The Ginzburg-Landau equations
of superconductivity
and the one-phase Stefan problem [1]

by

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ABSTRACT. – We study the time-dependent Ginzburg-Landau model for type I superconductivity in a cylindrically symmetric setting. We show that under appropriate monotonicity properties for the initial data, the singular limit (as the penetration depth tends to zero and the Ginzburg-Landau parameter is kept fixed) is a classical one-phase Stefan problem for the magnetic field H. We combine energy methods with monotonicity properties obtained via maximum principles.

RÉSUMÉ. – Nous montrons, sous l'hypothèse de symétrie cylindrique, que le champ magnétique H satisfait le problème de Stefan à une phase, en prenant la limite singulière des équations de Ginzburg-Landau non stationnaire qui modélisent la supraconductivité de type I. Nous supposons que les conditions initiales sont « monotones » et nous nous servons de méthodes d'énergie combinées avec des propriétés de monotonicité obtenues via des principes du maximum.

SECTION 1

In this paper, we study asymptotic behaviour of the dimensionless time dependent Ginzburg-Landau equations for superconductivity,

$$-\alpha\xi^2\partial_t\psi - i\alpha\frac{\xi}{\lambda}\psi\phi + \left(\xi\nabla - \frac{i}{\lambda}A\right)^2\psi = \psi(|\psi|^2 - 1),$$

$$-\lambda^2\,\mathrm{curl}\,^2A = \lambda^2(\partial_t A + \nabla\phi) + i\frac{\xi\lambda}{2}(\psi^*\nabla\psi - \psi\nabla\psi^*) + |\psi|^2 A.$$

Here ψ is a complex order parameter, whose squared modulus measures the density of superconducting electrons, A is the magnetic vector potential and ϕ the electric scalar potential. From Maxwell's equation it follows that the potentials determine the magnetic field H and the electric field E through

$$H = \mathrm{curl}\,A \quad \text{and} \quad E = -A_t - \nabla\phi.$$

The coefficients α, λ, and ξ are positive material constants. The parameter λ is called the penetration depth while the parameter ξ is the coherence length, and both parameters are small. The ratio of these length scales $\kappa = \frac{\lambda}{\xi}$ is called the Ginzburg-Landau parameter. We shall consider the asymptotic behaviour $\lambda \to 0$ while keeping κ fixed and prove, in a cylindrically symmetric case, convergence to a classical one-phase Stefan problem for the magnetic field H.

Next we briefly describe certain aspects of the Ginzburg-Landau theory of superconductivity. We refer to the overview papers [CHO: 1992, DGP: 1992] and the reference therein for a broader introduction to this theory. Ginzburg and Landau, in their fundamental paper of 1950 [GL: 1950], introduced a phenomenological theory for (steady-state) superconductivity based on the Ginzburg-Landau energy density

$$W(|\psi|) + |H - H_{\mathrm{appl}}|^2 + \left|\xi\nabla\psi - \frac{i}{\lambda}A\psi\right|^2,$$

where the typical $W(|\psi|) = \frac{1}{4}(1 - |\psi|^2)^2$, and H_{appl} is the applied magnetic field on the boundary of the superconducting device. Their theory was largely accepted when Gor'kov [G:1959] showed formally that the Ginzburg-Landau theory can be derived in the limit of the microscopic BCS theory [BCS: 1957].

Later, the time dependent Ginzburg-Landau equations were written down by Gor'kov and Eliashberg [GE: 1968] using an averaging of the microscopic BCS theory. These equations are not obtained directly as the

gradient flow for the Ginzburg-Landau energy of superconductivity because they must satisfy gauge invariance, due to the coupling with Maxwell's equations. This means that if ψ, A and ϕ are solutions to the above equations, then

$$\bar{\psi} := \psi e^{iw}, \qquad \bar{A} := A + \lambda \xi \nabla w \qquad \text{and} \qquad \bar{\phi} := \phi - \lambda \xi \partial_t w$$

lead to the same evolution for the electromagnetic fields E, H and electron density $|\psi|^2$. By choosing different gauges, Chen, Hoffman and Liang [CHL: 1993], Du [D:1994] and Elliott and Tang [ET] have proven independently the well-posedness of the time dependent Ginzburg-Landau equations using different methods.

Quite different asymptotic behaviour (for small λ) is expected for different values of the Ginzburg-Landau parameter κ. In type II superconductors ($\kappa > \frac{1}{\sqrt{2}}$), one expects that vortices of "normal phase" penetrate the superconducting matrix as the strength of the applied magnetic field $|H_{\mathrm{appl}}|$ is increased through some critical value. In contrast, for type I superconductors ($\kappa < \frac{1}{\sqrt{2}}$), one expects a *region* of normal phase to penetrate the superconducting device as $|H_{\mathrm{appl}}|$ exceeds the critical value $H_c = \frac{1}{\sqrt{2}}$, and that a smooth interface separates the two regions. Keller [K: 1958] studied superconducting materials of "cylindrical" form, $\Omega \times \mathbf{R}$ where Ω is a bounded set in $\mathbf{R}^2$, with the applied magnetic field parallel to the cylinder. Using physical reasoning based on Maxwell's equations, he predicted that the interface separating the normal from the superconducting regions should evolve according to a classical one-phase Stefan problem for the magnetic field H. In the general case in $\mathbf{R}^3$, Chapman [C] used asymptotic expansion to show formally that as $\lambda \to 0$ and for $|H_{\mathrm{appl}}|$ bigger than H_c, the time dependent Ginzburg-Landau equations approximate the following "vectorial" one-phase Stefan problem

$$\begin{cases} \partial_t H = \Delta H & \text{in the normal region,} \\ H = 0 & \text{in the superconducting region,} \\ |H| = H_c & \text{on the interface } \Gamma, \\ \operatorname{curl} H \times n = -v_n H & \text{on } \Gamma, \end{cases}$$

where Γ is the interface separating the superconducting and the normal phases. Here n is the unit normal to Γ (which we choose to point towards the normal region) and v_n is the normal velocity to Γ (negative when the superconductor region is shrinking). We note that in a superconducting material the electric field $E = 0$ and the magnetic field is expelled, *i.e.* $H = 0$; this later fact is known as the Meissner effect [MO: 1933]. This vectorial Stefan problem can also be derived from Maxwell's equations

(see [CHO: 1992]) and reduces to the classical one-phase Stefan problem derived by Keller in the "cylindrical" case. The existence, uniqueness and long-time behaviour for this system has been studied by Friedman and Hu [FH:1991] under the assumption that H depends on one variable or that it satisfies radial symmetry.

In this paper, we present a rigorous justification of Chapman's asymptotics in the cylindrically symmetric case, that is we prove convergence of the time dependent Ginzburg-Landau equations to the classical one-phase Stefan problem derived by Keller. We assume that the superconducting device is an infinite round wire (i.e. of the form $B_1(0) \times \mathbf{R}$ with $B_1(0)$ the open unit ball in $\mathbf{R}^2$), that the external applied field is constant and parallel to the wire, and we impose appropriate monotone initial data (see (A1) to (A5) of Section 4). This monotonicity assumption ensures that the one-phase Stefan problem is well-posed, or stable, in the sense that the normal region expends into the superconducting region (see e.g. [M]). In particular, our result shows rigorously that the time dependent Ginzburg-Landau equations are in fact a valid approximation of the well established free boundary model for type I superconductors.

To our knowledge there are few methods available at the present time to prove convergence of systems in general settings. However, energy methods have been quite successful in studying systems in radially symmetric settings. These methods have the important advantages that they are direct, give clear explanation and rigorous justification for the formal asymptotic results in special settings. Some examples are vector-valued singular perturbation problems with potentials vanishing on concentric circles [BS] or the phase-field equations [S]. In these papers, energy estimates are combined with error estimates on the approximation by the first order terms in the asymptotic expansion. Here, the method is even more direct since it only relies on energy estimates and maximum principles. More precisely, the major difference to those results obtained in [BS] and [S] is that compactness cannot be obtained by energy bounds, but has to be derived from structural properties such as monotonicity. This reflects the fact that the surface tension is of order λ for the present system of equations and thus vanishes in the limit. We combine the energy bounds with the monotonicity properties to directly pass to the limit in the equations. In particular, we do this without using the first order expansion and thus the arguments are less technical.

In Section 2, we implement the gauge transformation originally done in [C] (cf [CHO: 1992]) in which the time dependent Ginzburg-Landau equations admit a Lyapunov functional. In this gauge, we obtain coupled

equations with real coefficients for some new vector potential Q, some new scalar potential Φ and for $f = |\psi|$. We then derive the equations we shall study in the cylindrical symmetric setting.

In Section 3, we derive the energy estimates (cf Lemma 3.6 to 3.8) and using maximum principle and invariant region arguments, we prove that the solutions stay monotone for all time (cf Lemma 3.1 to 3.5). In particular, this means that the system of Ginzburg-Landau equations has the same stability properties as the well-posed Stefan problem. As a first by-product of our energy estimates and monotonicity properties, we show that the radial problem (GL) of Section 3 is well-posed.

In Section 4, we prove the convergence to the classical one-phase Stefan problem (cf Proposition 4.1, 4.2 and Theorem 4.9). In particular, we must show that there is no "indetermined region": that is a normal region where the limiting magnetic field is zero (cf Propositions 4.6 and 4.8).

Finally, we note that our results hold true even when $\kappa > \frac{1}{\sqrt{2}}$ (type II superconductor). This reflects our assumptions of cylindrical symmetry and monotonicity. Indeed, in the radial case, vortices can only be at the origin, which we rule out by the monotonicity assumptions on the initial data. We note however that even for type II superconductors, phase transformation has numerically been observed, when a very strong field is applied to a superconducting device. At the beginning of the process a normally conducting phase develops at the boundary, penetrates into the wire, and eventually decomposes into the mixed state.

