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Asymptotic behaviour of solutions of some semilinear parabolic problems

by

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ABSTRACT. – We consider the Cauchy problem:

$$u_t - \Delta u + u^p = 0 \quad \text{for } x \in I\!R^N, \quad t > 0, \tag{0.1}$$

$$u(x,0) = u_o(x) \quad \text{for } x \in I\!R^N. \tag{0.2}$$

Here $p > 1$, N$\geq$ 1 and $u_o(x)$ is a continuous, nonnegative and bounded function such that:

$$u_o(x) \sim A|x|^{-\alpha}, \text{ as } |x| \to \infty, \tag{0.3}$$

for some $A > 0$ and $\alpha > 0$. In this paper we discuss the asymptotic behaviour of solutions to (0.1)-(0.3) in terms of the various values of the parameters p, N, α and A. A common pattern that emerges from our analysis is the existence of an external zone where $u(x,t) \sim u_o(x)$ and one (or several) internal regions, where the influence of diffusion and absorption is most strongly felt. We present a complete classification of the size of these regions, as well as that of the stabilization profiles that unfold therein, in terms of the aforementioned parameters.

Key words: Semilinear heat equations, absorption, self-similarity, asymptotic behaviour, matched asymptotic expansions, stabilization profiles.

RÉSUMÉ. – Nous considérons le problème de Cauchy:

$$u_t - \Delta u + u^p = 0 \quad \text{pour } x \in I\!R^N, \quad t > 0. \tag{0.1}$$

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$$u(x,0) = u_o(x) \quad \text{pour } x \in I\!R^N. \tag{0.2}$$

Où $p > 1$, N ≥ 1 et $u_o(x)$ est une fonction continue, nonnégative et bornée telle que :

$$u_o(x) \sim A|x|^{-\alpha}, \text{ quand } |x| \to \infty, \tag{0.3}$$

pour $A > 0$ et $\alpha > 0$. Dans cette note nous étudions le comportement asymptotique des solutions de (0.1)-(0.3) dépendant des valeurs des paramètres p, N, α et A. Nous prouvons l'existence d'une zone extérieure où $u(x,t) \sim u_o(x)$ et une (ou plusieurs) régions intérieures où l'influence de la diffusion et l'absorption est très forte. Nous présentons une classification complète de l'étendue de ces régions et des profils des solutions dépendant des paramètres.

1. INTRODUCTION AND DESCRIPTION OF RESULTS

Let $p > 1, N \geq 1$, and consider the following initial value problem:

$$u_t - \Delta u + u^p = 0 \quad \text{for } x \in R^N, \quad t > 0, \tag{1.1}$$

$$u(x,0) = u_o(x) \quad \text{for } x \in R^N, \tag{1.2}$$

where $u_o(x)$ is continuous, nonnegative and bounded, and:

$$u_o(x) \sim A|x|^{-\alpha} \text{ as } |x| \to \infty, \tag{1.3}$$

for some positive constants A and α. The existence of a unique, classical and global solution of (1.1)-(1.3) follows at once from standard theory for parabolic equations (see for instance [8]). The goal of this paper consists in discussing the asymptotic behaviour of such solutions in terms of the parameters N, p, α, and A. This is a basic question in the theory of semilinear parabolic equations, and as such it has deserved considerable attention (cf. for instance [2], [3], [5], [7], [12], [14], [15],...). To illustrate our approach, it will be convenient to describe at once one of our results. Let λ be a positive number such that $0 < \lambda < 1$, and consider the following boundary value problem:

$$\zeta''(r) + \left(\frac{N-1}{r} + \frac{r}{2}\right)\zeta'(r) = (1-\lambda)\zeta(r), \qquad \text{for } r > 0 \tag{1.4}$$

$$\zeta(r) \text{ bounded at } r = 0, \quad \zeta(r) \sim r^{2(1-\lambda)}, \quad \text{for } r \to \infty. \tag{1.5}$$

For any such λ a unique solution of (1.4)-(1.5) exists. As a matter of fact, such solution can be represented in terms of Kummer's hypergeometric functions (cf. Section 3 below for related results). Let us examine now the situation when:

$$\alpha < \frac{2}{p-1}. \tag{1.6}$$

Then the following result holds:

THEOREM 1. – *Let $u(x,t)$ be the solution of (1.1)-(1.3), and assume that (1.6) is satisfied. Then for sufficiently large times t we have that:*

$$u(x,t) = \begin{cases} t^{-\frac{1}{p-1}}\left(c_* - kt^{-\lambda}\zeta\left(\frac{|x|}{\sqrt{t}}\right)\right)(1+o(1)) & \text{for } |x| \le f(t), \\ ((p-1)t + A^{1-p}|x|^{\alpha(p-1)})^{-\frac{1}{p-1}}(1+o(1)) & \text{for } f(t) \le |x| \le g(t), \\ A|x|^{-\alpha}(1+o(1)) & \text{for } g(t) \le |x|, \end{cases} \tag{1.7}$$

where $f(t)$ and $g(t)$ are arbitrary functions such that, for $t \gg 1$:

$$t^{\frac{1}{2}} \ll f(t) \ll t^{\beta} \text{ with } \beta = \frac{1}{\alpha(p-1)}; \quad t^{\beta} \ll g(t), \tag{1.8}$$

and:

$\lambda = 1 - \frac{1}{2\beta}$, zeta$(r)$ is the solution of (1.4)-(1.5) corresponding to the previous value of λ, $c_* = (p-1)^{-\frac{1}{p-1}}$ and $k = A^{-(p-1)}$ $(p-1)^{-\frac{1}{p-1}-2}$. (1.9)

Here and henceforth we shall freely use the customary asymptotic notations: $\ll, \sim, O(\cdot)$, etc. Let us remark briefly on Theorem 1. To begin with, we point out the existence of three asymptotic regions: an external one, where the initial value remains virtually unchanged; an internal region, where the effect of the nonlinearity is strongly felt, and a transition zone, where the influence of the nonlinear sink gradually fades away, and changes in the initial daa are less and less felt. As a matter of fact, this highlights a common pattern in our discussion to follow. We shall look first for an external region where the initial value experiences but small variations, and then look for an internal one, where diffusion and/or absorption prevail in determining the stabilization profile, that has to match asymptotically with the initial value when the external region is reached. The influence of the absorption term is most dramatically seen in Theorem 1 above, but it will be apparent (both in determining the asymptotics and the size of the internal region) in the results to follow.

Concerning our assumptions here, (1.3) is certainly stronger that some of the hypotheses previously made in the literature (cf. for instance [12]). Our analysis yields, however, global asymptotic expansions, and allows for precise determination of the transition regions. This is in strong contrast with most of the previously known results, that are shown to hold in parabolic regions of the type $|x| \leq Ct^{\frac{1}{2}}$ only (as was the case in [12]). This fact is related to the type of techniques commonly used in the literature. In general, authors proceed by considering a family of rescaled solutions $u_k(x,t) = k^{\theta}u(\sqrt{k}x, kt)$ for some real θ, and then by analyzing the equation satisfied by a limit function $w(x,t) = \lim_{k\to\infty} u_k(x,t)$, whose existence is shown as a part of the analysis to be done. We shall depart from that approach here, to make use of a blend of matched asymptotic techniques, integral estimates and comparison methods to derive our results. As a matter of fact, it follows that parabolic regions of size $|x| = O(t^{\frac{1}{2}})$ are not the optimal ones when stabilization to the inner profiles is considered. Actually, this fact can be checked on the linear heat equation, as will be shown in Section 3 below.

To the best of our knowledge, the only case where global asymptotics are known is when self-similarity plays an essential role in the stabilization process. For $\mu > 0$, let us denote by $g_{\mu}(s)$ the solution of the problem:

$$g''(s) + \left(\frac{N-1}{s} + \frac{s}{2}\right)g'(s) + \frac{1}{p-1}g(s) - g(s)^p = 0, \text{ for } s > 0, \quad (1.10)$$

$$g(0) = \mu, \qquad g'(0) = 0. \quad (1.11)$$

We remark on pass that existence and uniqueness for this and other related ODE problems in the sequel will be recalled later where appropriate. For instance, concerning (1.10)-(1.11), an extensive analysis of solutions can be found in [7] and [5]. Let us write now:

$$w_{\mu}(x,t) = t^{-\frac{1}{p-1}} g_{\mu}(|x|/\sqrt{t}). \quad (1.12)$$

Then there holds:

THEOREM 2. – *Assume that $\alpha = \frac{2}{p-1}$, and let $u(x,t)$ be the solution of (1.1)-(1.3). Let $w_{\mu}(x,t)$ be the function given in (1.12). Then there exists a* *unique* $\mu = \mu(A)$ *such that:*

$$u(x,t) = w_\mu(x,t)(1+o(1)) \text{ as } t \to \infty \text{, uniformly for } x \in I\!R^N. \quad (1.13)$$

Theorem 2 can be considered as a refinement of Proposition 1 in [7], where it has been proved that:

$$\lim_{t\to\infty} t^{\frac{1}{p-1}}\big(\sup_{x\in I\!R^N} |u(x.t) - w_\mu(x,t)|\big) = 0,$$

provided that $1 < p < 1 + 2/N$. Notice that the space dimension N plays no role in the asymptotics described in our previous Theorems 1 and 2, which cover together the situation corresponding to $\alpha \leq \frac{2}{p-1}$.

When $\alpha > \frac{2}{p-1}$, the space dimension does have an impact in the asymptotics. Self-similarity still prevails when $\frac{2}{p-1} < \alpha < N$, as described in our next result.

THEOREM 3. – *Let* $u(x,t)$ *be the solution of (1.1)-(1.3), and suppose that* $\frac{2}{p-1} < \alpha < N$. *One then has that:*

$$u(x,t) = t^{-\frac{\alpha}{2}} g(|x|/\sqrt{t})(1+o(1)), \text{ as } t \to \infty, \text{ uniformly for } x \in I\!R^N, \quad (1.14)$$

where $g(s)$ *is the unique solution of:*

$$\begin{cases} g''(s) + \left(\dfrac{N-1}{s} + \dfrac{s}{2}\right) g'(s) + \dfrac{\alpha}{2} g(s) = 0 \text{ for } s > 0, \\ g'(0) = 0, \quad g(s) \sim A s^{-\alpha} \quad \text{when } s \to \infty. \end{cases} \quad (1.15)$$

Convergence towards the linear self-similar profile $t^{-\alpha} g(|x|/\sqrt{t})$ has been shown to hold in regions $|x| \leq C t^{\frac{1}{2}}$ in [14]. Notice that in the cases considered in Theorems 2 and 3 above, $u(x,t) \sim A|x|^{-\alpha}$ whenever $|x| \gg \sqrt{t} \gg 1$ (when $\alpha = \frac{2}{p-1}$, this is a consequence of the properties of solutions of (1.10)-(1.11) to be recalled later).

When $\frac{2}{p-1} < \alpha = N$, self-similarity is no longer relevant to describe the asymptotics. This last is now characterized in the following:

THEOREM 4. – *Let* $u(x,t)$ *be the solution of (1.1)-(1.3), and assume that* $\frac{2}{p-1} < \alpha = N$. *One then has that, as* $t \to \infty$:

$$u(x,t) = \begin{cases} \dfrac{A\omega_N}{2} \log t (4\pi t)^{-\frac{N}{2}} e^{-\frac{|x|^2}{4t}} (1+o(1)) \\ \quad when\ |x|^2 \leq Ct \log(\log t)\ with\ C < 4, \\ A|x|^{-\alpha}(1+o(1)) \\ \quad when\ |x|^2 \geq Ct \log(\log t)\ with\ C > 4, \end{cases} \quad (1.16)$$

where ω_N denotes the surface of the unit $(N-1)$-dimensional sphere in $I\!R^N$.

The case $\alpha = N$ examined above has been considered in [15], where porous-media equations with absorption are studied, and their asymptotics are obtained in regions that correspond in our case to the parabolic scale, $|x| \leq Ct^{\frac{1}{2}}, t \gg 1$. It is to be noticed that the behaviour described in Theorem 4 agrees with that of the solutions of the linear heat equation with same initial value (cf. Lemmata 3.2, and 3.3 in Section 3).

When α is larger than N and $\frac{2}{p-1}$, three different situations arise. The first of these is analyzed in the following:

THEOREM 5. – *Let $u(x,t)$ be the solution of (1.1)-(1.3), and assume that $\frac{2}{p-1} < N < \alpha$. One then has that:*

$$u(x,t) = \begin{cases} C_o(4\pi t)^{-\frac{N}{2}} e^{-\frac{|x|^2}{4t}}(1+o(1)) \\ \quad when\ \ |x|^2 \leq Ct\log t\ with\ C < 2(\alpha - N), \\ A|x|^{-\alpha}(1+o(1)) \\ \quad when\ \ |x|^2 \geq Ct\log t\ with\ C > 2(\alpha - N), \end{cases} \tag{1.17}$$

where:

$$C_o = \int_{I\!R^N} u_o(x)dx - \int_0^\infty \int_{I\!R^N} u^p(y,s)dyds > 0. \tag{1.18}$$

Concerning this result, it has been proved in [12] that $(u(x,t) - C_o(4\pi t)^{-\frac{N}{2}} e^{-\frac{|x|^2}{4t}})$ converges to zero uniformly on sets $|x| \leq C\sqrt{t}$ with $C > 0$ and $t \gg 1$. It was also shown there that when $\frac{2}{p-1} = N < \alpha$, constant C_o in (1.18) turns out to be zero, so that (1.17) no longer holds. The corresponding result reads now as follows:

THEOREM 6. – *Let $u(x,t)$ be the solution of (1.1)-(1.3), and suppose that $\frac{2}{p-1} = N < \alpha$. Then there holds :*

$$u(x,t) = \begin{cases} C_N(t\log t)^{-\frac{N}{2}} e^{-\frac{|x|^2}{4t}}(1+o(1)) \\ \quad when |x|^2 \leq Ct\log t\ with\ C < 2(\alpha - N), \\ A|x|^{-\alpha}(1+o(1)) \\ \quad when |x|^2 \geq Ct\log t\ with\ C > 2(\alpha - N), \end{cases} \tag{1.19}$$

where:

$$C_N = \left(\frac{N}{2}\left(1+\frac{2}{N}\right)^{\frac{N}{2}}\right)^{\frac{N}{2}}. \tag{1.20}$$

We should mention here that the convergence estimate $u(x,t) \sim C_N(t\log t)^{-\frac{N}{2}}e^{-\frac{|x|^2}{4t}}$ as $t \to \infty$, has been suggested in [10] for a particular case of rapidly decaying data such that $u_o(x) = o(e^{-\gamma|x|^2})$, when $|x| \to \infty$ for some $\gamma > 0$.

The reader will notice that the transition between internal and external regions in Theorems 5 and 6 takes place in both cases at distances $|x|^2 \sim C_L t\log t$ with $C_L = 2(\alpha - N)$, as it happens for the linear heat equation with same initial value (cf. Lemmata 3.2 and 3.3 in Section 3). The influence of the absorption term is displayed in the different asymptotic profiles exhibited by solutions at the corresponding internal regions. In the remaining case to be considered ($N < \frac{2}{p-1} < \alpha$), nonlinearity also modifies the value of the constant C_L above. More precisely, there holds:

THEOREM 7. – *Let $u(x,t)$ be the solution of (1.1)-(1.3), and assume that $N < \frac{2}{p-1} < \alpha$. One then has that:*

$$u(x,t) = \begin{cases} t^{-\frac{1}{p-1}} g^*\left(\frac{|x|}{\sqrt{t}}\right)(1+o(1)) \\ \quad \text{when } |x|^2 \leq Ct\log t \text{ with } C < 2\left(\alpha - \frac{2}{p-1}\right), \\ A|x|^{-\alpha}(1+o(1)) \\ \quad \text{when } |x|^2 \geq Ct\log t \text{ with } C > 2\left(\alpha - \frac{2}{p-1}\right), \end{cases} \tag{1.21}$$

where $g^*(s)$ solves:

$$\begin{cases} g''(s) + \left(\frac{N-1}{s} + \frac{s}{2}\right)g'(s) + \frac{1}{p-1}g(s) - g(s)^p = 0 \text{ for } s > 0, \\ g'(0) = 0, \quad g(s) \sim Ds^{\frac{2}{p-1}-N}e^{-\frac{s^2}{4}} \quad \text{when } s \to \infty, \end{cases} \tag{1.22}$$

for a positive constant D which is uniquely determined by N and p.

We point out that existence of solutions satisfying the problem (1.22) has been obtained in [5] (cf. Section 6 there).

It will be apparent from the forthcoming discussion that our approach allows for a number of extensions. For instance the case when $u_o(x)$ has asymptotic behaviour as $|x| \to \infty$ different from (1.3) can be dealt with by means of suitable modifications in the arguments to follow. On the other hand, higher-order expansions for solutions could also be obtained. Moreover, the regularity assumptions in (1.2) can be largely relaxed. However, to keep this work within reasonable bounds, we have refrained from detailing these possibilities here.

We conclude this Introduction by describing the plan of the paper. The results stated in Theorems 1 to 7 will be formally obtained in Section 2 by means of asymptotic techniques. Section 3 contains a number of preliminary results that are required to provide a rigorous proof of the above results. We shall gather therein a number of (rather classical) asymptotic results for the linear heat equation, as well as some existence results for ODE problems that play a role in the sequel. Theorem 1 will then be proved in Section 4, whereas the proofs of Theorems 2 and 3 are to be found in Section 5. Section 6 will be devoted to the proofs of Theorems 4, 5 and 6. A final Section 7 contains the proof of Theorem 7.

2. A FORMAL DERIVATION OF RESULTS

In this Section we show how to obtain the results described at the Introduction by means of asymptotic analysis. To begin with, we recall that there is a natural upper bound for solutions of our problem, namely:

$$u(x,t) \leq ((p-1)t)^{-\frac{1}{p-1}}, \tag{2.1}$$

which follows after integrating the equation obtained by dropping the laplacian operator in (1.1). It is then customary to introduce self-similar variables as follows:

$$u(x,t) = t^{-\frac{1}{p-1}}\Phi(y,\tau); \quad y = xt^{-\frac{1}{2}}, \quad \tau = \log t. \tag{2.2}$$

Equation (1.1) can then be recast in the form:

$$\Phi_\tau = \Delta\Phi + \frac{1}{2}y\nabla\Phi + \frac{1}{p-1}\Phi - \Phi^p \equiv A_o\Phi + \frac{1}{p-1}\Phi - \Phi^p. \tag{2.3}$$

We may then expect that, as $\tau \to \infty$, solutions of (2.3) will converge towards a global, bounded and nonnegative stationary solution of the same equation. If we discard for the moment nonconstant selfsimilar asymptotic limits, we then expect either:

$$\Phi(y,\tau) \to c_* \equiv (p-1)^{-\frac{1}{p-1}} \quad \text{as } \tau \to \infty \tag{2.4}$$

or

$$\Phi(y,\tau) \to 0 \quad \text{as } \tau \to \infty. \tag{2.5}$$

It is to be noticed, however, that to the best of our knowledge such asymptotic results do not follow from standard theory (as applied, for instance, in [9] for a related semilinear equation), due to the sign in the coefficient $y\nabla\Phi$ in (2.3). As a matter of fact, we shall show herein that those behaviours actually hold for the case of data $u_o(x)$ as in (1.3). A word of caution is also needed concerning the sense in which convergence is to be understood in (2.4) and (2.5). Actually, the precise meaning of such statements is described in detail in Theorems 1-7 at the Introduction, but we should mention here that, even at a formal level, we cannot expect (2.4) and (2.5) to hold uniformly on R^N. Indeed, these behaviours have to be compatible with the fact that, for $y = xt^{-\frac{1}{2}} \gg 1$, solutions should remain close to their initial values, i.e.

$$u(x,t) \sim A|x|^{-\alpha} \text{ as } t \to \infty \text{ for sufficiently large } y = xt^{-\frac{1}{2}}. \tag{2.6}$$

As a matter of fact, solutions of (1.1)-(1.3) will develop various regions where different asymptotic behaviours (matching (2.4) or (2.5) with (2.6)) will appear as $t \to \infty$. We shall see now that (2.4) corresponds to the case when:

$$\alpha < \frac{2}{p-1},$$

regardless of the value of the space dimension N. To this end, we argue as follows. Assume that (2.4) holds, and let us linearize around constant c_* by setting:

$$\Phi(y,\tau) = c_* + \Psi(y,\tau), \tag{2.7}$$

so that Ψ satisfies:

$$\Psi_\tau = \Delta\Psi + \frac{1}{2}\nabla\Psi - \Psi + F(\Psi), \tag{2.8}$$

where $F(\Psi) = O(\Psi^2)$ as $\Psi \to 0$. We neglect for the moment the nonlinear term $F(\Psi)$ in (2.8), and look for radial solutions of the linear equation thus obtained which are of the form:

$$\Psi(y,\tau) = ke^{-\lambda\tau}\zeta(|y|). \tag{2.9}$$

From (2.8) and (2.9) we obtain the eigenvalue equation:

$$\zeta''(s) + \left(\frac{N-1}{s} + \frac{s}{2}\right)\zeta'(s) = (1-\lambda)\zeta(s) \quad \text{for } s = |y| > 0. \tag{2.10}$$

