some direct consequences of the properties of ℓ applied to isolated invariant sets.

PROPOSITION 3.4. – *Let S be an isolated invariant set of the equivariant local flow φ on M.*

(a) $\ell(C(S)) := \ell(N/A, \text{pt}) = \ell(N, A)$ *is independent of the choice of an index pair (N, A) for S.*

(b) $\ell(S) \geq \ell(C(S)) \geq \ell(N) - \ell(A)$ *for any index pair (N, A) of S.*

(c) *If $(M(\pi) : \pi \in P)$ is an S^1-Morse decomposition of S then*

$$\ell(S) \leq \sum_{\pi \in P} \ell(M(\pi)).$$

Proof. – a) Follows from 3.1 and 3.3 a), g).

b) Follows from the fact that we may choose an index pair (N, A) with $\ell(N) = \ell(S)$ by 3.1 and the continuity of ℓ. Then apply 3.3 b) and f).

c) Can be deduced from 3.3 a)-c); *see* [B2], Theorem 6.1. □

4. THE LOCAL FLOW

In this section we construct an S^1-equivariant local flow φ on $U(0)$ which leaves the level surfaces $\mathcal{H}_0^{-1}(\lambda)$ invariant and whose stationary points on $\mathcal{H}_0^{-1}(\lambda)$ are close to the critical points of $\mathcal{A}_0|\mathcal{H}_0^{-1}(\lambda)$. To begin, recall that

$$\mathcal{H}_0(v) = q(v) + o(\|v\|^2) \quad \text{for } \|v\| \to 0.$$

We need the following S^1-equivariant version of the C^1-Morse lemma; *see* [BL] or [Ca] for a non-equivariant version.

LEMMA 4.1. – *There exists an S^1-equivariant diffeomorphism $\chi : U \to U(0)$ of class C^1, defined in a neighborhood U of 0 in V with $\chi(0) = 0$ and*

$$\mathcal{H}_0(\chi(v)) = q(v).$$

Proof. – Decompose $V \cong V^+ \oplus V^- \cong V_1 \oplus \ldots \oplus V_n$ into irreductible representations V_k of S^1. Here $\dim V = 2n$ and we may assume (for later purposes) that q is positive definite on V^+ which consists of the first n^+ summands and negative definite on V^- which consists of the last n^- summands. Here $n = n^+ + n^-$ and $2(n^+ - n^-)$ is the signature of q. Then

$$q(v) = \frac{1}{2}\sum_{k=1}^{n} \alpha_k |v_k|^2$$

with real numbers $\alpha_1 \geq \ldots \geq \alpha_{n^+} > 0 > \alpha_{n^+ + 1} \geq \ldots \geq \alpha_n$. The function

$$g : U(0) \to \mathbb{R}, \qquad g(v) = \begin{cases} (\mathcal{H}_0(v) - q(v))/\|v\|^2 & \text{if} \quad v \neq 0; \\ 0 & \text{if} \quad v = 0; \end{cases}$$

is invariant, continuous and of class C^1 in $U(0) - 0$. Setting

$$\psi_k(v) := v_k \left(1 + \frac{2g(v)}{\alpha_k}\right)^{1/2}, \qquad k = 1, \ldots, n,$$

we therefore obtain an equivariant map (making $U(0)$ smaller if necessary)

$$\psi = (\psi_1, \ldots, \psi_n) : U(0) \to V = V_1 \oplus \ldots \oplus V_n.$$

This satisfies $\psi(0) = 0$ and

$$\mathcal{H}_0(v) = \frac{1}{2}\sum_{k=1}^{n} \alpha_k |\psi_k(v)|^2 = q(\psi(v)).$$

Clearly, ψ is continuous and of class C^1 in $U(0) - \{0\}$. One can now check as in [BL] that ψ is even of class C^1 in $U(0)$ with $\psi'(0) = \mathrm{Id}$. The lemma follows with $\chi := \psi^{-1}$ which is well defined in a neighborhood of 0. □

Instead of looking for critical points of $\mathcal{A}_0 | \mathcal{H}_0^{-1}(\lambda)$ we set $f := \mathcal{A}_0 \circ \chi$ and $q = \mathcal{H}_0 \circ \chi$ and are left with the problem of finding critical points of $f | q^{-1}(\lambda)$ for λ near 0. After this C^1-change of coordinates around the origin of V the C^1-function $\mathcal{H}_0$ has been replaced by the quadratic form

$$q(v) = \frac{1}{2}\sum_{k=1}^{n} \alpha_k |v_k|^2$$

with is C^∞, of course.

Now we choose ε_1, $\lambda_1 > 0$ so that f is defined on the set

$$\Sigma := \{v \in V : \|v\|^2 < \varepsilon_1, \ |q(v)| \leq \lambda_1\}.$$

We think of Σ as being a space over $\Lambda = [-\lambda_1, \lambda_1]$ via the map $q : \Sigma \to \Lambda$. If λ_1 is small then q is surjective. It is a bundle map if one restricts q to the part over $\Lambda - \{0\}$. We write $\Sigma_\lambda = q^{-1}(\lambda)$ for the fibres, even for $\lambda = 0$. Clearly, the Σ_λ are manifolds with Σ_0 having a singularity at 0. We decompose $V = V^+ \oplus V^-$ with $\dim V^+ = 2n^+$, $\dim V^- = 2n^-$ as in the proof of Lemma 4.1 so that

$$\Sigma_\lambda \cong BV^+ \times SV^- \cong B^{2n^+} \times S^{2n^- - 1} \quad \text{for } \lambda < 0$$

and

$$\Sigma_\lambda \cong SV^+ \times BV^- \cong S^{2n^+ - 1} \times B^{2n^-} \quad \text{for } \lambda < 0.$$

All spaces are S^1-spaces and the diffeomorphisms are equivariant.