SECTION 2: THE RADIAL EQUATIONS OF SUPERCONDUCTIVITY

If $|\psi| \neq 0$ we may write $\psi = f e^{i\chi}$, and following [C: 1992] and [CHO: 1992], we let $\vec{Q} = \vec{A} - \lambda\xi\nabla\chi$ and $\Phi = \phi + \lambda\xi\partial_t\chi$, so that the time-dependent Ginzburg-Landau equations become

$$\alpha\xi^2\partial_t f - \xi^2\Delta f + W'(f) + \frac{1}{\lambda^2}f|\vec{Q}|^2 = 0,$$

$$\lambda^2\left(\partial_t\vec{Q} + \nabla\Phi + \operatorname{curl}^2\vec{Q}\right) + f^2\vec{Q} = 0,$$

$$\alpha f^2\Phi + \operatorname{div}(f^2\vec{Q}) = 0.$$

Here $W(f) = \frac{1}{4}(1 - f^2)^2$, so that $W'(f) = f^3 - f$ and $W(f) \geq 0$. Suitable boundary conditions are:

$$\partial_\nu f = 0, \quad \nu \cdot (f\vec{Q}) = 0, \quad \nu \times \operatorname{curl}\vec{Q} = \nu \times \vec{H}_D.$$

As we saw in the introduction, this is a well-posed problem for f, Φ and $\vec{Q}$, if suitable initial values are imposed. From these quantities the physical fields $\vec{E}$ and $\vec{H}$ may be calculated, since we have $\vec{H} = \operatorname{curl} \vec{A} = \operatorname{curl} \vec{Q}$ and $\vec{E} = -\partial_t \vec{A} - \nabla\phi = -\partial_t\vec{Q} - \xi\lambda\partial_t\nabla\chi - \nabla\phi = -\partial_t\vec{Q} - \nabla\Phi$.

In this paper we assume that the domain is an infinite wire

$$\{(x,y) \mid x \in B_1(0) \subset \mathbb{R}^2, y \in \mathbb{R}\},$$

and that the external field $\vec{H}_D$ is parallel to the wire. In this situation we seek a solution of the form $(f, \vec{Q}, \Phi)(t,x,y) = (f, \vec{Q}, \Phi)(t,x)$, with $\vec{Q} = Q(t,|x|)\begin{pmatrix} \frac{-x_2}{r} \\ \frac{x_1}{r} \\ 0 \end{pmatrix}$, and $f = f(t,|x|)$. We put $r = |x|$ and denote differentiation with respect to x and r by ∇ and $'$, respectively. Due to this choice of gauge we find $\Phi \equiv 0$ and Q and f solve

$$\frac{\lambda^2}{\kappa^2}\left(\alpha\partial_t f - f'' - \frac{1}{r}f'\right) + W'(f) + \frac{1}{\lambda^2}fQ^2 = 0,$$

$$\left(\partial_t Q - Q'' - \frac{1}{r}Q' + \frac{1}{r^2}Q\right) + \frac{1}{\lambda^2}f^2 Q = 0,$$

and $\vec{H} = \begin{pmatrix} 0 \\ 0 \\ H(t,r) \end{pmatrix}$ with $H = \frac{1}{r}(rQ)'$. We remark that $H' = \left(\frac{1}{r}(rQ)'\right)' = Q'' + \frac{1}{r}Q' - \frac{1}{r^2}Q$. We choose the boundary values (see Section 3 (BC) and the explanation below)

$$f'|_{r=0} = f'|_{r=1} = 0, \quad \left.\frac{1}{r}(rQ)'\right|_{r=1} = H_D, \quad Q|_{r=0} = 0,$$

where $H_D \geq \frac{1}{\sqrt{2}} = H_c$. In this case the associated energy is given by

$$E_\lambda(f,Q) := \int_0^1 \left[\frac{\lambda^2}{2\kappa^2}(f')^2 + W(f) + \frac{1}{2\lambda^2}f^2 Q^2 + \frac{1}{2}\left(Q' + \frac{1}{r}Q - H_D\right)^2\right] r\,dr$$

Our method applies as well to the following one-dimensional problem corresponding to an infinite wall:

$$\{(x,y) \mid x \in (0,1), y \in \mathbb{R}^2\}.$$

If all quantities only depend on x and $\vec{Q} = Q(t,x)\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, then $\Phi \equiv 0$

and Q and f solve

$$\frac{\lambda^2}{\kappa^2}(\alpha\partial_t f - f'') + W'(f) + \frac{1}{\lambda^2}fQ^2 = 0,$$

$$(\partial_t Q - Q'') + \frac{1}{\lambda^2}f^2 Q = 0,$$

and $\vec{H} = \begin{pmatrix} 0 \\ 0 \\ H(t,x) \end{pmatrix}$ with $H = Q'$. The boundary values are

$$f'|_{x=0} = f'|_{x=1} = Q'|_{x=0} = 0, \quad Q'|_{x=1} = H_D.$$

SECTION 3: THE RADIAL EQUATION; EXISTENCE, MAXIMUM PRINCIPLES, ENERGY BOUNDS

In the situation of an infinite wire we study the solution of the radial Ginzburg-Landau system

(GL) $\qquad \dfrac{\lambda^2}{\kappa^2}\left(\alpha\partial_t f - f'' - \dfrac{1}{r}f'\right) + W'(f) + \dfrac{1}{\lambda^2}fQ^2 = 0,$

$$\left(\partial_t Q - Q'' - \frac{1}{r}Q' + \frac{1}{r^2}Q\right) + \frac{1}{\lambda^2}f^2 Q = 0,$$

for $r \in (0,1)$ and $t \in (0,T)$, together with the boundary conditions

(BC) $\quad f'(t,0) = f'(t,1) = 0, \quad Q(t,0) = 0, \quad (Q + Q')(t,1) = H_D.$

We will refer to the last boundary condition as the mixed condition. This choice of boundary values is good for the following reason: we want the magnetic field $H = \frac{1}{r}(rQ)' = Q' + \frac{1}{r}Q$ to be a smooth radial function. Thus H may attain a nonzero value at the origin, but its derivative has to vanish. Once H is given, any representation of H in terms of Q is unique up to the addition of a term $\frac{1}{r}$. If we impose $Q(0) = 0$, we render this representation unique and at the same time, through the differential equation, we impose that $H' = Q'' + \frac{1}{r}Q' - \frac{1}{r^2}Q$ vanishes at the origin. In addition Q is bounded, but its derivative at the origin does not vanish necessarily.

We show that a solution of this type exists by the following approximation procedure. For some fixed $\rho > 0$ we solve the system of differential

equations (GL) in $(\rho, 1)$. This corresponds to solving (GL) in an annular domain. We impose the boundary data

$$f'(t, \rho) = f'(t, 1) = 0, \quad Q(t, \rho) = 0, \quad (Q + Q')(t, 1) = H_D.$$

For any C^∞-initial data that satisfy the compatibility conditions we obtain a C^∞-solution of this evolution problem. We will prove the following energy relation and maximum principles:

$$0 = \int_\rho^1 \left(\frac{\lambda^2}{\kappa^2} \alpha (\partial_t f)^2 + (\partial_t Q)^2 \right) r \, dr$$

$$+ \frac{d}{dt} \int_\rho^1 \left[\frac{\lambda^2}{2\kappa^2} (f')^2 + W(f) + \frac{1}{2\lambda^2} f^2 Q^2 + \frac{1}{2} \left(Q' + \frac{1}{r} Q - H_D \right)^2 \right] r \, dr.$$

If the initial data satisfy $0 \leq f \leq 1$ and $0 \leq Q \leq H_D$, then the same remains true for all positive times. If furthermore the initial data satisfy $f' \leq 0$ and $Q' \geq 0$, then the same remains true for all positive times. Finally, if in addition the initial data satisfy $\partial_t Q = Q'' + \frac{1}{r} Q' - \frac{1}{r^2} Q - \frac{1}{\lambda^2} f^2 Q \geq 0$ and $\alpha \partial_t f = f'' + \frac{1}{r} f' - \frac{\kappa^2}{\lambda^2} \left(W'(f) + \frac{1}{\lambda^2} f Q^2 \right) \leq 0$, then $\partial_t Q \geq 0$ and $\partial_t f \leq 0$ for all positive times. Using these facts, we may pass to the limit $\rho \to 0$ and obtain a solution of the system (GL), subject to the boundary conditions (BC).

We now proceed with this program.

MAXIMUM PRINCIPLES

LEMMA 3.1. – *Assume that (f, Q) is a C^∞-solution of the radial Ginzburg-Landau equations in $(\rho, 1)$. Assume that initially f and Q are nonnegative. Then f and Q remain nonnegative for all positive times.*

Proof. – Assume that Q attains a negative minimum at (t_0, r_0). Since $Q' + Q = H_D > 0$ at $r = 1$, we find $r_0 < 1$. Since $Q = 0$ at $r = \rho$, we find $r_0 > \rho$. Thus $\partial_t Q - Q'' - \frac{1}{r} Q' + \frac{1}{r^2} Q < 0$ at the minimal point and $f^2 Q \leq 0$. This contradicts the differential equation for Q and consequently $Q \geq 0$.

For f we proceed as follows. Choose $\mu > \frac{\kappa^2}{\lambda^2 \alpha}$ and assume that $v := exp(-\mu t) f$ assumes a negative minimum $-\delta$ at some point (t_0, r_0). Then $\alpha \partial_t v - v'' - \frac{1}{r} v' \leq 0$ and $v Q^2 \leq 0$ at (t_0, r_0), and thus the differential equation for f implies that $exp(-\mu t_0) W'(-\delta exp(\mu t_0)) \geq \frac{\lambda^2}{\kappa^2} \alpha \mu \delta$. Since $W'(f) = f^3 - f$, this implies that $-\delta^2 exp(2\mu t_0) \geq \frac{\lambda^2}{\kappa^2} \alpha \mu - 1 > 0$ by

the choice of μ. This is a contradiction and consequently $v \geq 0$ which implies $f \geq 0$.