We consider (2.10) together with boundary conditions:

$$\zeta(s) \text{ bounded at } s = 0, \quad \zeta(s) \sim s^{2(1-\lambda)} \text{ as } s \to \infty. \tag{2.11}$$

For $0 < \lambda < 1$, problem (2.10)-(2.11) has a unique classical solution that can be written in terms of hypergeometric functions (cf. the proof of Lemma 3.3 in the next Section). Note that the behaviour at infinity of solutions of (2.10) can be derived from standard asymptotic methods (cf. [4]). From (2.4),(2.6) and (2.9)-(2.11), we would have an expansion of the following type:

$$\Phi(y,\tau) \sim c_* + ke^{-\lambda\tau}|y|^{2(1-\lambda)} \quad \text{as } |y| \to \infty. \tag{2.12}$$

Notice that constants k and λ remain undetermined as yet (except for the condition $0 < \lambda < 1$, that is required to solve (2.10)-(2.11)). Expansion (a12) cannot be expected to hold uniformly on $|y|$ and τ. As a matter of fact, the validity of (2.12) should break down when both terms on its right become of the same order, i.e, at distances $ke^{-\lambda\tau}|y|^{2(1-\lambda)} = O(1)$. This motivates the introduction of an intermediate variable:

$$\xi = ye^{-\frac{\lambda\tau}{2(1-\lambda)}}.$$

In the new variables, (2.3) reads:

$$\Phi_\tau = \frac{\xi\nabla\Phi}{2(1-\lambda)} + \frac{1}{p-1}\Phi - \Phi^p + e^{-\Gamma\tau}\Delta\Phi, \tag{2.13}$$

for some $\Gamma > 0$, where operators ∇ and Δ are now written with respect to the new variable ξ. When radial solutions are considered, it is natural to expect that $\Phi_\tau = o(1)$ and $e^{-\Gamma\tau}\Delta\Phi = o(1)$ as $\tau \to \infty$, and therefore (2.13) reduces asymptotically to:

$$\frac{\xi \cdot \Phi'(\xi)}{2(1-\lambda)} + \frac{1}{p-1}\Phi - \Phi^p = 0,$$

a Bernoulli equation, whose general solution is given by:

$$\Phi(\xi) = ((p-1) + C\xi^{2(1-\lambda)})^{-\frac{1}{p-1}}, \tag{2.14}$$

C being an arbitrary constant. In view of (2.6), (2.12) and (2.14), we are led to guessing the following asymptotic behaviour for the solution $u(x,t)$ under consideration when $t \gg 1$:

$$u(x,t) \sim \begin{cases} t^{-\frac{1}{p-1}}(c_* + ke^{-\lambda\tau}\zeta(|y|) & \text{for } \xi \ll 1, \\ t^{-\frac{1}{p-1}}((p-1) + C\xi^{2(1-\lambda)})^{-\frac{1}{p-1}} & \text{for } \xi = O(1), \\ A|x|^{-\alpha} & \text{for } \xi \gg 1 \end{cases} \tag{2.15}$$

Constants k, λ and C in (2.15) are now determined by matching considerations. For instance, on imposing that:

$$t^{-\frac{1}{p-1}}((p-1)+C\xi^{2(1-\lambda)})^{-\frac{1}{p-1}} \sim A|x|^{-\alpha} \quad \text{for } \xi \gg 1,$$

we obtain:

$$C = A^{-(p-1)}, \quad \lambda = 1 - \frac{1}{2\beta} \quad \text{with } \beta = \frac{1}{\alpha(p-1)}. \tag{2.16}$$

Notice that we need $\alpha < \frac{2}{p-1}$ in order to have $0 < \lambda < 1$ in (2.16). Once this condition has been set, we proceed to match the first and second regions in (2.15). On assuming that for $y \gg 1$ and $\xi \gg 1$:

$$t^{-\frac{1}{p-1}}(c_* + ke^{-\lambda\tau}\zeta(|y|)) \sim t^{-\frac{1}{p-1}}((p-1)+C\xi^{2(1-\lambda)})^{-\frac{1}{p-1}},$$

we arrive at:

$$k = -C(p-1)^{-\frac{1}{p-1}-2}. \tag{2.17}$$

Summing up (2.15)-(2.17), the behaviour described in Theorem 1 has been obtained.

Consider now the case when (2.5) holds. A crucial role in the corresponding analysis is then played by the spectral properties of operator A_0 in (2.3). Let us define the function spaces:

$$L^2_\omega(\mathbb{R}^N) = \left\{ f \in L^2_{loc}(\mathbb{R}^N) : \int_{\mathbb{R}^N} |f(s)|^2 e^{\frac{|s|^2}{4}} ds < \infty \right\},$$

$$H^k_\omega(\mathbb{R}^N) = \left\{ f \in L^2_{loc}(\mathbb{R}^N) : \frac{\partial^\alpha f}{\partial x^\alpha} \in L^2_\omega(\mathbb{R}^N) \text{ for } |\alpha| \le k \right\},$$

where k is a positive integer, and for $\alpha = (\alpha_1, ...\alpha_N)$, $|\alpha| = \alpha_1 + ... + \alpha_N$, and $\frac{\partial^\alpha f}{\partial x^\alpha} = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} ... \partial x_N^{\alpha_N}}$. Clearly, $L^2_\omega(\mathbb{R}^N)$ (resp. $H^2_\omega(\mathbb{R}^N)$) is a Hilbert space when endowed with the norm:

$$||f||_2^2 = \int_{\mathbb{R}^N} |f(s)|^2 e^{\frac{|s|^2}{4}} ds \equiv < f, f >,$$

(resp. $||f||_{k,2}^2 = ||f||_2^2 + \sum_{|\alpha|=1}^{|k|} ||\frac{\partial^\alpha f}{\partial x^\alpha}||_2^2$). Operator A_0 is selfadjoint in $L^2_\omega(\mathbb{R}^N)$, with domain $D(A_0) = H^2_\omega(\mathbb{R}^N)$. Its spectrum consists of the eigenvalues $\lambda_k = -(\frac{N}{2} + \frac{|k|}{2})$, and the corresponding eigenfunctions are given by:

$$\psi_k(y) = (H_{k_1}(y_1) \cdots H_{k_N}(y_N))e^{-\frac{|y|^2}{4}},$$

where $k = (k_1, .., k_N)$, $|k| = k_1 + ... + k_N$, $y = (y_1, .., y_N)$, and $H_n(x) = c_n \mathcal{H}_n(\frac{x}{2})$, $\mathcal{H}_n(y)$ being the standard n^{th}-Hermite polynomial. The normalization constant c_n is selected so as to have $||\psi_k||_2 = 1$ (i.e, $c_n = (2^{\frac{N}{2}}(4\pi)^{\frac{1}{4}}(n!)^{\frac{1}{2}})^{-1}$). The reader is referred to [6], [13] and [16] for details about these results. It is then natural to look for solutions of (2.3) in the form:

$$\Phi(y, \tau) = \sum_{k=0}^{\infty} a_k(\tau)\psi_k(y). \tag{2.18}$$

Plugging (2.18) into (2.3), we readily see that the k^{th}- Fourier coefficient $a_k(\tau)$ satisfies:

$$\dot{a}_k = \left(\lambda_k + \frac{1}{p-1}\right)a_k - < \Phi^p, \psi_k > .$$

We shall focus our attention in the equation satisfied by the first such mode, namely:

$$\dot{a}_o = \left(-\frac{N}{2} + \frac{1}{p-1}\right)a_o - < \Phi^p, \psi_o > . \tag{2.19}$$

Several situations may now appear. Suppose first that:

$$N > \frac{2}{p-1}. \tag{2.20}$$

If we discard the nonlinear interaction term on the right of (2.19), a linear equation for $a_o(\tau)$ is obtained that yields:

$$a_o(\tau) \sim e^{(\frac{1}{p-1} - \frac{N}{2})\tau} \text{ for } \tau \gg 1.$$

Assume now that the evolution of $u(x, t)$ is driven by its first mode as $\tau \to \infty$, i.e, suppose that $u(x, t) \sim t^{-\frac{1}{p-1}} a_o(\tau)\psi_o(y)$ for $\tau \gg 1$ in a suitable inner region. In our current case, we should then have that:

$$u(x, t) \sim Ct^{-\frac{N}{2}} e^{-\frac{|x|^2}{4t}} \quad \text{for } t \gg 1 \text{ in such inner region.} \tag{2.21}$$

Since we also expect (2.6) to hold, there must be a transition zone where (2.21) and (2.6) become of the same order, so that one has there that:

$$Ct^{-\frac{N}{2}}e^{-\frac{|x|^2}{4t}} \sim A|x|^{-\alpha},$$

whence:

$$\log C - \frac{N}{2}\log t - \frac{|x|^2}{4t} \sim \log A - \alpha \log |x|. \tag{2.22}$$

Assume now that in such a region:

$$\log |x| = \frac{1}{2}\log t + \text{ (lower order terms)} \qquad \text{for } t \gg 1. \tag{2.23}$$

Then (2.22) gives:

$$|x|^2 \sim 2(\alpha - N)t\log t \qquad \text{if } \alpha > N. \tag{2.24}$$

Note that (2.6), (2.21) and (2.24) provide the asymptotics described in Theorem 5 – except for the precise value of constant C in (2.21), that cannot be determined from local analysis only.

If (2.20) continues to hold, but $\alpha = N$, condition (2.24) does not make sense anymore. We then expect that the presence of a comparatively large initial value will trigger a resonance effect. More precisely, on setting $\gamma = \frac{1}{p-1} - \frac{N}{2}$, we guess that (2.19) can be written in the form:

$$\dot{a}_o = \gamma a_o + g(\tau), \tag{2.25}$$

where $g(\tau)$ will be of same order as the solution of the homogeneous equation associated to (2.25), i.e, $g(\tau) \sim Ce^{\gamma\tau}$, for $\tau \gg 1$. We then would have that:

$$a_o(\tau) \sim e^{\gamma\tau}\int_o^\tau e^{-\gamma s}g(s)ds \sim C\tau e^{\gamma\tau} \qquad \text{for } \tau \gg 1.$$

That amounts to an inner expansion of the type:

$$u(x,t) \sim Ct^{-\frac{N}{2}}\log t \cdot e^{-\frac{|x|^2}{4t}}. \tag{2.26}$$

Assuming that (2.23) continues to hold, matching (2.26) with (2.6) gives:

$$|x|^2 \sim 4t\log(\log t) \qquad \text{for } t \gg 1$$

(compare with Theorem 4 in the Introduction). A slower rate of decay at infinity of the initial value $u_o(x)$ leads us now to examine the case where

$2/(p-1) \leq \alpha < N$. The asymptotics is then self-similar, as described in Theorems 2 and 3 at the Introduction.

Suppose now that:

$$N = \frac{2}{p-1}. \tag{2.27}$$

Then the nonlinear term becomes crucial in (2.19). Assuming that $< \Phi^p, \psi_o > \sim a_o(\tau)^p < \psi^p, \psi > \equiv D_o a_o(\tau)^p$, for $\tau \gg 1$ and some explicit positive constant D_o, one obtains after integration:

$$a_o(\tau) \sim ((p-1)D_o\tau)^{-\frac{1}{p-1}} \qquad \text{for } \tau \gg 1,$$

whence:

$u(x,t) \sim C_N(t\log t)^{-\frac{N}{2}} e^{-\frac{|x|^2}{4t}}$ at some inner region, for C_N as in (1.20).

Recalling (2.23), matching with (2.6) yields a transition region given by:

$$|x|^2 \sim 2(\alpha - N)t\log t \qquad \text{for } t \gg 1,$$

provided that $\alpha > N$ (cf. Theorem 6). The only case that has no been considered as yet corresponds to the range of parameters:

$$N < \frac{2}{p-1} < \alpha.$$

Here again self-similar solutions will play a role in describing the asymptotics. Namely, we will have that, at the corresponding inner region:

$$u(x,t) \sim t^{-\frac{1}{p-1}} g(|x|/\sqrt{t}) \qquad \text{for } t \gg 1,$$

where $g(s)$ is a solution of the ODE (1.10)-(1.11) satisfying $g'(0) = 0$. Matching with (2.6) now requires to select a quickly decaying behaviour for $g(s)$ as $s \to \infty$, namely, that indicated in (1.22). This amounts to take $g(s)$ as being a fast orbit, in the terminology of [5]. Keeping to assumption (2.23), transition to the external profile (2.6) is now seen to occur at distances:

$$|x|^2 \sim 2\left(\alpha - \frac{2}{p-1}\right)t\log t \qquad \text{for } t \gg 1, \tag{2.28}$$

as indicated in Theorem 7. Notice that the constant in (2.28) is different from that one in (2.24), and is therefore not merely prescribed from linear considerations.

3. PRELIMINARY RESULTS

In this Section we shall gather a number of auxiliary facts on the asymptotics of solutions of the linear heat equation. These are of a rather classical nature, but we have found it convenient to state (and prove) them here for later reference.

Consider the Cauchy problem:

$$v_t = \Delta v \qquad \text{for } x \in I\!R^N, \quad t > 0, \tag{3.1}$$

$$v(x,0) = v_o(x) \qquad \text{for } x \in I\!R^N, \tag{3.2}$$

where $v_o(x)$ is a continuous, nonnegative and bounded function satisfying (1.3). Then there holds:

LEMMA 3.1. – *Let $v(x,t)$ be the solution of (3.1)-(3.2). One then has that:*

$$v(x,t) \sim A|x|^{-\alpha} \textit{ as } t \to \infty,$$
$$\textit{uniformly on sets } \Sigma = \{(x,t) : |x| \geq Ct^\nu\},$$
$$\textit{where } C > 0 \textit{ is arbitrary and } \nu > \frac{1}{2}. \tag{3.3}$$

Proof. – By linearity, it suffices to consider the case when $A = 1$. We have thus to show that:

$$||x|^\alpha v(x,t) - 1| = o(1), \qquad \text{uniformly for } |x| \geq Ct^\nu \text{ and } t \gg 1. \tag{3.4}$$

To derive (3.4), we first observe that for any given constant $M > 0$, one has that:

$$||x|^\alpha v(x,t) - 1| = |(4\pi t)^{-\frac{N}{2}} \int_{I\!R^N} \exp\left(-\frac{|x-y|^2}{4t}\right)(v_o(y)|x|^\alpha - 1)dy|$$

$$\leq (4\pi t)^{-\frac{N}{2}} \left(\int_{|y|\geq M} e^{-\frac{|x-y|^2}{4t}} |v_o(y)|x|^\alpha - 1|dy + \int_{|y|\leq M} e^{-\frac{|x-y|^2}{4t}} |v_o(y)|x|^\alpha - 1|dy\right)$$

$$\equiv I + J. \tag{3.5}$$

From now on, we shall denote by C a positive generic constant (possibly changing from line to line) depending only on N and p. Since

$|x-y|^2 \geq |x|^2 - 2M|x|$ whenever $|y| \leq M$, we readily see that, on setting $\eta = |x|t^{-\frac{1}{2}}$,

$$J \leq C|x|^\alpha t^{-\frac{N}{2}} e^{-\frac{|x|^2}{4t}} \int_{|y|\leq M} e^{\frac{2M|x|}{4t}} dy \leq C\eta^\alpha t^{\frac{\alpha-N}{2}} \exp\left(-\frac{\eta^2}{4} + \frac{M\eta}{2\sqrt{t}}\right). \tag{3.6}$$

If $\alpha \leq N$, (3.6) gives that $J \leq C\eta^{-a}$ for any given $a > 0$ and $t \gg 1$. On the other hand, if $\alpha > N$ one deduces from the fact that $|x| \geq Ct^\nu$ with $\nu > \frac{1}{2}$ that $J \leq C\eta^b \exp\left(-\frac{\eta^2}{4} + \frac{M\eta}{2\sqrt{t}}\right)$ with $b = \alpha + \frac{(\alpha-N)}{2}(\nu - \frac{1}{2})^{-1}$. In either case one has that:

$$J = O(\eta^{-a}) \qquad \text{for any given } a > 0 \text{ provided that } |x| \geq Ct^\nu \text{ and } t \gg 1. \tag{3.7}$$

To estimate the first term on the right of (3.5), we proceed as follows. Set $R(y) = v_o(y) - |y|^{-\alpha}$. Then there holds:

$$|v_o(y)|x|^\alpha - 1| \leq \left|\left(\frac{|x|}{|y|}\right)^\alpha - 1\right| + |R(y)||x|^\alpha,$$

whence:

$$\begin{aligned} I &\leq (4\pi t)^{-\frac{N}{2}} \left(\int_{|y|\geq M} e^{-\frac{|x-y|^2}{4t}} \left|\left(\frac{|x|}{|y|}\right)^\alpha - 1\right| dy \right. \\ &\qquad \left. + \int_{|y|\geq M} e^{-\frac{|x-y|^2}{4t}} |R(y)||x|^\alpha dy\right) \\ &\equiv I_1 + I_2. \end{aligned} \tag{3.8}$$

We now split further I_1 into two parts. Let us define sets Σ_1 and Σ_2 as follows: $\Sigma_1 = \{(y,t) : |y| \geq M \text{ and } |x-y| \geq 2\eta^{\frac{1}{2}}\sqrt{t}\}$, $\Sigma_2 = \{(y,t) : |y| \geq M \text{ and } |x-y| < 2\eta^{\frac{1}{2}}\sqrt{t}\}$. We now set:

$$I_1 = I_{11} + I_{12}, \tag{3.9}$$

where I_{11} (resp. I_{12}) is given by the corresponding expression in (3.8) with integration restricted to Σ_1(resp. to Σ_2). A quick computation reveals now that:

$$\begin{aligned} I_{11} &\leq C\eta^\alpha t^{\frac{\alpha-N}{2}} \int_{\frac{|x-y|}{2\sqrt{t}}\geq\sqrt{\eta}} \exp\left(-\frac{|x-y|^2}{4t}\right) dy \\ &\leq C\eta^\sigma \int_{|s|\geq\sqrt{\eta}} e^{-s^2} ds = O\left(\eta^{\sigma+\frac{N-2}{2}} e^{-\eta}\right), \end{aligned} \tag{3.10}$$

where $\sigma = 2\alpha\nu(2\nu-1)^{-1}$. On the other hand, setting $s = (y-x)(2\sqrt{t})^{-1}$, we obtain at once that, for

$$|s| \le \eta^{\frac{1}{2}}, \left|\left(\frac{|x|}{|y|}\right)^{\alpha} - 1\right| = \left|\left(\frac{|\eta|}{|2s+\eta|}\right)^{\alpha} - 1\right| = o(\eta^{-\frac{1}{2}}) \quad \text{for } \eta \gg 1,$$

where upon we derive that:

$$I_{12} = O(\eta^{-\frac{1}{2}}) \qquad \text{for } t \gg 1. \tag{3.11}$$

It remains yet to bound I_2 in (3.8)-(3.9). To this end, we observe that without loss of generality we may assume that:

$$R(y) = |y|^{-\alpha} g(|y|), \tag{3.12}$$

for some function $g(s)$ which is nonnegative, nonincreasing and such that $\lim_{s\to\infty} g(s) = 0$. We then split I_2 just as we did for I_1, thus obtaining:

$$I_2 = I_{21} + I_{21}.$$

Since $|R(y)|$ is bounded, we may estimate I_{21} exactly as I_{11} (cf. (3.10)). Finally, to deal with I_{22} we observe that, if $|x-y|/(2\sqrt{t}) < \eta^{\frac{1}{2}}$, then $|y| > |x| - 2|x|^{\frac{1}{2}}t^{\frac{1}{4}} > |x|/2$ provided that $\eta \gg 1$ and $t \gg 1$. Hence:

$$R(y) \le |x|^{-\alpha} g(|x|/2) \le |x|^{-\alpha} g(\eta/2),$$

and therefore:

$$I_{22} \le g(|x|/2\sqrt{t}) = o(1) \qquad \text{for } \eta \gg 1 \quad \text{and } t \gg 1,$$

so that:

$$I_2 = o(1) \qquad \text{for } |x| \ge Ct^{\nu} \quad \text{and } t \gg 1. \tag{3.13}$$

