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On the minimizers of the Ginzburg-Landau energy for high kappa: the axially symmetric case

by

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ABSTRACT. – The Ginzburg-Landau theory of superconductivity is examined in the case of a special geometry of the sample, the infinite cylinder. We restrict to axially symmetric solutions and consider models with and without vortices. First putting the Ginzburg-Landau parameter κ formally equal to infinity, the existence of a minimizer of this reduced Ginzburg-Landau energy is proved. Then asymptotic behaviour for large κ of minimizers of the full Ginzburg-Landau energy is analyzed and different convergence results are obtained. Our main result states that, when κ is large, the minimum of the energy is reached when there are about κ vortices at the center of the cylinder. Numerical computations illustrate the various behaviours.

RÉSUMÉ. – Le modèle de Ginzburg-Landau des supraconducteurs est étudié dans le cas d'une géométrie cylindrique pour des solutions radiales, avec et sans vortex. Quand le paramètre de Ginzburg-Landau κ est infini, on prouve l'existence d'un minimum pour une énergie réduite. Puis on étudie le comportement asymptotique des minimiseurs pour κ grand et on obtient différents résultats de convergence. On montre en particulier que l'énergie est minimum pour une configuration où le nombre de vortex au centre du cylindre est de l'ordre de κ. Des calculs numériques illustrent les divers comportements.

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1. INTRODUCTION

The superconductivity of certain metals is characterized at very low temperatures by the loss of electrical resistance and the expulsion of the exterior magnetic field. Superconducting currents in the material, which exclude the magnetic field, are due to the existence of pairs of electrons of opposite sign and momentum, the Cooper pairs. In the Ginzburg-Landau model, the electromagnetic properties of the material are completely described by the magnetic potential vector $\mathbf{A}$ ($\mathbf{H}$ = curl $\mathbf{A}$ being the magnetic field) and the complex-valued order parameter ψ (see [11], [13] or [16] for instance). In fact, ψ is an averaged wave function of the superconducting electrons; its phase is related to the current in the superconductor and its modulus to the density of superconducting carriers: $|\psi| = 0$ when the sample is wholly normal and $|\psi| = 1$ when it is wholly superconducting. The basic thermodynamic postulate of the Ginzburg-Landau theory says that a stable superconducting sample is in a state such that its Gibbs free energy is a minimum. The nondimensionalized form of this energy is given by:

$$E_\kappa(\psi, \mathbf{A}) = \int_\Omega \left| \left(\frac{1}{\kappa} \nabla - i\mathbf{A} \right) \psi \right|^2 + \frac{1}{2}(|\psi|^2 - 1)^2 + (\text{curl } \mathbf{A} - \mathbf{H}_0)^2 \, d\Omega. \quad (1)$$

where Ω is a domain in $\mathbb{R}^n$ (n = 1, 2 or 3) representing the region occupied by the superconducting sample, $\mathbf{H}_0$ is the given applied magnetic field, and κ is a material parameter called the Ginzburg-Landau parameter. This parameter is the ratio of λ, the penetration depth of the magnetic field to ξ, the coherence length, which is the characteristic length of variation of ψ. The value of κ determines the type of superconductor: $\kappa \leq 1/\sqrt{2}$ describes what is known as a type I superconductor and $\kappa \geq 1/\sqrt{2}$ as a type II. Type I superconductors are either normally conducting (normal state) or superconducting according to the value of the magnetic field, while for type II, a third state appears, called the mixed state: in the mixed state, the superconducting and the normal states coexist in what is usually called filaments or vortices. At the center of the vortex, the order parameter vanishes, so the material is normal; the vortex is circled by a superconducting current carrying with it a quantized amount of magnetic flux. Macroscopic models of superconducting vortices have been formulated in [9], [10] and the existence and behaviour of vortex solutions to the Ginzburg-Landau equations have been widely studied. See [2], [4], [5], [7], [8], [12], [14] for instance.

In this paper, we study the minimization of the Ginzburg-Landau energy when the parameter κ is large, for a special geometry of the sample, the infinite cylinder. Thus, we restrict to axially symmetric solutions. The previous paper [1] deals with the one-dimensional case. Our motivation was the study made in [3] of the solutions of the Ginzburg-Landau system with infinite κ.

This paper is organized as follows. First of all, we study the asymptotic behaviour of minimizers under the constraint that vortices do not exist: we put formally κ equal to infinity in the energy and study the minimizers of this reduced form; this will enable us to show convergence of Ginzburg-Landau energy minimizers as κ tends to infinity. We especially prove a uniqueness property of the solutions of the system for infinite κ, which extends a result of [3]. The section ends with numerical results. The same type of work is made in the next section for the model with vortices. Our main result states that when κ is large the minimum of the energy is reached when there are about κ vortices at the center of the cylinder.

Let us recall from [13] the main properties of the two dimensional Ginzburg-Landau model. We assume that the superconducting sample is an infinite cylinder. When the magnetic field is parallel to the axis of the cylinder, $\mathbf{H}_0 = (0, 0, H_0)$, one can assume that both ψ and $\mathbf{A}$ are uniform in the z-direction. The state of the superconductor is described by the pair $(\psi, \mathbf{A})$ that minimizes E_κ, where Ω is a regular bounded domain in $\mathbb{R}^2$. Let us define

$$H^1_n(div) = \{\mathbf{A} \in H^1(\Omega, \mathbb{R}^3),\ \text{div}\ \mathbf{A} = 0 \text{ and } \mathbf{A}.\mathbf{n} = 0\}.$$

We can choose $||\text{curl}\ \mathbf{A}||_{L^2}$ as a norm on $H^1_n(div)$.

DEFINITION 1.1. – *$(\psi, \mathbf{A})$ and $(\phi, \mathbf{B})$ are said to be gauge-equivalent if there exists $\theta \in H^2(\Omega, \mathbb{R})$ such that $\psi = \phi e^{i\theta}$ and $\mathbf{B} = \mathbf{A} - (1/\kappa)\nabla\theta$. Then, the energy E_κ is preserved by this transformation.*

THEOREM 1.2. – *A minimizer of E_κ over $H^1(\Omega, \mathbb{R}) \times H^1(\Omega, \mathbb{R}^3)$ is gauge equivalent to a minimizer of E_κ over $H^1(\Omega, \mathbb{R}) \times H^1_n(div)$.*

THEOREM 1.3. – *There exists a minimum $(\psi, \mathbf{A})$ of E_κ over $H^1 \times H^1_n(div)$. It is a solution of:*

$$\left(\frac{1}{\kappa}\nabla - i\mathbf{A}\right)^2 \psi = \psi(1 - |\psi|^2) \quad in \quad \Omega, \tag{2}$$

$$\Delta\mathbf{A} = \frac{i}{2\kappa}(\psi^*\nabla\psi - \psi\nabla\psi^*) + \mathbf{A}|\psi|^2 \quad in \quad \Omega, \tag{3}$$

$$\frac{\partial\psi}{\partial n} = 0 \quad on \quad \partial\Omega, \tag{4}$$

$$curl\ \mathbf{A} = \mathbf{H}_0 \quad on \quad \partial\Omega. \tag{5}$$

PROPOSITION 1.4. – *If* $(\psi, \mathbf{A})$ *is a minimizer of* E_κ *then* $|\psi| \leq 1$ *a.e.*

If Ω is a general domain, the gauge-invariance property does not remain in E_∞, (the energy obtained when κ is formally put equal to infinity), so that we do not have a global minimum of the energy. Hence we shall now restrict ourselves to a ball.