LEMMA 3.2. – *Assume that* (f, Q) *is a* C^∞-*solution of the radial Ginzburg-Landau equations in* $(\rho, 1)$. *Assume that initially* $0 \leq f \leq 1$ *and* $0 \leq Q \leq H_D$. *Then* $f \leq 1$ *and* $Q \leq H_D$ *for all positive times.*

Proof. – Since f is nonnegative by Lemma 3.1, f is a subsolution of the differential equation $\frac{\lambda^2}{\kappa^2}\left(\alpha\partial_t f - f'' - \frac{1}{r}f'\right) + W'(f) = 0$. Since $W'(f) > 0$ whenever $f > 1$, the classical maximum principle implies $f \leq 1$.

Since Q remains nonnegative by Lemma 3.1, we have $\partial_t Q - Q'' - \frac{1}{r}Q' \leq 0$. Thus the maximum principle implies that $Q \leq \bar{Q}$, where $\bar{Q}$ is a solution of $\partial_t \bar{Q} - \bar{Q}'' - \frac{1}{r}\bar{Q}' = 0$ with $\bar{Q}(t, \rho) = 0$ and with the mixed condition at $r = 1$ and the same initial data as Q. But $\bar{Q}$ attains its maximum on the boundary $r = 1$, and thus at the maximum point $\bar{Q}' \geq 0$. Thus the mixed boundary condition implies that the maximal value of $\bar{Q}$ is less than H_D. This implies the lemma.

LEMMA 3.3. – *Assume that* (f, Q) *is a* C^∞-*solution of the radial Ginzburg-Landau equations in* $(\rho, 1)$. *Assume that initially* f *is strictly positive and* $0 \leq Q \leq H_D$. *Then* $f(t, r) \geq \min f(0, r) \exp\left(-\frac{H_D^2 \kappa^2}{\lambda^4 \alpha}t\right)$.

Proof. – By Lemma 3.1 and Lemma 3.2 we have $0 \leq Q \leq H_D$ and $0 \leq f \leq 1$, and thus $W'(f) \leq 0$. Consequently $\frac{\lambda^2}{\kappa^2}\left(\alpha\partial_t f - f'' - \frac{1}{r}f'\right) + \frac{1}{\lambda^2}f H_D^2 \geq 0$. By the maximum principle $f \geq \bar{f}$, where $\bar{f}$ is a spatially constant function with $\frac{\lambda^2}{\kappa^2}\alpha\partial_t \bar{f} + \frac{1}{\lambda^2}\bar{f}H_D^2 = 0$ and $0 < \bar{f}(0) \leq \min f(0, r)$.

LEMMA 3.4. – *Assume that* (f, Q) *is a* C^∞-*solution of the radial Ginzburg-Landau equations in* $(\rho, 1)$. *Assume that initially* f *and* Q *are nonnegative and that* $Q \leq H_D$. *Furthermore assume that initially* f' *is nonpositive and* Q' *is nonnegative. Then this remains true for all positive times.*

Proof. – We show that $\{(t, r) : f'(t, r) \leq 0, \quad Q'(t, r) \geq 0\}$ is an invariant region. First we differentiate the equations (GL) with respect to r and find

$$\frac{\lambda^2}{\kappa^2}\left(\alpha\partial_t f' - f''' - \frac{1}{r}f'' + \frac{1}{r^2}f'\right) + W''(f)f' + \frac{1}{\lambda^2}\left(f'Q^2 + 2fQQ'\right) = 0,$$

$$\left(\partial_t Q' - Q''' - \frac{1}{r}Q'' + \frac{2}{r^2}Q' - \frac{2}{r^3}Q\right) + \frac{1}{\lambda^2}\left(2ff'Q + f^2Q'\right) = 0.$$

Next we define $w := exp(-\mu t)f'$ and $v := exp(-\mu t)Q'$ for some positive number μ. Then (w, v) satisfies the same system of differential equations as (f', Q'), with the right hand side substituted

by $-\frac{\lambda^2}{\kappa^2}\alpha\mu w$ and $-\mu v$ respectively. We show that for any $\delta > 0$ the set $\{(t,r) : w(t,r) \leq \delta, \quad v(t,r) \geq -\delta\}$ is an invariant region. Initially and on the boundary $r = \rho$ or $r = 1$ the pair (w,v) lies strictly in this set. Thus, if (w,v) leaves the region under consideration, then either $w = \delta$ with $\alpha\partial_t w - w'' - \frac{1}{r}w' \geq 0$ and $v \geq -\delta$, or $v = -\delta$ with $\partial_t v - v'' - \frac{1}{r}v' \leq 0$ and $w \leq \delta$. In the first case the differential equation for w implies that $-W''(f) + \frac{2}{\lambda^2}fQ \geq \frac{\lambda^2}{\kappa^2}\alpha\mu$, and in the second case the differential equation for v implies that $\frac{2}{\lambda^2}fQ \geq \mu$. This implies a contradiction if we choose $\mu = \mu(\lambda, H_D, \alpha, \kappa)$ big enough. Thus we conclude that $w \leq \delta$ and $v \geq -\delta$ for all times. Letting δ converge to zero, we conclude that $v \geq 0$ and $w \leq 0$, which is the assertion of this Lemma.

LEMMA 3.5. – *Assume that (f, Q) is a C^∞-solution of the radial Ginzburg-Landau equations in $(\rho, 1)$. Assume that initially f and Q are nonnegative. Furthermore assume that initially $\partial_t f$ is nonpositive and $\partial_t Q$ is nonnegative. Then this remains true for all positive times.*

Proof. – We differentiate the system of differential equations with respect to time and obtain

$$\frac{\lambda^2}{\kappa^2}\left(\alpha\partial_t\partial_t f - \partial_t f'' - \frac{1}{r}\partial_t f'\right) + W''(f)\partial_t f + \frac{1}{\lambda^2}\left(\partial_t f Q^2 + 2fQ\partial_t Q\right) = 0,$$

$$\left(\partial_t\partial_t Q - \partial_t Q'' - \frac{1}{r}\partial_t Q' + \frac{1}{r^2}\partial_t Q\right) + \frac{1}{\lambda^2}\left(2f\partial_t fQ + f^2\partial_t Q\right) = 0.$$

In addition we have the mixed condition $\partial_t Q + (\partial_t Q)' = 0$ on the boundary $r = 1$, and the Dirichlet condition $\partial_t Q = 0$ on the boundary $r = \rho$. For $\partial_t f$ we have a Neumann condition on the boundary. This implies that $\{\partial_t f \leq 0, \partial_t Q \geq 0\}$ is an invariant region, following the lines of the proof of Lemma 3.4.

ENERGY ESTIMATES

LEMMA 3.6. – *Assume that (f, Q) is a C^∞-solution of the radial Ginzburg-Landau equations in $(\rho, 1)$. Then*

$$0 = \int_\rho^1 \left(\frac{\lambda^2}{\kappa^2}\alpha(\partial_t f)^2 + (\partial_t Q)^2\right) r \, dr$$

$$+ \frac{d}{dt}\int_\rho^1 \left[\frac{\lambda^2}{2\kappa^2}(f')^2 + W(f) + \frac{1}{2\lambda^2}f^2Q^2 + \frac{1}{2}\left(Q' + \frac{1}{r}Q - H_D\right)^2\right] r \, dr.$$

Proof. – The result follows by multiplying the equation for f by $r\partial_t f$ and the equation for Q by $r\partial_t Q$ and integration by parts.

LEMMA 3.7. – *Assume that (f, Q) is a C^∞-solution of the radial Ginzburg-Landau equations in $(\rho, 1)$. Assume that $0 \le f \le 1$ and $0 \le Q \le H_D$ initially. Then*

$$
\frac{1}{2}\frac{d}{dt}\left\{ \int_\rho^1 \left((Q')^2 r + Q^2 \frac{1}{r} + \frac{\lambda^2 \alpha}{\kappa^2}(f')^2 r \right) dr + (Q - H_D)^2_{|r=1} \right\}
$$
$$
+ \int_\rho^1 \left(\left| \left(\frac{1}{r}(rQ)'\right)' \right|^2 r + \frac{\lambda^2}{\kappa^2}\left| (rf')' \right|^2 \frac{1}{r} \right) dr
$$
$$
+ \frac{1}{\lambda^2}\int_\rho^1 \left(f^2(Q')^2 r + \frac{1}{2}(f')^2 Q^2 r + \frac{1}{r}f^2 Q^2 \right) dr
$$
$$
\le \frac{1}{\lambda^2}\left(H_D^2 + 8\int_\rho^1 |Q'|^2 r\, dr + \lambda^2 \int_\rho^1 |f'|^2 r\, dr \right).
$$

Proof. – We multiply the differential equation for f by $-(rf')'$ and the differential equation for Q by $-(\frac{1}{r}(rQ)')'/r$, integrate over $r \in (\rho, 1)$ and add the results. This implies that

$$
\frac{1}{2}\frac{d}{dt}\left\{ \int_\rho^1 \left((Q')^2 r + Q^2 \frac{1}{r} + \frac{\lambda^2 \alpha}{\kappa^2}(f')^2 r \right) dr + (Q - H_D)^2_{|r=1} \right\}
$$
$$
+ \int_\rho^1 \left(\left| \left(\frac{1}{r}(rQ)'\right)' \right|^2 r + \frac{\lambda^2}{\kappa^2}\left| (rf')' \right|^2 \frac{1}{r} \right) dr
$$
$$
+ \frac{1}{\lambda^2}\int_\rho^1 \left(f^2(Q')^2 r + (f')^2 Q^2 r + \frac{1}{r}f^2 Q^2 \right) dr
$$
$$
= \frac{1}{\lambda^2}f^2 Q(H_D - Q)(t, 1) - \frac{4}{\lambda^2}\int_\rho^1 ff'QQ'r\, dr - \int_\rho^1 W''(f)(f')^2 r\, dr
$$
$$
\le \frac{1}{\lambda^2}f^2 Q(H_D - Q)(t, 1) + \frac{1}{2\lambda^2}\int_\rho^1 (f')^2 Q^2 r\, dr
$$
$$
+ \frac{8}{\lambda^2}\int_\rho^1 f^2(Q')^2 r\, dr + \max(-W'')\int_\rho^1 (f')^2 r\, dr.
$$

Since $f \le 1$ and $Q \le H_D$ and $\max(-W'') = 1$, this proves the assertion in combining the first integral term of the right hand side with the corresponding term of the left hand side.