Putting together (3.5)-(3.13), the result follows. ■

We next describe how the precise size of the region where solutions remain stationary depends on the value of the parameter α. More precisely, the following result holds:

LEMMA 3.2. – *Let $v(x,t)$ be the solution of (3.1)-(3.2). One then has that:*
a) If $\alpha > N$, then:

$$\begin{aligned} &v(x,t) \sim A|x|^{-\alpha} \text{ as } t \to \infty, \text{ uniformly on sets} \\ &S_1 = \{|x|^2 \ge Ct\log t\} \quad \text{with } C > 2(\alpha - N). \end{aligned} \tag{3.14}$$

b) If $\alpha = N$*, then:*

$$\begin{aligned} &v(x,t) \sim A|x|^{-\alpha} \text{ as } t \to \infty\,, \text{ uniformly on sets} \\ &S_2 = \{|x|^2 \geq Ct\log(\log t)\} \quad \text{with } C > 4. \end{aligned} \tag{3.15}$$

c) If $\alpha < N$*, then:*

$$\begin{aligned} &v(x,t) \sim A|x|^{-\alpha} \text{ as } t \to \infty\,, \text{ uniformly on sets} \\ &S_3 = \{|x|^2 \geq g(t)\},\ \text{where } g(t) \text{ is any nonnegative and} \\ &\text{smooth function such that } g(t) \gg \sqrt{t} \quad \text{as } t \to \infty. \end{aligned} \tag{3.16}$$

Proof. – We shall proceed by modifying, where required, the corresponding argument in the proof of Lemma 3.1. Consider first the case $\alpha > N$. As in our previous result, we need to show that both terms I and J on the right of (3.5) are of order $o(1)$ as $t \to \infty$ in the region described in (3.14). Since $t \leq \exp(\eta^2/C)$ when $|x|^2 \geq Ct\log t$, (3.6) gives at once $J \leq C\eta^\alpha \exp\left(-\frac{1}{2}(\frac{1}{2} - \frac{(\alpha-N)}{C})\eta^2 + \frac{M\eta}{2\sqrt{t}}\right)$. On taking $C > 2(\alpha - N)$, we thus obtain:

$$J = o(1) \qquad \text{on } S_1 \quad \text{as } t \to \infty. \tag{3.17}$$

As to the term I, we split it as in (3.8)-(3.9), and consider first the quantity I_{11} thus obtained. Let $\varepsilon \in (0,1)$ be given, and denote by Σ_{11} (resp. by Σ_{12}) the subset of Σ_1 where $|y| \geq \varepsilon|x|$ (resp. where $|y| < \varepsilon|x|$). We then split I_{11} in the form:

$$\begin{aligned} I_{11} &= (4\pi t)^{-\frac{N}{2}}\left(\int_{\Sigma_{11}} e^{-\frac{|x-y|^2}{4t}}\big||x|^\alpha|y|^{-\alpha} - 1\big|dy \right.\\ &\quad \left.+ \int_{\Sigma_{12}} e^{-\frac{|x-y|^2}{4t}}\big||x|^\alpha|y|^{-\alpha} - 1\big|dy\right) \\ &\equiv I_{111} + I_{112}. \end{aligned}$$

Notice that in Σ_{11}, $|y| \leq |x| - 2|x|^{\frac{1}{2}}t^{\frac{1}{4}} \leq |x|$. Hence:

$$\begin{aligned} &I_{111} \leq (\varepsilon^{-\alpha} - 1)(4\pi t)^{-\frac{N}{2}}\left(\int_{|x-y|\geq 2\sqrt{\eta t}} \exp\left(-\frac{|x-y|^2}{4t}\right)dy\right) \\ &\leq C\int_{|s|\geq\sqrt{\eta}} e^{-s^2}ds = O(\eta^{\frac{N-2}{2}}e^{-\eta}) = o(1) \qquad \text{in } S_1 \quad \text{for } t \gg 1. \end{aligned}$$

On the other hand, if $|y| < \varepsilon|x|$ we have that:

$$|x-y|^2 \geq (1-\sqrt{\varepsilon})|x|^2 + (1-\frac{1}{\sqrt{\varepsilon}})|y|^2$$
$$= (1+\varepsilon^2-\sqrt{\varepsilon}-\varepsilon^{\frac{3}{2}})|x|^2 = (1-\varepsilon_1)|x|^2, \qquad (3.18)$$

where $\varepsilon_1 = \varepsilon_1(\varepsilon) > 0$ is such that $\lim_{\varepsilon\to 0}\varepsilon_1 = 0$. By selecting $\varepsilon > 0$ sufficiently small, we thus obtain that if $|x|^2 > Ct\log t$ with $C = 2(\alpha - N) + \delta$, and $\delta > 0$,

$$I_{112} \leq Ct^{-\frac{N}{2}}|x|^\alpha \exp(-(1-\varepsilon_1)|x|^2/4t)\int_{|y|\geq M}|y|^{-\alpha}dy$$
$$\leq C\eta^\alpha t^{\frac{\alpha-N}{2}}\exp\left(-(1-2\varepsilon_1)\left(\frac{2(\alpha-N)+\delta}{4}\right)\log t\right)\exp\left(-\frac{\varepsilon_1\eta^2}{4}\right)$$
$$\leq C\eta^\alpha\exp\left(-\frac{\varepsilon_1\eta^2}{4}\right).$$

Hence:

$$I_{11} = o(1) \qquad \text{in } S_1 \qquad \text{for } t \gg 1. \qquad (3.19)$$

As to the terms I_2 and I_{12} arising from (3.8)-(3.9), arguing as in the previous Lemma we obtain:

$$I_{12} + I_2 = o(1) \qquad \text{in } S_1 \qquad \text{for } t \gg 1. \qquad (3.20)$$

From (3.17)-(3.20), the proof of (3.14) follows. Let us examine now the case $\alpha = N$. The only term that needs particular attention is I_{11} in (3.9). To estimate it, we write Σ_1^1 (resp. Σ_1^2) to denote the subset of Σ_1 where $|y| \geq \sqrt{t}$ (resp. where $|y| < \sqrt{t}$). We then split I_{11} in the form:

$$I_{11} = I_{11}^1 + I_{11}^2, \qquad (3.21)$$

where I_{11}^1 (resp. I_{11}^2) consists of the part of I_{11} where integration is restricted to the set Σ_1^1 (resp. to Σ_1^2). Then there holds:

$$I_{11}^1 \leq Ct^{-\frac{N}{2}}\eta^N\int_{|x-y|\geq 2\sqrt{\eta t}}\exp\left(-\frac{|x-y|^2}{4t}\right)dy = O(\eta^{N+\frac{N-2}{2}}e^{-\eta}). \qquad (3.22)$$

When $|x|^2 \geq Ct\log(\log t)$, one certainly has that, if $|y| < \sqrt{t}$, then $|y| < \varepsilon|x|$ for any given $\varepsilon > 0$, provided that t is large enough. Recalling

(3.18), we then have that:

$$I_{11}^2 \le Ct^{-\frac{N}{2}}|x|^N \left(\int_{\Sigma_1^2} \exp\left(-(1-\varepsilon)\frac{|x-y|^2}{4t}\right) \right.$$
$$\left. \exp\left(-\varepsilon\frac{|x-y|^2}{4t}\right)|y|^{-N}dy \right) \tag{3.23}$$
$$\le Ct^{-\frac{N}{2}}|x|^N \exp\left(-(1-\varepsilon)(1-\varepsilon_1)\frac{|x|^2}{4t}\right)e^{-\varepsilon\eta}$$
$$\int_{M<|y|<\sqrt{t}} |y|^{-N}dy. \tag{3.24}$$

Set now $\delta = (1-\varepsilon)(1-\varepsilon_1)$. Since $C > 4$ in (3.14), it turns out that if $\varepsilon > 0$ is sufficiently small, we then have that:

$$I_{11}^2 \le C\eta^N e^{-\varepsilon\eta}(\log t)^{1-\frac{C\delta}{4}} \le C\eta^N e^{-\varepsilon\eta}. \tag{3.25}$$

Therefore (3.22) and (3.25) provide a suitable bound for I_{11}, and (3.15) follows.

Finally, the case $\alpha < N$ is straightforward, and we shall omit further details. ■

Lemma 3.2 describes the size of the regions where solutions of (3.1)-(3.2) remain asymptotically close to their initial values. We next turn our attention to the complementary regions, where diffusion-driven profiles set in. The following result holds:

LEMMA 3.3. – *Let $v(x,t)$ be the solution of (3.1)-(3.2). One then has that:*
a) If $\alpha > N$, then:

$$v(x,t) = \|v_o\|_1(4\pi t)^{-\frac{N}{2}}e^{-\frac{|x|^2}{4t}}(1+o(1))$$
$$\textit{as } t \to \infty, \textit{uniformly on sets} \tag{3.26}$$
$$\{|x|^2 \le Ct\log t\}(1+o(1)) \textit{ with } C < 2(\alpha - N),$$
$$\textit{where } \|v_o\|_1 = \int_{\mathbb{R}^N} v_o(x)dx.$$

b) If $\alpha = N$, then:

$$v(x,t) = \frac{A\omega_N}{2}\log t(4\pi t)^{-\frac{N}{2}}e^{-\frac{|x|^2}{4t}}(1+o(1)) \textit{ as } t \to \infty, \tag{3.27}$$
uniformly on sets
$\{|x|^2 \le Ct\log(\log t)\}$ *with* $C < 4$, *where* ω_N *denotes*
the surface of the $(N-1)$*-dimensional sphere in* $\mathbb{R}^N$.

c) If $\alpha < N$, then:

$$v(x,t) = t^{-\frac{\alpha}{2}} g(|x|/\sqrt{t})(1+o(1)) \text{ as } t \to \infty, \text{ unifomly in } \mathbb{R}^N, \tag{3.28}$$

where $g(s)$ satisfies:

$$\begin{cases} g'' + \left(\dfrac{N-1}{s} + \dfrac{s}{2}\right)g' + \dfrac{\alpha}{2}g = 0 \text{ for } s > 0, \\ g'(0) = 0, \quad g(s) \sim As^{-\alpha} \text{ as } s \to \infty. \end{cases} \tag{3.29}$$

Proof. – We shall begin by the case $\alpha > N$. As a matter of fact, we will prove a slightly more general version of (3.26). Namely, it will be shown that, if $v_o(x) \geq 0$ is such that $\int_{\mathbb{R}^N} v_o(y)dy < \infty$, then (3.26) holds uniformly on sets $\{|x|^2 \leq tf(t)\}$ when $t \gg 1$, where $f(t)$ is any function that is smooth for large t, is such that $c_1 < f(t) \ll t$ for some $c_1 > 0$ when $t \gg 1$, and:

$$e^{\frac{f(t)}{4}} \int_{|y|>D\sqrt{\frac{t}{f(t)}}} v_o(y)dy \ll 1 \qquad \text{as } t \to \infty, \tag{3.30}$$

for every given constant $D > 0$. A quick computation reveals now that (3.30) yields (3.26) when $v_o(x) \sim A|x|^{-\alpha}$ for large $|x|$ and $f(t) = C\log t$ with $C < 2(\alpha - N)$. The sought-for result will be obtained as soon as we prove that:

$$I \equiv \int_{\mathbb{R}^N} \exp\left(-\frac{|x-y|^2}{4t} + \frac{|x|^2}{4t}\right) v_o(y)dy \sim \| v_o\|_1, \tag{3.31}$$

whenever $\{|x|^2 \leq tf(t)\}$ and $t \gg 1$, with $f(t)$ as before. To obtain (3.30), we argue as follows. Set $H = \int_{\mathbb{R}^N} \exp\left(-\frac{|y|^2}{4t} + \frac{|x||y|}{2t}\right)v_o(y)dy$. For any given $\varepsilon > 0$, one then has that, whenever $|x|^2 \leq tf(t)$, there holds:

$$\begin{aligned} H &= \int_{|y|\leq\varepsilon\sqrt{\frac{t}{f(t)}}} e^{-\frac{|y|^2}{4t}+\frac{|x||y|}{2t}} v_o(y)dy + \int_{|y|>\varepsilon\sqrt{\frac{t}{f(t)}}} e^{-\frac{|y|^2}{4t}+\frac{|x||y|}{2t}} v_o(y)dy \\ &\leq e^{\frac{\sqrt{C}\varepsilon}{2}} \int_{|y|\leq\varepsilon\sqrt{\frac{t}{f(t)}}} v_o(y)dy + e^{\frac{f(t)}{4}} \int_{|y|>\varepsilon\sqrt{\frac{t}{f(t)}}} v_o(y)dy. \end{aligned} \tag{3.32}$$

Write now $J = \int_{\mathbb{R}^N} \exp\left(-\frac{|y|^2}{4t} - \frac{|x||y|}{2t}\right) v_o(y)dy$. Under the previous assumptions, we now have that:

$$J \geq \int_{|y| \leq \varepsilon\sqrt{\frac{t}{f(t)}}} e^{-\frac{|y|^2}{4t} - \frac{|x||y|}{2t}} v_o(y)dy \geq e^{-\frac{\varepsilon}{2} - \frac{\varepsilon^2}{4f(t)}} \int_{|y| \leq \varepsilon\sqrt{\frac{t}{f(t)}}} v_o(y)dy$$
$$\geq e^{-\varepsilon\left(\frac{1}{2} + \frac{1}{4c_1}\right)} \int_{|y| \leq \varepsilon\sqrt{\frac{t}{f(t)}}} v_o(y)dy. \qquad (3.33)$$

Since $J \leq I \leq H$, (3.30) follows now from (3.32) and (3.33) upon letting first $t \to \infty$ and then $\varepsilon \to 0$ in these inequalities.

Assume now that $\alpha = N$. Then (3.27) will follow provided that:

$$I \equiv \int_{\mathbb{R}^N} \exp\left(\frac{|x|^2 - |x-y|^2}{4t}\right) v_o(y)dy \sim \frac{\omega_N}{2} \log t,$$
whenever $\{|x|^2 \leq Ct\log(\log t)\}$ with $t \gg 1$ and $C < 4$. (3.34)

Let H and J be as in the previous case, and set $f(t) = C\log(\log t)$. Then for $|x|^2 \leq tf(t)$ and any $M > 0$ given, one has that:

$$H \leq \exp\left(\frac{M}{2}\left(\frac{f(t)}{t}\right)^{\frac{1}{2}}\right) \int_{\mathbb{R}^N} dy$$
$$+ \int_{|y|>M} \exp\left(-\frac{|y|^2}{4t} + \frac{|x||y|}{2t}\right) v_o(y)dy$$
$$\leq CM^N \exp\left(\frac{M}{2}\left(\frac{f(t)}{t}\right)^{\frac{1}{2}}\right)$$
$$+ \int_{|y|>M} \exp\left(-\frac{|y|^2}{4t} + \frac{1}{2}\left(\frac{f(t)}{t}\right)^{\frac{1}{2}} |y|\right) |y|^{-N} dy$$
$$+ \int_{|y|>M} \exp\left(-\frac{|y|^2}{4t} + \frac{1}{2}\left(\frac{f(t)}{t}\right)^{\frac{1}{2}} |y|\right) (v_o(y) - |y|^{-N}) dy$$
$$\equiv CM^N \exp\left(\frac{M}{2}\left(\frac{f(t)}{t}\right)^{\frac{1}{2}}\right) + H_1 + H_2. \qquad (3.35)$$

To estimate H_1 above, we use polar coordinates and change then variables in the form $r = 2u\left(\frac{t}{f(t)}\right)^{\frac{1}{2}}$. We thus obtain:

$$H_1 = \omega_N \int_{u\frac{M}{2}\sqrt{\frac{t}{f(t)}}} \exp\left(u - \frac{u^2}{f(t)}\right) u^{-1} du.$$

Let now $R > 0$, and let t be such that $\frac{M}{2}\sqrt{\frac{t}{f(t)}} < R < f^2(t)$. We then split H_1 as follows:

$$\begin{aligned} H_1 &= \omega_N \int_{\frac{M}{2}\sqrt{\frac{t}{f(t)}}}^{R} \exp\left(u - \frac{u^2}{f(t)}\right) u^{-1} du \\ &+ \omega_N \int_{R}^{f^2(t)} \exp\left(u - \frac{u^2}{f(t)}\right) u^{-1} du \\ &+ \omega_N \int_{f^2(t)}^{\infty} \exp\left(u - \frac{u^2}{f(t)}\right) u^{-1} du \equiv H_{11} + H_{12} + H_{13}. \end{aligned} \quad (3.36)$$

Application of L'Hopital rule readily gives that:

$$H_{11} \sim \frac{\omega_N}{2} \log t \qquad \text{as } t \to \infty. \quad (3.37)$$

On the other hand, since $-\frac{u^2}{f} + u \le \frac{f}{4}$, recalling that $C < 4$ we derive:

$$H_{12} \le \omega_N e^{\frac{f(t)}{4}} \int_R^{f^2(t)} u^{-1} du \le C\omega_N e^{\frac{f(t)}{4}} f^2(t) \ll \log t, \quad (3.38)$$

whenever $t \gg 1$. To estimate H_{13}, we just observe that $g(u) = u - u^2/2f$ is such that $g'(u) < 0$ for $u > f$. It then turns out that:

$$\begin{aligned} H_{12} &= \omega_N \int_{f^2(t)}^{\infty} e^{u - \frac{u^2}{2f}} e^{-\frac{u^2}{2f}} u^{-1} du \le \omega_N e^{f^2 - \frac{f^3}{2}} \int_{f^2(t)}^{\infty} e^{-\frac{u^2}{2f}} u^{-1} du \\ &\le C\omega_N \exp\left(f^2 - \frac{f^3}{2}\right) \le C \qquad \text{for } t \gg 1. \end{aligned} \quad (3.39)$$

From (3.37)-(3.39) it follows that:

$$H_1 \le \overline{H}_1(t), \qquad \text{where } \overline{H}_1(t) \sim \frac{\omega_N}{2} \log t. \quad (3.40)$$

To bound H_2 in (3.36), we first make use of (3.12) and then repeat our previous steps to obtain that, on splitting the corresponding integral into two terms as in (3.36), there holds:

$$H_{21} \le \frac{\omega_N}{2} g\left(2M\sqrt{\frac{t}{f(t)}}\right) \log t. \quad (3.41)$$

Since this quantity is $o(\log t)$ when $t \gg 1$, the corresponding upper bound in (3.27) follows now from (3.35), (3.40) and (3.41). Finally, a suitable lower bound is easily obtained by noting that:

$$\begin{aligned} J &\geq \omega_N \int_{\frac{M}{2}\sqrt{\frac{f(t)}{t}}}^{R} \exp\left(-\frac{u^2}{f(t)} - u\right) u^{-1} du \\ &\geq \omega_N e^{-R\left(1+\frac{R}{f^2(t)}\right)} \int_{\frac{M}{2}\sqrt{\frac{f(t)}{t}}}^{R} u^{-1} du \sim \frac{\omega_N}{2} e^{-R\left(1+\frac{R}{f^2(t)}\right)} \log t, \end{aligned} \tag{3.42}$$

which holds for any $R > 0$ such that $R > \frac{M}{2}\sqrt{\frac{f(t)}{t}}$. This concludes the proof of the case $\alpha = N$. We finally consider the case of slowly decaying data, when $\alpha < N$. For any given $\varepsilon > 0$, we denote by $g_\varepsilon(s)$ the solution of the following problem:

$$\begin{cases} g'' + \left(\dfrac{N-1}{s} + \dfrac{s}{2}\right) g' + \dfrac{\alpha}{2} g = & \text{for } s > 0, \\ g'(0) = 0, \quad g(s) \sim (A+\varepsilon) s^{-\alpha} & \text{as } s \to \infty. \end{cases} \tag{3.43}$$

Existence and uniqueness for (3.43) follows at once from classical results. As a matter of fact, when $d < \frac{N}{2}$, the general solution of:

$$g''(s) + \left(\frac{N-1}{s} + \frac{s}{2}\right) g'(s) + d g(s) = 0, \text{ for } s > 0, \tag{3.44}$$

can be written in the form:

$$g(s) = C_1 \varphi_1(s) + C_2 \varphi_2(s), \tag{3.45}$$

where C_1, C_2 are arbitrary constants, and:

$$\begin{aligned} \varphi_1(s) &= e^{-\frac{s^2}{4}} M(N/2 - d, N/2; s^2/4); \\ \varphi_2(s) &= e^{-\frac{s^2}{4}} U(N/2 - d, N/2; s^2/4). \end{aligned} \tag{3.46}$$