2. THE CASE WITHOUT VORTICES

When Ω is assumed to be a ball, we use polar coordinates and restrict to radial solutions. In this section, we make the extra assumption that no vortices exist (so ψ is never equal to zero) and we look for:

$$\psi(r,\theta) = f(r) \quad f(r) \in \mathbb{R}_+^*, \quad \text{and} \quad \mathbf{A}(r,\theta) = Q(r)\mathbf{e}_\theta.$$

We automatically have div $\mathbf{A} = 0$, which is the major interest of this gauge. So $\|\text{curl } \mathbf{A}\|_{L^2(B_R)} = \|(1/r)(rQ)'\|_{L^2(B_R)}$ is a norm on $H^1(B_R)$. Since we want $(\psi, \mathbf{A})$ in $H^1 \times H^1$, we define:

$$D_f = \{f \text{ radial},\ f \geq 0 \ a.e.,\ f \in H^1(B_R, \mathbb{R})\},$$
$$D_Q = \{\mathbf{Q} \in H^1(B_R, \mathbb{R}^3),\quad \exists Q(r) \text{ such that } \mathbf{Q} = Q(r)\mathbf{e}_\theta\}.$$

From now on, when we write that Q is in D_Q, it will mean that $\mathbf{Q} = Q(r)\mathbf{e}_\theta$ is in D_Q. Notice that $\|f\|_{H^1(B_R)}$ is a norm on D_f and $\|(1/r)(rQ)'\|_{L^2(B_R)}$ is a norm on D_Q. The energy can be rewritten as follows:

$$E_\kappa(f,Q) = \int_0^R \left(\frac{1}{\kappa^2} f'^2 + \frac{1}{2}(f^2-1)^2 + f^2Q^2 + \left(\frac{1}{r}(rQ)' - H_0\right)^2 \right) rdr. \tag{6}$$

The same method as in [1] allows us to say that:

- there exists a minimizer of E_κ over $D_f \times D_Q$,
- it is a solution of

$$\frac{1}{r\kappa^2}(rf')' = f(f^2+Q^2-1) \quad \text{in} \quad (0,R), \tag{7}$$

$$\left(\frac{1}{r}(rQ)'\right)' = f^2Q \quad \text{in} \quad (0,R), \tag{8}$$

$$f'(R) = 0, \tag{9}$$

$$\frac{1}{R}(rQ)'(R) = H_0. \tag{10}$$

Equations (7)-(8) can be rewritten using the laplacian in polar coordinates:

$$\frac{1}{\kappa^2}\Delta f = f(f^2 + Q^2 - 1) \quad \text{in} \quad B_R \setminus \{0\}, \tag{11}$$

$$\Delta \mathbf{Q} = f^2 \mathbf{Q} \quad \text{in} \quad B_R \setminus \{0\}. \tag{12}$$

2.1. Properties of solutions

First of all, we notice, as in [4], that if $\mathbf{Q}$ is in D_Q, it implies that $\sqrt{r}Q \in H^1(0,R)$, Q is continuous on $[0,R]$ and $Q(0) = 0$. More precisely, $D_Q \subset L^\infty(B_R)$ with

$$\|Q\|_{L^\infty} \leq \frac{1}{\sqrt{2\pi}} \left\| \frac{1}{r}(rQ)' \right\|_{L^2(B_R)}.$$

THEOREM 2.1. – *If (f,Q) is a minimizer of E_κ, then f and Q are in $C^\infty[0,R]$.*

Proof. – The previous remark and Proposition 1.4 give that (f,Q) is in $L^\infty(B_R)$. Then the result follows from classical elliptic estimates on the formulation given by equations (11)-(12). □

Remark. – As f is regular in 0, equation (7) implies that $f'(0) = 0$.

PROPOSITION 2.2. – *If (f,Q) is a minimizer of E_κ, then either $f \equiv 0$ or f is never equal to zero.*

From now on, we shall assume that $f > 0$ on $(0,R]$ for a minimizer of E_κ. The solution $f_0 \equiv 0$ and $Q_0(r) = H_0 r/2$ is called the normal state.

THEOREM 2.3. – *If (f,Q) is a minimizer of E_κ, then Q is nondecreasing.*

Proof. – Let us first show that $Q \geq 0$ on $[0,R]$. If Q reached a negative minimum at $r = r_0$ on $(0,R)$, then we would have $Q < 0$ and (8) would mean

$$\Delta Q = Q\left(f^2 + \frac{1}{r^2}\right) \leq 0$$

on a small intervall around r_0; this is in contradiction with the Maximum Principle. Similarly, if Q reached a negative minimum at $r = R$, the Hopf Lemma would imply that $Q'(R) < 0$ but then condition (10) would not be satisfied.

Now if Q was not nondecreasing, we would have the existence of r_1 and r_2 in $(0,R)$ such that $r_1 < r_2$ and $Q(r_1) > Q(r_2)$. It would imply

Q reached a positive maximum on (r_1, r_2). This contradicts the Maximum Principle since $\Delta Q \geq 0$ on $(0, R)$. □

2.2. Infinite Ginzburg-Landau Parameter

We put κ equal to infinity in the energy. We define

$$E_\infty(f, Q) = \int_0^R \left(\frac{1}{2}(f^2 - 1)^2 + f^2 Q^2 + \left(\frac{1}{r}(rQ)' - H_0 \right)^2 \right) rdr, \quad (13)$$

where $f \in L^4_{rad}(B_R, \mathbb{R}) = \{f \text{ radial}, \ f \geq 0, \ f \in L^4(B_R, \mathbb{R})\}$ and $Q \in D_Q$.

THEOREM 2.4. – *The minimum of E_∞ over $L^4_{rad}(B_R) \times D_Q$ exists and is attained. Moreover it satisfies:*

$$Q''(r) + \frac{1}{r}Q'(r) - \frac{1}{r^2}Q(r) = Q(r)(1 - Q^2(r))\mathbf{1}_{|Q| \leq 1} \quad in \quad B_R, \quad (14)$$

$$Q'(r) + \frac{1}{r}Q(r) = H_0 \quad for \quad r = R \quad and \quad |Q(r)| \geq 1, \quad (15)$$

$$f^2(r) = (1 - Q^2(r))\mathbf{1}_{|Q| \leq 1} \quad in \quad B_R, \quad (16)$$

where $\mathbf{1}_{|Q| \leq 1}(x) = 1$ if $|Q(x)| \leq 1$, and 0 otherwise.

The proof is almost the same as in [1]. It comes from the fact that when Q is fixed, the minimum of $(1/2)(1 - f^2)^2 + f^2 Q^2$ is attained for $f = 0$ if $|Q| \geq 1$, and $f^2 = 1 - Q^2$ if $|Q| \leq 1$.

THEOREM 2.5. – *There exists H_0^c such that*

(i) *for $H_0 \leq H_0^c$, there exists a unique solution Q of* (14)-(15) *in D_Q and it remains smaller than* 1 *in $(0, R)$. If we call $\alpha(R, H_0) = Q(R)$, then $\alpha(R, H_0)$ is a continuous and increasing function of R and H_0. Moreover, $\alpha(R, H_0)$ reaches* 1 *when $H_0 = H_0^c$,*

(ii) *for $H_0 > H_0^c$, there is no solution that remains smaller than* 1.