In the same "over Λ" spirit we write f_λ for the restriction of f to Σ_λ. These functions induce a vector field over Λ on $\Sigma - \{0\}$ which we denote by $\nabla_\Lambda f$ defined as follows. For $v = 0$ we set $\nabla_\Lambda f(0) = 0$ and for $v \in \Sigma_\lambda - \{0\}$ we set

$$\nabla_\Lambda f(v) := (\nabla f_\lambda)(v) \in T_v \Sigma_\lambda.$$

We write $K := \{v \in \Sigma : \nabla_\Lambda f(v) = 0\}$ for the zero set of this vector field. *From now on we assume that* $0 \in \Sigma_0$ *is an isolated zero of* ∇f_0. If this is not the case then part (i) of Theorem 1.1 holds and we are done. We may also assume that ε_1 is so small that 0 is the only zero of ∇f_0 in Σ_0, *i.e.* $K_0 = \{0\}$. Upon making λ_1 smaller if necessary we may assume that $\|v\| < \varepsilon_1/4$ for $v \in K$. Finally, we assume that $\|\nabla_\Lambda f(v)\| < 1$ for every $v \in \Sigma$.

Unfortunately, f is only of class C^1, so $\nabla_\Lambda f$ is only continuous and may not be integrable. We replace it by a locally Lipschitz continuous vector field η over Λ for f whose zero set is close to K. Compared with the usual construction of a pseudo-gradient vector field as in [R], for instance, there are two differences. First, we want the vector field to be locally Lipschitz on all of Σ, not just on $\Sigma - K$. The reason for this is that we do not have a minimax description of the critical values of f_λ but instead apply ideas from Conley index theory. Second, the new vector field η has to respect the fibres Σ_λ, that is, $\eta(v) \in T_v \Sigma_\lambda$ for $v \in \Sigma_\lambda - \{0\}$. This has to be done with some care near the singularity 0. We construct a function $\varepsilon : \Lambda \to (0, \varepsilon_1/4)$ which is continuous in $\Lambda - \{0\}$ and satisfies $\varepsilon(\lambda) \to 0$ as $\lambda \to 0$. We may also assume that $\|v\| < \varepsilon_1/4$ for $v \in U_{\varepsilon(\lambda)}(K_\lambda)$ where

$U_{\varepsilon(\lambda)}(K_\lambda) \subset \Sigma_\lambda$ is the closed $\varepsilon(\lambda)$-neighborhood of K_λ in Σ_λ. Moreover, we can choose $\varepsilon(0)$ so that

$$|f(v)| < \varepsilon_1 \cdot \mu/6 \quad \text{for every } v \in \Sigma_0 \quad \text{with } \|v\| \leq \varepsilon(0) \tag{4.2}$$

where

$$\mu := \min\{\|\nabla_\Lambda f(v)\|^2 : v \in \Sigma_0,\ \varepsilon_1/3 \leq \|v\| \leq \varepsilon_1/2\} > 0. \tag{4.3}$$

In addition, if for $\lambda \neq 0$, K_λ consists of finitely many S^1-orbits $o_1, \ldots, o_r$ then we require that $\varepsilon(\lambda)$ has the following two properties.

$$\begin{gathered}\max f(U_{\varepsilon(\lambda)}(o_i)) < \min f(U_{\varepsilon(\lambda)}(o_j)) \\ \text{for } i, j \in \{1, \ldots, r\} \quad \text{with } f(o_i) < f(o_j);\end{gathered} \tag{4.4}$$

$$\begin{gathered}\operatorname{dist}(U_{\varepsilon(\lambda)}(o_i), U_{\varepsilon(\lambda)}(o_j)) > \max f|U_{\varepsilon(\lambda)}(o_i) - \min f|U_{\varepsilon(\lambda)}(o_j) \\ \text{if } f(o_i) = f(o_j).\end{gathered} \tag{4.5}$$

The reason for the choice of $\varepsilon(0)$ is the following. If $v(t)$ solves $\dot{v}(t) = -\nabla_\Lambda f(v(t))$ and if for the time $t_1 < t_2 < t_3$ we have $\|v(t_1)\| = \|v(t_3)\| = \varepsilon_1/3$, $\|v(t_2)\| = \varepsilon_1/2$ then $t_3 - t_1 > \varepsilon_1/3$ because $\|\nabla_\Lambda f(v(t))\| < 1$ for $v \in \Sigma$. From this it follows that $f(v(t_1)) - f(v(t_3)) > \varepsilon_1 \cdot \mu/3$. Consequently, an orbit of the negative gradient flow of f_0 that connects two points of Σ_0 with norm $\varepsilon(0)$ cannot approach the boundary of Σ_0 in between. Similarly, $\varepsilon(\lambda)$ is chosen such that there cannot exist a solution $v(t)$ of $\dot{v}(t) = -\nabla_\Lambda f(v(t))$ with $\alpha(v(0)) \subset U_{\varepsilon(\lambda)}(o_i)$ and $\omega(v(0)) \subset U_{\varepsilon(\lambda)}(o_j)$ if $f(o_i) \leq f(o_j)$.