Here $M(a, b; z)$ and $U(a, b; z)$ are Kummer's functions of first and second kind respectively. Functions M and U possess the following asymptotic behaviours:

$$M(a, b; z) = \frac{\Gamma(b)}{\Gamma(a)} e^z z^{a-b} \left(1 + \frac{(b-a)(1-a)}{z} + O\left(\frac{1}{z^2}\right)\right) \quad \text{as } \operatorname{Re} z \to +\infty, \tag{3.47}$$

$$U(a, b; z) = z^{-a} \left(1 - \frac{a(1+a-b)}{z} + O\left(\frac{1}{z^2}\right)\right) \quad \text{as } \operatorname{Re} z \to +\infty, \tag{3.48}$$

$$M(a, b; z) = 1 + \frac{az}{b} + \frac{a(a+1)}{b(b+1)} z^2 + O(z^3) \quad \text{as } |z| \to 0, \tag{3.49}$$

where $\Gamma(s)$ is Euler's gamma function (cf. [1], [11]). Define now:

$$\overline{v}_\varepsilon(r,t) = t^{-\frac{N}{2}} g_\varepsilon(r/\sqrt{t}). \tag{3.50}$$

We now claim that $\overline{v}_\varepsilon \in C^\infty$, and:

$$(\overline{v}_\varepsilon)_t = (\overline{v}_\varepsilon)_{rr} + \left(\frac{N-1}{r}\right)(\overline{v}_\varepsilon)_r \quad \text{for } r > 0 \text{ and } t > 0, \tag{3.51}$$

$$\overline{v}_\varepsilon(r,t_o) \geq v(x,t_o) \quad \text{for any } x \in I\!R^N \text{ and some } t_o > 0. \tag{3.52}$$

Of these statements, only (3.52) requires of some explanation. Set now $w = (v - \overline{v}_\varepsilon)_+$, where $r_+ = \max\{r, 0\}$. Then w has bounded support at some $t_o \gg 1$ because of Lemma 3.2. Moreover, w is subcaloric, i.e, $w_t \leq \Delta w$ in some appropriate weak sense. This is readily seen by using Kato's inequality, i.e, the fact that setting $\text{sgn}^+ f = 1$ for $f > 0$ and $\text{sgn}^+ f = 0$ otherwise, one then has that $\Delta f \ \text{sgn}^+ f \leq \Delta f^+$ in $\mathcal{D}'$, whenever f and Δf are in L^1_{loc}. Standard estimates for the heat equation yield then that $w(x,t) \leq C_1(t - t_1)^{-\frac{N}{2}}$ for some $C_1 > 0$ and any $t > t_1$. We then deduce that:

$$v(x,t) \leq \overline{v}_\varepsilon(|x|,t) + C_1(t-t_1)^{-\frac{N}{2}} \quad \text{for } r \leq h(t) \text{ and } t > t_1, \tag{3.53}$$

where $h(t)$ is a positive function such that $\sqrt{t} \ll h(t) \ll t^{\frac{N}{2\alpha}}$. (Note that by Lemma 3.2 we have the result for $r \geq h(t)$.) We next observe that, on taking t_1 large enough, we also have

$$\overline{v}_\varepsilon(r,t) + C_1(t-t_1)^{-\frac{N}{2}} \leq \overline{v}_{2\varepsilon}(r,t) \quad \text{for } r \leq h(t) \text{ and } t > t_1. \tag{3.54}$$

To check (3.54), we first remark that such inequality holds at $r = h(t)$. Indeed, one has that:

$$\begin{aligned} \overline{v}_\varepsilon(h(t),t) + C_1(t-t_1)^{-\frac{N}{2}} &= t^{-\frac{\alpha}{2}} \overline{g}_\varepsilon\left(\frac{h(t)}{\sqrt{t}}\right) + C_1(t-t_1)^{-\frac{N}{2}} \\ &\sim (A+\varepsilon)h(t)^{-\alpha} + C_1(t-t_1)^{-\frac{N}{2}} \\ &\leq (A+2\varepsilon)h(t)^{-\alpha}. \end{aligned}$$

We may restate that estimate by saying that $\overline{v}_{2\varepsilon}(r,t) - \overline{v}_\varepsilon(r,t) \geq C_1(t-t_1)^{-\frac{N}{2}}$ for $r = h(t)$ and $t \gg t_1$. Since on the other hand $\overline{v}_{2\varepsilon}(r,t_2) - \overline{v}_\varepsilon(r,t_2) \geq O(\varepsilon)(h(t_2))^{-\alpha} \geq C_1(t_2 - t_1)^{-\frac{N}{2}}$, for $r \leq h(t_2)$ and t_2 large enough, it follows from the maximum principle applied to the caloric function $(v_{2\varepsilon} - v_\varepsilon)$ that (3.54) necessarily holds. This fact, together with Lemma 3.2, gives that $u(x,t) \leq u_{2\varepsilon}(x,t)$ for all $x \in I\!R^N$, which provides the required upper bound on replacing everywhere ε by $\varepsilon/2$. The corresponding lower bound is similarly obtained. ■

4. THE CAUCHY PROBLEM WHEN $\alpha < \dfrac{2}{p-1}$

This Section is devoted to the proof of Theorem 1 at the Introduction. To this end, we shall proceed in several steps.

4.1. The external region

Our main result here reads as follows:

LEMMA 4.1. – *Let $u(x,t)$ be the solution of (1.1)-(1.2), and assume that $\alpha < \frac{2}{p-1}$. One then has that:*
$u(x,t) \sim A|x|^{-\alpha}$ as $t \to \infty$, uniformly on sets $\Sigma = \{(x,t) : |x| \geq g(t)\}$, where $g(t)$ is any function such that $g(t) \gg t^{\beta}$ as $t \to \infty$, and $\beta = \frac{1}{\alpha(p-1)}$. (4.1)

Proof. – Let $v(x,t)$ be the solution of (3.1)-(3.2), and let us denote by $S(t)$ the semigroup associated to the heat equation in R^N, so that $v(x,t) = S(t)u_o(x)$. Our starting point is the well known variation of constants formula:

$$u(x,t) = S(t)u_o(x) - \int_o^t S(t-s)u^p(\cdot,s)ds. \tag{4.2}$$

Since $\frac{1}{\alpha(p-1)} > \frac{1}{2}$, the result would follow from Lemma 3.1 as soon as we prove that:

$$J \equiv |x|^{\alpha} \int_o^t S(t-s)u^p(\cdot,s)ds \ll 1 \quad \text{in } \Sigma \quad \text{as } t \to \infty. \tag{4.3}$$

To obtain (4.3), we take $\varepsilon \in (0, \frac{1}{\sqrt{2}})$ and split the integral in (4.3) as follows:

$$J \equiv |x|^{\alpha} \int_o^t (4\pi(t-s))^{-\frac{N}{2}} \int_{|y| \leq \varepsilon|x|} \exp\left(-\frac{|x-y|^2}{4(t-s)}\right) u^p(y,s)dyds$$
$$+|x|^{\alpha} \int_o^t (4\pi(t-s))^{-\frac{N}{2}} \int_{|y| > \varepsilon|x|} \exp\left(-\frac{|x-y|^2}{4(t-s)}\right) u^p(y,s)dyds \equiv J_1 + J_2.$$

Since u is bounded, we then have that:

$$
\begin{aligned}
J_1 \leq C|x|^{\alpha} \int_o^t (4\pi(t-s))^{-\frac{N}{2}} \int_{|y|\leq\varepsilon|x|} \exp\left(-\frac{|x-y|^2}{8t}\right) \\
\exp\left(-\frac{|x-y|^2}{8(t-s)}\right) dyds \\
\leq C|x|^{\alpha} \exp\left(-\frac{|x|^2}{16t}\right) \int_o^t (4\pi(t-s))^{-\frac{N}{2}} \int_{|y|\leq\varepsilon|x|} \exp\left(\frac{|y|^2}{8t}\right) \\
\exp\left(-\frac{|x-y|^2}{8(t-s)}\right) dyds \\
\leq C|x|^{\alpha} \exp\left(-\frac{1}{8}\left(\frac{1}{2}-\varepsilon^2\right)\frac{|x|^2}{t}\right) \int_o^t (4\pi(t-s))^{-\frac{N}{2}} \int_{\mathbb{R}^N} \\
\exp\left(-\frac{|x-y|^2}{8(t-s)}\right) dyds \\
\leq C|x|^{\alpha} t \exp\left(-\frac{1}{8}\left(\frac{1}{2}-\varepsilon^2\right)\frac{|x|^2}{t}\right).
\end{aligned}
$$

Setting again $\eta = |x|t^{-\frac{1}{2}}$ and observing that whenever $|x| \geq Ct^{\beta}$ one has that $t \leq C_1\eta^{\gamma}$ with $\gamma = (\beta - \frac{1}{2})^{-1}$ and $C_1 = C^{-\gamma}$, we have obtained that, for some $b > 0$,

$$
J_1 \leq C\eta^{\alpha} \exp\left(-\frac{1}{8}\left(\frac{1}{2}-\varepsilon^2\right)\eta^2\right) \quad \text{in } \Sigma \quad \text{as } \to \infty. \tag{4.4}
$$

As to J_2, we just observe that, when $t \gg 1$,

$$
\begin{aligned}
J_2 \leq C|x|^{\alpha} \int_o^t (4\pi(t-s))^{-\frac{N}{2}} \int_{|y|>\varepsilon|x|} \exp\left(-\frac{|x-y|^2}{4(t-s)}\right) |y|^{-\alpha p} dyds \\
\leq C|x|^{-\alpha(p-1)} t,
\end{aligned} \tag{4.5}
$$

and this quantity is of order $o(1)$ in Σ as $t \to \infty$. From (4.4) and (4.5), (4.3) follows, and the proof is complete. ∎

4.2. The comparison argument: Obtaining a suitable supersolution

To continue with the proof of the Theorem, a number of auxiliary functions will be obtained. Let λ be a real parameter, and consider the boundary value problem:

$$
\zeta''(s) + \left(\frac{N-1}{s} + \frac{s}{2}\right)\zeta'(s) = (1-\lambda)\zeta(s), \qquad \text{for } s > 0, \tag{4.6}
$$

$$\zeta(s) \text{ bounded at } s=0, \qquad \zeta(s) \sim s^{2(1-\lambda)}, \qquad \text{for } s \to \infty. \tag{4.7}$$

As it has been seen in the preceding Section, there exists a unique solution of (4.6)-(4.7). Moreover, one then has that:

$$\zeta(0) = 4^{1-\lambda}\Gamma\left(\frac{N}{2}+1-\lambda\right)\left(\Gamma\left(\frac{N}{2}\right)\right)^{-1},$$

where $\Gamma(s)$ denotes Euler's gamma function. For $\varepsilon > 0$ given, and for $i = 1, 2, 3$, we now define functions $u_i(r,t)$ as follows:

$$u_1(r,t) = (c_*+\varepsilon)t^{-\frac{1}{p-1}} - k_\varepsilon t^{-\lambda-\frac{1}{p-1}} \cdot \zeta\left(\frac{r}{\sqrt{t}}\right) + B_\varepsilon t^{-2\lambda-\frac{1}{p-1}} \cdot \zeta^2\left(\frac{r}{\sqrt{t}}\right) - R(t), \tag{4.8}$$

where:

$$\begin{aligned}
&c_* = (p-1)^{-\frac{1}{p-1}}, \quad \lambda = 1 - \frac{1}{\alpha(p-1)}, \quad k_\varepsilon = (A+\varepsilon)^{1-p}\frac{(c_*+\varepsilon)^p}{p-1},\\
&B_\varepsilon = p(p-1)^{-2}(c_*+\varepsilon)^{2p-1}(A+\varepsilon)^{-2(p-1)},\\
&R(t) = (c_*+\varepsilon)t^{-\frac{1}{p-1}} - k_\varepsilon t^{-\lambda-\frac{1}{p-1}} \cdot \zeta\left(\frac{f(t)}{\sqrt{t}}\right) + B_\varepsilon t^{-2\lambda-\frac{1}{p-1}} \cdot \zeta^2\left(\frac{f(t)}{\sqrt{t}}\right)\\
&\qquad - \left((c_*+\varepsilon)^{1-p}t + (A+\varepsilon)^{1-p} f(t)^{\alpha(p-1)}\right)^{-\frac{1}{p-1}},
\end{aligned} \tag{4.9}$$

and $f(t)$ is as in the statement of Theorem 1,

$$u_2(r,t) = \left((c_*+\varepsilon)^{1-p}t + (A+\varepsilon)^{1-p} r^{\alpha(p-1)}\right)^{-\frac{1}{p-1}}, \tag{4.10}$$

$$\begin{aligned}
u_3(r,t) &= (A+\varepsilon)r^{-\alpha} - G(t) \equiv (A+\varepsilon)r^{-\alpha}\\
&\quad - \left((A+\varepsilon)g(t)^{-\alpha}\left((c_*+\varepsilon)^{1-p}t + (A+\varepsilon)^{1-p}g(t)^{\alpha(p-1)}\right)^{-\frac{1}{p-1}}\right),
\end{aligned} \tag{4.11}$$

$g(t)$ being as in the statement of Theorem 1. Let now $\overline{u}_\varepsilon(r,t)$ be given by:

$$\overline{u}_\varepsilon(r,t) = \begin{cases} u_1(r,t) & \text{if} \quad r \le f(t), \\ u_2(r,t) & \text{if} \quad f(t) \le r \le g(t), \\ u_3(r,t) & \text{if} \quad g(t) \le r. \end{cases} \tag{4.12}$$

Then there holds:

LEMMA 4.2. – *Let $\overline{u}_\varepsilon(r,t)$ be the function defined in (4.12). Then $\overline{u}_\varepsilon(r,t)$ is continuous for $x \in I\!R^N$ and $t > 0$, and there exists $t_o \gg 1$ such that, for $t > t_o$:*

$$\frac{\partial u_1}{\partial r}(r,t) \geq \frac{\partial u_2}{\partial r}(r,t) \quad \text{at } r = f(t), \tag{4.13}$$

$$\frac{\partial u_2}{\partial r}(r,t) \geq \frac{\partial u_3}{\partial r}(r,t) \quad \text{at } r = g(t), \tag{4.14}$$

$$\frac{\partial}{\partial t}(\overline{u}_\varepsilon) - \frac{\partial^2}{\partial r^2}(\overline{u}_\varepsilon) - \left(\frac{N-1}{r}\right)\frac{\partial \overline{u}_\varepsilon}{\partial r} + (\overline{u}_\varepsilon)^p \geq 0 \quad \text{in } \mathcal{D}'(I\!R^N). \tag{4.15}$$

Moreover, $\overline{u}_\varepsilon(r,t)$ is asymptotically equivalent to the function given in the right-hand side of (1.7). (4.16)

Proof. – The continuity of $\overline{u}_\varepsilon$ is straightforward. To check (4.13), we first observe that, by the asymptotic properties of the solution $\zeta(s)$ of (4.6)-(4.7), one has that:

$$\begin{aligned}
\frac{\partial u_1}{\partial r}(f(t),t) &\sim -k_\varepsilon \alpha(p-1)t^{-\lambda-\frac{p+1}{2(p-1)}}\left(\frac{f(t)}{\sqrt{t}}\right)^{\alpha(p-1)-1} \\
&\quad + 2\alpha B_\varepsilon(p-1)t^{-\lambda-\frac{p+1}{2(p-1)}}\left(\frac{f(t)}{\sqrt{t}}\right)^{2\alpha(p-1)-1} \\
&= -k_\varepsilon \alpha(p-1)f(t)^{\alpha(p-1)-1}t^{-\frac{p}{p-1}} \\
&\quad + 2\alpha B_\varepsilon(p-1)f(t)^{2\alpha(p-1)-1}t^{-\frac{p}{p-1}-1} \\
&\equiv -a_1 + a_2.
\end{aligned} \tag{4.17}$$

On the other hand, recalling that $f(t) \ll t^\beta$ for $t \gg 1$, and using the fact that $(A+Br)^{-\gamma} \sim r^{-\gamma}(B - \gamma\frac{A}{r})$ when $r \gg 1$, one readily sees that, for large enough t:

$$\begin{aligned}
\frac{\partial u_2}{\partial r}(f(t),t) &\sim -\alpha(A+\varepsilon)^{1-p}(c_*+\varepsilon)^p f(t)^{\alpha(p-1)-1}t^{-\frac{p}{p-1}} \\
&\quad + \alpha(A+\varepsilon)^{2(1-p)}(c_*+\varepsilon)^{2p-1}p(p-1)^{-1}f(t)^{2\alpha(p-1)-1}t^{-\frac{p}{p-1}-1} \\
&\equiv -a_1 + b_2.
\end{aligned} \tag{4.18}$$

A quick check reveals now that $b_2 \leq a_2$, whereupon (4.13) follows. The proof of (4.14) is similar, and will be omitted. To show (4.15), it will be convenient to use self-similar variables Φ, τ, given by:

$$\Phi(y,\tau) = t^{\frac{1}{p-1}} u(x,t); \quad y = xt^{-\frac{1}{2}}, \quad \tau = \log t. \tag{4.19}$$

Equation (1.1) is then transformed into:

$$\Phi_\tau = \Delta\Phi + \frac{1}{2} y\nabla\Phi + \frac{1}{p-1}\Phi - \Phi^p. \tag{4.20}$$

Set now $\Psi_1 = t^{-\lambda}\zeta$. Function $u_1(r,t)$ in (4.12) reads now:

$$\Phi_1(y,\tau) = (c_* + \varepsilon) - k_\varepsilon \Psi_1(y,\tau) + B_\varepsilon(\Psi_1(y,\tau))^2 - \overline{R}(\tau),$$

where $\overline{R}(\tau) = e^{\frac{\tau}{p-1}} R(t)$. We next observe that:

$$((\Psi_1)_y)^2 < (1-\lambda)(\Psi_1)^2. \tag{4.21}$$

To check (4.21), we just multiply both sides of (4.6) by $\zeta'(s)$, and then integrate twice there. Consider now the region where $r < f(t)$ and $t \gg 1$. A routine computation reveals then that:

$$\begin{aligned}
\mathcal{L}(\Phi_1) &\equiv (\Phi_1)_\tau - \Delta\Phi_1 - \frac{1}{2} y\nabla\Phi_1 - \frac{1}{p-1}\Phi_1 + \Phi_1^p \\
&\geq ((c_* + \varepsilon) - k_\varepsilon \Psi_1 + B_\varepsilon \psi_1^2 - \overline{R})^p - \frac{(c_* + \varepsilon)}{p-1} + \frac{pk_\varepsilon \Psi}{p-1} \\
&\quad - \left(2 + \frac{1}{p-1} + 2(1-\lambda)\right)\Psi_1^2 - \overline{R}_\tau + \frac{\overline{R}}{p-1} \\
&\sim (c_* + \varepsilon)^p - \frac{(c_* + \varepsilon)}{p-1} + H(y,\tau),
\end{aligned} \tag{4.23}$$

where $H(y,\tau) \ll \left((c_* + \varepsilon)^p - \frac{(c_*+\varepsilon)}{p-1}\right)$ in the region under consideration. We have thus obtained that:

$$\mathcal{L}(\Phi) \geq 0 \quad \text{for } r \leq f(t) \text{ and } t \gg 1. \tag{4.24}$$

We now consider the region where $f(t) < r < g(t)$ and $t \gg 1$. We shall perform there an analysis quite similar to that just made above. The role of the new scales is best highlighted by changing variables in the form :

$$u(x,t) = (t\Psi(y,\tau))^{-\frac{1}{p-1}} \qquad \text{with } \xi = xt^{-\beta}. \tag{4.25}$$

As a matter of fact, all the following comparison arguments can be checked on the operator $\mathcal{L}(\Phi_1)$ in (4.22). Instead of adapting our previous results, however, we shall formally proceed with the new variables as follows. Set $s = |\xi|$. Then the sought-for inequality (4.15) reads:

$$\mathcal{L}_2(\Psi) = \Psi_\tau - \beta s \Psi_s + \Psi - (p-1) + e^{-\Gamma\tau} \left(\frac{p}{p-1} \frac{(\Psi_s)^2}{\Psi} - \left(\Psi_{ss} + \frac{N-1}{s} \Psi_s \right) \right) \leq 0, \tag{4.26}$$

where $\Gamma = 2\beta - 1$. Consider now the auxiliary function:

$$\overline{\Psi}(s,\tau) = (c_* + \varepsilon)^{1-p} + (A+\varepsilon)^{1-p} s^{\alpha(p-1)},$$

which coincides with $u_2(r,t)$ in (4.12) when written in the former variables. A straightforward computation reveals now that:

$$\begin{aligned} \mathcal{L}_2(\overline{\Psi}) &\leq (c_* + \varepsilon)^{1-p} - (p-1) + e^{-\Gamma\tau} \\ &\quad \left(\frac{p}{p-1} \frac{(\overline{\Psi}_s)^2}{\overline{\Psi}} - \left(\overline{\Psi}_{ss} + \frac{N-1}{s} \overline{\Psi}_s \right) \right) \\ &\equiv (c_* + \varepsilon)^{1-p} - (p-1) + F(s,\tau), \end{aligned}$$

where $F(s,\tau) = o(1)$ as $\tau \to \infty$ in the region under consideration. This gives (4.15) when $f(t) < r < g(t)$ and $t \gg 1$. To conclude, we now observe that $u_3(r,t)$ satisfies (4.15) when $r > g(t)$ and $t \gg 1$, since function $G(t)$ in (4.11) is such that $G(t) = o(1)$ and $G'(t) \leq 0$ there. We therefore deduce that (4.15) holds for large enough t, except perhaps when $r = f(t)$ or $r = g(t)$. Along these curves, however, we may take advantage of (4.13) and (4.14) to obtain the desired result. ■