Proof. – We proceed as in [3], using a shooting method from the boundary $r = R$. For a given $\alpha \in (0, 1)$, there is a unique solution Q_α of

$$\begin{cases} Q''_\alpha + \dfrac{1}{r}Q'_\alpha - \dfrac{1}{r^2}Q_\alpha = Q_\alpha(1 - Q^2_\alpha) \quad \text{on} \quad (0, R), \\ Q_\alpha(R) = \alpha \quad \text{and} \quad Q'_\alpha(R) = H_0 - \alpha/R. \end{cases} \quad (17)$$

We check that (15) is satisfied when $r = R$. The three possible behaviours of Q_α are described by the following sets:

$$I(R, H_0) = \{\alpha \in (0,1),\ Q_\alpha(r) \in (0,1) \text{ on } (0,R)\},$$
$$I_0(R, H_0) = \{\alpha \in (0,1),$$
$$\exists \zeta \in (0,R) \text{ st } Q_\alpha(r) \in (0,1) \text{ on } (\zeta, R) \text{ and } Q_\alpha(\zeta) = 0\},$$
$$I_1(R, H_0) = \{\alpha \in (0,1),$$
$$\exists \zeta \in (0,R) \text{ st } Q_\alpha(r) \in (0,1) \text{ on } (\zeta, R) \text{ and } Q_\alpha(\zeta) = 1\}.$$

We recall from [3] that

- $I(R, H_0)$ corresponds to regular solutions, that is $Q_\alpha(0) = 0$,
- $I_0(R, H_0)$ and $I_1(R, H_0)$ are open sets,
- $I_1(R, H_0) \neq \emptyset$ when $RH_0 < 1$, so it implies that $I(R, H_0) \neq \emptyset$ too,
- $(0, \frac{H_0}{R+2/R}) \subset I_0(R, H_0)$, so for $H_0 > R + 2/R$, $I(R, H_0) = \emptyset$.

We may also notice that Q_α cannot reach a positive local maximum while it remains between 0 and 1, since (17) can be rewritten

$$\Delta Q_\alpha - Q_\alpha\left(1 + \frac{1}{r^2} - Q_\alpha^2\right) = 0.$$

So Q_α is increasing on $(r_{0,\alpha}, R)$ where $r_{0,\alpha}$ is such that $Q_\alpha(r_{0,\alpha}) = 0$.

It is proved in [3] that there exists H_0^* such that for $H_0 < H_0^*$, (14)-(15) has a solution that remains smaller than 1; H_0^* is obtained as the maximum of H_0 over the points (R, H_0) in the connected component of I_1 which contains the set $\{(R, H_0),\ \text{st } RH_0 < 1\}$. The uniqueness result there is only obtained for $R \leq 1/\sqrt{2}$. We are going to prove that in fact $I_0(R, H_0)$ and $I_1(R, H_0)$ are connected and that $I(R, H_0)$ has at most one point, using an idea inspired by [17].

Let u and v be two different regular solutions of

$$Q'' + \frac{1}{r}Q' - Q\left(1 + \frac{1}{r^2} - Q^2\right) = 0 \quad \text{on} \quad (0, R). \tag{18}$$

We already know that u and v are increasing so they can be inverted. We denote the inverse functions by $r(x)$ and $s(x)$. Using as new variables

$$x = u(r) \quad \text{and} \quad U(x) = \frac{1}{2}r^2(x)u'^2(r(x)),$$
$$y = v(s) \quad \text{and} \quad V(y) = \frac{1}{2}s^2(y)v'^2(s(y)),$$

we can rewrite (18) as

$$U'(x) = x(1 + r^2(x)(1 - x^2)) \quad x \in (0, u(R)),$$
$$V'(y) = y(1 + s^2(y)(1 - y^2)) \quad y \in (0, v(R)).$$

We assume $u < v \leq 1$ on $(0,\sigma)$, so that $s(x) < r(x)$ for $x \in (0, u(\sigma))$ and $u'(0) \leq v'(0)$. We obtain $(U-V)'(x) > 0$ and since $U(0) = V(0) = 0$, we have $U(x) > V(x)$, which can be rewritten $r(x)u'(r(x)) > s(x)v'(s(x))$ for $x \in (0, u(\sigma))$. As a consequence, for all $y \in (0, u(\sigma))$,

$$\int_0^y \frac{x^2(1-x^2)}{r(x)u'(r(x))}\,dx < \int_0^y \frac{x^2(1-x^2)}{s(x)v'(s(x))}\,dx, \tag{19}$$

$$\text{or} \quad \int_0^{r(y)} \frac{u^2(r)(1-u^2(r))}{r}\,dr < \int_0^{s(y)} \frac{v^2(s)(1-v^2(s))}{s}\,ds. \tag{20}$$

For any solution Q of (18), we now introduce

$$F_Q(r) = \frac{1}{2}\left(\frac{1}{r}(rQ)'\right)^2 + \frac{1}{4}(1-Q^2)^2.$$

A straightforward computation, using (18), gives $F_Q'(r) = (1/r)Q^2(1-Q^2)$. Thanks to an integration from 0 to σ of $F_Q'(r)$, thanks to (20) and because $u'(0) \leq v'(0)$, we get

$$\left(u'(\sigma)+\frac{u(\sigma)}{\sigma}\right)^2 - \left(v'(\sigma)+\frac{v(\sigma)}{\sigma}\right)^2 + \frac{1}{2}(1-u(\sigma)^2)^2 - \frac{1}{2}(1-v(\sigma)^2)^2 < 0. \tag{21}$$

Therefore, two regular solutions cannot intersect before reaching 1, otherwise we would have $u < v$ on $(0,\sigma)$ and $u(\sigma) = v(\sigma)$ with $u'(\sigma) > v'(\sigma)$, which contradicts (21).

We are now able to show uniqueness of regular solutions. Let R be fixed. Let us assume Q_1 (resp. Q_2) is a solution of (17) with $\alpha = \alpha_1$ (resp. α_2) and $H_0 = H_1$ (resp. H_2). If $\alpha_1 < \alpha_2$, since two regular solutions do not intersect before reaching 1, we have $Q_1 < Q_2$ on $(0, R)$. We infer from (21) that $H_1 < H_2$. So there is a unique $\alpha(R, H_0)$ in $I(R, H_0)$ for each R and H_0 and $\alpha(R, H_0)$ is an increasing function of H_0. An immediate consequence of the uniqueness is that $I_0(R, H_0)$ and $I_1(R, H_0)$ are connected sets.

Moreover, $\alpha(R, H_0)$ is an increasing function of R. Indeed, let $\alpha_1(R_1, H_0) < \alpha_2(R_2, H_0)$ be the initial data for the two regular solutions Q_1 and Q_2. We know that Q_1 and Q_2 do not intersect in $(0, \min(R_1, R_2))$. Let us assume $Q_1 < Q_2$ on $(0, \min(R_1, R_2))$; we call r_2 the point where $Q_2(r) = \alpha_1$. We now use (20) with $y = \alpha_1$ which provides the following comparison result $F_{Q_1}(R_1) - F_{Q_1}(0) < F_{Q_2}(r_2) - F_{Q_2}(0)$. This yields $H_0 < Q_2'(r_2) + (1/r_2)Q_2(r_2)$. But it is impossible since

$Q_2'(r) + (1/r)Q_2(r)$ is increasing and reaches H_0 when $r = R_2$. So $Q_2 < Q_1$ on $(0, \min(R_1, R_2))$ and it implies $R_1 < R_2$.

The continuity of $\alpha(R, H_0)$ follows from the fact that $I_0(R, H_0)$ and $I_1(R, H_0)$ are open and the continuous dependence of Q with respect to R and H_0 on any interval that does not contain zero. Indeed, if $\alpha_0 < \alpha(R, H_0) < \alpha_1$, we have $Q_{\alpha_0}(r_0) < 0$ and $Q_{\alpha_1}(r_1) > 1$ for some r_0 and r_1 smaller than R. But these inequalities remain true for all $(\hat{R}, \hat{H}_0)$ in a neighbourhood of (R, H_0), which shows $\alpha_0 < \alpha(\hat{R}, \hat{H}_0) < \alpha_1$.

We are now able to conclude the proof. Let R be fixed. For H_0 small, we know that $I_1(R, H_0)$ is nonempty, so $I(R, H_0)$ is nonempty too. We call

$$H_0^c = \max\{H_0 \text{ such that for } H < H_0,\ I_1(R, H) \neq \emptyset\}.$$

It can easily be seen from the continuous dependence of $I_1(R, H_0)$ with respect to H_0 that when H_0 reaches H_0^c, $\alpha(R, H_0)$ reaches 1. As $\alpha(R, H_0)$ is an increasing function of H_0, it implies $I(R, H_0)$ is empty for $H_0 > H_0^c$. □

PROPOSITION 2.6. – *We have $H_0^c(R)$ is a decreasing function of R. Moreover*

$$\lim_{R \to \infty} H_0^c(R) = \frac{1}{\sqrt{2}}.$$

Remark. – This is the same limit as in the one-dimensional case.