DEFINITION. – *In view of Lemma 3.7 we define the weighted Sobolev-spaces*

$$V := \left\{ (f, Q) \in \left(H^1(0, 1) \right)^2 \; : \; \int_0^1 \left((Q')^2 r + \frac{Q^2}{r} + (f')^2 r \right) dr < \infty \right\}$$

and

$$W := \left\{ (f, Q) \in \left(H^2(0, 1) \right)^2 \; : \right.$$

$$\left. \int_0^1 \left(\left| \left(\frac{1}{r}(rQ)' \right)' \right|^2 r + |(rf')'|^2 \frac{1}{r} \right) dr < \infty \right\}.$$

Remark. – Any $(f, Q) \in V$ necessarily satisfies $Q \in C^{1/2}(0, 1)$ with $Q(0) = 0$.

LEMMA 3.8. – *Assume that (f, Q) is a C^∞-solution of the radial Ginzburg-Landau equations in $(\rho, 1)$. Assume that f, $Q \geq 0$, $\partial_t f \leq 0$ and $\partial_t Q \geq 0$ initially. Then $(Q' + \frac{1}{r}Q)' = Q'' + \frac{1}{r}Q' - \frac{1}{r^2}Q \geq 0$ and*

$$\int_\rho^1 \left| \left(Q' + \frac{1}{r}Q \right)' \right| dr \leq H_D.$$

Proof. – By Lemma 3.1 and Lemma 3.5 both $\partial_t Q$ and Q remain nonnegative. Thus the differential equation for Q implies that $(Q' + \frac{1}{r}Q)' = Q'' + \frac{1}{r}Q' - \frac{1}{r^2}Q \geq 0$. Thus $\int_\rho^1 |(Q' + \frac{1}{r}Q)'| dr = (Q' + \frac{1}{r}Q)|_\rho^1 = H_D - Q'(t, \rho)$. Since Q attains its minimum at $r = \rho$, we have $Q'|_{r=\rho} \geq 0$, which implies the result.

LEMMA 3.9. – *Assume that (f, Q) is a C^∞-solution of the radial Ginzburg-Landau equations in $(\rho, 1)$. Assume that f, $Q \geq 0$, $\partial_t f \leq 0$ and $\partial_t Q \geq 0$ initially. Then*

$$\frac{1}{\lambda^2} \int_\rho^1 f^2 Q \, dr \leq H_D.$$

Proof. – By Lemma 3.1 and Lemma 3.5 both Q and $\partial_t Q$ remain positive. We integrate the differential equation for Q in space and find $\int_\rho^1 \partial_t Q \, dr + \frac{1}{\lambda^2} \int_\rho^1 f^2 Q \, dr = H_D - Q'(t, \rho) \leq H_D$.

PROPOSITION 3.10. – *For any initial data $(f_{in}, Q_{in}) \in V$ with $0 \leq f_{in} \leq 1$ and $0 \leq Q_{in} \leq H_D$ there exists a solution*

$$(f, Q) \in L^\infty(0, T; V) \cap L^2(0, T; W) \cap L^\infty((0, T) \times (0, 1))$$

with

$$(\partial_t f, \partial_t Q)\sqrt{r} \in L^2((0,T) \times (0,1))$$

of the radial Ginzburg-Landau equations (GL) satisfying the boundary conditions (BC).

The claim follows from the estimates of Lemma 3.6 and Lemma 3.7.

Remark. – Via approximation all the preceding Lemmata hold true in the limit $\rho \to 0$ under the weaker regularity properties given in Proposition 3.10. The hypothesises and the assertions have then to be satisfied or are then true, respectively, in the almost everywhere sense.

Remark. – The physical situation we have in mind is the following: an external magnetic field $H_D > \frac{1}{\sqrt{2}}$ is applied, and a normally conducting region penetrates the originally superconducting wire.

We may construct initial data as follows: choose a real number β with $\max(1/\kappa, 2\kappa) \leq \beta \leq \infty$ and a smooth, increasing function h on the real axis with $h(z) = 0$ for all $z \leq 0$ and $h_\infty := \lim_{z \to \infty} h(z) \in (0, \infty)$. Then define for $0 \leq r \leq 1$ the initial data

$$f_\lambda(0,r) = f(r) := f_0(z) \quad \text{and} \quad Q_\lambda(0,r) = Q(r) := \lambda Q_0(z),$$

where $z := (r - r_0)/\lambda$ with some fixed $0 < r_0 < 1$, and where (f_0, Q_0) solves the system of ordinary differential equations

$$f_0' = -\kappa f_0 Q_0, \quad Q_0' = \beta\sqrt{W(f_0)} + h$$

on the real axis with $f_0(0) = 1$ and $Q_0(0) = 0$.

Such a solution exists and has the following properties: $f_0(z) = 1$ and $Q_0(z) = 0$ for all $z \leq 0$, $f_0(z) > 0$ for all z (since once Q_0 is locally given, f_0 solves an ODE with a Lipschitz continuous right hand side which has 0 as equilibrium value), $Q_0(z)$, $Q_0'(z) \geq 0$ and $f_0'(z) \leq 0$ for all z. In addition $Q_0(z) \geq \int_0^z h(\xi)\, d\xi > 0$ for all z in the nonempty interior of the support of h. Next we find that $\lim_{z \to \infty} f_0(z) = 0$, since otherwise f_0' would be strictly negative for large z, and thus f_0 would attain negative values. Using this, we may determine $\lim_{z \to \infty} Q_0'(z) = \frac{1}{2}\beta + h_\infty$. The main point is now that

$$\frac{1}{\kappa^2} f_0'' - W'(f_0) - f_0 Q_0^2 = -W'(f_0)\left(1 - \frac{\beta}{2\kappa}\right) - \frac{1}{\kappa} f_0 h \leq 0,$$
$$Q_0'' - f_0^2 Q_0 = f_0^2 Q_0(\kappa\beta - 1) + h' \geq 0$$

by the assumptions on the parameter β and on the function h.

As a consequence of these properties of f_0 and Q_0 and by definition, (f, Q) satisfies

- $0 < f \leq 1$ and $0 \leq Q \leq Q(1)$,
- $f' \leq 0$ and $Q' \geq 0$,
- $\frac{\lambda^2}{\kappa^2}(f'' + \frac{1}{r}f') - W'(f) - \frac{1}{\lambda^2}fQ^2 \leq 0$ and $Q'' + \frac{1}{r}Q' - \frac{1}{r^2}Q - \frac{1}{\lambda^2}f^2Q \geq 0$.

In the last inequality, we have used that $Q''(r) = \frac{1}{\lambda}Q_0''(z) \geq 0$ and that for some $\xi \in (r_0, r)$, we have $Q'(r) - \frac{1}{r}Q(r) = \frac{1}{r}(rQ'(r) - (r - r_0)Q'(\xi)) \geq Q'(r) - Q'(\xi) \geq 0$.

Concerning the boundary data we find by this construction that

- $Q(r) = 0$ and $f(r) = 1$ for all $0 \leq r \leq r_0$,
- $H_D := Q(1) + Q'(1) \to [\beta/2 + h_\infty](2 - r_0)$ and $f(1)$ and $f'(1)$ are exponentially small as $\lambda \to 0$.

Since h_∞ may be arbitrarily small and $2 - r_0$ may be arbitrarily close to 1, the range of boundary data H_D that may be attained using this construction is given by $2H_D \geq \beta$, and is thus restricted by the conditions on β. In addition we remark that the Neumann condition for f at $r = 1$ is only attained up to an exponentially small negative term. We may remove this defect by adding an exponentially small term to f with support only close to the fixed boundary. In addition we point out that Lemmata 3.1 to 3.5 remain true if constant negative Neumann data are imposed for all times at $r = 1$.

We finally mention that the energy of these initial data is uniformly bounded in λ and that Q attains some nontrivial limit as λ tends to 0. For this last claim we point out that

$$\frac{d}{d\lambda}Q(r) = Q_0(z) - \frac{r - r_0}{\lambda}Q_0'(z) = \frac{r - r_0}{\lambda}(Q_0'(\xi) - Q_0'(z)) \leq 0$$

for some $\xi \in (0, z)$. Thus Q attains a pointwise limit, which is nontrivial as long as $H_D \neq 0$.

SECTION 4: PASSAGE TO THE LIMIT

We assume that (f_λ, Q_λ) is a solution of the radial Ginzburg-Landau equations (GL) with r ranging in $(0, 1)$, with time t ranging in $[0, T]$ and with the boundary conditions (BC) in the sense of Proposition 3.10.

We recall that we use x for the spatial variable ranging in $B_1(0) \subset \mathbf{R}^2$ and $r = |x|$ for the radial variable ranging in $(0, 1)$, and that we denote differentiation with respect to x and r with ∇ and $'$, respectively.

We assume that the initial values satisfy the following:

(A1) $0 \le f_\lambda(0, \cdot) \le 1$ and $0 \le Q_\lambda(0, \cdot) \le H_D$,

(A2) $f'_\lambda(0, \cdot) \le 0$ and $Q'_\lambda(0, \cdot) \ge 0$,

(A3) $\partial_t f_\lambda(0, \cdot) \le 0$ and $\partial_t Q_\lambda(0, \cdot) \ge 0$, *i.e.*

$$\left(f''_\lambda + \frac{1}{r}f'_\lambda - \frac{\kappa^2}{\lambda^2}W'(f_\lambda) - \frac{\kappa^2}{\lambda^4}f_\lambda Q_\lambda^2 \right)(0, \cdot) \le 0$$

and

$$\left(Q''_\lambda + \frac{1}{r}Q'_\lambda - \frac{1}{r^2}Q_\lambda - \frac{1}{\lambda^2}f_\lambda^2 Q_\lambda \right)(0, \cdot) \ge 0,$$

(A4) there exists a constant C_0 such that for all λ the initial energy satisfies $E_\lambda(f_\lambda, Q_\lambda)(0) \le C_0$.