We next show:

LEMMA 4.3. – *Let $\overline{u}_\varepsilon(r,t)$ be the function defined in (4.12). Then there exists $t_o \gg 1$ such that:*

$$u(x,t) \leq \overline{u}_\varepsilon(|x|,t) \quad \textit{for } x \in \mathbb{R}^N \quad \textit{and } t > t_o. \tag{4.27}$$

Proof. – We shall divide it into several steps:

Step 1. – Let $\varepsilon > 0$ be given. Then there exists $M_1 = M_1(\varepsilon)$ such that:

$$u(x,t) \leq \overline{u}_\varepsilon(|x|,t) \quad \text{for } |x| \geq M_1 t^\beta \quad \text{and } t \gg 1. \tag{4.28}$$

To obtain (4.28), we first observe that for any given $\varepsilon > 0$ there exist a constant $M > 0$ and a time $t_o > 0$ (both depending on ε) such that:

$$u(x,t) \leq (A+\varepsilon)|x|^{-\alpha} \quad \text{for } |x| > M \text{ and } t > t_o. \tag{4.29}$$

Indeed, consider the auxiliary function $w(r,t)$ defined in the following manner: $w(r,t) = (A+\varepsilon)M^{-\alpha}$ if $r \le M, w(r,t) = (A+\varepsilon)r^{-\alpha}$ if $r > M$. Set now $\mathcal{L}w \equiv w_t - w_{rr} - \left(\frac{N-1}{r}\right)w_r + w^p$. One readily sees that, if $N-\alpha-2 \ge 0$, then $\mathcal{L}w \ge 0$ when $r > M$ and $t > 0$ for any given $M > 0$. On the other hand, when $N-\alpha-2 < 0$, one makes use of the assumption $\alpha < \frac{2}{p-1}$ to derive that $\mathcal{L}w \ge 0$ for $r > M$ with $M \gg 1$. Since $\mathcal{L}w \ge 0$ for $r < M$ and w_r has a negative jump at $r = M$, one readily sees that $\mathcal{L}w \ge 0$ in $\mathcal{D}'(I\!R^N)$, which is enough for our comparison purposes. Recalling the upper bound (2.1), one certainly has that $u(x,t) \le (A+\varepsilon)M^{-\alpha}$ for all $x \in I\!R^N$ and $t \gg 1$. Summing these facts up, (4.29) follows. We then observe that $(A+\frac{\varepsilon}{2})r^{-\alpha} \le ((c_*+\varepsilon)^{1-p}t+(A+\varepsilon)^{1-p}r^{\alpha(p-1)})^{-\frac{1}{p-1}}$ whenever $r \ge Ct^{\beta}$ with $C = C(\varepsilon) \gg 1$. Taking advantage of this inequality, together with (4.29) (with ε replaced by $\varepsilon/2$ there), (4.28) follows.

Step 2. – For any given $C_1, \gamma > 0$ one has that:

$$u(x,t) \le \overline{u}_{\frac{\varepsilon}{2}}(|x|,t) + \gamma t^{-\frac{1}{p-1}} \quad \text{for } |x| \ge C_1 t^{\beta} \quad \text{and } t \gg 1. \tag{4.30}$$

To check (4.30), we set $v = (u - \overline{u}_{\frac{\varepsilon}{2}})_+$. Using Kato's inequality, one readily sees that:

$$v_t \le \Delta v - \left(\frac{u^p - \overline{u}^p_{\varepsilon/2}}{u - \overline{u}_{\varepsilon/2}}\right) v \le \Delta v.$$

Let now $t_o \gg 1$ be such that (4.28) holds, and write $M = M_1 t_o^{\beta}$. Since v defined above is subcaloric, for $t > t_o$ we may then write:

$$\begin{aligned} v(x,t) &\le C(t-t_o)^{-\frac{N}{2}} \int_{|y|\le M} \exp\left(-\frac{|x-y|^2}{4(t-t_o)}\right) dy \\ &\le C(t-t_o)^{-\frac{N}{2}} \exp\left(-\frac{|x|^2}{8(t-t_o)}\right) \int_{|y|\le M} \exp\left(\frac{|y|^2}{4(t-t_o)}\right) dy. \end{aligned}$$

Take now $t \gg t_o$ and $|x| \ge C_1 t^{\beta}$. Since $2\beta > 1$, we have that:

$$v(x,t) \le C(t-t_o)^{-\frac{N}{2}} \exp\left(-\frac{C_1 t^{\beta}}{8(t-t_o)}\right) \ll \gamma t^{-\frac{1}{p-1}}.$$

and (4.30) follows.

Step 3. - Assume that C_1, C_2 are two given positive constants satisfying $0 < C_1 < C_2$. Then there exists $\delta = \delta(\varepsilon, C_1, C_2) > 0$ such that:

$$\overline{u}_{\varepsilon}(r,t) - \overline{u}_{\frac{\varepsilon}{2}}(r,t) \ge \delta t^{-\frac{1}{p-1}} \quad \text{for } C_1 t^{\beta} \le r \le C_2 t^{\beta} \quad \text{and } t \gg 1. \tag{4.31}$$

Actually, a straight forward computation gives:

$$\frac{\partial \overline{u}_\varepsilon}{\partial \varepsilon} = ((c_* + \varepsilon)^{-p} t + (A+\varepsilon)^{-p} r^{\alpha(p-1)})((c_* + \varepsilon)^{1-p} t + (A+\varepsilon)^{1-p} r^{\alpha(p-1)})^{-\frac{p}{p-1}} > 0.$$

Since $\overline{u}_\varepsilon(r,t) = \overline{u}_o(r,t) + \left(\frac{\partial \overline{u}_\varepsilon}{\partial \varepsilon}\,|_{\varepsilon=0}\right) \cdot \varepsilon + O(\varepsilon^2)$, it then turns out that:

$$\overline{u}_\varepsilon(r,t) - \overline{u}_{\frac{\varepsilon}{2}}(r,t) = \frac{\varepsilon}{2}(c_*^{-p} + A^{-p} r^{\alpha(p-1)} t^{-1}) ({c_*}^{1-p} + A^{1-p} r^{\alpha(p-1)} t^{-1})^{-1} t^{-\frac{1}{p-1}} + O(\varepsilon^2),$$

whereupon (4.31) follows.

Step 4: *End of the Proof.* – We first remark that by (4.30) and (4.31), we have that, for any given $C_1 > 0$,

$$u(x,t) \le \overline{u}_\varepsilon(|x|,t) \quad \text{for } |x| \ge C_1 t^\beta \text{ and } t \gg 1. \tag{4.32}$$

Let us write now $z(\xi,\tau) \equiv z(xt^{-\beta}, \log t) = t^{\frac{1}{p-1}}(u(x,t) - \overline{u}_\varepsilon(|x|,t))_+$. A quick check reveals that z satisfies:

$$z_\tau \le e^{-\Gamma\tau}\Delta z + \beta\xi\nabla z + A(\xi,\tau) z, \tag{4.33}$$

where $A(\xi,\tau) = (\Phi^p - \Phi_\varepsilon^p)(\Phi - \Phi_\varepsilon)^{-1}$ if $\Phi \ne \Phi_\varepsilon$ and $A(\xi,\tau) = 0$ if $\Phi = \Phi_\varepsilon$, $\Phi = e^{\tau/(p-1)}u$ (resp. $\Phi_\varepsilon = e^{\tau/(p-1)}u_\varepsilon$), $\Gamma = (2/\alpha(p-1)) - 1 > 0$, and differentiation on the right of (4.33) is performed with respect to ξ. We shall now argue by contradiction. To this end, we may assume that:

$$u(x,t) > \overline{u}_\varepsilon(|x|,t) \quad \text{for } |x| \le C_1 t^\beta \text{ and } t \text{ large enough.}$$

Since $(a^p - b^p)(a-b)^{-1} > pb^{p-1}$ whenever $a > b$ and $p > 1$, we then have that:

$$z_\tau \le e^{-\Gamma\tau}\Delta z + \beta\xi\nabla z + F(\xi,\tau) z, \tag{4.34}$$

where $F(\xi,\tau) = \frac{1}{p-1} - p\Phi_\varepsilon^{p-1}$. Notice that:

$$F(\xi,\tau) < 0 \quad \text{if } |\xi| \le C_o \equiv ((p-1)^2 (A+\varepsilon)^{p-1})^{\frac{1}{\alpha(p-1)}}. \tag{4.35}$$

As a matter of fact, at the inner region where $\overline{u}_\varepsilon(r,t) = u_1(r,t)$ (cf. (4.12)), one has that $\Phi_\varepsilon = (c_* + \varepsilon) + o(1)$, so that $F \sim -1$. When $\overline{u}_\varepsilon(r,t) = u_2(r,t)$, then :

$$\Phi_\varepsilon = ((c_* + \varepsilon)^{1-p} + (A+\varepsilon)^{1-p}|\xi|^{\alpha(p-1)})^{-\frac{1}{p-1}},$$

and (4.35) follows. Setting $C_1 = C_o$, we thus obtain:

$$z_\tau \leq e^{-\Gamma\tau}\Delta z + \beta\xi\nabla z \quad \text{for all } \xi \text{ and } \tau \gg 1. \tag{4.36}$$

The change of variables $w(x,t) = z(\xi,\tau)$ transforms (4.36) into:

$$w_t \leq \Delta w \quad \text{for all } x \in I\!R^N \text{ and } t \gg 1. \tag{4.37}$$

By our previous results, $w(x,t_1)$ has compact support for some t_1 sufficiently large. Hence for $t > 2t_1$:

$$w(x,t) \leq C(t-t_1)^{-\frac{N}{2}}.$$

Summing up, we have obtained that:

$$u(x,t) - \overline{u}_{\frac{\varepsilon}{2}}(|x|,t) \leq C(t-t_1)^{-\frac{N}{2}} t^{-\frac{1}{p-1}} \leq \overline{u}_\varepsilon(|x|,t)\overline{u}_{\frac{\varepsilon}{2}}(|x|,t)$$

for $|x| \leq C_1 t^\beta$ and $t \gg 1$,

and the proof is now concluded. ■

4.3. A subsolution

To complete our comparison argument, we shall make use of the following auxiliary function. For given $T > 0$ and $0 < \varepsilon \ll 1$, let us write:

$$u_\varepsilon(r,t) = \begin{cases} u_1(r,t) & \text{when r} \leq f(t), \\ u_2(r,t) & \text{when r} \geq f(t), \end{cases} \tag{4.38}$$

where $f(t)$ is as in Lemma 4.2,

$$u_1(r,t) = (c_* - \varepsilon)(t+T)^{-\frac{1}{p-1}} - k_\varepsilon (t+T)^{-\lambda-\frac{1}{p-1}} \cdot \zeta\left(\frac{r}{\sqrt{t}}\right) - R(t), \tag{4.39}$$

$$u_2(r,t) = (1-\varepsilon)\Big((c_* - \varepsilon)^{1-p}(t+T) + (A-\varepsilon)^{1-p} r^{\alpha(p-1)}\Big)^{-\frac{1}{p-1}}, \tag{4.40}$$

and λ, $\zeta(s)$ are as in (4.6)-(4.7) and (4.8)-(4.9) respectively, $k_\varepsilon = (A-\varepsilon)^{1-p}(c_* - \varepsilon)^p (p-1)^{-1}$, and

$$\begin{aligned} R(t) = {} & (c_* - \varepsilon)(t+T)^{-\frac{1}{p-1}} - k_\varepsilon (t+T)^{-\lambda-\frac{1}{p-1}} \cdot \zeta\left(\frac{f(t)}{\sqrt{t}}\right) \\ & - (1-\varepsilon)\Big((c_* - \varepsilon)^{1-p}(t+T) + (A-\varepsilon)^{1-p} f(t)^{\alpha(p-1)}\Big)^{-\frac{1}{p-1}}. \end{aligned}$$

Then the following result holds:

LEMMA 4.4. – *Let $u_\varepsilon(r,t)$ be the function defined in (4.38)-(4.39)-(4.40). Then $u_\varepsilon(r,t)$ is continuous for all r and $t > 0$. Moreover, there exist T, $t_o > 0$ such that, for $t > t_o$:*

$$\frac{\partial u_1}{\partial r}(r,t) \le \frac{\partial u_2}{\partial r}(r,t) \text{ at } r = f(t), \tag{4.41}$$

$$\frac{\partial \overline{u}_\varepsilon}{\partial t} - \frac{\partial^2 \overline{u}_\varepsilon}{\partial r^2} - \left(\frac{N-1}{r}\right)\frac{\partial u_\varepsilon}{\partial r} + \overline{u}_\varepsilon^p \le 0 \quad \text{in } \mathcal{D}'(I\!R^N), \tag{4.42}$$

$$u(x,t_o) \ge u_\varepsilon(|x|,t_o) \quad \text{for all } x \in I\!R^N, \tag{4.43}$$

$$u_\varepsilon(r,t) \text{ is asymptotically equivalent to the function in (1.7).} \tag{4.44}$$

Proof. – Checking that u_ε is continuous and that (4.44) holds is straightforward. Inequality (4.41) is obtained as the corresponding result in Lemma 4.2, and a routine computation reveals that u_1 and u_2 given in (4.39) and (4.40) both satisfy (4.43) when $\varepsilon > 0$ is sufficiently small. Finally, (4.43) is obtained as follows. Let $g(t)$ be as in the statement of Lemma 4.1. Then, by (4.1) there exist t_o and T positive such that $u_\varepsilon(r,t) \equiv u_\varepsilon(r,t;T) \le u(x,t)$ for $r = |x| \ge g(t)$ and $t \ge t_o$. Since $u_\varepsilon(r,t;T)$ is decreasing on T, we observe that, after possibly selecting a smaller parameter T, we also have that $u(x,t_o) \ge C(t_o) \ge u_\varepsilon(|x|,t_o;T)$ for $|x| \le g(t_o)$. The sought-for conclusion follows now by the maximum principle. ∎

5. SELF-SIMILAR ASYMPTOTICS

In this Section we shall prove Theorems 2 and 3. To obtain the first of these results, it will be convenient to consider separately the cases $N > \alpha = \frac{2}{p-1}$ and $N \le \alpha = \frac{2}{p-1}$.

5.1. The case $N > \alpha = \dfrac{2}{p-1}$.

The crux of the forthcoming arguments consists in making use of rather classical results concerning the asymptotics of solutions of some (linear and nonlinear) ODE's. To begin with, we recall that classical ODE techniques show that there exists a unique, smooth and positive solution of the problem:

$$g''(s) + \left(\frac{N-1}{s} + \frac{s}{2}\right)g'(s) + \frac{1}{p-1}g(s) - g(s)^p = 0 \quad \text{for } s > 0, \tag{5.1}$$

$$g'(0) = 0, \quad g(s) \sim As^{-\frac{2}{p-1}} \quad \text{for } s \to \infty \tag{5.2}$$

(cf. [7]). To obtain Theorem 2 under our current assumptions, we introduce self-similar variables:

$$u(x,t) = (t+1)^{-\frac{1}{p-1}}\Phi(y,\tau); \quad y = x(t+1)^{-\frac{1}{2}}. \quad \tau = \log(t+1). \quad (5.3)$$

(cf. (2.2)). For any given $\varepsilon > 0$ such that $\frac{1}{p-1} + \varepsilon < \frac{N}{2}$, we now define auxiliary functions $\Phi^+(y,\tau)$, $\Phi^-(y,\tau)$ as follows:

$$\Phi^+(y,\tau) = (1+\varepsilon)g(y) + M_1 e^{-\varepsilon\tau}G(y), \quad (5.4)$$

$$\Phi^-(y,\tau) = \left((1-\varepsilon)g(y) - M_2 e^{-\varepsilon\tau}G(y)\right)_+, \quad (5.5)$$

where $G(y) \equiv G_\varepsilon(y)$ is the solution of (3.44) with $d = \frac{1}{p-1} + \varepsilon$, such that:

$$G(0) \text{ is positive and bounded, and } G(s) \sim s^{-2\left(\frac{1}{p-1}+\varepsilon\right)} \text{ for } s \to \infty. \quad (5.6)$$

Then there holds:

LEMMA 5.1. – *For any given positive constants M_1 and M_2, $\Phi^+(y,\tau)$ (resp. $\Phi^-(y,\tau)$) is a supersolution of (4.20) (resp. is a subsolution).*

Proof. – Let us write $\mathcal{L}_o\Phi \equiv \Phi_{yy} + \left(\frac{N-1}{y} + \frac{y}{2}\right)\Phi_y + \frac{1}{p-1}\Phi$. Then $\mathcal{L}_o((1+\varepsilon)g) = (1+\varepsilon)g(s)^p \le ((1+\varepsilon)g(s))^p$. Since $\mathcal{L}_o(M_1G(s)) = -M_1\varepsilon G(s) < 0$, one readily sees that the differential operator $\mathcal{L}_o(\Phi)$ given above is such that $\mathcal{L}_o((1+\varepsilon)g(s) + M_1e^{-\varepsilon\tau}G(s)) \ge 0$. Since on the other hand $((1-\varepsilon)g)^p \ge ((1-\varepsilon)g(s) - M_2e^{-\varepsilon\tau}G)^p$ provided that $(1-\varepsilon)g(s) - M_2e^{-\varepsilon\tau}G(s) > 0$, one has that $\mathcal{L}_o\ ((1-\varepsilon)g(s) - M_2e^{-\varepsilon\tau}G(s)) \le 0$ in such a region. Finally, the subsolution statement follows by observing that the jump of Φ^-_{yy} has the right sign at points where $(1-\varepsilon)g(s) = M_2e^{-\varepsilon\tau}G(s)$.■

End of the proof of Theorem 2 when $N > \alpha = 2/(p-1)$:

The desired result will follow from the inequalities:

$$\Phi^-(y,0) \le \Phi(y,0) \le \Phi^+(y,0). \quad (5.7)$$

Actually, the fact that (5.7) holds for $|y| \ge M$ with $M \gg 1$ is a consequence of (5.2). Since $u_o(y) = \Phi(y,0)$ is bounded, we have that $\Phi(y,0) \le k$ for some $k > 0$ and $|y| \le M$. We then select M_1 large enough so that $k \le M_1G(M)$, and take $M_2 = (1-\varepsilon)g(0)/G(M)$, in which case $(1-\varepsilon)g(y) \le M_2G(y)$ for $|y| \le M$, whereupon (5.7) holds. By the maximum principle, we then have that:

$$\Phi^-(y,\tau) \le \Phi(y,\tau) \le \Phi^+(y,\tau) \quad \text{for y} \in I\!R^N, \quad \tau > 0. \quad (5.8)$$

and since $g(y) \gg e^{-\varepsilon\tau}G(y)$ as $\tau \to \infty$ for any $y \in I\!R^N$, the conclusion follows by taking $\varepsilon > 0$ arbitrarily small. ■

5.2. The case $N \leq \alpha = \dfrac{2}{p-1}$.

Under the assumptions above, it is no longer true that (3.44) and (5.6) have a positive solution. Instead, we have the following result:

Let $\varepsilon > 0$ be such that $\frac{1}{p-1} + \varepsilon - \frac{N}{2} \neq 0, 1, 2, 3, \ldots$ Then there exists a unique solution of (3.44) with $d = \frac{1}{p-1} + \varepsilon$ such that $G(0) = 1, G'(0) = 0$, and $G(y) \sim C_\varepsilon y^{-2(\frac{1}{p-1}+\varepsilon)}$ as $y \to \infty$,
where $C_\varepsilon = \Gamma(N/2)(\Gamma((N/2) - 1/(p-1) - \varepsilon))^{-1}$.
Moreover $G(y)$ has a finite number of zeroes (that depend on ε). (5.9)

