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Decay estimates for the critical semilinear wave equation

by

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ABSTRACT. – In this paper we prove that finite energy solutions (with added regularity) to the critical wave equation $\Box u + u^5 = 0$ on $\mathbb{R}^3$ decay to zero in time. The proof is based on a global space-time estimate and dilation identity.

RÉSUMÉ. – Dans cet article, on montre que les solutions à énergie finie (avec régularité ajoutée) de l'équation des ondes critique $\Box u + u^5 = 0$, dans $\mathbb{R}^3$ décroissent vers zéro en temps. La démonstration est basée sur une estimée temps-espace globale et une identité de dilatation.

1. INTRODUCTION

In this note we will show that the solutions of the critical semilinear wave equation with finite energy initial data

$$u_{tt} - \Delta u + u^5 = 0 \quad \text{on} \quad \mathbb{R}^3 \times \mathbb{R} \tag{1.1}$$

$$u_{|t=0} = u_0, \quad u_{t|t=0} = u_1 \quad \text{on} \quad \mathbb{R}^3 \tag{1.2}$$

$$(u_0, u_1) \in \dot{H}^1 \times L^2 \tag{1.3}$$

are globally in the space

$$(1.4) \qquad (u, u_t) \in C\Big(\mathbb{R}, \dot{H}^1 \times L^2\Big) \cap L^4\Big(\mathbb{R}, \dot{B}^{1/2} \times \dot{B}^{-1/2}\Big)$$

The study of the general semilinear wave equations dates back to the early works of Segal [7], Jörgen [5], Strauss [10]. For a detailed bibliography, see Zuily [16]. For nonlinearities that are subcritical with respect to the H^1 norm, Ginibre and Velo [3] have shown global existence and uniqueness of solutions in the space defined by (1.4), using a subtle improvement of the Strichartz ([13], [14]) estimates for the wave equations. For the critical problem, when the initial data are only of finite energy, Shatah and Struwe [9] have shown global existence and uniqueness of solutions in the space

$$(1.5) \qquad (u, u_t) \in C(\mathbb{R}, H^1 \times L^2) \cap L^4_{\text{loc}}\Big(\mathbb{R}, \dot{B}^{1/2} \times \dot{B}_4^{-1/2}\Big)$$

Their approach hinges on showing that the energy and the Morawetz identity hold for weak solutions, and thus are able to prove non-concentration of the energy of solutions. This identity was used originally to prove the existence of globally smooth solutions by Struwe [15], and Grillakis [3]. In the radial case, Ginibre, Soffer and Velo [1] have shown that the solutions are in the space defined by (1.4). The proof of (1.4) that we present here is a consequence of the decay estimate

$$g(t) = \frac{1}{6}\int_{\mathbb{R}^3} |u(t, x)|^6 \, dx \underset{|t|\to+\infty}{\longrightarrow} 0$$

which is obtained using the methods of Shatah and Struwe [9]. These decay estimates are used by Bahouri and Gerard [1] to prove scattering of solutions to the above equation with finite energy initial data.

2. STUDY OF THE FUNCTION g

LEMMA 2.1. – *Let u be the solution of the Cauchy problem (1.1), (1.2), (1.3), then*

$$(2.1) \qquad g(t) = \frac{1}{6}\int_{\mathbb{R}^3} |u(t, x)|^6 \, dx \underset{|t|\to+\infty}{\longrightarrow} 0$$

Proof. – For any $\varepsilon_0 > 0$ we have to show the existence of T_0 such that

$$(2.2) \qquad \forall t > T_0 \quad , \qquad |\, g(t) \,| \le \varepsilon_0 \, .$$

Since the initial data has finite energy, we have for R large enough

$$\int_{|x|\geq R} e(u)\,(0,\,x)\,dx \;\leq\; \frac{\varepsilon_0}{8} \tag{2.3}$$

where

$$e(u) \;=\; \frac{1}{2}\,(|u_t|^2+|\nabla_x u|^2) \;+\; \frac{1}{6}\,|u|^6 \tag{2.4}$$

denotes the energy density.

The classical energy-conservation law on the exterior of a truncated forward light cone (see Strauss [11]) implies that

$$\int_{|x|>R+t} e(u)\,dx + \frac{1}{\sqrt{2}}\;\text{flux}\;(0,t) \;\leq\; \frac{\varepsilon_0}{8} \tag{2.5}$$

where the flux on the mantle is given by

$$\text{flux}\;(a,b) \overset{\text{def}}{=} \int_{M_b^a} \left\{\frac{1}{2}\left|\frac{x}{|x|}u_t+\nabla u\right|^2+\frac{|u|^6}{6}\right\}d\sigma \tag{2.6}$$

$$M_a^b \overset{\text{def}}{=} \{(x,t)\in\mathbb{R}^3\times[a,b]\,;\quad |x|=R+t\} \tag{2.7}$$