In addition we assume that as $\lambda \to 0$

(A5) $Q_\lambda(0, \cdot) \to Q_0^0$ in $L^1(B_1(0))$ with $Q_0^0 \not\equiv 0$ and $H_0^0 := \text{div}(Q_0^0 \frac{x}{|x|}) \in L^1(B_1(0))$.

Remark. – We say that a constant only depends on the data if it can be determined a priori from the above constant C_0, the Dirichlet value H_D, the time T as well as the parameters κ and α and is independent of λ.

PROPOSITION 4.1. – *There exists a subsequence $\lambda = \lambda_n \to 0$ $(n \to \infty)$ and some $Q_0 \in H^{1,2}((0,T) \times B_1(0))$ such that $Q_\lambda \rightharpoonup Q_0$ weakly in $H^{1,2}((0,T) \times B_1(0))$ and almost everywhere in $(0,T) \times B_1(0)$.*

Furthermore $Q_\lambda \longrightarrow Q_0$ strongly in $L^p(0,T;C^0([0,1]))$ for all $1 < p < \infty$ and $Q'_\lambda \longrightarrow Q'_0$ strongly in $L^2((0,T) \times (0,1))$ and almost everywhere in $(0,T) \times B_1(0)$.

In addition $Q_0 \in L^\infty(0,T;H^{1,\infty}(0,1))$ and $(Q'_0 + \frac{1}{r}Q_0)' \in (L^p(0,T;C^0([0,1])))^*$, with $0 \le Q'_0 + \frac{1}{r}Q_0 \le H_D$ and $(Q'_0 + \frac{1}{r}Q_0)' \ge 0$ as well as $Q'_0, \partial_t Q_0 \ge 0$ in the distributional sense.

Proof. – The energy estimate of Lemma 3.6 and assumption (A4) imply that Q_λ is uniformly bounded in $H^{1,2}((0,T) \times B_1(0))$, and therefore a subsequence is precompact in the weak topology of $H^{1,2}((0,T) \times B_1(0))$. We denote its limit by Q_0. Lemma 3.8 together with the assumptions (A1) and (A3) imply that $Q''_\lambda + \frac{1}{r}Q'_\lambda - \frac{1}{r^2}Q_\lambda = (Q'_\lambda + \frac{1}{r}Q_\lambda)' \ge 0$. Thus $Q'_\lambda + \frac{1}{r}Q_\lambda$ is increasing and thus bounded by the boundary data H_D. Since both Q'_λ and Q_λ are positive by Lemma 3.1 and 3.4 and the assumptions (A1) and (A2), this implies that $Q'_\lambda \le H_D$. By Lemma 3.2 and assumption (A1) we have as well that $Q_\lambda \le H_D$. Thus Q_λ is bounded in $L^\infty(0,T;H^{1,\infty}(0,1))$.

Now we apply the following compactness result (cf Lions [L]): if a reflexive Banach space B_0 is compactly embedded into a Banach space

B, which is itself embedded into another reflexive Banach space B_1, then the embedding from $W := \{v \in L^{p_0}(0, T; B_0); \ \partial_t v \in L^{p_1}(0, T; B_1)\}$ into $L^{p_0}(0, T; B)$ is compact for all $1 < p_0, p_1 < \infty$.

We choose $B_0 = H^{1,q}(B_1(0))$ with $q > 2$, and $B = C^0(\overline{B_1(0)})$ and $B_1 = L^2(B_1(0))$. These spaces satisfy the assumptions. We choose $p_1 = 2$ and $1 < p = p_0 < \infty$ can be arbitrary. Since Q_λ is bounded in $H^{1,2}((0, T) \times B_1(0))$ and $L^\infty(0, T; H^{1,\infty}(B_1(0)))$, the compactness result implies strong convergence in $L^p(0, T; C^0(\overline{B_1(0)}))$.

By Lemma 3.8 together with the assumptions (A1) and (A3) we know that $(Q'_\lambda + \frac{1}{r}Q_\lambda)'$ is bounded in $L^\infty(0, T; L^1(0, 1)) \subset (L^p(0, T; C^0([0, 1])))^*$. Since $L^p(0, T; C^0([0, 1]))$ is separable, this implies that $(Q'_\lambda + \frac{1}{r}Q_\lambda)'$ converges to some μ in the weak $*$ topology of $(L^p(0, T; C^0([0, 1])))^*$. Since Q'_λ and Q_λ converge weakly in $L^2((0, T) \times (0, 1))$, we obtain that μ is the distributional derivative of $Q'_0 + \frac{1}{r}Q_0$: $< \zeta, \mu > = -\int_0^T \int_0^1 (Q'_0 + \frac{1}{r}Q_0)\zeta' \, dr \, dt$ for all $\zeta \in L^p(0, T; C^0([0, 1])) \cap L^2(0, T; \overset{\circ}{H}{}^{1,2}(0, 1))$. Thus we may calculate

$$\int_0^T \int_0^1 |Q'_\lambda|^2 \zeta \, dr \, dt$$

$$= \int_0^T \int_0^1 \left(Q'_\lambda \left(Q'_\lambda + \frac{1}{r}Q_\lambda \right)\zeta - Q'_\lambda \frac{1}{r}Q_\lambda \zeta \right) dr \, dt$$

$$= -\int_0^T \int_0^1 Q_\lambda \left(Q'_\lambda + \frac{1}{r}Q_\lambda \right)' \zeta \, dr \, dt$$

$$\quad - \int_0^T \int_0^1 \left(Q_\lambda \left(Q'_\lambda + \frac{1}{r}Q_\lambda \right)\zeta' + Q'_\lambda \frac{1}{r}Q_\lambda \zeta \right) dr \, dt$$

$$\longrightarrow - < Q_0 \zeta, \mu > - \int_0^T \int_0^1 \left(Q_0 \left(Q'_0 + \frac{1}{r}Q_0 \right)\zeta' + Q'_0 \frac{1}{r}Q_0 \zeta \right) dr \, dt$$

$$= \int_0^T \int_0^1 |Q'_0|^2 \zeta \, dr \, dt$$

for all smooth ζ with compact support in $[0, T] \times (0, 1)$. Since Q'_λ is uniformly bounded we can derive the same result for $\zeta \equiv 1$, and thus the L^2-norm of Q'_λ converges to the L^2-norm of Q'_0. This together with the weak convergence in L^2 implies the strong convergence.

PROPOSITION 4.2. – *There exists a further subsequence $\lambda = \lambda_{n'} \to 0$ ($n' \to \infty$) and some $f_0 \in BV((0, T) \times (0, 1)) \cap L^\infty((0, T) \times (0, 1))$, such that $f_\lambda \overset{*}{\rightharpoonup} f_0$ in the weak $*$ topology of $BV((0, T) \times (0, 1))$ and $f_\lambda \longrightarrow f_0$ strongly in $L^p((0, T) \times (0, 1))$ for any $1 \leq p < \infty$ and almost everywhere in $(0, T) \times (0, 1)$.*

Furthermore $\lambda \nabla f_\lambda$ *converges to* 0 *in* $L^2((0,T) \times B_1(0))$ *and* $\lambda^2 \Delta f_\lambda$ *converges to* 0 *weakly in* $L^2((0,T) \times B_1(0))$.

In addition $\partial_t f_0, f_0' \leq 0$ *in the distributional sense.*

Proof. – By Lemma 3.1, 3.2, 3.4 and 3.5 and the assumptions (A1), (A2) and (A3), the sequence f_λ is bounded in $L^\infty((0,T) \times (0,1))$ and $\partial_t f_\lambda \leq 0$ and $f_\lambda' \leq 0$, so that f_λ is bounded in $BV((0,T) \times (0,1))$. Hence f_λ is compact in the weak $*$ topology of $BV((0,T) \times (0,1))$ and compact in $L^p((0,T) \times (0,1))$.

The first equation of (GL) together with Lemmata 3.6 and 3.7 and assumption (A4) imply that $\lambda^2 \Delta f_\lambda$ is bounded in $L^2((0,T) \times B_1(0))$. Thus a subsequence of $\lambda^2 \Delta f_\lambda$ has a weak limit g_0 in $L^2((0,T) \times B_1(0))$. But for any smooth ζ with compact support in the unit ball $\int_0^T \int_{B_1(0)} \lambda^2 \Delta f_\lambda \zeta = -\int_0^T \int_{B_1(0)} \lambda^2 \nabla f_\lambda \cdot \nabla \zeta \longrightarrow 0$, since $\lambda \nabla f_\lambda$ is bounded in $L^2((0,T) \times B_1(0))$ by the energy estimate of Lemma 3.6. This implies that g_0 vanishes. Now we may calculate

$$\int_0^T \int_{B_1(0)} \lambda^2 |\nabla f_\lambda|^2 \, dx = -\int_0^T \int_{B_1(0)} \lambda^2 \Delta f_\lambda f_\lambda \, dx$$

$$\longrightarrow -\int_0^T \int_{B_1(0)} g_0 f_0 \, dx = 0,$$

since $f_\lambda \longrightarrow f_0$ strongly in $L^2((0,T) \times B_1(0))$.

Remark. – We restrict all further discussions to the subsequence selected in Propositions 4.1 and 4.2.

LEMMA 4.3. – *There exists a constant* C, *that only depends on the data, such that*

$$\int_0^T \|f_\lambda^2 Q_\lambda^2 r\|_{L^\infty(0,1)} \, dt \leq C\lambda,$$

$$\int_0^T \int_0^1 \left(\frac{f_\lambda}{\lambda} Q_\lambda^2\right)^2 r \, dr \, dt \leq C\lambda^2,$$

$$\int_0^T \int_0^1 \frac{f_\lambda^3 Q_\lambda^2}{\lambda^2} \sqrt{r} \, dr \, dt \leq C\sqrt{\lambda}.$$

Proof. – We estimate $(f_\lambda Q_\lambda)^2(t,r)r \leq 2\int_0^1 |(f_\lambda Q_\lambda)(f_\lambda Q_\lambda)'|s\,ds + \int_0^1 (f_\lambda Q_\lambda)^2\,ds$. Integration in time and the Hölder inequality then implies

$$\int_0^T \sup_r \left(f_\lambda^2 Q_\lambda^2 r\right) dt \leq \left(\int_0^T \int_0^1 (f_\lambda Q_\lambda)^2 s\,ds\,dt\right)^{\frac{1}{2}}$$

$$\times \left(4\int_0^T \int_0^1 |(f_\lambda Q_\lambda)'|^2 s\,ds\,dt + \int_0^T \int_0^1 |(f_\lambda Q_\lambda)|^2 \frac{1}{s}\,ds\,dt\right)^{\frac{1}{2}}.$$

Now, since $\int_0^T \int_0^1 |(f_\lambda Q_\lambda)'|^2 r\,dr\,dt \leq 2\int_0^T \int_0^1 f_\lambda^2 (Q_\lambda')^2 r\,dr\,dt + 2\int_0^T \int_0^1 (f_\lambda')^2 Q_\lambda^2 r\,dr\,dt$, the energy estimate of Lemma 3.6 and the estimate of Lemma 3.7 imply the first assertion.