Actually, $G(y) = e^{-\frac{y^2}{4}} M\Big(\frac{N}{2} - \frac{1}{p-1} - \varepsilon, \frac{N}{2}; \frac{y^2}{4}\Big)$, and the statements in (5.9) follow from classical results on Kummer's functions (cf. [1], [11], ...) Take now $\varepsilon_1, \varepsilon_2$ such that $0 < 2\varepsilon_1 < \varepsilon_2$ for $i = 1, 2$, ε_i satisfies the requirement made in (5.9), and $\Gamma(\frac{N}{2} - \frac{1}{p-1} - \varepsilon_i) > 0$. We now define:

$$F(y, \tau) = 2e^{-\varepsilon_1 \tau} G_{\varepsilon_1}(y) - e^{-\varepsilon_2 \tau} G_{\varepsilon_2}(y).$$

We then have that:

$$F(y, \tau) > 0 \quad \text{for all } y \in I\!R^N \text{ and } \tau > 0. \tag{5.10}$$

To check (5.10), we merely observe that $F(y, \tau) \geq e^{-\varepsilon_1 \tau}(2G_{\varepsilon_1}(y) - G_{\varepsilon_2}(y))$ where function $H(y) = 2G_{\varepsilon_1}(y) - G_{\varepsilon_2}(y)$ satisfies:

$$H''(y) + \left(\frac{N-1}{y} + \frac{y}{2}\right) H'(y) = (\varepsilon_2 - 2\varepsilon_1) H(y) \quad \text{for } y > 0, \tag{5.11}$$

$$H(0) = 1, \quad H'(0) = 0, \quad H(y) > 0 \quad \text{for } y > 0, \tag{5.12}$$

$$H(y) \sim D_{\varepsilon_1} y^{-2(\frac{1}{p-1}+\varepsilon_1)} \quad \text{for } y \to \infty, \text{ for some } D_{\varepsilon_1} > 0. \tag{5.13}$$

We now define:

$$\Phi^+(y, \tau) = (1 + \varepsilon) g(y) + M_1 F(y, \tau),$$
$$\Phi^-(y, \tau) = ((1 - \varepsilon) g(y) - M_2 F(y, \tau))_+$$

(compare with (5.4)-(5.5)). One readily sees that:

For any positive numbers M_1 and M_2, one has that Φ^+ (resp. Φ^-) is a supersolution of (4.20) (resp. is a subsolution). (5.14)

To check (5.14), we merely observe that $\mathcal{L}_o F = 0$, where $\mathcal{L}_o$ is the differential operator defined in the proof of Lemma 5.1. It then suffices to repeat the argument there to derive the sought-for result. Since $F(y,\tau)$ is everywhere positive (cf. (5.10)), we now obtain (5.7) exactly as before. By the maximum principle, (5.8) continues to hold. The conclusion follows now by observing that $e^{-\varepsilon_1\tau}(2G_1 - G_2) \equiv He^{-\varepsilon_1\tau} \ll g$ as $\tau \to \infty$, since there exists $C > 0$ such that $g(y) \geq CH(y)$ for all $y \in I\!R^N$. ■

5.3. The proof of Theorem 3

A first step towards deriving such result is the following:

LEMMA 5.2. – *Let $u(x,t)$ be the solution of (1.1)-(1.3) with $\frac{2}{p-1} < \alpha < N$. One then has that:*

$$u(x,t) \sim A|x|^{-\alpha}, \text{ as } t \to \infty, \text{ uniformly on sets} \tag{5.15}$$

$$\Lambda = \{|x| \geq g(t)\} \text{ with } \gg \sqrt{t} \text{ for } t \gg 1.$$

Proof. – Since $u(x,t) \leq v(x,t)$, where $v(x,t)$ is the solution of (3.1)-(3.3), it follows from (3.16) that $u(x,t)$ satisfies in Λ an upper bound as that in the right of (5.15). Let now $\overline{p}$ be such that $\frac{2}{\overline{p}-1} = \alpha$. Clearly, $\overline{p} < p$, and therefore $\overline{u}(x,t) \leq u(x,t)$ in Λ for $t \gg 1$, where $\overline{u}$ denotes the solution of (1.1)-(1.3) with p replaced by $\overline{p}$. By Theorem 2, we then have that $\overline{u}$ satisfies (5.15), and the proof is concluded. ■

End of the proof of Theorem 3:

Let $\varepsilon > 0$ be given, and denote by $g_\varepsilon^+(r)$ the solution of (5.1) with coefficient $\left(\frac{1}{p-1} + \varepsilon\right)$ replaced there by $\frac{\alpha}{2}$, and where the second condition in (5.2) is replaced by $G(s) \sim (A+\varepsilon)s^{-\alpha}$ for $s \to \infty$. Arguing as in part c) in Lemma 3.3, we readily see that $u_\varepsilon^+(r,t) = t^{-\frac{\alpha}{2}} g_\varepsilon^+(r/\sqrt{t})$ is a supersolution for (1.1) when $t \gg 1$. Consider now the auxiliary function:

$$u_\varepsilon^- = t^{-\frac{\alpha}{2}}\left(g_\varepsilon^-(r/\sqrt{t}) + \theta\Big(g_\varepsilon^-(r/\sqrt{t})\Big)^p t^{-\frac{\alpha(p-1)}{2}}\right),$$

where $g_\varepsilon^-(s)$ denotes now the solution of (3.44) with $d = \frac{\alpha}{2}$, with boundary conditions (5.2) (with A replaced by $(A - \varepsilon)$ there). A routine computation reveals that u_ε^- satisfies all the required properties for a subsolution if θ and t are sufficiently large. Furthermore, for $t \gg 1$ one also has that $u_\varepsilon^-(r,t) \leq u_{\frac{\varepsilon}{2}}^-(r,t)$ for all $r > 0$. On the other hand, considering now $f(t)$ such that $t^{\frac{1}{2}} \ll f(t) \ll t^{\frac{N}{2\alpha}}$, then by Lemma 5.2, $\left(u_{\frac{\varepsilon}{2}}^-(r,t) - u(r,t)\right)_+ = 0$ for $r \geq f(t)$. This gives at once that:

$$u_{\frac{\varepsilon}{2}}^-(r,t) \leq u(r,t) + C(t - t_o)^{-\frac{N}{2}} \qquad \text{for } t > t_o.$$

Hence:

$$t^{-\frac{\alpha}{2}}\left(g_\varepsilon^-(r/\sqrt{t}) - C(t-t_o)^{\frac{\alpha-N}{2}}\right) \leq u(r,t),$$

and the result follows by observing that $g_\varepsilon^-(r/\sqrt{t}) \gg t^{\frac{\alpha-N}{2}}$, as $f(s) \ll s^{\frac{N}{2\alpha}}$. ■

6. THE PROOFS OF THEOREMS 4, 5 AND 6

For convenience, our discussion will be split into a number of subcases.

6.1. The case $\frac{2}{p-1} < N < \alpha$

Our first result is:

LEMMA 6.1. – *Let $u(x,t)$ be the solution of (1.1)-(1.3), and assume that $\frac{2}{p-1} \leq N < \alpha$. One then has that:*

$$u(x,t) \sim A|x|^{-\alpha} \text{ as } t \to \infty, \text{ uniformly on sets } \{|x|^2 \geq kt\log t\} \text{ with } k > 2(\alpha - N). \tag{6.1}$$

Proof. – It follows by adapting the argument already used in the proof of Lemma 5.2. We first notice that an upper bound alike to that in (6.1) holds in view of the corresponding result for the heat equation (cf. (3.14) in Lemma 3.2). We then select $\overline{p}$ exactly as in Lemma 5.1, and observe that the required lower bound holds for $|x| \geq g(t)$ with $g(t) \gg t^{\frac{1}{2}}$ when $t \gg 1$, whence in particular in the region considered in (6.1). ■

We are now in a position to show:

LEMMA 6.2. – *Let $u(x,t)$ be the solution of (1.1)-(1.3), and assume that $\frac{2}{p-1} < N < \alpha$. Then the following estimate holds:*

$$u(x,t) = C_o(4\pi t)^{-\frac{N}{2}} \exp\left(-\frac{|x|^2}{4t}\right)(1+o(1)) \qquad \text{as } t \to \infty,$$

uniformly on regions $\{|x|^2 \leq kt\log t\}$ with $k < 2(\alpha - N)$, where

$$C_o = \int_{I\!R^N} u_o(x)dx - \int_o^\infty \int_{I\!R^N} u^p(y,s)dyds > 0. \tag{6.2}$$

Proof. – Recalling part a) in Lemma 3.3, the proof will follow as soon as we show that:

$$I \equiv \int_o^t \left(\frac{t}{t-s}\right)^{\frac{N}{2}} ds \int_{\mathbb{R}^N} \exp\left(\frac{|x|^2}{4t} - \frac{|x-y|^2}{4(t-s)}\right) u^p(y,s)dy \to C_1,$$
$$\text{as } t \to \infty, \qquad \text{where } C_1 = \int_o^\infty \int_{\mathbb{R}^N} u^p(y,s)dyds. \tag{6.3}$$

To derive (6.3), we proceed in several steps, the first of which is:

Step 1. – Set:

$$I_{o,\varepsilon} \equiv \int_o^{\sqrt{t}} \int_{|y|\le\varepsilon\sqrt{t/\log t}} P(x,t;y,s)u^p(y,s)dyds.$$

where $\varepsilon > 0$ is fixed and $P(x,t;y,s) = \left(\frac{t}{t-s}\right)^{\frac{N}{2}} \exp\left(\frac{|x|^2}{4t} - \frac{|x-y|^2}{4(t-s)}\right)$. Then there holds:

$$I \ge I_{o,\varepsilon} \quad \text{and} \quad I_{o,\varepsilon} = (1+O(\varepsilon))C_1 \quad \text{as } t \to \infty. \tag{6.4}$$

To check (6.4), we observe that when $s < \sqrt{t}$ one has that:

$$\frac{|x|^2}{4t} - \frac{|x-y|^2}{4(t-s)} \ge -\frac{|x|^2\sqrt{t}}{4t(t-\sqrt{t})} - \frac{|y|^2}{4(t-\sqrt{t})} - \frac{|x||y|}{2(t-\sqrt{t})}.$$

Thus, when $|x|^2 < Ct\log t$ and $|y| \le \varepsilon\sqrt{t/\log t}$, it turns out that:

$$\frac{|x|^2}{4t} - \frac{|x-y|^2}{4(t-s)} \ge -\frac{\sqrt{C}\varepsilon}{2} \quad \text{for } t \gg 1,$$

whereupon (6.4) follows. Write now:

$$I \equiv \int_o^{h(t)} \int_{\mathbb{R}^N} Pu^p dyds + \int_{h(t)}^t \int_{\mathbb{R}^N} Pu^p dyds \equiv I_1 + I_2, \tag{6.5}$$

where $h(t) = t^{1-\varepsilon}$ and $\varepsilon \in (0,1)$ will be selected presently,

$$I_1 = \int_o^{h(t)} \int_{|y|\le M(s)} Pu^p dyds + \int_o^{h(t)} \int_{|y|>M(s)} Pu^p dyds \equiv I_{11} + I_{12}, \tag{6.6}$$

where $M(s) = (Ms\log s)^{\frac{1}{2}}$ with $M > 2(\alpha - N)$.

Step 2. – There holds:

$$I_{11} \leq (1+o(1)) \int_o^{h(t)} \int_{|y|\leq M(s)} u^p(y,s)dyds \qquad \text{for } t \gg 1. \qquad (6.7)$$

To check (6.7), we notice that $|x|^2/4t - |x-y|^2/4(t-s) \leq |x|^2/4t - |x-y|^2/4t \leq |x||y|/2t$. Therefore, for $|x|^2 \leq Ct\log t$ and $|y| \leq M(s)$, we have that:

$$\frac{|x||y|}{2t} \leq \frac{1}{2t}(Ct\log t)^{\frac{1}{2}} M(h(t)) \sim \frac{C}{2t}(t\log t)^{\frac{1}{2}}(Mh(t)\log t)^{\frac{1}{2}} \equiv G(t),$$

where $\lim\limits_{t\to\infty} G(t) = 0$. It then turns out that, as $t \to \infty$:

$$I_{11} \leq e^{G(t)} \int_o^{h(t)} \left(\frac{t}{t-s}\right)^{\frac{N}{2}} \int_{|y|\leq M(s)} u^p(y,s)dyds,$$

whence (6.7). Let us examine now I_{12} in (6.6). We shall prove that:

Step 3. – There holds:

$$I_{11} \leq (1+o(1)+O(\varepsilon)) \int_o^{h(t)} \int_{|y|>M(s)} u^p(y,s)dyds \qquad \text{for } t \gg 1. \qquad (6.8)$$

To derive (6.8), we proceed as follows. Firstly, we observe that for t large enough and small $\varepsilon > 0$:

$$\begin{aligned} I_{12} &\leq (1+o(1)) \int_o^{h(t)} \int_{|y|>M(s)} P(x,t;y,s)u^p(y,s)dyds \\ &\equiv (1+o(1)) \left(\int_o^{h(t)} \int_{|y|>M(s),|y|>\varepsilon\sqrt{t/\log t}} Pu^p dyds \right. \\ &\qquad \left. + \int_o^{h(t)} \int_{|y|>M(s),|y|\leq\varepsilon\sqrt{t/\log t}} Pu^p dyds \right) \\ &\equiv I_{121} + I_{122}. \end{aligned} \qquad (6.9)$$

It is easy to see now that I_{122} satisfies (6.8), since $|x|^2/4t - |x-y|^2/4(t-s) \leq |x||y|/4t \leq M\varepsilon$. To estimate I_{121}, we take $\alpha_1 > 1$ and $\alpha_2 > 0$ such that $\alpha_1 + \alpha_2 = p$. Taking advantage of Lemma 6.1, we then have that:

$$\begin{aligned} I_{121} &\leq C\exp\left(\frac{|x^2|}{4t}\right) \int_o^{h(t)} (s+T)^{-\frac{\alpha_2}{p-1}} \int_{|y|\leq\varepsilon\sqrt{t/\log t}} |y|^{-\alpha\alpha_1} dyds \\ &\leq C\exp\left(\frac{|x^2|}{4t}\right) t^{\left(1-\frac{\alpha_2}{p-1}\right)(1-\varepsilon)} \left(\frac{t}{\log t}\right)^{\frac{N-\alpha\alpha_1}{2}} \end{aligned}$$

Since $|x|^2 \geq kt\log t$ with $k < 2(\alpha - N)$, $\exp\left(\frac{|x|^2}{4t}\right) \leq t^{\frac{\alpha - N}{2}}$, whence:

$$I_{121} \leq Ct^{-\gamma}(\log t)^C,$$

for $\gamma = \frac{\alpha}{2}(\alpha_1 - 1) - \left(1 - \frac{\alpha_2}{p-1}\right)(1 - \varepsilon)$, and $C = \frac{\alpha\alpha_1 - N}{2}$. A quick check reveals that $\gamma = \left(\frac{\alpha}{2} - \frac{1}{p-1}\right)(\alpha_1 - 1) + O(\varepsilon)$, which can be selected positive for $\alpha_1 > 1$ and $0 < \varepsilon \ll 1$, in which case $I_{121} = o(1)$ as $t \to \infty$ and (6.8) follows.

We have yet to deal with I_2 in (6.5). Analysis of that integral will require of further splittings in its domain of integration. To this end, we shall write:

$$I_2 = \int_{h(t)}^{t} \int_{|y| \leq M(s)} Pu^p + \int_{h(t)}^{t} \int_{|y| > M(s)} Pu^p \equiv I_{21} + I_{22},$$

$$I_{22} = \int_{h(t)}^{t} \int_{|y| > M(s), |y| \geq \varepsilon\sqrt{t/\log t}} Pu^p$$

$$+ \int_{h(t)}^{t} \int_{|y| > M(s), |y| < \varepsilon\sqrt{t/\log t}} Pu^p \equiv I_{221} + I_{222},$$

$$I_{222} \equiv I_{2221} + I_{2222}, \tag{6.10}$$

where integration in I_{2221} (resp. in I_{2222}) is performed in the region where $|x - y| \geq \varepsilon$ (resp. where $|x - y| < \varepsilon$). We then prove:

Step 4. – The following inequality holds true:

$$I_2 \leq (1 + o(1) + O(\varepsilon)) \int_{h(t)}^{t} \int_{\mathbb{R}^N} u^p(y, s)dyds \quad \text{for } t \gg 1. \tag{6.11}$$

To check (6.11) we first remark that:

$$I_{2221} \leq (1 + o(1) + O(\varepsilon)) \int_{h(t)}^{t} \int_{|y| > M(s)} u^p(y, s)dyds \quad \text{for } t \gg 1. \tag{6.12}$$

As a matter of fact, (6.12) follows from the fact that $(t - s)^{-\frac{N}{2}} \exp\left(-\frac{|x-y|^2}{4t}\right) \leq t^{-\frac{N}{2}} \exp\left(-\frac{|x-y|^2}{4t}\right)$ since $|x - y| \neq 0$. One thus has that $P(x, t; y, s) \leq \exp\left(\frac{|x|^2}{4t} - \frac{|x-y|^2}{4t}\right)$, and the result is obtained by repeating the argument already used to estimate I_{122} in Step 3. We next show that:

$$I_{2222} = o(1) \quad \text{as } t \to \infty. \tag{6.13}$$

To derive (6.13), we note that $|y| < \varepsilon(t(\log t)^{-1})^{\frac{1}{2}}$ and $|x-y| < \varepsilon$ give at once $|x| < \varepsilon(t(\log t)^{-1})^{\frac{1}{2}}$, whence $\exp\left(\frac{|x|^2}{4t}\right) \leq C < \infty$. Take now γ_1 and γ_2 positive and such that $\gamma_1 < 1$, $\gamma_2 > p-1, \gamma_1 + \gamma_2 = p$. Denoting by Σ the region of variation of y, we then have that:

$$I_{2222} \leq Ct^{\frac{N}{2}} \int_{t^{1-\varepsilon}}^{t} (t-s)^{-\frac{N}{2}} s^{-\frac{\gamma_2}{p-1}} ds \int_{\Sigma} \exp\left(-\frac{|x-y|^2}{4(t-s)}\right) |y|^{-\gamma_1 \alpha} dy.$$

Since $|y| \geq M(s) \geq M(t^{1-\varepsilon})$, it turns out that $|y|^{-\alpha\gamma_1} \leq Ct^{-\frac{\alpha\gamma_1}{2}(1-\varepsilon)}(\log t)^C$. Therefore, for $t \gg 1$:

$$\begin{aligned} I_{2222} &\leq Ct^{\frac{N}{2}-\frac{\alpha\gamma_1}{2}(1-\varepsilon)}(\log t)^C \int_{t^{1-\varepsilon}}^{t} (t-s)^{-\frac{N}{2}} s^{-\frac{\gamma_2}{p-1}} ds \int_{\Sigma} \exp\left(-\frac{|x-y|^2}{4(t-s)}\right) dy \\ &\leq Ct^{\frac{N-\alpha\gamma_1(1-\varepsilon)}{2}}(\log t)^C \int_{t^{1-\varepsilon}}^{t} s^{-\frac{\gamma_2}{p-1}} ds \leq Ct^{\frac{N-\alpha\gamma_1(1-\varepsilon)}{2}}(\log t)^C = o(1), \end{aligned}$$

provided that $\varepsilon > 0$ is small enough. On the other hand, arguing as for I_{121} (cf. (6.9)) we easily see that:

$$I_{221} = o(1) \quad \text{as } t \to \infty. \tag{6.14}$$

Let us write now $\Sigma_1 = \{(y,s) : |y| \leq M(s), |y|^2 \leq 2(\alpha - N)(1-\varepsilon)s \log s\}$, $\Sigma_2 = \{(y,s) : 2(\alpha - N)(1-\varepsilon)s\log s \leq |y|^2 \leq (M(s))^2\}$. We next set:

$$I_{21} = \int_{h(t)}^{t} \int_{\Sigma_1} Pu^p + \int_{h(t)}^{t} \int_{\Sigma_2} Pu^p \equiv I_{211} + I_{212}. \tag{6.15}$$

To bound I_{212}, we make use of the estimate:

$$u^p(y,s) \leq Ct^{\left(\frac{\varepsilon}{2}(\alpha-N)p-\frac{\alpha}{2}\right)p(1-\varepsilon)} \quad \text{when } (y,s) \in \Sigma_2 \text{ and } h(t) \leq s \leq t. \tag{6.16}$$

Inequality (6.16) is derived as follows. By (3.26), one certainly has that $u(y,s) \leq Cs^{-\frac{N}{2}} e^{-\frac{|y|^2}{4s}}$ at $|y| = (2(\alpha - N)(1-\varepsilon)s\log s)^{\frac{1}{2}}$, whence $u^p(y,s) \leq C|y|^{-\alpha p} \leq Cs^{\left(\frac{\varepsilon(\alpha-N)}{2}-\frac{\alpha}{2}\right)p} \equiv f_1(s)$ there. On the other hand, by (e1), $u(y,s) \leq C|y|^{-\alpha} \leq C(s\log s)^{-\frac{\alpha}{2}} \equiv f_2(s)$ at $|y| = M(s)$. Since $f_2(s) \leq f_1(s)$ for $s \gg 1$, it follows from the maximum principle that $f_1(s)$ provides a bound for $u^p(y,s)$ in all Σ_2, whereupon (6.16) follows by recalling that $s \geq h(t) \equiv t^{1-\varepsilon}$ under our current assumptions.