Now we set

$$Z_\lambda := U_{\varepsilon(\lambda)}(K_\lambda) \subset \Sigma_\lambda \quad \text{for } \lambda \in \Lambda$$

and

$$Z := \bigcup_{\lambda \in \Lambda} Z_\lambda \subset \Sigma.$$

Then Z is a closed subset of Σ lying in a small neighborhood of K; in fact, $Z = U_\varepsilon(K)$ in the "over Λ" sense.

LEMMA 4.6. – *There exists a continuous equivariant vector field η on Σ with the following properties.*

a) *η is a vector field over Λ*, i.e. $\eta(v) \in T_v \Sigma_\Lambda$ *for* $v \in \Sigma_\lambda - \{0\}$

b) $\eta(v) = 0 \quad \Leftrightarrow \quad v \in Z$

c) $\|\eta(v)\| \leq 1$ *for every* $v \in Z$

d) $\langle \nabla_\Lambda f(v), \eta(v) \rangle > 0$ *for every* $v \in \Sigma - Z$

e) $\langle \nabla_\Lambda f(v), \eta(v) \rangle \geq \frac{1}{2} \|\nabla_\Lambda f(v)\|^2$ *for* $v \in \Sigma$ *with* $\varepsilon_1/3 \leq \|v\| < \varepsilon_1$

f) $\eta | \Sigma_\lambda$ *is locally Lipschitz continuous for every* $\lambda \in \Lambda$.

Proof. – We first construct an equivariant pseudo gradient vector field η_1 for f over Λ. This is a vector field over Λ which is locally Lipschitz continuous in $\Sigma - K$ and satisfies

(i) $\|\eta_1(v)\| < 1$ for $v \in \Sigma$;

(ii) $\langle \nabla_\Lambda f(v), \eta_1(v) \rangle > \frac{1}{2} \|\nabla_\Lambda f(v)\|^2$ for $v \in \Sigma - K$.

Each $v \in \Sigma - \{0\}$ has a neighborhood N^v which is isomorphic over Λ to $U(\lambda) \times \mathbb{R}^k$ where $U(\lambda)$ is a neighborhood of $\lambda = q(v)$ in Λ. The isomorphism $\kappa^v : N^v \to U(\lambda) \times \mathbb{R}^k$ is a smooth map over Λ such that the restriction $\kappa^v_\lambda : N^v_\lambda \to \{\lambda\} \times \mathbb{R}^k$ is a diffeomorphism with $\kappa^v_\lambda(v) = (\lambda, 0)$. In our situation $k = \dim \Sigma_\lambda = 2n - 1$. We define $\eta_2(v) \in \mathbb{R}^k$ by

$$d\kappa^v_\lambda(v)(\nabla_\Lambda f(v)) = (\lambda, \eta_2(v)) \in \{\lambda\} \times \mathbb{R}^k = \{\lambda\} \times T_0 \mathbb{R}^k.$$

Now we set

$$\eta^v_3(u) := (d\kappa^v_\mu(u))^{-1}(\mu, \eta_2(v))$$

for $u \in N^v$ with $\mu = q(u)$. Then $\eta^v_3(v)$ is a pseudo-gradient vector for $\nabla_\Lambda f(v)$ if $\nabla_\Lambda f(v) \neq 0$; that is, the inequalities (i) and (ii) are satisfied for $\eta^v_3(v)$ if $v \in \Sigma - K$. If N^v is small enough then (i) and (ii) are also satisfied for $\eta^v_3(u)$ for all $u \in N^v$ provided $v \in \Sigma - K$. Moreover, η^v_3 is Lipschitz continuous. Next we choose a locally finite partition of unity $\{\pi_i : \Sigma \to [0, 1] | i \in I\}$ subordinated to the covering $\{N^v : v \in \Sigma - K\}$:

$$\pi_i^{-1}(0, 1] \subset N^{v_i} \quad \text{for } i \in I.$$

We also assume that each π_i is locally Lipschitz continuous. Then we obtain a pseudo-gradient vector field η_4 over Λ for f by setting

$$\eta_4(v) := \sum_{i \in I} \pi_i(v) \eta^{v_i}_3(v) \quad \text{for } v \in \Sigma - K$$

and

$$\eta_4(v) := 0 \quad \text{for } v \in K.$$

Finally we define for $v \in \Sigma - 0$ and $\lambda = q(v)$

$$\eta_1(v) := \int_0^1 (d\theta(v))^{-1} \eta_4(v_\theta) \, d\theta \in T_v \Sigma_\lambda.$$

Here $d\theta(v) : T_v \Sigma_\lambda \to T_{v_\theta} \Sigma_\lambda$ is the derivative of the action of θ on Σ_λ. This is the required equivariant pseudo-gradient vector field over Λ for f. Alternatively, one can construct η^v_3 to be equivariant, that

is, N^v is an invariant neighborhood of the orbit $\{v_\theta : \theta \in S^1\}$ and $\eta_3^v(u_\theta) = d\theta(v)\,\eta_3^v(u)$. Then also each π_i can be chosen to be invariant. This makes $\eta_1 = \eta_4$ automatically equivariant.