The differential equation for f_λ implies that $\frac{f_\lambda Q_\lambda^2}{\lambda} = \lambda h_\lambda$, with $h_\lambda = -\frac{\lambda^2}{\kappa^2}(\alpha\partial_t f_\lambda - \Delta f_\lambda) - W'(f_\lambda)$.

The energy estimate of Lemma 3.6 and the bound of Lemma 3.7 imply that h_λ is bounded in $L^2((0,T) \times B_1(0))$. We estimate $\int_0^1 \frac{f_\lambda^3 Q_\lambda^2}{\lambda^2} \sqrt{r}\,dr \leq \sup_r(f_\lambda Q_\lambda \sqrt{r}) \int_0^1 \frac{f_\lambda^2 Q_\lambda}{\lambda^2}\,dr$. We apply Lemma 3.9 and the first part of this lemma to obtain the last result of this lemma.

PROPOSITION 4.4. – *The limit f_0 attains the values 0 or 1 almost everywhere in $(0,T) \times (0,1)$.*

Proof. – The sequence f_λ converges to f_0 pointwise and thus for all $\varepsilon > 0$ there exists $A_\varepsilon \subset (0,T) \times B_1(0)$ such that f_λ converges uniformly to f_0 in A_ε and $|(0,T) \times B_1(0)\backslash A_\varepsilon| \leq \varepsilon$. For any positive number c we consider the set $B_{\varepsilon,c} := A_\varepsilon \cap \{f_0 > c\}$. We integrate the differential equation for f_λ over this set. Since $\lambda\partial_t f_\lambda$ is bounded in $L^2((0,T) \times B_1(0))$ by the energy estimate of Lemma 3.6, and since $\lambda^2\Delta f_\lambda$ converges weakly in $L^2((0,T) \times B_1(0))$ to zero by Proposition 4.2, we find that the terms involving time or spatial derivatives of f_λ converge to 0. Since f_λ converges uniformly in $B_{\varepsilon,c}$, the W' term converges to $\int_B W'(f_0)r\,dr$. We estimate the remaining term: we have $f_\lambda \geq \frac{c}{2}$ in B, and thus $\int_B \frac{f_\lambda Q_\lambda^2}{\lambda^2}\,dx \leq \frac{4}{c^2}\int_B \frac{f_\lambda^3 Q_\lambda^2}{\lambda^2}\,dx$, which converges to 0 by Lemma 4.3. Consequently $\int_B W'(f_0)\,dx = 0$, and thus $f_0 = 1$ almost everywhere in $B_{\varepsilon,c}$. Letting c converge to 0 implies that $f_0 = 1$ almost everywhere in $A_\varepsilon \cap \{f_0 > 0\}$. Letting ε tend to zero implies $f_0 = 1$ almost everywhere in $\{f_0 > 0\}$.

PROPOSITION 4.5. – *The limit $f_0 Q_0$ vanishes identically and in addition $\frac{1}{\lambda}f_\lambda Q_\lambda$ converges to 0 in $L^2((0,T) \times B_1(0))$ as $\lambda \to 0$.*

Proof. – The energy bound of Lemma 3.6 implies that $f_\lambda Q_\lambda$ converges to 0 in $L^2((0,T) \times B_1(0))$. Since by Propositions 4.1 and 4.2 both f_λ and Q_λ converge pointwise to their respective limits, the first result is obvious.

Next we multiply the differential equation for f_λ by f_λ and integrate over $(0,T) \times B_1(0)$. This implies $\frac{\lambda^2 \alpha}{2\kappa^2} \int f_\lambda^2 \, dx \big|_0^T + \frac{\lambda^2}{\kappa^2} \int_0^T \int |\nabla f_\lambda|^2 \, dx \, dt + \int_0^T \int W'(f_\lambda) f_\lambda \, dx + \frac{1}{\lambda^2} \int_0^T \int f_\lambda^2 Q_\lambda^2 \, dx = 0$. Since $f_\lambda \leq 1$, and $W'(f_\lambda) f_\lambda \to W'(f_0) f_0 = 0$ in $L^1((0,T) \times B_1(0))$ and $\lambda \nabla f_\lambda \to 0$ in $L^2((0,T) \times B_1(0))$ by Propositions 4.2 and 4.4, this implies $\lim_{\lambda \to 0} \frac{1}{\lambda^2} \int_0^T \int_{B_1(0)} f_\lambda^2 Q_\lambda^2 \, dx = 0$.

DEFINITION. – *We define the free boundary* $r_0 : [0,T] \to [0,1]$ *by* $r_0(t) := \text{ess sup} \{r \in [0,1] : Q_0(t,r) = 0\}$ *and its extinction time by* $T^* := \inf\{t \in [0,T] : r_0(t) = 0\}$.

Remark. – Since $\partial_t Q_0 \geq 0$ (cf (A3) and Proposition 4.1), the free boundary r_0 is decreasing in time, $\lim_{\tau \uparrow t} r_0(\tau) = r_0(t)$, and $r_0(t) = 0$ for all $t \geq T^*$. Since $Q_0' \geq 0$ (cf (A3) and Proposition 4.1), we have $Q_0(t,r) = 0$ for $r \leq r_0(t)$ and $Q_0(t,r) > 0$ for $r > r_0(t)$.

We may as well define the free boundary for f_0 by $s_0(t) := \text{ess sup} \{r \in [0,1] : f_0(t,r) = 1\}$ and its extinction time by $t^* := \inf\{t \in [0,T] : s_0(t) = 0\}$. Following Proposition 4.4 and 4.5, we always have $s_0(t) \leq r_0(t)$ and $t^* \leq T^*$. We will show later that they agree.

PROPOSITION 4.6. – *For almost all* $t \in (0,t^*)$ *we have* $s_0(t) = r_0(t)$ *and there exists a constant* C *only depending on the data and a function* $D \in L^2(0,T)$, *such that*

$$\left| \frac{1}{2} |Q_0'(t,r)|^2 - \frac{1}{4} \right| \leq C \frac{r - r_0(t)}{r_0(t)^2} + D(t) \sqrt{\frac{r - r_0(t)}{r_0(t)}}$$

for almost all $t \in (0,t^*)$ *and* $r > r_0(t)$, *and*

$$|Q_0'(t,r)|^2 \leq C \frac{r - r_0(t)}{r_0(t)^2} + 2D(t) \sqrt{\frac{r - r_0(t)}{r_0(t)}}$$

for almost all $t \in (t^*, T^*)$ *and* $r > r_0(t)$.

Proof. – We multiply the differential equation for Q_λ by Q_λ' and the differential equation for f_λ by f_λ', integrate over $\rho \in (s,r) \subset (0,1)$ and add the results. This yields

$$\left[\frac{\lambda^2}{2\kappa^2} |f_\lambda'|^2 - W(f_\lambda) - \frac{1}{2\lambda^2} f_\lambda^2 Q_\lambda^2 + \frac{1}{2} |Q_\lambda'|^2 \right](t,r)$$

$$= \left[\frac{\lambda^2}{2\kappa^2} |f_\lambda'|^2 - W(f_\lambda) - \frac{1}{2\lambda^2} f_\lambda^2 Q_\lambda^2 + \frac{1}{2} |Q_\lambda'|^2 \right](t,s)$$

$$+ \int_s^r \left(\frac{\lambda^2}{\kappa^2} \alpha \partial_t f_\lambda f_\lambda' - \frac{\lambda^2}{\kappa^2} |f_\lambda'|^2 \frac{1}{\rho} \right.$$

$$\left. + \partial_t Q_\lambda Q_\lambda' - |Q_\lambda'|^2 \frac{1}{\rho} + Q_\lambda Q_\lambda' \frac{1}{\rho^2} \right) d\rho. \qquad (\star)$$

According to Propositions 4.1, 4.2 and 4.5, $|Q_\lambda'|^2 \to |Q_0'|^2$ and $W(f_\lambda) \to W(f_0)$ for almost all (t,r), and possibly selecting a further subsequence λ' we may assume that $(\lambda')^2 |f_{\lambda'}|^2 \to 0$ and $\frac{1}{(\lambda')^2} f_{\lambda'}^2 Q_{\lambda'}^2 \to 0$ for almost all (t,r), as well as $\int_s^r (\lambda')^2 \partial_t f_{\lambda'} f_{\lambda'}' d\rho \to 0$ and $\int_s^r (\lambda')^2 |f_{\lambda'}'|^2 d\rho \to 0$ for almost all t and all r,s. In addition, since Q_λ' and Q_λ are bounded by H_D

$$\left| \int_s^r \left(\partial_t Q_\lambda Q_\lambda' - |Q_\lambda'|^2 \frac{1}{\rho} + Q_\lambda Q_\lambda' \frac{1}{\rho^2} \right) d\rho \right|$$

$$\leq C \left(\int_0^1 |\partial_t Q_\lambda|^2 \rho d\rho \right)^{\frac{1}{2}} \sqrt{\frac{r-s}{s}} + C \frac{r-s}{s^2}.$$

We now choose a further subsequence λ'' of λ', which might depend on the time t, such that

$$\lim_{\lambda'' \to 0} \int_0^1 |\partial_t Q_{\lambda''}|^2 \rho d\rho = \liminf_{\lambda' \to 0} \int_0^1 |\partial_t Q_{\lambda'}|^2 \rho d\rho =: D^2(t).$$

By lower semicontinuity and the energy bound of Lemma 3.6 we have that $D \in L^2(0,T)$.