Proof. – We have shown in the proof of Theorem 2.5 that for $R_1 < R_2$, $\alpha(R_1, H_0) < \alpha(R_2, H_0)$. Let $H_0 = H_0^c(R_2)$, so that $\alpha(R_2, H_0^c(R_2)) = 1$. It means $H_0^c(R_1) > H_0^c(R_2)$, which is the desired monotonicity property.

Let Q be the regular solution of (17) with $H_0 = H_0^c(R)$, so that $\alpha = 1$. As F_Q is increasing, we obtain $F_Q(0) \leq F_Q(R)$, that is $H_0^c(R) > 1/\sqrt{2}$. In order to get the estimate on the other side, we introduce a new energy:

$$G_1(r) = \frac{1}{2}Q'^2(r) + \frac{1}{4}(1 - Q^2(r))^2 - \frac{1}{2r^2}Q^2(r).$$

A simple computation, using (17) gives $G_2(r) = r^3 G_1'(r) = -r^2 Q'^2(r) + Q^2(r)$. We now differentiate G_2 and get $G_2'(r) = -2r^2 Q(r) Q'(r)(1 - Q^2(r))$. So G_2 is decreasing and since $G_2(0) = 0$, it means $G_1(0) > G_1(R)$ which can be rewritten

$$H_0^c(R) < \frac{1}{R} + \sqrt{\frac{1}{2} + \frac{1}{R^2}}.$$

□

THEOREM 2.7. – *There exists a unique minimizer* (f_∞, Q_∞) *of* E_∞. *For* $H_0 \le H_0^c$, Q_∞ *remains smaller than 1 and* $f_\infty^2 = 1 - Q_\infty^2$. *For* $H_0 > H_0^c$, *there exists a unique* $R_\infty < R$ *such that in* $(0, R_\infty)$, Q_∞ *remains smaller than* 1 *and* $f_\infty^2 = 1 - Q_\infty^2$ *while in* (R_∞, R), $Q'_\infty + (1/r)Q_\infty = H_0$ *and* $f_\infty \equiv 0$.

Proof. – Theorem 2.5 gives that for $H_0 \le H_0^c$, there is a unique (f_∞, Q_∞) solution of (14)-(15)-(16) and Q_∞ remains smaller than 1. For $H_0 > H_0^c$, as we want a solution in D_Q, it means $Q_\infty(0) = 0$ so Q_∞ is a solution of (18) on $(0, R_\infty)$, with $Q_\infty(R_\infty) = 1$. The radius R_∞ is unique because $\alpha(R, H_0)$ is an increasing function of R. So there is a unique solution of (14)-(15)-(16) with $f_\infty \not\equiv 0$. What only remains to show is that this solution is the minimizer of E_∞, or more precisely that the normal state (any solution defined by $f_0 \equiv 0$ and $Q'_0 + Q_0/r = H_0$) has a higher energy. We introduce a new energy

$$E(r) = \frac{1}{2}(rQ'_\infty)^2 - \frac{1}{2}Q_\infty^2 + \frac{1}{4}r^2(1 - Q_\infty^2)^2 \quad \text{on} \quad (0, R_\infty),$$

and we call $R_\infty = R$ in the case $H_0 \le H_0^c$ with $\alpha = Q_\infty(R_\infty)$. Computing $E'(r)$, using equation (14), gives

$$\frac{1}{2}\int_0^{R_\infty} r(1 - Q_\infty^2)^2 \, dr = \frac{1}{4}R_\infty^2(1 - \alpha^2)^2 + \frac{1}{2}H_0^2 R_\infty^2 - H_0 R_\infty \alpha.$$

This and an integration by parts on $\int_0^R (1/r)((rQ_\infty)')^2$ enable us to estimate

$$E_\infty(f_\infty, Q_\infty) = \frac{1}{4}R^2 - \frac{1}{4}R_\infty^2(1 - \alpha^2)^2 - \int_0^{R_\infty} rQ_\infty^2(1 - Q_\infty^2).$$

So $E_\infty(f_\infty, Q_\infty) < R^2/4 = E_\infty(f_0, Q_0)$. □

2.3. Convergence of minimizers

PROPOSITION 2.8. – *For all* H_0, *there exists* κ_0 *such that for* $\kappa \ge \kappa_0$, *the normal state is not a minimizer of* E_κ.

The proof relies on energy comparisons as in [1].

THEOREM 2.9. – *The whole sequence* (f_κ, Q_κ) *of minimizers of* E_κ *converges to the unique minimizer* (f_∞, Q_∞) *of* E_∞. *More precisely, when* $\kappa \to \infty$,

$$f_\kappa \to f_\infty \quad \text{in} \quad L^p(B_R) \quad \forall p \quad 1 \le p < \infty \text{ and weakly in } H^1(B_R),$$
$$Q_\kappa \to Q_\infty \quad \text{in} \quad C^{1,\alpha}(B_R) \quad \forall \alpha < 1.$$

Proof. – It is almost the same as in [1]. We only mention the main ideas.

For $H_0 < H_0^c$, as $|Q_\infty| < 1$, f_∞ is in $H^1(B_R)$ and we can test it in E_κ. We infer that f_κ is bounded in $H^1(B_R)$ and $E_\kappa(f_\kappa, Q_\kappa)$ tends to $E_\infty(f_\infty, Q_\infty)$. As a consequence, Q_κ is bounded in $H^1(B_R)$. So for a subsequence,

$$f_\kappa \to f \quad \text{in} \quad L^p(B_R) \quad \forall p \quad 1 \leq p < \infty \text{ and weakly in } H^1(B_R),$$
$$Q_\kappa \to Q \quad \text{in} \quad L^p(B_R) \quad \forall p \quad 1 \leq p < \infty \text{ and weakly in } H^1(B_R).$$

Using equation (12), we can improve the convergence on Q_κ: classical elliptic estimates and Sobolev embeddings give that $\mathbf{Q}_\kappa$ is bounded in $W^{2,p}(B_R)$ for all finite p, and Q_κ converges in $C^{1,\alpha}(B_R)$ for $\alpha < 1$. We see that $E_\infty(f, Q) = E_\infty(f_\infty, Q_\infty)$ and Theorem 2.7 gives the conclusion.

For $H_0 \geq H_0^c$, we have seen that f_∞ is not in $H^1(B_R)$, but as in the one-dimensional case, we can find g_κ in $H^1(B_R)$ such that $\lim_{\kappa\to\infty} E_\kappa(g_\kappa, Q_\infty) = E_\infty(f_\infty, Q_\infty)$. Energy comparisons give:

$$\lim_{\kappa\to\infty} \int_{B_R} \frac{1}{\kappa^2} f'^2_\kappa = 0 \text{ and } \lim_{\kappa\to\infty} E_\infty(f_\kappa, Q_\kappa) = E_\infty(f_\infty, Q_\infty).$$

As in the one-dimensional case, up to the extraction of a subsequence,

$$f_\kappa^2 \rightharpoonup f^2 \quad \text{weakly in } L^2(B_R),$$
$$Q_\kappa \to Q \quad \text{in} \quad C^{1,\alpha}(B_R),$$

for $(f, Q) \in L^4 \times D_Q$. Lower semi-continuity yields: $\liminf_{\kappa\to\infty} E_\kappa(f_\kappa, Q_\kappa) \geq E_\infty(f, Q)$, so that (f, Q) is the minimizer of E_∞. Theorem 2.7 allows us to know the properties of f: there exists R_∞ with $f \equiv 0$ on $(0, R_\infty)$ and $f^2 = 1 - Q^2$ on (R_∞, R). Then the result of convergence follows as in the one-dimensional case. □