We obtain a vector field η satisfying a)-f) by defining $\eta(0) := 0$ and

$$\eta(v) := \alpha(v) \cdot \eta_1(v) \quad \text{for } v \in \Sigma - 0$$

where $\alpha : \Sigma \to [0, 1]$ is a smooth S^1-invariant function with $\alpha^{-1}(0) = Z$ and $\alpha(v) = 1$ if $\varepsilon_1/3 \leq \|v\| < \varepsilon_1$. □

Let η be a vector field as in 4.5. For each $\Lambda \in \Lambda$ we obtain a local flow $\varphi_\lambda : \mathcal{O}_\lambda \to \Sigma_\lambda$ on Σ_λ which consists of the maximal solution of the differential equation

$$\frac{d}{dt}\varphi_\lambda^t = -\eta \circ \varphi_\lambda^t,$$
$$\varphi_\lambda^0 = \mathrm{id}_{\Sigma_\lambda}.$$

Clearly φ_λ is gradient-like with Lyapunov function f_λ because for $v \in \Sigma_\lambda - Z_\lambda$

$$\begin{aligned}\frac{d}{dt} f_\lambda \circ \varphi_\lambda^t(v) &= \left\langle \nabla f_\lambda\,(\varphi_\lambda^t(v)),\ \frac{d}{dt}\varphi_\lambda^t(v) \right\rangle \\ &= \langle \nabla f_\lambda\,(\varphi_\lambda^t(v)),\ -\eta \circ \varphi_\lambda^t \rangle \\ &< 0.\end{aligned}$$

Thus for $v \in \Sigma_\lambda$ with $\eta(v) \neq 0$ the map $t \mapsto f \circ \varphi_\lambda^t(v)$ is strictly decreasing. Here we used the properties of η as stated in Lemma 4.5.

We set

$$\varphi : \mathcal{O} := \bigcup_{\lambda \in \Lambda} \mathcal{O}_\lambda \to \Sigma, \qquad \varphi|\mathcal{O}_\lambda := \varphi_\lambda.$$

LEMMA 4.7. – *φ is an equivariant local flow over Λ. It is gradient-like with Lyapunov function f.*

Proof. – We only have to show that $\mathcal{O}$ is an open subset of $\mathbb{R} \times \Sigma$ and that φ is continuous. The other conditions for φ to be an equivariant gradient-like local flow over Λ with Lyapunov function f follow immediately from the fact that these properties hold for each φ_λ. Since η is continuous and $\eta|\Sigma_\lambda$ is locally Lipschitz continuous for each λ according to 4.5 f), we see that $\mathcal{O}' := \mathcal{O} \cap \mathbb{R} \times (\Sigma - \{0\})$ is an open subset of $\mathbb{R} \times (\Sigma - \{0\})$ and $\varphi|\mathcal{O}'$ is continuous. It remains to prove that $\mathcal{O}$ is an open neighborhood of $\mathbb{R} \times \{0\}$ in $\mathbb{R} \times \Sigma$ and that φ is continuous at each point of $\mathbb{R} \times \{0\}$.

If $v \in \Sigma$ we write $(T^-(v), T^+(v))$ for the maximal interval on which $\varphi^t(v)$ is defined. Then $\mathcal{O}$ is an open subset of $\mathbb{R} \times \Sigma$ if the function $T^+ : \Sigma \to (0, \infty]$ is lower semi-continuous and $T^- : \Sigma \to [-\infty, 0)$ is upper semi-continuous. We only treat the case of T^+, the other case being analogous. Clearly, $T^+(0) = \infty$ and we have to show that $T^+(v_n) \to \infty$ for every sequence $v_n \to 0$. Suppose to the contrary that $T^+(v_n)$ remains bounded for some sequence $v_n \to 0$. Then there exists a sequence $t_n \in (0, T^+(v_n))$ such that $\|\varphi^{t_n}(v_n)\| = \varepsilon_1/4 =: \delta$ and such that $\|\varphi^t(v_n)\| \leq \delta$ for every $t \in [0, t_n]$ for n large. We may assume that $t_n \to t_0$ as $n \to \infty$. In addition we have

$$u_n := \varphi^{t_n}(v_n) \to u \in \Sigma_0 \quad \text{along some subsequence}$$

and $\|u\| = \delta$. Clearly $-t_n \in (T^-(u_n), 0)$ for n large and we claim that $-t_0 \in (T^-(u), 0]$. If $-t_0 \leq T^-(u)$ then there exists $t \in (T^-(u), 0]$ with $\|\varphi^t(u)\| = 2\delta$. Then $\varphi^t(u_n)$ is defined for n large because T^- is upper semi-continuous in $\Sigma - 0$. Moreover $\varphi^t(u_n) \to \varphi^t(u)$, which is not possible because

$$\|\varphi^t(u_n)\| = \|\varphi^{t+t_n}(v_n)\| \leq \delta < 2\delta = \|\varphi^t(u)\|.$$

This shows that $-t_0 \in (T^-(u), 0]$. Therefore

$$v_n = \varphi^{-t_n}(v_n) \to \varphi^{-t_0}(u) \in \Sigma_0 - 0$$

contradicting the facts that $v_n \to 0$ and that φ is continuous at the point $(-t_0, u)$. Thus we have proved that $\mathcal{O}$ is an open subset of $\mathbb{R} \times \Sigma$.