Now passage to the limit $\lambda'' \to 0$ in $(\star)$ implies

$$\left| \left[\frac{1}{2} |Q_0'|^2 - W(f_0) \right]_{(t,s)}^{(t,r)} \right| \leq C \frac{r-s}{s^2} + D(t) \sqrt{\frac{r-s}{s}}. \qquad (\star\star)$$

Now assume that $t^* \leq t \leq T^*$. Then by the definition of t^* and Proposition 4.4 we have $f_0 \equiv 0$ and hence $(\star\star)$ implies

$$\left| |Q_0'|^2(t,r) - |Q_0'|^2(t,s) \right| \leq C \frac{r-s}{s^2} + 2D(t) \sqrt{\frac{r-s}{s}}.$$

We now choose a sequence of points $s_n(t) < r_0(t)$ with $s_n(t) \to r_0(t)$ and use that $Q_0'(t, s_n(t)) = 0$ by the definition of $r_0(t)$, to prove the second claim of this proposition.

Next assume that $0 \leq t \leq t^*$ and that $s_0(t) < r_0(t)$. Then we choose two sequences of points $0 < s_n(t) < s_0(t) < r_n(t) < r_0(t)$ with $r_n(t) - s_n(t) \to 0$. Since $Q_0'(t, s_n(t)) = 0$ and $Q_0'(t, r_n(t)) = 0$ as well as $f_0(t, s_n(t)) = 1$ and $f_0(t, r_n(t)) = 0$ we conclude from $(\star\star)$ that $\frac{1}{4} = 0$. Since this is impossible we find that r_0 and s_0 agree in $(0, t^*)$.

To finish the proof, we now choose a sequence of points $s_n(t) \to r_0(t)$ with $s_n(t) < r_0(t) = s_0(t)$. Since then $\left(-W(f_0) + \frac{1}{2}|Q_0'|^2\right)(t, s_n(t)) = 0$, the identity $(\star\star)$ implies the first claim of this proposition.

PROPOSITION 4.7. – *The free boundary r_0 is locally Hölder-continuous of exponent $\frac{1}{3}$ in $\{t \mid 0 < r_0(t) \leq 1\} \cap [0, t^*]$.*

Proof. – We choose $t^* \geq t > \tau$ and use the identity $\int_{r_0(t)}^{r_0(\tau)} Q_0(t, r) r \, dr = \int_{r_0(t)}^{r_0(\tau)} Q_0(\tau, r) r \, dr + \int_\tau^t \int_{r_0(t)}^{r_0(\tau)} \partial_t Q_0(\sigma, r) r \, dr \, d\sigma$. Since r_0 is decreasing, $\int_{r_0(t)}^{r_0(\tau)} Q_0(\tau, r) r \, dr = 0$. Since $\partial_t Q_0 \sqrt{r} \in L^2((0, T) \times (0, 1))$, the bulk integral of the right hand side is estimated by $C|r_0(t) - r_0(\tau)|^{\frac{1}{2}}|t - \tau|^{\frac{1}{2}} r_0(t)^{\frac{1}{2}}$. Since $\left(\frac{1}{r}(r Q_0)'\right)' \geq 0$ by Proposition 4.1 and $Q_0'(t, r_0(t)) = \frac{1}{\sqrt{2}}$ by Proposition 4.6, it follows that $(Q_0(t, r) r)' \geq \frac{1}{\sqrt{2}} r \geq \frac{1}{\sqrt{2}} r_0(t)$ for $r \geq r_0(t)$. Thus for $r > r_0(t)$ integration yields $Q_0(t, r) r \geq \frac{r_0(t)}{\sqrt{2}}(r - r_0(t))$. Hence the integral of the left hand side is bounded below by $\frac{r_0(t)}{2\sqrt{2}}(r_0(t) - r_0(\tau))^2$. This implies the assertion.

PROPOSITION 4.8. – *The limit Q_0 statisfies the differential equation*

$$\partial_t Q_0 - Q_0'' - \frac{1}{r} Q_0' + \frac{1}{r^2} Q_0 = 0$$

in the open set

$$\Omega_T^{normal} := \{(t, r) \mid t \in (0, T), \ r_0(t) < r < 1\}$$

together with the (natural) mixed boundary condition

$$Q_0'(t, 1) + Q_0(t, 1) = H_D \quad \text{for all } t \in (0, T)$$

and the Dirichlet condition

$$Q_0(t, 0) = 0 \quad \text{for almost all } t.$$

The initial data are given by $Q_0(0, \cdot) = Q_0^0$.

In addition $t^* = T^*$ and $s_0 \equiv r_0$.

Furthermore the limit Q_0 is analytic in the open set Ω_T^{normal}.

Proof. – For $t \in [0, T]$ we define the c-level-sets $r_c : [0, T] \to [0, 1]$ of Q_0 by $Q_0(t, r_c(t)) = c$, if such an $r_c(t)$ exists, or else by $r_c(t) = 1$. Since Q_0' is nonnegative this is well defined. In addition, since Q_0 is continous for almost all t, we have $r_c(t) \to r_0(t)$ almost everywhere as $c \to 0$.

We now prove that $\int_0^T \int_{r_c(t)}^1 \frac{1}{\lambda^2} f_\lambda^2 Q_\lambda \, dr \, dt \to 0$ for a subsequence $\lambda \to 0$. According to Proposition 4.5, we know that for a subsequence $\lambda_{n''}$ ($n'' \to \infty$), the integral $\int_0^1 \frac{1}{\lambda^2} f_\lambda^2 Q_\lambda^2(t, r) \, dr \to 0$ for almost all t. (This is a further subsequence of the subsequence $\lambda_{n'}$ selected in Proposition 4.2. Choosing this further subsequence does not alter the limit but helps to identify the limit equation.) Since Q_λ is increasing we have $\int_{r_c(t)}^1 \frac{1}{\lambda^2} f_\lambda^2 Q_\lambda(t, r) \, dr \leq \int_{r_c(t)}^1 \frac{1}{\lambda^2} f_\lambda^2 Q_\lambda^2 \, dr \frac{1}{Q_\lambda(t, r_c(t))}$ and according to Proposition 4.1, $Q_\lambda(t, r_c(t)) \to Q_0(t, r_c(t))$ for almost all t, and thus $\int_0^1 \frac{1}{\lambda^2} f_\lambda^2 Q_\lambda(t, r) \, dr \to 0$ for almost all t. But Lemma 3.9 implies a uniform bound for $\int_0^1 \frac{1}{\lambda^2} f_\lambda^2 Q_\lambda(t, r) \, dr$, and the Dominant Convergence Theorem implies the assertion.

We now consider the set $(0, t^*]$. Arguments similar to those in the proof of Proposition 4.7 imply that $0 \leq r_c(t) - r_0(t) \leq 2\sqrt{2}c$. Thus $r_c(t)$ converges uniformly to $r_0(t)$ for $t \in [0, t^*]$ as $c \to 0$. Now we choose a smooth test function ζ with compact support in $\{(t, r) \mid t \in (0, t^*], \ r_0(t) < r \leq 1\}$. Then due to the uniform convergence of r_c as $c \to 0$ we may conclude that $supp \, \zeta \subset \{(t, r) \mid t \in (0, T^*), \ r_c(t) \leq r \leq 1\}$ for some positive c. We use ζ as a test function in the differential equation for Q_λ, integrate by parts and pass to the limit, making use of the strong convergence of Q_λ and Q_λ' and the weak convergence of $\partial_t Q_\lambda$. The limit of the nonlinear part of the equation is 0, as shown in the first step of this proof. We obtain

$$
\int_0^T \int_0^1 \left(\partial_t Q_0 \zeta + \left(Q_0' + \frac{1}{r} Q_0 \right) \zeta' \right) dr \, dt - \int_0^T H_D \zeta(t, 1) \, dt = 0.
$$

Thus we have obtained a weak formulation of the differential equation for Q_0 in the set $\{(t, r) \mid t \in (0, t^*], \ r_0(t) < r \leq 1\}$ and the mixed boundary condition at $r = 1$.

Next we assume that $t^* < T^*$ and derive a contradiction. We define $\eta_{\delta,c}$ to be a continuous cut-off function with

$$
\eta_{\delta,c}(t, r) = 0 \ \text{ if } r \leq r_c(t),
$$

$$
\eta_{\delta,c}'(t, r) = \frac{1}{\delta} \ \text{ if } r_c(t) \leq r \leq r_c(t) + \delta,
$$

$$
\eta_{\delta,c}(t, r) = 1 \ \text{ if } r_c(t) + \delta \leq r.
$$

We choose a smooth test function ζ with compact support in $[t^*, T^*] \times (0, 1]$ and use $\zeta \eta_{\delta,c}$ as a test function in the differential equation for Q_λ, integrate by parts and pass to the limit as above and obtain

$$\int_0^T \int_0^1 \left(\partial_t Q_0 \eta_{\delta,c} \zeta + \left(Q_0' + \frac{1}{r} Q_0 \right) (\eta_{\delta,c} \zeta)' \right) dr \, dt$$
$$- \int_0^T H_D \eta_{\delta,c} \zeta(t, 1) \, dt = 0.$$

Now, since r_c converges to r_0 as $c \to 0$ in any L^p, we may substitute c by 0 in the above equality.