Therefore to prove the lemma, it suffices to show the existence of T_0 such that

$$\forall t \;>\; T_0 \quad , \quad \frac{1}{6}\int_{|x|\leq R+t} |u(t,\,x)|^6\,dx \;\leq\; \frac{\varepsilon_0}{2}$$

and by translating time $t \longrightarrow t+R$ it is sufficient to prove

$$\forall t \;>\; T_0 \quad , \quad \frac{1}{6}\int_{|x|\leq t} |u(t,\,x)|^6\,dx \;\leq\; \frac{\varepsilon_0}{2}. \tag{2.8}$$

Proceeding exactly as in Shatah and Struwe [8], multiply equation (1.1) by $tu_t + x\cdot\nabla u + u$ to obtain the identity

$$\partial_t(tQ_0 \;+\; u_t u) - \;\text{div}\;(tP_0) \;+\; R_0 \;=\; 0 \tag{2.9}$$

where

$$Q_0 = e \;+\; u_t\Big(\frac{x}{t}\cdot\nabla u\Big),$$

$$P_o = \frac{x}{t}\left(\frac{|u_t|^2-|\nabla u|^2}{2} \;-\; \frac{|u|^6}{6}\right) \;+\; \nabla u\Big(u_t \;+\; \frac{x}{t}\;\cdot\;\nabla u \;+\; \frac{u}{t}\Big)\;,$$

$$R_0 \;=\; \frac{|u|^6}{3}$$

Integrating equation (2.9) over the truncated cone $K_{T_1}^{T_2} = \{z = (x, t) \big| |x| < t, T_1 \leq t \leq T_2\}$, where $0 < T_1 < T_2$, we obtain by Stokes formula

$$(2.10)\quad \int_{D(T_2)} (T_2 Q_0 + u_t u)\, dx - \int_{D(T_1)} (T_1 Q_0 + u_t u)\, dx \\ - \frac{1}{\sqrt{2}} \int_{M_{T_1}^{T_2}} \left(t Q_0 + u_t u + t P_0 \cdot \frac{x}{|x|} \right) d\sigma + \int_{K_{T_1}^{T_2}} \frac{|u|^6}{3} dx\, dt \\ = \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV} = 0$$

where

$$D(T_i) = \{x \in \mathbb{R}^3 \big| |x| \leq T_i\}$$

denotes space-like sections for $i \in \{1, 2\}$, and $M_{T_1}^{T_2}$ denotes the truncated mantle. On $M_{T_1}^{T_2}$, we have $|x| = t$, therefore we can rewrite the term III using spherical coordinates

$$\mathrm{III} = -\frac{1}{\sqrt{2}} \int_{M_{T_1}^{T_2}} \{r(u_t + u_r)^2 + u(u_t + u_r)\} d\sigma$$

Parametrizing $M_{T_1}^{T_2}$ via $y \to (|y|, y)$, and setting $v(y) = u(|y|, y)$ we find

$$\mathrm{III} = -\int_{T_1}^{T_2} \int_{S^2} r \left(v_r + \frac{v}{r} \right)^2 r^2\, dr\, d\omega \\ + \int_{T_1}^{T_2} \int_{S^2} \frac{1}{2} \left(r^2 v^2\right)_r dr\, d\omega$$

Integrating by parts, we obtain

$$(2.11)\quad \mathrm{III} = -\int_{T_1}^{T_2} \int_{S^2} r \left(v_r + \frac{v}{r} \right)^2 r^2\, dr\, d\omega \\ + \frac{1}{2} \int_{S^2} T_2^2\, v^2\, (T_2\, \omega)\, d\omega - \frac{1}{2} \int_{S^2} T_1^2 v^2\, (T_1\, \omega)\, d\omega$$

To estimate the first term we have

$$\mathrm{I} = \int_{D(T_2)} \left\{ T_2 \left(\frac{|u_t|^2}{2} + \frac{1}{2} \left(u_r + \frac{1}{r} u \right)^2 + \frac{1}{2r^2} |\nabla_\omega u|^2 \right.\right. \\ \left.\left. + \frac{1}{6} |u|^6 \right) + r \left(u_r + \frac{1}{r} u \right) u_t \right\} dx \\ - \frac{1}{2} \int_0^{T_2} \int_{S^2} T_2 \left(r u^2\right)_r dr\, d\omega$$

Integrating by parts in the last term of the expression for I yields

$$(2.12)\ \mathrm{I} = \int_{D(T_2)} \left\{ T_2 \left(\frac{|u_t|^2}{2} + \frac{1}{2}\left(u_r + \frac{1}{r}u\right)^2 + \frac{1}{2r^2} |\nabla_\omega u|^2 + \frac{1}{6}|u|^6 \right) + r\left(u_r + \frac{1}{r}u\right) u_t \right\} dx - \frac{1}{2}\int_{S^2} T_2^2 \ v^2 \ (T_2\omega)\, d\omega$$