In order to see that φ is continuous at the points of $\mathbb{R} \times \{0\}$ we again argue indirectly. Suppose there are sequences $v_n \to 0$ and t_n bounded with $\varphi^{t_n}(v_n)$ bounded away from 0. Making $|t_n|$ smaller if necessary we may assume that $u_n := \varphi^{t_n}(v_n) \to u \in \Sigma_0$ with $\|u\| = \delta \in (0, \varepsilon_1/2)$ and such that $\|\varphi^t(v_n)\| \leq \delta$ for every $t \in [0, t_n]$ for n large. This leads to a contradiction as above. □

5. PROOF OF THEOREM 1.1

We study the local flow φ on Σ using Conley's index theory and Borel cohomology as described in § 3.

LEMMA 5.1. – *For any* $\varepsilon_2 \in (\varepsilon_1/2, \varepsilon_1)$ *the set*

$$N_0 := \{v \in \Sigma_0 : \|v\| \leq \varepsilon_2\} \subset \Sigma_0$$

is an isolating neighborhood of the flow φ_0 *on* Σ_0.

Proof. – We have to show that the set

$$S_0 := \operatorname{inv} N_0 = \{v \in \Sigma_0 : \varphi^t(v) \in N_0 \text{ for all } t \in \mathbb{R}\}$$

is obtained in the interior of N_0 (relative to Σ_0). The only stationary flow orbits of φ_0 are the points in $Z_0 = U_{\varepsilon(0)}(0)$. Since φ_0 is gradient-like a point $v \in S_0$ must satisfy

$$\alpha(v) \cup \omega(v) \subset Z_0 \subset U_{\varepsilon_1/4}(0). \tag{5.2}$$

We claim that $\|\varphi^t(v)\| < \varepsilon_1/2$ for any $t \in \mathbb{R}$. Suppose to the contrary that there exist times $t_1 < t_2 < t_3$ with

$$\|\varphi^{t_1}(v)\| \leq \|\varphi^{t_3}(v)\| = \varepsilon_1/3, \qquad \|\varphi^{t_2}(v)\| = \varepsilon_1/2$$

and

$$\|\varphi^t(v)\| \in [\varepsilon_1/3, \varepsilon_1] \quad \text{for all } t \in [t_1, t_3].$$

Since $\frac{d}{dt}\varphi^t = -\eta \circ \varphi^t$ and $\|\eta(\varphi^t(v))\| \leq 1$ for $t \in [t_1, t_3]$ by Lemma 4.6c) this yields $t_3 - t_1 \geq \varepsilon/3$. Therefore we obtain

$$\begin{aligned}
f(\varphi^{t_1}(v)) - f(\varphi^{t_3}(v)) &= -\int_{t_1}^{t_3} \frac{d}{dt} f \circ \varphi^t(v)\, dt \\
&= -\int_{t_1}^{t_3} \left\langle \nabla_\Lambda f(\varphi^t(v)), \frac{d}{dt}\varphi^t(v) \right\rangle dt \\
&= -\int_{t_1}^{t_3} \left\langle \nabla_\Lambda f(\varphi^t(v)), \eta \circ \varphi^t(v) \right\rangle dt \\
&\geq \int_{t_1}^{t_3} \|\nabla_\Lambda f(\varphi^t(v))\|^2 dt \\
&\geq \varepsilon_1 \cdot \mu/3
\end{aligned}$$

where μ is defined as in (4.3). Now using (4.2) and (5.2) we obtain $|f(\varphi^t(v))| < \varepsilon_1 \cdot \mu/6$ for all $t \in \mathbb{R}$. This leads to the contradiction

$$\begin{aligned}
\varepsilon_1 \cdot \mu/3 &> \lim_{t \to -\infty} f(\varphi^t(v)) - \lim_{t \to \infty} f(\varphi^t(v)) \\
&> f(\varphi^{t_1}(v)) - f(\varphi^{t_3}(v)) \\
&\geq \varepsilon_1 \cdot \mu/3. \quad \square
\end{aligned}$$

LEMMA 5.3. – *There exists $\lambda_2 > 0$ such that for $|\lambda| \leq \lambda_2$ the set*

$$U_\lambda = \{v \in \Sigma_\lambda : \|v\| \leq \varepsilon_2\}$$

is an isolating neighborhood of $S_\lambda := \operatorname{inv}(U_\lambda)$.

Proof. – We have to show that $S_\lambda \subset \operatorname{int}(U_\lambda)$ for $|\lambda|$ small. If this is not the case then there exists a sequence $\lambda_k \to 0$ and points

$$v_k \in \operatorname{inv}(U_{\lambda_k}) \cap \partial U_{\lambda_k} = \operatorname{inv}(U_{\lambda_k}) \cap \{v : \|v\| = \varepsilon_2\}.$$