2.4. Numerical Study

We want to compute solutions of the Ginzburg-Landau system such that f is positive. Instead of solving the system (7)-(8)-(9)-(10), we define $S(r) = rQ(r)$, choose ϵ small and solve:

$$-\frac{\partial f}{\partial t} + \frac{1}{\kappa^2} r^2 \frac{\partial^2 f}{\partial r^2} + r\frac{\partial f}{\partial r} = f(f^2 r^2 + S^2 - r^2) \quad \text{in} \quad (\epsilon, R),$$
$$\frac{\partial f}{\partial r}(\epsilon) = 0 \quad \text{and} \quad \frac{\partial f}{\partial r}(R) = 0,$$
$$r\frac{\partial^2 S}{\partial r^2} - \frac{\partial S}{\partial r} = r f^2 S \quad \text{in} \quad (\epsilon, R),$$
$$S(\epsilon) = 0 \quad \text{and} \quad \frac{1}{R}\frac{\partial S}{\partial r}(R) = H_0,$$

using the same scheme as in [1], that is implicit discretisation in time. It can be shown as in [6], thanks to the Maximum Principle, that a minimizer of the Ginzburg-Landau energy is an asymptotically stable solution of this problem. We compute the solutions with $R = 1$. We study the convergence of f_κ when κ tends to infinity. Figure 1 ($H_0 = 1$) and figure 2 ($H_0 = 3$) illustrate the two different behaviours described in Theorem 2.7.

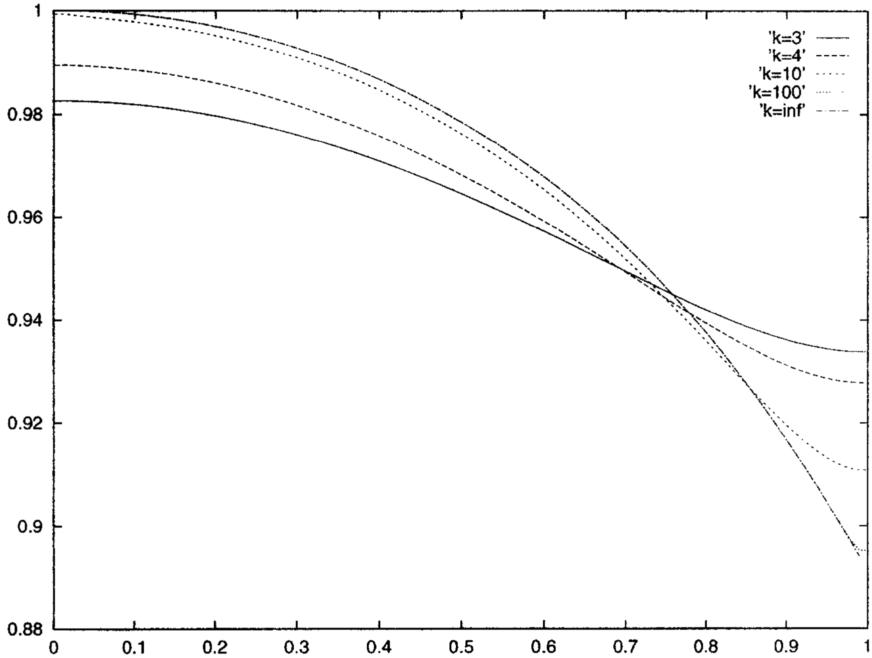

Figure 1. – Radial solutions $f(r)$ for $R = 1$, $H_0 = 1$.

Remark. – It would be interesting to show that f_κ is nonincreasing and improve the convergence of the sequence.

3. THE CASE WITH VORTICES

In this section, we allow N vortices to appear in the center of the ball and intend to minimize the energy over this number N. We are going to show that $N \neq 0$ for the minimizer, and more precisely that N/κ has a limit when κ tends to infinity. The existence of vortices at the center of the ball can be described mathematically by introducing, as in [4], solutions

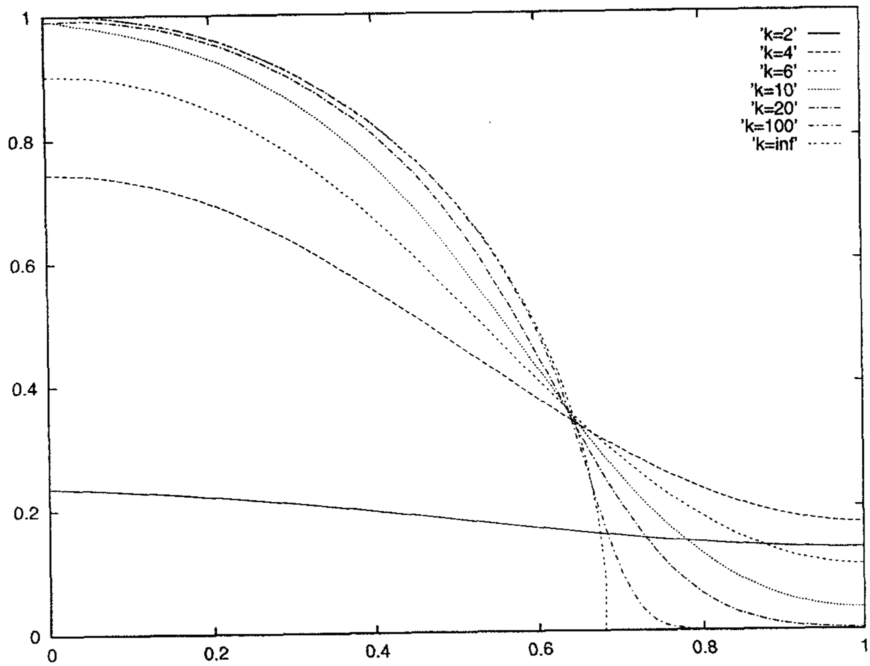

Figure 2. – Radial solutions $f(r)$ for $R = 1$, $H_0 = 3$.

$(\psi, \mathbf{A})$ such that

$$\psi(r, \theta) = f(r)e^{iN\theta} \quad f(r) \in \mathbb{R}, \quad \text{and} \quad \mathbf{A}(r, \theta) = A(r)\mathbf{e}_\theta,$$

and A will be regular at the origin. We may notice that $(\psi, \mathbf{A})$ is gauge equivalent (see Definition 1.1) to $(f, \mathbf{Q})$ with

$$\mathbf{Q}(r) = \frac{1}{r}\left(S(r) - \frac{N}{\kappa}\right)\mathbf{e}_\theta,$$

where $S(r)/r$ is regular at the origin.

In this situation, the Ginzburg-Landau energy is the following:

$$E_\kappa(N, f, S) = \int_{B_R} \left(\frac{1}{\kappa^2} f'^2 + \frac{1}{2}(f^2 - 1)^2 + \frac{1}{r^2}\left(S - \frac{N}{\kappa}\right)^2 f^2 + \left(\frac{1}{r}S' - H_0\right)^2\right) r dr d\theta. \quad (22)$$

3.1. Finite Ginzburg-Landau Parameter

The introduction of N in the energy comes from a number of vortices, that is an integer. But in fact, the definition (22) has a meaning for any real number N. So for mathematical purposes, from now on, we will allow N to vary in $\mathbb{R}$. We will see that the convergence properties and the limit would not be affected by restricting N to lie in $\mathbb{Z}$, if a judiciously sequence of κ is chosen.

3.1.1. Fixed number of vortices

To start with, we fix N and want to minimize E_κ on $D_f \times D_S$ where

$$D_f = \{f \text{ radial, } f \geq 0 \text{ a.e. } f \in H^1(B_R)\},$$

$$D_S = \left\{S \text{ radial, } \frac{1}{r}S \in L^2(B_R) \text{ and } \frac{1}{r}S' \in L^2(B_R)\right\}.$$

With the norm $\|(1/r)S'\|_{L^2}$, D_S is a Hilbert space (see [4]). We will not give the proof of the next three Theorems as it is almost the same as in the case treated in [4].