A quick check reveals now that, if we set $b_1 = \frac{p}{2}(\varepsilon(\alpha - N) - \alpha)p(1-\varepsilon)$ and recall that $t^{\frac{N}{2}} \exp\left(\frac{|x|^2}{4t}\right) < t^{\frac{\alpha}{2}}$ for $|x|^2 < 2(\alpha - N)t\log t$, then:

$$\begin{aligned} I_{212} &\le Ct^{\frac{\alpha}{2}+b_1} \int_o^t (t-s)^{-\frac{N}{2}} \int_{\mathbb{R}^N} \exp\left(-\frac{|x-y|^2}{4(t-s)}\right) dyds \\ &\le Ct^{\frac{\alpha}{2}+b_1+1} = o(1) \qquad \text{for } t \gg 1, \end{aligned} \tag{6.17}$$

since $\frac{\alpha}{2} + b_1 + 1 = 1 - \frac{\alpha}{2}(p-1) + O(\varepsilon)$ for $0 < \varepsilon \ll 1$. To conclude with the proof, we have yet to show that:

$$I_{211} = o(1) \qquad \text{as } t \to \infty. \tag{6.18}$$

To obtain (6.18), we take advantage of the fact that, for $|y|^2 \le 2(\alpha - N)(1-\varepsilon)s\log s$ with $h(t) \le s \le t$, one has that:

$$u^p(y,s) \le Cs^{-\frac{Np}{2}} \exp\left(-\frac{p|y|^2}{4s}\right) \le Ct^{-\frac{N}{2}p(1-\varepsilon)} \exp\left(-\frac{p|y|^2}{4t}\right).$$

We next take a, b positive and such that $a + b = \frac{1}{4}$. Clearly,

$$\exp\left(\frac{|x|^2}{4t} - \frac{|x-y|^2}{4(t-s)}\right) = \exp\left(\frac{|x|^2}{4t} - (a+b)\frac{|x-y|^2}{(t-s)}\right),$$

and:

$$\begin{aligned} \exp\left(\frac{|x|^2}{4t} - a\frac{|x-y|^2}{(t-s)}\right) u^p(y,s) &\le Ct^{-\frac{N}{2}p(1-\varepsilon)} \\ \exp\left(\frac{|x|^2}{4t} - a\frac{|x-y|^2}{(t-s)} - \frac{|y|^2 p}{4t}\right). \end{aligned} \tag{6.19}$$

We then have that:

$$A(t) \equiv \left(\frac{|x|^2}{4t} - a\frac{|x-y|^2}{(t-s)} - \frac{|y|^2 p}{4t}\right) = o(1) \quad \text{as } t \to \infty,$$

whenever $|y| \le M(s)$ and $|x|^2 < 2(\alpha - N)t\log t$, with $t^{1-\varepsilon} \le s \le t$. (6.20)

To keep the main flow of the arguments here, we shall assume (6.20) for a moment to conclude the proof. From (6.19) and (6.20), it follows that:

$$\begin{aligned} I_{211} &\le Ct^{-\frac{N}{2}p(1-\varepsilon)} \exp(A(t)) t^{\frac{N}{2}} \int_o^t (t-s)^{-\frac{N}{2}} \int_{\mathbb{R}^N} \exp\left(-\frac{b|x-y|^2}{(t-s)}\right) dyds \\ &\le Ct^{-\frac{N}{2}(p-1)+1+O(\varepsilon)} = o(1) \qquad \text{as } t \gg 1. \end{aligned}$$

Finally, to derive (6.20) we proceed as follows. Suppose first that $|x-y| \leq r(t) \ll \sqrt{t}$, as $t \to \infty$. Then, since $|x|^2 \leq (|x-y|+|y|)^2$, we readily see that:

$$\begin{aligned} A(t) &\leq \frac{1}{t}\left(\left(\frac{1}{4}-a\right)|x-y|^2 + \frac{|y||x-y|}{2}\right) \\ &\leq \frac{1}{t}\left(\left(\frac{1}{4}-a\right)r^2(t) + \frac{M(t)r(t)}{2}\right) \\ &= o(1) \qquad \text{as } t \to \infty. \end{aligned}$$

On the other hand, if $|x-y| \geq r(t)$ then $|y| \geq r(t) - |x|$, and

$$\begin{aligned} A(t) &\leq \frac{1}{t}\left(\frac{|x|^2}{4}(1-p) - a|x-y|^2 - \frac{r^2(t)p}{4} + p\frac{r(t)|x|}{2}\right) \leq p\frac{r(t)|x|}{2t} \\ &= o(1) \qquad \text{as } t \to \infty. \end{aligned}$$

From Lemmata 6.1 and 6.2, it follows at once that Theorem 5 holds. ■

6.2. The case $\alpha = N > \dfrac{2}{p-1}$

We now consider the situation where the asymptotics is encoded in that of the corresponding linear heat equation. The behaviour in external regions is as follows:

LEMMA 6.3. – *Let $u(x,t)$ be the solution of (1.1)-(1.3), and assume that $\alpha = N > \frac{2}{p-1}$. One then has that:*

$$\begin{aligned} &u(x,t) \sim A|x|^{-\alpha} \quad \textit{as } t \to \infty, \\ &\textit{uniformly on sets } \{|x|^2 \geq kt\log t\} \textit{ with } k > 4. \end{aligned} \tag{6.21}$$

Proof. – It is entirely similar to that of Lemma 6.1, and will therefore be omitted. ■

Stabilization on internal regions is described in the following:

LEMMA 6.4. – *Let $u(x,t)$ be the solution of (1.1)-(1.3), and assume that $\frac{2}{p-1} < N = \alpha$. Then the following estimate holds:*

$$u(x,t) = \frac{A\omega_N}{2}\log t(4\pi t)^{-\frac{N}{2}}\exp\left(-\frac{|x|^2}{4t}\right)(1+o(1)) \qquad \textit{as } t \to \infty, \tag{6.22}$$

uniformly on sets $|x|^2 \leq kt\log(\log t)$ with $0 < k < 4$.

Proof. – Recalling the derivation of (3.36) in Lemma 3.3, it will suffice to show that:

$$I \equiv \int_o^t \left(\frac{t}{t-s}\right)^{\frac{N}{2}} ds \int_{\mathbb{R}^N} \exp\left(\frac{|x|^2}{4t} - \frac{|x-y|^2}{4(t-s)}\right) u^p(y,s) dy$$

is such that (6.23)

$$I \ll \log t \quad \text{as } t \to \infty \quad \text{on } \Sigma. \tag{6.24}$$

To this end we take up the argument in Lemma 6.2 and adapt it as follows. We first split I exactly as in (6.5), and subdivide again the term I_1 thus obtained as in (6.6), but with a different choice of $M(s)$. Namely, we now take $M(s) = (Ms \log\log s)^{\frac{1}{2}}$ with $M > 4$. One then readily checks that:

$$I_{11} = O(1) \quad \text{as } t \to \infty \ \text{ on } \Sigma. \tag{6.25}$$

To estimate I_{12}, we split it in the form:

$$\begin{aligned} I_{12} = \int_o^{h(t)} \int_{\mathbb{R}^N} Pu^p dyds = \int_o^{h(t)} \int_{\Lambda_1} Pu^p dyds \\ + \int_o^{h(t)} \int_{\Lambda_2} Pu^p dyds \equiv I_{121} + I_{122} \end{aligned} \tag{6.26}$$

where now $\Lambda_1 = \{(y,s) : |y| > M(s) \text{ and } |y| > \varepsilon(s(\log(\log s))^{-1})^{\frac{1}{2}}\}$ and $\Lambda_2 = \{(y,s) : |y| > M(s) \text{ and } |y| \le \varepsilon(s(\log(\log s))^{-1})^{\frac{1}{2}}\}$. Just as in our previous Lemma, it is easy to check that:

$$I_{122} = O(1) \quad \text{as } t \to \infty \ \text{ in the region under consideration.} \tag{6.27}$$

To bound I_{121}, we select α_1 and α_2 as when obtaining the corresponding estimate in Lemma 6.2. Arguing as in that result, and observing that now $\frac{|x|^2}{4t} \le C\log(\log t)$, we now obtain:

$$\begin{aligned} I_{121} &\le C \exp\left(\frac{|x|^2}{4t}\right) t^{\left(1-\frac{\alpha_2}{p-1}\right)(1-\varepsilon)} \left(\frac{t}{\log\log t}\right)^{\frac{N}{2}(1-\alpha_1)} \\ &\le Ct^{\sigma}(\log t)^{C_1}(\log\log t)^{C_2}, \end{aligned}$$

where $\sigma = \left(1 - \frac{\alpha_2}{p-1}\right) + \frac{N}{2}(1-\alpha_1) = (\alpha_1 - 1)\left(\frac{1}{p-1} - \frac{N}{2}\right) < 0$ provided that $\alpha_1 > 1$. We have thus obtained that:

$$I_{121} = o(1) \qquad \text{as } t \to \infty \qquad \text{for } (x,t) \text{ as in (4.36).} \tag{6.28}$$

We next set out to control I_2. This we shall do by splitting that integral as indicated in (6.10)-(6.10) with the modifications already introduced:

namely, we take now $M(s) = (Ms\log s)^{\frac{1}{2}}$ with $M > 4$, and in (6.10) we replace $|y| \geq \varepsilon(t(\log t)^{-1})^{\frac{1}{2}}$ (resp. $|y| < \varepsilon(t(\log\log t)^{-1})^{\frac{1}{2}}$) by $|y| \geq \varepsilon(t(\log\log t)^{-1})^{\frac{1}{2}}$ (resp. $|y| < \varepsilon(t(\log\log t)^{-1})^{\frac{1}{2}}$). As before, we readily check that $I_{2221} = o(1)$ in the region under consideration. The corresponding estimate for I_{2222} is a minor modification of that obtained in the previous Lemma. Namely, we now derive:

$$\begin{aligned} I_{2222} &\leq Ct^{\frac{N}{2}(1-\gamma_1(1-\varepsilon))}(\log\log t)^C \cdot t^{1-\frac{\gamma_2}{p-1}} \\ &\leq Ct^{-\chi}(\log\log t)^C, \end{aligned}$$

where $\chi = \frac{N}{2}(\gamma_1(1-\varepsilon)-1) - 1 + \frac{p-\gamma_1}{p-1} = \left(\frac{N}{2} - \frac{1}{p-1}\right)(\gamma_1 - 1) < 0$, provided that $\gamma_1 < 1$ as in the previous case. The term I_{221} is once again estimated as in Lemma 6.2, to yield that $I_{221} = o(1)$ for large t in the region under consideration. Finally, I_{21} is first splitted as in (6.15), with obvious modifications in the definition of sets Σ_1 and Σ_2. The contribution arising from I_{211} is of similar order (and is analogously obtained) as in the previous Lemma. Finally, I_{212} is easily shown to be $o(1)$ as $t \to \infty$ in the region considered by using the bound:

$$u(y,s) \leq s^{-\frac{N}{2}}(\log s)^{\varepsilon_1} \quad \text{for some } \varepsilon_1, \text{ whenever } (y,s) \in \Sigma_1,$$

which is derived in a similar way as (6.16). Summing up, we have obtained that:

$$I_2 = o(1) \quad \text{as } t \to \infty \text{ in the region under consideration.} \tag{6.29}$$

Putting together (6.25)-(6.29), (6.22) follows, and the proof of the Lemma is now complete. ■

Theorem 4 is now a consequence of Lemmata 6.3 and 6.4.

6.3. The proof of Theorem 6

We note that the result for the outer region follows by arguing as in the previous cases. To obtain the corresponding behaviour for the inner region, we shall proceed in several steps. To begin with, we obtain the following estimate:

LEMMA 6.5. – *Let $u(x,t)$ be the solution of (1.1)-(1.3), and assume that $\alpha > N = \frac{2}{p-1}$. Then there exist positive constants δ, M such that for t large:*

$$\delta \leq u(x,t)(t\log t)^{\frac{N}{2}}e^{\frac{|x|^2}{4t}} \leq M, \text{ on sets } \{(x,t) : |x| \leq C\sqrt{t}\} \text{ with } C > 0. \tag{6.30}$$

Proof. – We define for fixed positive T the functions:

$$\underline{u}(x,t) = \delta((t+T)\log(t+T))^{-\frac{N}{2}} e^{-\frac{|x|^2}{4(t+T)}},$$

and

$$\overline{u}(x,t) = \begin{cases} M((t+T)\log(t+T))^{-\frac{N}{2}} e^{-\frac{|x|^2}{4(t+T)(1+(\log(t+T))^{-1})}} \\ \quad + E(t+T)^{-\frac{\alpha+N}{4}} \text{ if } |x| \le \sqrt{t+T}, \\ M((t+T)\log(t+T))^{-\frac{N}{2}} e^{-\frac{|x|^2}{4(t+T)(1+(\log(t+T))^{-1})}} \\ \quad + E|x|^{-\alpha}(t+T)^{\frac{\alpha-N}{4}} \text{ if } |x| \ge \sqrt{t+T}, \end{cases} \tag{6.31}$$

for δ small and positive, and for M, E large. We now apply the maximum principle in $|x| \le C\sqrt{t+T}$ for any C larger than a certain constant (this can be seen by using the technique developed in Section 4, together with Lemma 6.1), and hence we obtain the proof. ■

To proceed further, a suitable subsolution will be derived on regions of the type $|x| \ge C(T+t)^{\frac{1}{2}}$ with $C \gg 1$. Namely, we will prove:

LEMMA 6.6. – *Let $T \ge 1$ be given. For any $M > 0$ and for any θ with $1 < \theta < 2$, there exists $R_o = R_o(M, \theta, N, p) > 0$ such that the function $\underline{u}(x,t)$ given by:*

$$\underline{u}(x,t) = M((t+T)\log(t+T))^{-\frac{N}{2}} \exp\left(-\left(\frac{1}{4(t+T)} + \frac{B}{(t+T)^\theta}\right)|x|^2\right), \tag{6.32}$$

satisfies:

$$(\underline{u})_t \le \Delta\underline{u} - \underline{u}^p \qquad \textit{for} \quad |x| \ge R_o(t+T)^{\frac{1}{2}} \textit{ and } t \ge 0, \tag{6.33}$$

for any nonnegative value of the free parameter B. Pick now ε such that $0 < \varepsilon \ll 1$, and let $\underline{u}$ be the function in (6.32) with $M_1 = \delta - \varepsilon$, where δ is given in the previous Lemma. Let $u(x,t)$ be the solution of (1.1)-(1.3). Then there exists $t_o \gg 1$ such that:

$$\underline{u}(x,t_o) \le u(x,t_o) \quad \textit{for } |x| \ge R_o(t_o+T)^{\frac{1}{2}}, \tag{6.34}$$

$$\underline{u}(x,t) \le u(x,t) \quad \textit{at } |x| = R_o(t+T)^{\frac{1}{2}} \textit{ with } t \ge t_o. \tag{6.35}$$

Proof. – Checking that $\underline{u}(x,t)$ is a subsolution in the region under consideration is made by means of a direct computation. To derive (6.34), we first select $t_o \gg 1$ so that:

$$u(x,t_o) \ge (\delta - \frac{\varepsilon}{2})((t_o+T)\log(t_o+T))^{-\frac{N}{2}} \exp\left(-\frac{|x|^2}{4(t_o+T)}\right), \tag{6.36}$$

for $|x| = R_o(t_o+T)^{\frac{1}{2}}$ and $t \geq t_o$, where T is fixed (this can be done by (6.30)). We then take $B \gg 1$ such that for $|x| \geq R_o(t_o+T)^{\frac{1}{2}}$:

$$u(x,t_o) \geq (\delta-\varepsilon)((t_o+T)\log(t_o+T))^{-\frac{N}{2}} \exp\left(-\left(\frac{1}{4(t_o+T)}+\frac{B}{(t_o+T)^{\theta}}\right)|x|^2\right). \quad (6.37)$$

To obtain (6.37) we take advantage of the fact that $u(x,t_o) > 0$ by the strong maximum principle. From this and the external Lemma 6.1, we merely have to observe that the quantity on the right in (6.37) decreases exponentially in B to derive the sought-for inequality. Finally, to derive (6.35) it suffices to check that:

$$(\delta-\frac{\varepsilon}{2})((t+T)\log(t+T))^{-\frac{N}{2}}e^{-\frac{|x|^2}{4(t+T)}} \geq (\delta-\varepsilon)((t_o+T)\log(t_o+T))^{-\frac{N}{2}}e^{-\left(\frac{1}{4(t_o+T)}+\frac{B}{(t_o+T)^{\theta}}\right)|x|^2},$$

whenever $|x| = R_o(t+T)^{\frac{1}{2}}$ with $t \geq t_o \gg 1$. Such inequality holds whenever $\exp\left(-\frac{BR^2}{(t+T)^{\theta-1}}\right) \leq 1 \leq \frac{\delta-\frac{\varepsilon}{2}}{\delta-\varepsilon}$, and the proof is now complete.■

As a natural complement to Lemma 6.6, a suitable supersolution in the same type of external zones is provided in the following:

LEMMA 6.7. – *Let $T \geq 1$ be given. For any $M_1 > 0, \delta > 0$ and $\gamma > \frac{N}{2}$, there exists $R_o > 0$ and $t_o > 0$ (both depending on γ, δ, M_1, N and p) such that the function $\overline{u}(x,t)$ given by:*

$$\overline{u}(x,t) = (t+T)^{-\frac{N}{2}}\Big(M_1(\log(t+T))^{-\frac{N}{2}} +M_2(\log(t+T))^{-\gamma}\Big)e^{-\frac{|x|^2}{4(t+T)\left(1+\frac{1}{\log(t+T)}\right)}} + M_3|x|^{-\alpha}(t+T)^{\delta} \equiv \overline{u}_1(x,t)+\overline{u}_2(x,t)+\overline{u}_3(x,t), \quad (6.38)$$

satisfies:

$$\overline{u}_t \geq \Delta\overline{u} \qquad \text{for } |x| \geq R_o(t+T)^{\frac{1}{2}} \text{ and } t \geq t_o, \quad (6.39)$$

for any value of the free parameters M_2 and M_3. Take now ε such that $0 < \varepsilon \ll 1$, and let $\overline{u}(x,t)$ be the function in (6.38) with $M_1 = M+\varepsilon$, where M is the constant of Lemma 6.5, and $M_3 = 2A$. Let $u(x,t)$ be the solution of (1.1)-(1.3). Then there exists a constant C^* such that:

$$u(x,t_o) \leq \overline{u}(x,t_o) \quad \text{for } |x| \geq R_o(t_o+T)^{\frac{1}{2}}, \quad (6.40)$$

$$u(x,t) \leq \overline{u}(x,t) \quad \text{at } |x| = R_o(t+T)^{\frac{1}{2}} \text{ with } t \geq t_o. \quad (6.41)$$

Proof. – As in our previous result, the fact that $\overline{u}(x,t)$ given in (6.38) satisfies (6.39) follows from a direct computation. Fix now $T \geq 1$. As to (6.40) we observe that, by Lemma 6.5 for $|x| = R_o(t+T)^{\frac{1}{2}}$ and $t \geq t_o \gg 1$:

$$u(x,t) \leq (M + \frac{\varepsilon}{2})((t+T)\log(t+T))^{-\frac{N}{2}} \exp\left(-\frac{|x|^2}{4(t+T)}\right). \qquad (6.42)$$

By Lemma 6.1, (6.40) certainly holds for $|x| \geq S(t_o)$, where function $S(t)$ is selected so that $S(t) \gg (t\log t)^{\frac{1}{2}}$ when $t \gg 1$: we just have to observe that $u(x,t_o) \leq \overline{u}_3(x,t_o)$ in such regions. On the other hand, when $R_o(t_o+T)^{\frac{1}{2}} \leq |x| \leq S(t_o)$ we take $M_2 \gg 1$ in $\overline{u}_2(x,t)$ to derive that $u(x,t_o) \leq \overline{u}_2(x,t_o)$ there. Summing these results up, (6.40) follows. To check (6.41), we merely have to remark that $\overline{u}_1(x,t)$ is larger than the term coming from the right of (6.30) for $t \gg 1$. We shall omit further details. ■

End of the proof of Theorem 6:

By the maximum principle, we now have that, for any $\varepsilon > 0$:

$$\underline{u}(x,t) \leq u(x,t) \leq \overline{u}(x,t) \qquad \text{for } |x| \geq R(t+T)^{\frac{1}{2}} \text{ and } t \geq t_o.$$

where $T \geq 1$ is fixed, and R and t_o are large enough. We note that $\underline{u}(x,t) \sim (\delta - \varepsilon)(t\log t)^{-\frac{N}{2}} e^{-\frac{|x|^2}{4t}}$ for $t \gg 1$ and $\overline{u}(x,t) \sim (M+\varepsilon)(t\log t)^{-\frac{N}{2}} e^{-\frac{|x|^2}{4t}}$ when $R\sqrt{t} \leq |x| \leq D\sqrt{t\log t}$, provided that $D < \sqrt{2(\alpha - N)}$. The bound on the value of D is readily obtained by looking for the region where $\overline{u}_1 \sim \overline{u}_3$.