Consequently $\varphi^t(v_k) \in U_{\lambda_k}$ for all $t \in \mathbb{R}$. By compactness we may assume that v_k converges towards $v \in \partial U_0$ along a subsequence. Moreover, the continuity of φ implies $\varphi^t(v) \in U_0$ for all $t \in \mathbb{R}$. This means

$$v \in \operatorname{inv}(U_0) \cap \partial U_0 = S_0 \cap \partial U_0 = \emptyset,$$

a contradiction. □

With ε_2 and λ_2 as above we set

$$U := \{v \in \Sigma : \|v\| \leq \varepsilon_2,\ |q(v)| \leq \lambda_2\}.$$

This is an isolating neighborhood with respect to the flow φ restricted to the part over $[-\lambda_2, \lambda_2]$. Let $S := \operatorname{inv}(U)$, so that $S_\lambda = \operatorname{inv}(U_\lambda) = S \cap q^{-1}(\lambda)$. Choose an index pair (N, A) for S in U. Clearly, (N_λ, A_λ) is an index pair for S_λ, $|\lambda| \leq \lambda_2$. By the continuity of the length ℓ we can find $\lambda_0 = 0$ such that $\ell(A_\lambda) \leq \ell(A_0)$ for $|\lambda| \leq \lambda_0$. because A_λ is contained in an arbitrarily small neighborhood of A_0 in N if λ is small. Proposition 3.5 implies

$$\ell(S_\lambda) \geq \ell(\mathcal{C}(S_\lambda)) \geq \ell(N_\lambda) - \ell(A_\lambda) \geq \ell(N_\lambda) - \ell(A_0).$$

Observe that

$$A_0 \subset N_0 - \{0\} \subset \{v \in V - \{0\} : q(v) = 0\} \simeq SV^+ \times SV^-.$$

Thus there exist equivariant maps $A_0 \to SV^+$ and $A_0 \to SV^-$. According to Proposition 3.3 a), h) we obtain

$$\ell(A_0) \leq \min\{\ell(SV^+), \ell(SV^-)\} = \min\{n^+, n^-\}.$$

For $\lambda > 0$ small we have

$$SV^+ \tilde{\simeq} \Sigma_\lambda \cap (V^+ \times \{0\}) \subset N_\lambda \subset \Sigma_\lambda \tilde{\simeq} SV^+ \times BV^- \tilde{\simeq} SV^+$$

hence, $\ell(N_\lambda) = \ell(SV^+) = n^+$. Analogously we obtain $\ell(N_\lambda) = n^-$ for $\lambda < 0$ close to 0. Finally this yields for $|\lambda|$ small

$$\ell(S_\lambda) \geq \ell(N_\lambda) - \ell(A_0) \geq |n^+ - n^-| = \frac{1}{2}|\sigma(\tau_0)| \qquad \text{if} \quad \lambda\sigma(\tau_0) > 0. \tag{5.4}$$

To deduce Theorem 1.1 recall that the set $Z_\lambda = U_{\varepsilon(\lambda)}(K_\lambda)$ of stationary orbits of φ_λ has the following property. If K_λ consists only of finitely

many S^1-orbits $o_1, \ldots, o_r$ of critical points then Σ_λ is the disjoint union of the $\varepsilon(\lambda)$-neighborhoods of $o_1, \ldots, o_r$. In that case we obtain an S^1-Morse decomposition $(M_1, \ldots, M_r)$ of S_λ with $M_i := U_{\varepsilon(\lambda)}(o_i)$. Here we use that $\varepsilon(\lambda)$ satisfies (4.4) and (4.5). Now Propositions 3.4c) and 3.3a), h) yield

$$\ell(S_\lambda) \leq \sum_{i=1}^{r} \ell(M_i) = r. \tag{5.5}$$

Theorem 1.1 follows from (5.4) and (5.5). □

Observe that the same argument yields more solutions if $\ell(A_0) < \min\{n^+, n^-\}$. In fact, then we would get $n^+ - \ell(A_0)$ S^1-orbits of stationary points on Σ_λ for $\lambda > 0$ small; and we would obtain $n^- - \ell(A_0)$ stationary S^1-orbits on Σ_λ for $\lambda < 0$ close to 0. Therefore the number

$$\ell^u(S_0) := \min\{\ell(A_0) : (N_0, A_0) \text{ is an index pair for } S_0\}$$

is of interest. It is called the exit-length of S_0 with respect to the flow φ_0 on Σ_0. It has been introduced and studied in [B2], Chapter 7. In certain situations it is invariant under continuation; *see* [B2], Theorem 7.4. In particular, if one considers a one-parameter family of Hamiltonian functions $H_\mu \in C^2(\mathbb{R}^{2N}, \mathbb{R})$, $0 \leq \mu \leq 1$, with $H_\mu(0) = 0$, $H'_\mu(0) = 0$, $H''_\mu(0)$ non-degenerate, then the exit-length $\ell^u(S_0) = \ell^u(\{0\})$ does not depend on the Hamiltonian H_μ to which one applies the constructions of this paper, at least as long as there are no other periodic solutions on $H_\mu^{-1}(0)$ near the origin. Potentially this continuation invariance of $\ell^u(S_0)$ provides means for its computation. Improvements of Theorem 1.1 in this direction would depend on higher order terms of H.