THEOREM 3.1. – *If $E_\kappa(N, f, S) < \infty$, then $f \in H^1(0, R)$, $(1/r)S' \in L^2(0, R)$ and $(1/r)(S(r) - S(0)) \in L^2(0, R)$.*

THEOREM 3.2. – *We have the following regularity properties:*

(i) $f \in D_f \Rightarrow f \in C^0(0, R)$,

(ii) $S \in D_S \Rightarrow S \in C^0(0, R)$ *and* $S(0) = 0$,

(iii) $f \in D_f$, $S \in D_S$ *and* $E_\kappa(N, f, S) < \infty \Rightarrow f \in C^0[0, R]$ *and* $f(0) = 0$.

THEOREM 3.3. – *There exists a minimizer (f, S) of E_κ. It is in $(C^\infty(B_R - \{0\}) \cap C^2(B_R))^2$ and is a solution of:*

$$\frac{1}{\kappa^2}f'' + \frac{1}{r\kappa^2}f' = f\left(f^2 + \frac{1}{r^2}\left(\frac{N}{\kappa} - S\right)^2 - 1\right) \quad in \quad (0, R), \tag{23}$$

$$f'(R) = 0, \tag{24}$$

$$S'' - \frac{1}{r}S' = \left(S - \frac{N}{\kappa}\right)f^2 \quad in \quad (0, R), \tag{25}$$

$$\frac{1}{R}S'(R) = H_0. \tag{26}$$

Remark. – With the expression of the laplacian in the radial case, equations (23) and (25) can be rewritten:

$$\frac{1}{\kappa^2}\Delta f = f\left(f^2 + \frac{1}{r^2}\left(S - \frac{N}{\kappa}\right)^2 - 1\right) \quad \text{in} \quad B_R, \tag{27}$$

$$\Delta S - \frac{2}{r^2}(x_1\partial_1 S + x_2\partial_2 S) = \left(S - \frac{N}{\kappa}\right)f^2 \quad \text{in} \quad B_R. \tag{28}$$

THEOREM 3.4. – *If* (f, S) *is a minimizer of* E_κ *such that* f *is not identically zero, then* $0 < f \leq 1$ *on* $(0, R]$ *and* $S > 0$ *on* $(0, R)$. *Moreover, neither function is constant on a subinterval of* $(0, R)$.

Proof. – Since the solutions are regular by Theorem 3.3, if they are constant on a subinterval, it means they are constant on $[0, R]$. But the boundary condition $S(0) = 0$ and (23)-(25) show that S cannot be constant and the only possibility for f is 0.

If (f, S) is a minimizer of E_κ, so is $(|f|, S)$. Hence $(|f|, S)$ is a solution of the Ginzburg-Landau equations and $|f|$ is C^∞ on $(0, R)$. So either $f \equiv 0$ or f is never equal to zero.

Let us assume $\max_{r\in[0,R]} f(r) = f(r_0) > 1$. As $f(0) = 0$, $r_0 \in (0, R]$. Equation (27) can be rewritten

$$\frac{1}{\kappa^2}\Delta f - c(r)f = 0$$

on an interval around r_0, which does not contain 0, and on which $f \geq 1$ so that $c(r)$ is positive. We choose the largest interval possible. Necessarily, either $f = 1$ on the boudary or $r = R$ is the right end. The strong Maximum Principle implies that the maximum is reached on the boundary. But because of condition (26), it cannot be reached on $r = R$ and otherwise $f = 1$ on the boundary, which is the minimum. So that f remains smaller than 1.

We know that $S(0) = 0$. Let us assume that there exists r_0 in $(0, R]$ such that $\max_{r\in[0,R]} S(r) = S(r_0) \leq 0$. Equation (28) gives:

$$\Delta S - \frac{2}{r^2}(x_1\partial_1 S + x_2\partial_2 S) - Sf^2 \leq 0.$$

We again apply the Maximum Principle on an interval around r_0. The minimum is reached on the boundary which means $r_0 = R$. But condition (26) gives that $S'(R) > 0$ which contradicts the Hopf Lemma. So $S > 0$ on $(0, R]$. □

THEOREM 3.5. – *If* (f, S) *is a minimizer of* E_κ, *then* S *is non decreasing.*

Proof. – We rewrite equation (28) as follows:

$$\Delta\left(S - \frac{N}{\kappa}\right) - \frac{2}{r^2}\left(x_1\partial_1\left(S - \frac{N}{\kappa}\right) + x_2\partial_2\left(S - \frac{N}{\kappa}\right)\right) - \left(S - \frac{N}{\kappa}\right)f^2 = 0.$$

The Maximum Principle implies that $S - (N/\kappa)$ cannot reach any positive maximum nor any negative minimum in the interior. Since $S(0) = 0$, there are two possibilities:

- $S \leq (N/\kappa)$ on $(0, R)$.
  So S has no local minimum in $(0, R)$. As $S'(R) > 0$, it implies S is non decreasing.
- $S(r) > (N/\kappa)$ for some r in $(0, R)$.
  Let r_0 be the first point where $S = \frac{N}{\kappa}$. Necessarily, $S'(r_0) > 0$ by the Hopf Lemma and as in the previous case, S is non decreasing on $(0, r_0)$. Then the Maximum principle gives that $S - (N/\kappa)$ has no positive maximum in (r_0, R). So S is non decreasing on (r_0, R). □

Remark. – It would be interesting to study the monotonicity of f.

3.1.2. Minimization of E_κ

From now on, we intend to minimize E_κ over $\mathbb{R} \times D_f \times D_S$, that is to find the best number of vortices to put at the origin. We will show that the presence of vortices lower the energy.

THEOREM 3.6. – *There exists a minimizer* (N, f, S) *of* E_κ; (f, S) *is in* $(C^\infty(B_R \setminus \{0\}) \cap C^2(B_R))^2$ *and is a solution of* (23)-(24)-(25)-(26) *with*

$$\int_{B_R} \frac{1}{r^2}\left(S - \frac{N}{\kappa}\right)f^2 = 0. \tag{29}$$

Proof. – Let (n_i, f_{n_i}, S_{n_i}) be a minimizing sequence. For each n_i, we can always replace f_{n_i} and S_{n_i} by a minimizer of E_κ with fixed $N = n_i$ as in the previous section. Then f_{n_i} is bounded in $H^1(B_R) \cap L^\infty(B_R)$ and S_{n_i} is bounded in D_S. So, up to the extraction of a subsequence,

$$f_{n_i} \rightarrow f \quad \text{weakly in } H^1, \text{ a.e. and strongly in } L^p \text{ for all finite } p,$$

$$\frac{1}{r}S'_{n_i} \rightarrow \frac{1}{r}S' \quad \text{weakly in } L^2,$$

$$\frac{1}{r}S_{n_i} \rightarrow \frac{1}{r}S \quad \text{strongly in } L^p,$$

where $(f, S) \in D_f \times D_S$. Thanks to lower semi-continuity, we have

$$\int_{B_R} \frac{1}{\kappa^2} f'^2 + \frac{1}{2}(f^2 - 1)^2 + \left(\frac{1}{r}S' - H_0\right)^2$$
$$\leq \liminf_{i\to\infty} \int_{B_R} \frac{1}{\kappa^2} f_{n_i}'^2 + \frac{1}{2}(f_{n_i}^2 - 1)^2 + \left(\frac{1}{r}S_{n_i}' - H_0\right)^2. \quad (30)$$

If the limit f is identically zero, equation (30) implies that the normal state (f_0, S_0), with $f_0 \equiv 0$ and $S_0'(r) = H_0 r$, is a minimizer of E_κ. Then the number N of vortices does not intervene.