In the same manner, the second term can be written

$$(2.13)\ \mathrm{II} = -\int_{D(T_1)} \left\{ T_1 \left(\frac{|u_t|^2}{2} + \frac{1}{2}\left(u_r + \frac{1}{r}u\right)^2 + \frac{1}{2r^2} |\nabla_\omega u|^2 + \frac{1}{6}|u|^6 \right) + r\left(u_r + \frac{1}{r}u\right) u_t \right\} dx + \frac{1}{2}\int_{S^2} T_1^2 \ v^2 \ (T_1\omega)\, d\omega$$

Let $T_2 = T > 0$, and $T_1 = \varepsilon T$ for some $0 < \varepsilon < 1$. Substituting equations (2.11), (2.12), and (.213) into equation (2.10), and using Hardy's inequality

$$(2.14)\qquad \int \frac{|u|^2}{|x|^2}\, dx \ \leq \ C \int |\nabla u|^2 dx$$

we deduce

$$(2.15)\ T \int_{D(T)} \frac{|u|^6}{6}\, dx \ \leq \ C\,\varepsilon\, T E \ + \ \int_{\varepsilon T}^{T} \int_{S^2} T\left(v_r + \frac{v}{r}\right)^2 r^2\, dr\, d\omega$$

where C is a constant and

$$E = \int_{\mathbb{R}^3} e(u)(t,\, x)\, dx$$

denotes the energy.

Dividing by T, we obtain

$$\int_{D(T)} \frac{|u|^6}{6}\, dx \ \leq \ C\,\varepsilon\, E \ + \ \int_{\varepsilon T}^{\infty} \int_{S^2} \left(v_r + \frac{v}{r}\right)^2 r^2\, dr\, d\omega$$

Choose ε such that

$$C\,\varepsilon\, E \ = \ \frac{\varepsilon_0}{4}$$

From Hardy's inequality and the energy inequality (2.5) there exists a T_0 such that

$$\int_{\varepsilon T_0}^{\infty} \int_{S^2} \left(v_r + \frac{v}{r}\right)^2 r^2\, dr\, d\omega \ \leq \ 2\,\mathrm{flux}\ (\varepsilon T_0, \infty) \ < \ \frac{\varepsilon_0}{4}.$$

This proves inequality (2.8) which implies the L^6 norm decay of solutions.

3. STATEMENT AND PROOF OF THE THEOREM

THEOREM 3.1. – *The Cauchy problem (1.1), (1.2), (1.3) has a unique global solution* u *in the space*

$$(u, u_t) \in C\big(\mathbb{R}, H^1 \times L^2\big) \cap L^4\Big(\mathbb{R}, \dot{B}_4^{1/2} \times \dot{B}_4^{-1/2}\Big)$$

Proof. – We have to show that $u \in L^4\Big([T_0, \infty[, \dot{B}_4^{1/2}\Big)$, for some T_0. Fix $\varepsilon_0 > 0$ sufficiently small and choose T_0 such that (2.2) is satisfied.

Following the proof of the proposition 1.4 of Shatah and Struwe [8], we find, for $T > T_0$

$$(3.1) \qquad \|u\|_{L^4\left([T_0, T];\, \dot{B}_4^{1/2}(\mathbb{R}^3)\right)} \leq C\Big\{E^{1/2} + \|u\|^3_{L^4\left([T_0, T];\, \dot{B}_4^{1/2}(\mathbb{R}^3)\right)} \sup_{T_0 \leq t \leq T} \|u(t, \cdot)\|^2_{L^6(\mathbb{R}^3)}\Big\}$$

where E denotes the energy and C is a constant independent of T. Using the L^6 decay of solutions we obtain for arbitrary small ε_0

$$(3.2) \quad \|u\|_{L^4\left([T_0, T];\, \dot{B}_4^{1/2}(\mathbb{R}^3)\right)} \leq CE^{1/2} + \varepsilon_0^{1/3} \|u\|^3_{L^4\left([T_0, T];\, \dot{B}_4^{1/2}(\mathbb{R}^3)\right)}$$

Choose ε_0 sufficiently small, then the above inequality implies

$$(3.3) \qquad \|u\|_{L^4\left([T_0, T];\, \dot{B}_4^{1/2}(\mathbb{R}^3)\right)} \leq 2C\, E^{1/2}$$

for all $T > T_0$, and letting $T \to \infty$ finishes the proof of the theorem.

Remark 3.1. – The same result holds in n dimensions. The proof is identical and uses the n dimensional result of Shatah and Struwe [9].

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