By Proposition 4.6 we know that $Q_0'(t, r_0(t)) = 0$ in $[t^*, T^*)$, and due to the Hölder-type estimate for Q_0' of Proposition 4.6, we may let δ converge to zero. For this we note that $\text{supp}\,\zeta \subset [t^*, T^*] \times [c_0, 1]$ for some positive c_0, and thus the most difficult term $\int_0^T \int_0^1 Q_0' \eta_{\delta,0}' \zeta \, dr \, dt = \int_0^T \int_{r_0(t)}^{r_0(t)+\delta} \frac{1}{\delta} Q_0' \zeta \, dr \, dt$ is estimated by $\|\zeta\|_\infty \int_{r_0(t) \geq c_0} \left(C \sqrt{\delta} \frac{1}{r_0(t)} + \delta^{1/4} \sqrt{D(t)} \frac{1}{r_0(t)^{1/4}} \right) dt$ and converges to zero. Since $\partial_t Q_0$ and Q_0' vanish almost everywhere in $r < r_0(t)$, we finally obtain

$$\int_0^T \int_0^1 \left(\partial_t Q_0 \zeta + \left(Q_0' + \frac{1}{r} Q_0 \right) \zeta' \right) dr \, dt - \int_0^T H_D \zeta(t, 1) \, dt = 0.$$

This implies that Q_0 satisfies a regular parabolic differential equation in the strip $[t^*, T^*] \times (0, 1]$. Then the strong maximum principle implies that Q_0 is strictly positive in the interior of this strip. This in return implies that $r_0(t) = 0$ for all t in $[t^*, T^*]$, and thus $t^* = T^*$. This in particular implies that $s_0 \equiv r_0$.

In the time interval $t > T^*$, we proceed in a similar way to obtain the differential equation and the mixed boundary condition at $r = 1$.

Thus we have shown the first part of this Proposition. We have found that Q_0 is a weak solution in the open set Ω_T^{normal} of a differential equation with analytic coefficients. Thus due to parabolic regularity theory Q_0 is itself analytic in this open set and is a strong solution of the differential equation. In addition Q_0 satisfies the weak equation up to the fixed boundary, and is thus regular up to the fixed boundary.

Obviously $Q_0(t, 0) = 0$, since Q_λ converges to Q_0 in $L^p(0, T; C(\overline{B_1(0)}))$.

Since $\partial_t Q_\lambda$ is bounded in $L^2((0, T) \times B_1(0))$ and $Q_\lambda(0, \cdot) \to Q_0^0$ in $L^1(B_1(0))$ by assumption (A5), the attainment of initial data is immediate.

Remark. – We have derived two conditions on the free boundary, namely $Q_0(t, r_0(t)) = 0$ and $Q_0'(t, r_0(t)) = \frac{1}{\sqrt{2}}$. Formally differentiating the first equation with respect to time, and then substituting the second relation, implies $\partial_t Q_0(t, r_0(t)) + \frac{1}{\sqrt{2}} \dot{r}_0(t) = 0$. Formally assuming that the differential equation for Q_0 holds up to the boundary $r_0(t)$ implies that $-\frac{1}{\sqrt{2}} \dot{r}_0(t) = (\frac{1}{r}(rQ)')'(t, r_0(t))$. We recall that $\frac{1}{r}(rQ)' = H$, and we have thus formally derived

$$H = \frac{1}{\sqrt{2}} \quad \text{and} \quad H' = -\frac{1}{\sqrt{2}} \dot{r}_0 \quad \text{on } r = r_0(t).$$

Differentiating the differential equation for Q_0 implies in addition, that

$$\partial_t H - H'' - \frac{1}{r} H' = 0 \quad \text{in } \{r > r_0(t)\} = \{H > 0\}.$$

We now turn this formal argument into a rigorous proof.

THEOREM 4.9. – *Let f_λ and Q_λ be the solution of the radial Ginzburg-Landau equations with boundary condition (BC) in the sense of Proposition 3.10. Assume (A1) - (A5) for the initial data. Then (f_λ, Q_λ) converges to (f_0, Q_0) in the topology of Proposition 4.1 and Proposition 4.2. In addition we may define the limit magnetic field $H_0 := Q_0' + \frac{1}{r} Q_0 = \text{div } Q_0 \frac{x}{|x|}$ and the limit order parameter $\varphi_0 := 1 - f_0$. With $u_0 := H_0 - \frac{1}{\sqrt{2}} \varphi_0$ we have*

$$\nabla u_0 \in L^2((0, T) \times B_1(0))$$

and $u_0 = 0$ in $\{(t, x) : \varphi_0(t, x) = 0\}$ and $u_0 > 0$ in $\{(t, x) : \varphi_0(t, x) = 1\}$.

In addition the pair (u_0, φ_0) is the unique distributional solution of

$$\partial_t \left(u_0 + \frac{1}{\sqrt{2}} \varphi_0 \right) - \Delta u_0 = 0 \quad \text{in } (0, T) \times B_1(0)$$

with Dirichlet condition $u_0(t, \cdot)_{|\partial B_1(0)} = H_D - \frac{1}{\sqrt{2}}$ for almost all t and initial values $u_0 + \frac{1}{\sqrt{2}} \varphi_0 = H_0^0$. Moreover $\varphi_0(0, \cdot)$ is the characteristic function of $\{Q_0^0 > 0\}$.

Remark. – The above Theorem implies that u_0 is the solution of the classical one-phase Stefan problem. In terms of H_0 it implies that $H_0 = 0$ in the superconducting region, and that $\partial_t H_0 - \Delta H_0 = 0$ in the normal region, with interfacial condition $H_0 = \frac{1}{\sqrt{2}}$ and $\nabla H_0 \cdot \nu = -\frac{1}{\sqrt{2}} V$ on the interface separating normal and superconducting regions. Here ν denotes the normal to the interface pointing into the normal region, and V is the

normal velocity of the interface. The boundary data for H_0 are given by H_D and the initial data by H_0^0 (see (A5)).

Proof. – Under the hypothesis of this theorem we may select a subsequencs $\lambda_n \to 0$ and a limit order paramater f_0 and a limit magnetic potential Q_0 such that $(f_{\lambda_n}, Q_{\lambda_n})$ converge to (f_0, Q_0) in the topology made precise in Proposition 4.1 and Proposition 4.2. In addition the whole analysis of this section applies to this subsequence and these limits.

In particular, by Proposition 4.1 we have $0 \le H_0 \le H_D$. In addition $H_0 = 0$ in $\{(t, r) : r < r_0(t)\}$ and $H_0' = \partial_t Q_0$ in $\{(t, r) : \varphi_0(t, r) = 1\}$ by Proposition 4.8 and $H_0(t, r_0(t)+) = \frac{1}{\sqrt{2}}$ by Proposition 4.6 for almost all $t \in (0, T^*)$. Now we may calculate

$$
\begin{aligned}
\int_0^T \int_0^1 dr \, dt u_0 \zeta' &= \int_0^{T^*} \int_{r_0(t)}^1 \left(H_0 \zeta' - \frac{1}{\sqrt{2}} \zeta' \right) dr \, dt \\
&\quad + \int_{T^*}^T \int_0^1 \left(H_0 \zeta' - \frac{1}{\sqrt{2}} \zeta' \right) dr \, dt \\
&= - \int_0^{T^*} \int_{r_0(t)}^1 H_0' \zeta \, dr \, dt - \int_0^{T^*} \left(H_0(t, r_0(t)) + \frac{1}{\sqrt{2}} \right) \zeta(t, r_0(t)) \, dt \\
&\quad - \int_{T^*}^T \int_0^1 H_0' \zeta \, dr \, dt \\
&= - \int_0^T \int_0^1 \partial_t Q_0 \zeta \, dr \, dt
\end{aligned}
$$

for all smooth test functions ζ with compact support in $(0, T) \times (0, 1)$. Thus $\nabla u_0 \in L^2((0, T) \times B_1(0))$ and $\nabla u_0 = \partial_t Q_0 \frac{x}{|x|}$. Next we calculate for any smooth test function with compact support in $[0, T) \times B_1(0)$

$$
\begin{aligned}
\int_0^T \int_\Omega \nabla u_0 \cdot \nabla \zeta \, dx \, dt &= \int_0^T \int_\Omega \partial_t Q_0 \frac{x}{|x|} \cdot \nabla \zeta \, dx \, dt \\
&= - \int_0^T \int_\Omega Q_0 \frac{x}{|x|} \cdot \nabla \partial_t \zeta \, dx \, dt - \int_\Omega Q_0^0 \frac{x}{|x|} \cdot \nabla \zeta(0, x) \, dx \\
&= \int_0^T \int_\Omega \operatorname{div} \left(Q_0 \frac{x}{|x|} \right) \partial_t \zeta \, dx \, dt + \int_\Omega \operatorname{div} \left(Q_0^0 \frac{x}{|x|} \right) \zeta(0, x) \, dx \\
&= \int_0^T \int_\Omega H_0 \partial_t \zeta \, dx \, dt + \int_\Omega H_0^0 \zeta(0, x) \, dx \\
&= \int_0^T \int_\Omega \left(u_0 + \frac{1}{\sqrt{2}} \varphi_0 \right) \partial_t \zeta \, dx \, dt + \int_\Omega H_0^0 \zeta(0, x) \, dx.
\end{aligned}
$$

In addition u_0 attains the boundary data $H_D - \frac{1}{\sqrt{2}}$ following Proposition 4.8. This is the distributional formulation of the assertion. Uniqueness follows by the weak maximum principle for the one-phase Stefan problem (cf [O]).

Thus u_0 and φ_0 are uniquely given. In return H_0 and f_0 are uniquely given. Again as a consequence Q_0 is uniquely given via the differential equation by $H_0 = Q_0' + \frac{1}{r} Q_0$ if we impose in addition that Q is bounded and that $Q_0(t, 0) = 0$. Thus the limit (f_0, Q_0) is uniquely given. But this in return implies that the whole sequence converges to this limit in the topology of Proposition 4.1 and Proposition 4.2. This finishes the proof.

5. CONCLUSION

We have shown that the solutions of the radial Ginzburg-Landau equations in $\mathbb{R}^2$ approximate the solution of the classical one-phase Stefan problem, as the penetration depth converges to 0 and the Ginzburg-Landau parameter κ is kept fixed. We have shown this in the stable situation of a normal region growing into a superconducting wire. We have established energy-type a priori estimates and used invariant region principles to obtain compactness and suitable bounds. To deduce the free boundary problem we have shown that the system of equations behaves to leading order like a Hamiltonian system.

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(Manuscript received January 30, 1996.)