When f is not identically zero, the sequence n_i is bounded. Indeed, we have

$$\left(\frac{n_i}{\kappa}\right)^2 \int_{B_R} \frac{1}{r^2} f_{n_i}^2 \leq 2 \int_{B_R} \frac{1}{r^2}\left(\frac{n_i}{\kappa} - S_{n_i}\right)^2 f_{n_i}^2 + 2\int_{B_R} \frac{1}{r^2} S_{n_i}^2 f_{n_i}^2,$$

and since S_{n_i} is in D_S, it implies $(1/r)S \in L^2(B_R)$. As a consequence, we can assume $n_i \to N$, and (N, f, S) is a minimizer of E_κ because of (30) and

$$\forall \alpha \quad \int_{B_R\setminus B_\alpha} \frac{1}{r^2}\left(S - \frac{N}{\kappa}\right)^2 f^2 \leq \liminf_{i\to\infty} \int_{B_R\setminus B_\alpha} \frac{1}{r^2}\left(S_{n_i} - \frac{n_i}{\kappa}\right)^2 f_{n_i}^2.$$

□

COROLLARY 3.7. – *When the normal state is not a minimizer, then the minimizing solution has vortices at the origin.*

Proof. – When $f \not\equiv 0$, equation (29) gives that $S - N/\kappa$ changes sign. The study in section 2 implies that N is positive.

3.2. Infinite Ginzburg-Landau Parameter

We let formally $\kappa = \infty$ in the energy. The difference with the one dimensional case is that we shall have to find the constant C (which comes from the term N/κ) which minimizes the energy. We define

$$E_\infty(C, f, S) = \int_{B_R} \frac{1}{2}(f^2 - 1)^2 + \frac{1}{r^2}(S - C)^2 f^2 + \left(\frac{1}{r}S' - H_0\right)^2 \, r dr d\theta. \quad (31)$$

THEOREM 3.8. – *There exists a minimizer $(C_\infty, f_\infty, S_\infty)$ of E_∞ over $\mathbb{R}\times L^4_{rad}(B_R; \mathbb{R}^+)\times D_S$. S_∞ is in $C^2(B_R))$ and $Q(r) = (S_\infty(r) - C_\infty)/r$*

satisfies

$$Q'' + \frac{1}{r}Q' - \frac{1}{r^2}Q = Q(1-Q^2)\mathbf{1}_{|Q|\le 1}, \tag{32}$$

$$Q'(r) + \frac{1}{r}Q(r) = H_0 \quad \textit{when} \quad |Q(r)| \ge 1 \quad \textit{and} \quad r = R, \tag{33}$$

$$\int_{B_R} Q(1-Q^2)\mathbf{1}_{|Q|\le 1} = 0, \tag{34}$$

$$f_\infty^2 = (1-Q^2)\mathbf{1}_{|Q|\le 1}. \tag{35}$$

Proof. – We call (C_n, f_n, S_n) a minimizing sequence and define $Q_n(r) = (S_n(r) - C_n)/r$. We proceed as in the one dimensional case: replacing f_n^2 by $(1-Q_n^2)\mathbf{1}_{|Q_n|\le 1}$ can only lower the energy so Q_n is a minimizing sequence of

$$J(Q) = \int_{B_R} \left(\frac{1}{2} - \frac{1}{2}(1-Q^2)^2\mathbf{1}_{|Q|\le 1}\right) + \left(\frac{1}{r}(rQ)' - H_0\right)^2 r dr d\theta. \tag{36}$$

Since $(1/r)(rQ)'$ is bounded in L^2, $(1/r)S_n'$ is bounded in L^2 too, so it implies S_n is bounded in L^∞. As we are only concerned with r such that $|Q_n(r)| \le 1$, we can also assume C_n is bounded. Then we can extract a subsequence that will converge to a minimizer of E_∞. Equations (32) and (34) are the corresponding Euler-Lagrange equations for the variations of S and C.

THEOREM 3.9. – *Any minimizer $(C_\infty, f_\infty, S_\infty)$ of E_∞ is such that $f_\infty \not\equiv 0$. More precisely, let $Q_\infty(r) = (S_\infty(r) - C_\infty)/r$. Then Q_∞ is increasing on $(0, R)$, $|Q_\infty|$ remains smaller than one in an annulus (r_1, R) with $Q_\infty(r_1) = -1$. On $(0, r_1)$, $f_\infty \equiv 0$ and $Q_\infty(r) = (1/2)H_0 r - (1/r)C_\infty$, where $C_\infty = r_1(1 + H_0 r_1/2)$.*

Proof. – We are going to study the shape of solutions of (32)-(33)-(34)-(35). An easy computation shows that if $Q'(r) + (1/r)Q(r) = H_0$, then $Q(r) = (1/2)H_0 r - (1/r)C$ for a given constant C. As (32) can be rewritten

$$\Delta Q - Q\left(1 + \frac{1}{r^2} - Q^2\right) = 0 \quad \text{when} \quad |Q| \le 1, \tag{37}$$

the Maximum Principle implies that Q can neither reach a positive local maximum nor a negative local minimum while it remains between -1 and 1. Moreover (34) implies that Q changes sign. So any solution of (32)-(33)-(34)-(35) with $f \not\equiv 0$ is such that Q is increasing, $|Q| < 1$ in an annulus and outside the annulus, the solution is defined by (33) and the continuity condition on the boundary. There are two possible types of solution:

- type a: $|Q| < 1$ on (r_1, R) with $Q(r_1) = -1$ and $Q(R) = \alpha \in (0,1)$,
- type b: $|Q| < 1$ on (r_1, r_2) with $Q(r_1) = -1$, $Q(r_2) = 1$ and $r_2 < R$.

We are going to show that type b solutions cannot occur. Let us define

$$F(r) = \frac{1}{2}\left(Q'(r) + \frac{1}{r}Q(r)\right)^2 + \frac{1}{4}(1 - Q^2(r))^2,$$

on the annulus where $|Q| < 1$. Thanks to (32), we see that $F'(r) = (1/r)Q^2(1-Q^2)$, so F is increasing on the annulus. But this is impossible in the case of a type b solution since $F(r_1) = F(r_2) = H_0^2/2$. So we have to investigate the existence of type a solutions. The proof now relies on a shooting method: for a given $\alpha \in (0,1)$, there is a unique solution Q_α of

$$\begin{cases} Q''_\alpha + \frac{1}{r}Q'_\alpha - \frac{1}{r^2}Q_\alpha = Q_\alpha(1 - Q_\alpha^2) \quad \text{on} \quad (0, R), \\ Q_\alpha(R) = \alpha \quad \text{and} \quad Q'_\alpha(R) = H_0 - \frac{\alpha}{R}. \end{cases} \tag{38}$$

We check that (33) is satisfied when $r = R$. We are interested in the interval where Q_α remains smaller than 1. We introduce the same sets as in the proof of Theorem 2.5: $I(R, H_0)$, $I_0(R, H_0)$ and $I_1(R, H_0)$. Notice that the Maximum Principle applied to (37) implies that when $\alpha \in I_0(R, H_0)$ there exists $r_{1,\alpha} \in (0, R)$ with $Q_\alpha(r_{1,\alpha}) = -1$ and Q_α is increasing on $(r_{1,\alpha}, R)$. For $\alpha \in I_0(R, H_0)$, we define

$$f_R(\alpha) = H_0 + \frac{1}{r_{1,\alpha}} - Q'_\alpha(r_{1,\alpha}). \tag{39}$$

We notice thanks to (32) that

$$f_R(\alpha) = \int_{B_R} Q_\alpha(1 - Q_\alpha^2)\mathbf{1}_{|Q_\alpha| \le 1} = \int_{r_{1,\alpha}}^R rQ_\alpha(1 - Q_\alpha^2)\, dr. \tag{40}$$

Classical ODE theory implies that f_R is a continuous function of α.