6. REMARKS

In the situation of Theorem 1.1 one obtains more periodic solutions if one knows more about the signatures $\sigma_i := \sigma(\tau_0/i)$ of Q restricted to the spaces $E_i := E(\tau_0/i)$, $i \in \mathbb{N}$, of τ_0/i-periodic solutions of (LHS). Clearly $E_i \subset E_j$ if j divides i, in particular $E_i \subset E_1 = E$ for any $i \in \mathbb{N}$. Set

$$\sigma^+ := \frac{1}{2} \max\{\sum_{i \in I} \sigma_i : I \subset \mathbb{N},\ \sigma_i \geq 0$$
$$\text{for every } i \in I,\ E_i \cap E_j = 0 \text{ for all } i \neq j \in I\}$$

and

$$\sigma^- := \frac{1}{2} \max \{ \sum_{i \in I} |\sigma_i| \ : \ I \subset \mathbb{N},\ \sigma_i \leq 0$$
$$\text{for every } i \in I,\ E_i \cap E_j = 0 \text{ for all } i \neq j \in I \}.$$

Theorem 1.1 yields the following

COROLLARY 6.1. – *In the situation of Theorem* 1.1 *assume that case* (i) *does not apply, that is, there are no periodic solutions of* (HS) *on* $H^{-1}(0)$ *near* 0 *with periods close to* τ_0. *Then for each* $\lambda > 0$ (*respectively* $\lambda < 0$) *close to* 0 *there exist at least* $\frac{1}{2}\sigma^+$ (*respectively* $\frac{1}{2}\sigma^-$) *geometrically different periodic solutions on* $H^{-1}(\lambda)$ *with periods close to* τ_0.

We conjecture that this result is optimal in the following sense. Let Q be a non-degenerate quadratic form on $\mathbb{R}^{2N}$ and consider the linear Hamiltonian system

$$\dot{x} = JQ'(x) = JQ''(0)\,x. \qquad \text{(LHS)}$$

Fix some period $\tau_0 > 0$ of non-trivial periodic solutions of (LHS). Suppose $\mathbb{R}^{2N} = E \oplus F$ splits as in the introduction into two linear subspaces invariant under $JQ''(0)$. As above we write $E_i = E(\tau_0/i)$ for the space of periodic solutions of (LHS) with (not necessarily minimal) period τ_0/i, so $E = E_1$. And we write σ_i for the signature of Q restricted to E_i. Then we conjecture that there exists a polynomial function $H(x) = Q(x) + o(\|x\|^2)$ such that the Hamiltonian system $\dot{x} = JH'(x)$ has for $\lambda > 0$ precisely $\frac{1}{2}\sigma^+$ geometrically different periodic solutions on $H^{-1}(\lambda)$ with periods near τ_0; and it should have precisely $\frac{1}{2}\sigma^-$ such periodic solutions on $H^{-1}(\lambda)$ for $\lambda < 0$. We shall only prove the following special case.

PROPOSITION 6.2. – *If the signature* σ_i *of* Q *restricted to* E_i *is* 0 *for all* $i \in \mathbb{N}$ *then there exists a polynomial function* $H : \mathbb{R}^{2N} \to \mathbb{R}$ *with* $H(x) = Q(x) + o\|x\|^2)$ *and such that the Hamiltonian system* $\dot{x} = JH'(x)$ *does not have any periodic orbits with periods near* τ_0 *except the equilibrium* 0.

Proposition 6.2 shows that the sequence $(\sigma_i = \sigma(\tau_0/i) \ : \ i \in \mathbb{N})$ of signatures is the only invariant of H which depends only on the second order terms of H and whose non-vanishing guarantees the existence of periodic solutions of (HS) near 0 with periods near τ_0.

Proof. – As in § 2 we may assume that

$$Q(x) = Q(x_E) + Q(x_F) \qquad \text{where} \quad x = x_E + x_F \in E \oplus F.$$

We shall find a polynomial H as required such that $H(x) = Q(x)$ depends only on $x_E \in E$. Therefore we may assume that $\mathbb{R}^{2N} = E$ and $x = x_E$. Let E^j be the space of periodic solutions of (LHS) corresponding to the eigenspace of the eigenvalue $2\pi j/\tau_0$. These solutions have minimal period τ_0/j (except 0, of course). Let σ^j be the signature of Q restricted to E^j. If $\sigma_i = 0$ for all i then $\sigma^j = 0$ for all j because

$$\sigma^j = \sum_{j|i} \mu(i/j)\,\sigma_i$$

where $\mu : \mathbb{N} \to \{0, \pm 1\}$ is the Möbius function (*cf.* [J], § 8.6). For each $j \in \mathbb{N}$ with $E^j \neq 0$ we shall find a polynomial P^j which depends only on $x \in E^j$ with $P^j(x) = o(\|x\|^2)$ and such that the Hamiltonian system $\dot{x} = J(Q + P^j)'(x)$ has no periodic solutions in E^j. Thus we may assume that $E = E^j$ and

$$Q(x) = \frac{\alpha}{2}(\|x^+\|^2 - \|x^-\|^2) \quad \text{for } x = x^+ + x^- \in E \cong E^+ \oplus E^-$$

Here $\alpha := 2j\pi/\tau_0$ and $E \cong E^+ \oplus E^-$ is a direct sum decomposition of E into subspaces E^+ and E^- on which Q is positive respectively negative definite. Clearly, $\dim E^+ = \dim E^- =: 2d$ since $\sigma^j = 0$. We introduce symplectic coordinates in $E^\pm$ so that $x^\pm = q^\pm + p^\pm$. Now we define (*cf.* [MW], Example 9.2)

$$\begin{aligned} H(x) &= Q(x) + \|x\|^2 \cdot (q^+ q^- - p^+ p^-) \\ &= Q(x) + \|x\|^2 \cdot \sum_{k=1}^{d} (q_k^+ q_k^- - p_k^+ p_k^-). \end{aligned}$$

If $x = x(t)$ is a solution of the associated Hamiltonian system (HS) then a straightforward computation shows that

$$\begin{aligned} \frac{d}{dt}(p^+ q^- - p^- q^+) &= \frac{d}{dt}\left(\sum_{k=1}^{d} (p_k^+ q_k^- - p_k^- q_k^+)\right) \\ &= 4(q^+ q^- - p^+ p^-)^2 + \|x\|^4 \end{aligned}$$

If $x \neq 0$ then $p^+ q^- - p^- q^+$ is strictly increasing, so x cannot be periodic. This proves Proposition 6.2. □

The methods of this paper together with the length for arbitrary compact Lie groups as defined in [B2], Chapter 4, can also be used to treat more general non-linear eigenvalue problems.