We now return to the spectral analysis which was already introduced in Section 2. We recall the self-similar variables:

$$u(x,t) = t^{-\frac{1}{p-1}}\Phi(y,\tau); \quad y = xt^{-\frac{1}{2}}, \quad \tau = \log t. \qquad (6.43)$$

With the new variables, equation (1.1) can be recast in the form:

$$\Phi_\tau = \Delta\Phi + \frac{1}{2}y\nabla\Phi + \frac{1}{p-1}\Phi - \Phi^p \equiv A_o\Phi + \frac{1}{p-1}\Phi - \Phi^p. \qquad (6.44)$$

Spectral properties of operator A_o were studied in Section 2 and will be used in the sequel. Because of the previous Lemmata, we have the bound:

$$(\delta - \varepsilon)\tau^{-\frac{N}{2}} e^{-\frac{|y|^2}{4}} \leq \Phi(y,\tau) \leq (M+\varepsilon)\tau^{-\frac{N}{2}} e^{-\frac{|y|^2}{4}}, \qquad (6.45)$$

uniformly in $\{|y| \leq C\sqrt{\tau}\}$ for $C < \sqrt{2(\alpha - N)}$. We now define the function:

$$\hat{\Phi}(y,\tau) = \Phi(y,\tau) f(y,\tau), \qquad (6.46)$$

where $f(y,\tau) \equiv f(|y|/\sqrt{\tau})$ is a smooth and nonnegative cut-off function such that $f(\xi) = 1$ if $\xi < C_1$ and $f(\xi) = 0$ if $\xi > C_2$, for some constants $0 < C_1 < C_2 < \sqrt{2(\alpha - N)}$. The previous bound (6.45) ensures that the new function $\hat{\Phi}(y,\tau)$ belongs to $L^2_\omega(I\!R^N)$ (cf. Section 2) for large τ. On the other hand, it can be easily seen that:

$$\hat{\Phi}_\tau = A_o\hat{\Phi} + \frac{1}{p-1}\hat{\Phi} - f\hat{\Phi}^p + G(y,\tau), \tag{6.47}$$

where $G(y,\tau) = \Phi[f_\tau - \Delta f - \frac{1}{2}y\nabla f] - 2\nabla\Phi\nabla f$. We next define:

$$\hat{\Phi} = g(\tau)\psi_o(y) + R(y,\tau), \tag{6.48}$$

where $g(\tau) = < \hat{\Phi}, \psi_o >$ and $\psi_o(y)$ is the first eigenfunction of the operator A_o. Our goal now is to prove that $\hat{\Phi} \sim g(\tau)\psi_o(y)$ as time grows, uniformly in some region. Function $R(y,\tau)$ satisfies the equation:

$$R_\tau - A_oR - \frac{1}{p-1}R = -g'(\tau)\psi_o(y) - f\hat{\Phi}^p + G(y,\tau) \equiv H(y,\tau). \tag{6.49}$$

From (6.45) it is easy to obtain the estimate $< H,H >= O(\tau^{-\frac{N}{2}-1})$. Therefore, using the variation of constants formula and the orthogonality of R to ψ_o, we obtain for some k (with $|k| \geq 1$):

$$< R,R > = O(e^{-\nu_k\tau}) + \int_{\tau_o}^{\tau} e^{-\nu_k(\tau-s)}O(s^{-\frac{N}{2}-1})ds = O(\tau^{-\frac{N}{2}-1}), \tag{6.50}$$

where $\nu_k = -\lambda_k - \frac{1}{p-1} = \frac{|k|}{2}$, λ_k being the eigenvalue of the leading mode. Therefore by regularizing effects for parabolic equations we have that $|R(y,\tau)| \leq C\tau^{-\frac{N}{2}-1}$ for $\tau \gg 1$ and $|y| \leq M$. In particular we obtain that $\Phi \sim g(\tau)\psi_o(y)$ as $\tau \gg 1$ uniformly in compact sets $|y| \leq R$.

Substituting now (6.48) into (6.47) and taking the scalar product with $\psi_o(y)$, we obtain:

$$\begin{aligned} g'(\tau) &= - < f\hat{\Phi}^p, \psi_o > + < G, \psi_o > \\ &= -\int_{|y|\leq M} \Phi^p\psi_o(y)e^{\frac{|y|^2}{4}}dy - \int_{|y|\geq M} f\hat{\Phi}^p\psi_o(y)e^{\frac{|y|^2}{4}}dy \\ &\quad + \int_{I\!R^N} G(y,\tau)\psi_o(y)e^{\frac{|y|^2}{4}}dy \\ &= -f_1(\tau) - f_2(\tau) + f_3(\tau). \end{aligned} \tag{6.51}$$

The convergence $\Phi \sim g(\tau)\psi_o(y)$ on compact sets gives that:

$$f_1(\tau) \sim -g^p(\tau) \int_{|y|\leq M} \psi_o^{p+1}(y) e^{\frac{|y|^2}{4}} dy.$$

We now have that by (6.45):

$$\begin{aligned} 0 < f_2(\tau) &\leq C \int_{M\leq|y|\leq C_2\sqrt{\tau}} \hat{\Phi}^p(u,\tau)dy \\ &\leq C\tau^{-\frac{N}{2}p} \int_{M\leq|y|} e^{-\frac{|y|^2}{4}p} = C_1 M^{-\frac{N}{2}} \tau^{-\frac{N}{2}p}, \end{aligned}$$

where C_1 is a constant which does not depend on M. Finally, on using the bound (6.45) for each one for the terms in $G(y,\tau)$, there holds:

$$|f_3(\tau)| \leq C \int_{C_1\sqrt{\tau}}^{C_2\sqrt{\tau}} |G(y,\tau)|dy = O(e^{-\varepsilon\tau}),$$

for some $\varepsilon > 0$. For instance, we have that for some $a, b > 0$ such that $a + b = \frac{1}{4}$:

$$\begin{aligned} \int_{C_1\sqrt{\tau}}^{C_2\sqrt{\tau}} |\Phi\Delta f|dy &\leq C \int_{|y|\geq C_1\sqrt{\tau}} \Phi dy \leq C \int_{|y|\geq C_1\sqrt{\tau}} \tau^{-\frac{N}{2}} e^{-\frac{|y|^2}{4}} dy \\ &\leq C\tau^{-\frac{N}{2}} \int_{|y|\geq C_1\sqrt{\tau}} e^{-a|y|^2-b|y|^2} dy \leq C\tau^{-\frac{N}{2}} e^{-aC_1^2\tau} = O(e^{-\varepsilon\tau}). \end{aligned}$$

From (6.51) we obtain, on taking the limit $M \to \infty$, that as $\tau \gg 1$:

$$g'(\tau) \sim -g^p(\tau) \int_{\mathbb{R}^N} \psi_o^{p+1}(y) e^{\frac{|y|^2}{4}} dy.$$

As $\psi_o(y) = (4\pi)^{-\frac{N}{4}} e^{-\frac{|y|^2}{4}}$ we have that $g(\tau) \sim (C_o(p-1)\tau)^{-\frac{1}{p-1}}$ for $C_o =< \psi_o^p, \psi_o >= (2\sqrt{\pi}p^{\frac{N}{2}})^{-1}$. Back to our original variables we obtain that $u(x,t) \sim C_N(t\log t)^{-\frac{N}{2}}$ in regions $|x|^2 \leq Ct$, where $C_N = (4\pi)^{-\frac{N}{2}}(C_o(p-1))^{-\frac{1}{p-1}} = (\frac{N}{2}(1+\frac{2}{N})^{\frac{N}{2}})^{\frac{N}{2}}$. To obtain the convergence in the regions $|x|^2 \leq Ct\log t$ for any $C < 2(\alpha - N)$ one only needs to apply the maximum principle to functions (6.32) and (6.38), taking now constants M and M_1 equal to $C_N \pm \varepsilon$, which completes the proof. ■

7. THE PROOF OF THEOREM 7

In this Section we shall be concerned with the case when:

$$N < \frac{2}{p-1} < \alpha. \tag{7.1}$$

Our first result is a non-optimal asymptotic estimate in an external region, namely:

LEMMA 7.1. – *Let $u(x,t)$ be the solution of (1.1)-(1.3), and assume that (7.1) holds. Then:*

$$\begin{aligned} &u(x,t) \sim A|x|^{-\alpha} \quad \text{as } t \to \infty \text{, uniformly on regions} \\ &\{(x,t) : |x|^2 \geq kt\log t\} \quad \text{with } k > 2(\alpha - N). \end{aligned} \tag{7.2}$$

Proof. – As we have done in the previous Section, we just compare $u(x,t)$ with the solution of (1.1)-(1.3) to derive a suitable upper bound, and then compare it from below with the solution $\underline{u}(x,t)$ of (1.1)-(1.3) with p replaced by $\overline{p} = 1 + \frac{2}{\alpha}$, this last function being estimated by means of Lemma 6.1. ■

A rather crude stabilization result in a parabolic inner region is provided by:

LEMMA 7.2. – *Let $u(x,t)$ be as in the statement of Lemma 7.1. One then has that:*

$$\begin{gathered} u(x,t) = t^{-\frac{1}{p-1}} g(x/\sqrt{t})(1+o(1)) \text{ as } t \to \infty, \\ \text{uniformly on sets } \{|x| \leq C\sqrt{t}\} \\ \text{with } C > 0, \text{ where } g \text{ is a nontrivial solution of (1.22).} \end{gathered} \tag{7.3}$$

Proof. – It follows from the results obtained in [7]. As a matter of fact, it has been proved there that under our assumptions:

$$\lim_{t\to\infty} t^{\frac{1}{p-1}} \Big(\sup_{x\in\mathbb{R}^N} |u(x.t) - t^{-\frac{1}{p-1}} g(x/\sqrt{t})| \Big) = 0,$$

whereupon (7.3) follows. ■

Let now γ be a positive real number, and let $F(s)$ be a solution of the following problem:

$$\begin{cases} F''(s) + \left(\dfrac{N-1}{s} + \dfrac{s}{2}\right)F'(s) + \left(\dfrac{1}{p-1} + \gamma\right)F(s) = 0 \quad \text{for } s > R_o, \\ F(R_o) = C, \quad F(s) > 0 \quad \text{for } s > R_o, \\ F(s) \sim s^{2(\frac{1}{p-1}+\gamma)-N} e^{-\frac{s^2}{4}} \quad \text{for } s \to \infty. \end{cases} \tag{7.4}$$

The existence of $F(s)$ for some positive constants C, R_o can be shown along the lines of (5.9), where the corresponding result for algebraically decreasing solutions was obtained. Recall now the change of variables given in (5.3). Then there holds:

LEMMA 7.3. – *For any positive numbers M, γ and ε, the function*

$$\underline{\Phi}(y,\tau) = (1-\varepsilon)g(y) - MF(y)e^{-\gamma\tau}, \tag{7.5}$$

is a subsolution of (4.20) in the region where Φ is nonnegative. Moreover, for any R_o large enough , there exist $\tau_o > R_o, M > 0$ such that:

$$\underline{\Phi}(y,\tau_o) \le \Phi(y,\tau_o) \qquad \text{for } R_o \le y \le k\tau_o, \text{ with } k > 2(\alpha - N), \tag{7.6}$$

$$\underline{\Phi}(y,\tau) \le \Phi(y,\tau) \qquad \text{for } y = R_o \text{ and } \tau \ge \tau_o, \tag{7.7}$$

$$\underline{\Phi}(y,\tau) \le \Phi(y,\tau) \qquad \text{for } y = k\tau \text{ and } \tau \ge \tau_o, \tag{7.8}$$

Proof. – Checking that (7.5) is a subsolution where positive follows from a straightforward computation. On the other hand, we have that $\underline{\Phi}(y,\tau) \sim (1-\varepsilon)g(y)$ for $y = \tau \gg 1$, whence (7.8) follows by the external Lemma 7.1. Take now $R_o \gg 1$ (in particular we need it to be greater than the largest zero of F). Then, by Lemma 7.2 one has that $\Phi(R_o,\tau_o) \ge (1-\varepsilon)g(R_o) \ge \underline{\Phi}(y,\tau_o)$ which gives (7.7). It just remains to obtain (7.6). To this end, we observe that for fixed $\tau_o \gg 1, \Phi(y,\tau_o) \ge (1-\varepsilon)g(y) \ge \underline{\Phi}(y,\tau_o)$ for $|y| \ge B = B(\tau_o)$ in view of Lemma 7.2. By selecting R_o sufficiently large, we may always assume that $F(y) \ge C_1 > 0$ for $R_o < y < B$. Inequality (f6) follows now on taking $M \gg 1$ in (7.5). ■

A second subsolution that will used presently is provided by the following:

LEMMA 7.4. – *Let $\varepsilon > 0$ be such that $0 < \varepsilon < A$. Then if $\alpha > N$ there exists a constant $R_o = R_o(A, N, p, \alpha, \varepsilon, \gamma) > 0$ such that, for $F(s)$ and γ as before and $M > 0$ arbitrary, the function:*

$$\underline{u}(x,t) = \left((A-\varepsilon)|x|^{-\alpha} - MF\left(\frac{|x|}{\sqrt{t}}\right)t^{-\frac{1}{p-1}-\gamma}\right)_+ \tag{7.9}$$

satisfies:

$$\underline{u}_t \le \Delta\underline{u} - \underline{u}^p \qquad \text{for } |x|^2 \ge R_o t, \quad t > 1. \tag{7.10}$$

Moreover, for $0 < \varepsilon \ll 1$ *and* $t_o \gg 1$ *one has that:*

$$\underline{u}(x, t_o) \leq u(x, t_o) \quad \textit{for } |x|^2 \geq R_o t_o, \tag{7.11}$$

$$\underline{u}(x, t) \leq u(x, t) \quad \textit{for } |x|^2 = R_o t, \ \ t \geq t_o. \tag{7.12}$$

Proof. – s in the case of our previous Lemma, a quick computation reveals that $\underline{u}(r, t)$ given in (7.9) satisfies (7.10). To proceed further, we observe that there exists a constant R_o large enough such that F is positive for $|x|^2 \geq R_o t_o$ and such that (7.12) holds. To see the second statement we note that if $|x|^2 = R_o t$, by the previous Lemma:

$$\underline{u}(x, t) \leq C t^{-\frac{\alpha}{2}} \ll (1 - \varepsilon) g(R_o) t^{-\frac{1}{p-1}} \leq u(x, t). \tag{7.13}$$

Finally, once t_o is chosen, we take M large enough to obtain (7.11). ∎

Let us summarize a bit. Putting together Lemmata 7.1-7.4 we have shown that we can obtain subsolutions that provide bounds for $u(x, t)$ exactly as required in Theorem 7. This can be seen, for instance, by checking that the second term in the right of (7.9) is negligible with respect to the first one there for $r \gg \sqrt{t}$. To conclude the proof of such result, we yet need to obtain supersolutions displaying the required behaviour. To this end, we consider the auxiliary function:

$$\overline{\Phi}(y, \tau) = (1 + \varepsilon) g_1(y) + g_2(y) + \frac{M}{\tau} e^{-\frac{|y|^2}{4}}, \tag{7.14}$$

where, as usual, ε is a fixed (but small), positive number, $g_1(\tau)$ is the solution of (1.22), $M > 0$ is a free parameter and $g_2(\tau)$ solves:

$$\begin{cases} g''(s) + \left(\dfrac{N-1}{s} + \dfrac{s}{2} \right) g'(s) + \dfrac{\alpha}{2} g(s) - g^p(s) = 0, \\ g'(0) = 0, \quad g(s) \sim (A + \varepsilon) s^{-\alpha} \text{ as s} \to \infty. \end{cases} \tag{7.15}$$

We now prove:

LEMMA 7.5. – *Function* $\overline{\Phi}(y, \tau)$ *given in (7.14) is a supersolution of (7.15) for* $|y| \geq R_o$ *and* $\tau \geq 1$, *where* R_o *is a positive number depending on* N *and* p. *Moreover, there exists* $\tau_o \gg 1$ *such that:*

$$\Phi(y, \tau_o) \leq \overline{\Phi}(y, \tau_o) \qquad \textit{for } y \geq R_o, \tag{7.16}$$

$$\Phi(y, \tau) \leq \overline{\Phi}(y, \tau) \qquad \textit{for } y = R_o \textit{ and } \tau > \tau_o. \tag{7.17}$$

Proof. – Since the sum of supersolutions is a new supersolution, the first statement in the Lemma is easily checked by noting that each one of the three terms on the right in (7.14) is itself a supersolution, which is straightforward. To obtain (7.16), we first select $\tau_o \gg 1$ so that $\Phi(y,\tau) \le (1+\frac{\varepsilon}{2})g_1(y)$ for $y = R_o$ and $\tau \ge \tau_o$ (this can be done by Lemma 7.2). By redefining τ_o if necessary, we next observe that the solution $u(x,t)$ under consideration is such that $u(x,t) \le (A + \frac{\varepsilon}{2})|x|^{-\alpha}$ for $|x|^2 \ge S(t)$, where $S(t) = kt\log t$ with $k > 2(\alpha - N)$. We next remark that for large enough t, $t^{-\frac{1}{p-1}} g_2\left(\frac{|x|}{\sqrt{t}}\right) \ge (A+\varepsilon)|x|^{-\alpha}$ for $|x|^2 \ge S(t)$ (this follows since $S(t) \gg t$ for $t \gg 1$). Denoting by $\overline{u}(x,t)$ the function corresponding to $\overline{\Phi}(y,\tau)$ under the self-similar rescaling, we then have that:

$$u(x,t_o) \le \overline{u}(x,t_o) \qquad \text{for } |x|^2 \ge S(t_o).$$

To take care of the region $R_o^2 \le |x|^2 \le S(t_o)$, we make use of the third term on the right of (7.14). More precisely, we select $M \gg 1$ there. Finally the proof of (7.17) is straightforward. ■

End of the proof of Theorem 7:

We deduce from Lemma 7.4 and the maximum principle that:

$$u(x,t) \le \overline{u}(x,t) \qquad \text{for } |x| \ge R_o\sqrt{t} \text{ and } t \ge t_o \gg 1. \tag{7.18}$$

Comparing the various behaviours of the functions appearing on the right of (7.14), we readily see that:

$$\overline{\Phi}(y,\tau) \sim (1+\varepsilon)g_1(y) \text{ for } 1 \ll y \le \sqrt{k\tau} \text{ with } k < 2(\alpha - \frac{2}{p-1}), \tag{7.19}$$

$$\overline{\Phi}(y,\tau) \sim g_2(y) \text{ for } y \ge \sqrt{k\tau} \text{ with } k > 2(\alpha - \frac{2}{p-1}). \tag{7.20}$$

It follows from (7.19)-(7.20) and Lemma 7.2 that, in order to obtain a supersolution exhibiting the required behaviour, the only region that needs of further analysis is:

$$\Sigma = \{(x,t) : Ct \le |x|^2 \le kt\log t\} \quad \text{with } C > 0 \text{ and } k < 2(\alpha - \frac{2}{p-1}), \tag{7.21}$$

since $g_1(\tau)$ and $g_2(\tau)$ are of the same order in Σ. To conclude the argument, we now consider an auxiliary function $\widetilde{\Phi}(y,\tau)$ given by $\overline{\Phi}(y,\tau)$ in (7.14) without the term $g_2(y)$ included there. Clearly, $\widetilde{\Phi}(y,\tau)$ is a supersolution. Moreover, the lower boundary corresponding to $t = t_1$ is taken care of by selecting $M \gg 1$ as before. As to the side conditions, $|x|^2 = Ct$ is included in Lemma 7.4, whereas $|x|^2 = kt\log t$ follows from the result obtained with the previous supersolution. This concludes the proof. ■

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