Let $\Phi, \Psi : X \to \mathbb{R}$ be of class C^2 defined on a Hilbert space X on which a compact Lie group G acts orthogonally. Suppose Φ and Ψ are G-invariant and consider the equation

$$\Phi'(x) = \tau \Psi'(x) \tag{P}$$

If $\Phi'(0) = 0 = \Psi'(0)$ and $\Psi''(0)$ is non-degenerate one can study the bifurcation of solutions on $\Psi^{-1}(\lambda)$ near $(\tau_0, 0)$ with the level λ as parameter. Equations of this type have been studied by many authors, but mainly in the special case $\Psi'(x) = x$ or if $\Psi''(0)$ is positive definite on the kernel of the linearization $\Phi''(0) - \tau_0 \Psi''(0)$. Our method allows to treat the case where $\Psi''(0)$ is indefinite on this kernel. We state only one result in this direction.

THEOREM 6.3. – *Let $\Phi, \Psi \in C^2(X, \mathbb{R})$ satisfy $\Phi(0) = \Psi(0) = 0$ and $\Phi'(0) = \Psi'(0) = 0$. Let $\tau_0 \in \mathbb{R}$ be a possible bifurcation value, that is, $V := \ker(\Phi''(0) - \tau_0 \Psi''(0)) \neq 0$. Assume moreover that the quadratic form $q(v) := \frac{1}{2}\langle \Psi''(0)v, v\rangle$ on V is nondegenerate with signature $\sigma \neq 0$.*

a) *A least one of the following holds.*

(i) *There exists a sequence $(\tau_k, x_k) \in \mathbb{R} \times \Psi^{-1}(0)$, $k \in \mathbb{N}$, of solutions of* (P) *which converges towards $(\tau_0, 0)$ as $k \to \infty$.*

(ii) *For each $\lambda > 0$ close to 0 there exists a solution $(\tau_\lambda, x_\lambda) \in \mathbb{R} \times \Psi^{-1}(\lambda)$ of* (P) *which converges towards $(\tau_0, 0)$ as $\lambda \to 0$.*

(iii) *The same statement as in* (ii) *holds for $\lambda < 0$ close to 0.*

b) *If Φ, Ψ are even functions then either* (i) *holds or*

(iv) *For each λ close to 0 with $\sigma \cdot \lambda > 0$ there exist at least $|\sigma|/2$ pairs $(\tau_{\lambda,i}, \pm x_{\lambda,i}) \in \mathbb{R} \times \Psi^{-1}(\lambda)$ of solutions of* (P) *which converge towards $(\tau_0, 0)$ as $\lambda \to 0$.*

One can construct examples where precisely one of the cases (i), (ii) or (iii) in 6.3 a) holds. It is interesting to observe that with the $\mathbb{Z}/2$ symmetry the direction of the bifurcating solutions is determined by the sign of σ which is not the case in general. Theorem 6.3 can be generalized to include other symmetry groups. For example, if a compact Lie group G acts orthogonally on X and Φ, Ψ are invariant with respect to this action then 6.3 a) holds if the unit spheres SV^+ and SV^- are not stably G-homotopy equivalent. Here V^+ and V^- are invariant subspaces of $V = V^+ \oplus V^-$ such that $\pm q$ is positive definit on $V^\pm$. Also 6.3 b) can be generalized to other symmetry groups. In fact, in this paper we essentially proved the S^1-version of 6.3 b). We refer to [B3] for details. It is interesting to compare the proof of Theorem 1.1 with the approach of Floer and Zehnder in [FZ] and of the author in [B2] who give new proofs of the result of

Fadell and Rabinowitz [FR] on the existence of periodic solutions of (HS) parametrized by the period. In [FZ] and [B2] both the Conley index and Borel cohomology are also used. The situation considered in these papers is somehow dual to the one considered here. There one considers a family N_τ of isolating neighborhoods but the topology of N_τ does not change and neither changes the exit-length $\ell^u(S_\tau)$ of $S_\tau = \mathrm{inv}(N_\tau)$. In fact, the family of flows $\varphi = (\varphi_\tau)$ provides a flow over the parameter space $[\tau_0 - \delta, \tau_0 + \delta]$ in the sense of [B1], contrary to the flow φ over $\Lambda = [-\lambda_1, \lambda_1]$ constructed in § 4. This implies in particular that also the Conley index of S_τ does not change. Instead for each τ the invariant set S_τ contains a "trivial" solution $x_\tau = 0$ which is isolated for $\tau \neq \tau_0$. The existence of non-trivial solutions follows from a change of the exit-length of $\{x_\tau\}$ as τ passes τ_0.

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