1st step. – Let R and H_0 be fixed. We are going to show that when $I(R, H_0) \neq \emptyset$, there is a minimizer of E_∞ of type a. In this case, we know from the proof of Theorem 2.5 that $I_0(R, H_0) = (0, \alpha^*)$ with $\alpha^* \in I(R, H_0)$. For $\alpha = 0$, $Q_\alpha < 0$ on $(r_{1,\alpha}, R)$ so (40) gives that $f_R(0) < 0$. As a consequence, f_R is negative for α small.

We let α tend to α^*. We call $r_{0,\alpha}$ the point where $Q_\alpha(r) = 0$. Up to the extraction of a subsequence, we have $r_{0,\alpha} \to r_0$. If $r_0 > 0$, then classical ODE estimates give that $Q_\alpha(r_{0,\alpha}) \to Q_{\alpha^*}(r_0)$. But as $\alpha^* \in I(R, H_0)$, we cannot have $Q_{\alpha^*}(r_0) = 0$ for $r_0 > 0$, so $r_0 = 0$. Since Q_α is increasing, $r_{1,\alpha} < r_{0,\alpha}$ and $r_{1,\alpha} \to 0$ too. We know that Q_α is bounded on $(r_{1,\alpha}, R)$, so we have

$$\lim_{\alpha\to\alpha^*} \int_{r_{1,\alpha}}^{r_{0,\alpha}} rQ_\alpha(1 - Q_\alpha^2)\, dr = 0 \quad \text{and} \quad \lim_{\alpha\to\alpha^*} \int_{r_{0,\alpha}}^{R} rQ_\alpha(1 - Q_\alpha^2)\, dr > 0.$$

It implies that $f_R(\alpha) > 0$ for α close to α^*. As $f_R(\alpha) < 0$ for small α, there exists $\beta \in (0, \alpha^*)$ such that $f(\beta) = 0$. So Q_β is a solution to (32)-(33)-(34).

We already know that the minimizer of E_∞ exists. Either it is the normal state, that is a solution (C_0, f_0, S_0) with $f_0 \equiv 0$ and $S_0' = H_0 r$, or a solution of type a. We may notice that once we have a solution Q of type a, we can go back to (C, f, S) thanks to (35), the continuity condition $H_0 r_1/2 - C/r_1 = -1$ and the definition of Q which gives $S(r) = C + rQ(r)$. Moreover, $E_\infty(C, f, S) = J(Q_\beta)$. Now let us show that Q_β provides a minimizer of E_∞. We introduce a new energy

$$E(r) = \frac{1}{2}(rQ_\beta')^2 - \frac{1}{2}Q_\beta^2 + \frac{1}{4}r^2(1 - Q_\beta^2)^2.$$

Computing $E'(r)$, using equation (32), gives

$$\frac{1}{2}\int_{r_{1,\beta}}^{R} r(1-Q_\beta^2)^2\, dr = \frac{1}{4}R^2(1-\beta^2)^2 + \frac{1}{2}H_0^2(R^2 - r_{1,\beta}^2) - H_0(R\beta + r_{1,\beta}).$$

This and an integration by parts on $\int_{r_{1,\beta}}^{R} (1/r)((rQ_\beta)')^2$ enable us to estimate $J(Q_\beta)$.

$$J(Q_\beta) = \frac{1}{4}R^2 - \frac{1}{4}R^2(1 - \beta^2)^2 - \int_{r_{1,\beta}}^{R} rQ_\beta^2(1 - Q_\beta^2). \tag{41}$$

Since $E_\infty(C_0, f_0, S_0) = R^2/4$, it means Q_β has a lower energy than the normal state.

2nd step. – We assume H_0 is fixed. Then for small R ($R \leq 1/H_0$ for instance), the first step indicates that there exist a minimizer of type a. We call

$$R_0 = \max\{R \text{ st the normal state is not a minimizer of } E_\infty.\} \tag{42}$$

Let us assume that R_0 is finite. A straightforward argument shows that if Q_{α_R} corresponds to a minimizer of E_∞ with $Q_{\alpha_R}(R) = \alpha_R$, and if $J_0 = R_0^2/4$ is the energy of the normal state for $R = R_0$, then

$$\lim_{R \to R_0} J(Q_{\alpha_R}) = J_0.$$

According to (41), it means that $\alpha_R \to 1$ and $\int_{r_{1,\alpha_R}}^R rQ_{\alpha_R}^2(1 - Q_{\alpha_R}^2) \to 0$. But this is impossible, so $R_0 = +\infty$. □

Remark. – Numerical computations show that f_R is increasing on $(0, 1)$ hence the minimizer of E_∞ is unique.

COROLLARY 3.10. – *For all H_0, there exists κ_0 such that for $\kappa \geq \kappa_0$, the normal state is not a minimizer of E_κ.*

The proof relies on energy comparisons as in [1].

PROPOSITION 3.11. – *When R is large, C_∞ is equivalent to $H_0 R^2/2$.*

Proof. – We already know that $C_\infty = r_1(1 + H_0 r_1/2)$, where r_1 is such that $Q(r_1) = -1$ and Q is as in Theorem 3.8 associated to the minimizer of E_∞. We only need to show that r_1 is equivalent to R when R is large. Let V be the solution of

$$\begin{cases} V'' = V(1 - V^2) \quad \text{on} \quad (0, R), \\ V(R) = \alpha \quad \text{and} \quad V'(R) = H_0 - \alpha/R, \end{cases}$$

where $\alpha = Q(R)$. It is easy to see that there exists ρ_1 in $(0, R)$ such that $V(\rho_1) = -1$, V is increasing on (ρ_1, R) and the energy $V'^2 + (1 - V^2)^2/2$ is preserved on (ρ_1, R). A straightforward computation gives

$$R - \rho_1 = \int_{-1}^{\alpha} \left(\left(H_0 - \frac{\alpha}{R} \right)^2 + \frac{1}{2}(1 - \alpha^2)^2 - \frac{1}{2}(1 - v^2)^2 \right)^{-\frac{1}{2}} dv. \quad (43)$$

So if we show that $r_1 > \rho_1$, the proof is over. We know that $Q < V$ for r close to R and we are going to show that Q and V cannot intersect before reaching -1. Let

$$E_1(r) = Q'^2 + \frac{1}{2}(1 - Q^2)^2.$$

We immediately get $E_1'(r) = (2/r^2)Q'(Q - rQ')$. Let r_0 be the point where Q crosses zero. Since Q is increasing, $E_1' < 0$ on (r_1, r_0). Let $E_2(r) = rQ - r2Q'$ on (r_0, R). Since $E_2'(r) = -r^2 Q(1 - Q^2)$, it implies

$E_2(r) < 0$ on (r_0, R) and E_1 is decreasing. Since for $r = R$ the energy E_1 is the same for Q and V, it means

$$Q'^2 + \frac{1}{2}(1 - Q^2)^2 > V'^2 + \frac{1}{2}(1 - V^2)^2 \quad \forall r \in (r_1, R).$$

So Q and V cannot intersect before reaching -1 and $r_1 > \rho_1$. □

3.3. Convergence of minimizers

THEOREM 3.12. – *Let $(N_\kappa, f_\kappa, S_\kappa)$ be a sequence of minimizers of E_κ. There exists $(C_\infty, f_\infty, S_\infty)$, a minimizer of E_∞, and a subsequence $(N_{\kappa_n}, f_{\kappa_n}, S_{\kappa_n})$, such that when κ_n tends to ∞*

$$\frac{N_{\kappa_n}}{\kappa_n} \to C_\infty,$$
$$f_{\kappa_n} \to f_\infty \quad \text{in} \quad L^p(B_R) \quad \text{for all finite } p,$$
$$\frac{1}{r}S_{\kappa_n} \to \frac{1}{r}S_\infty \quad \text{in} \quad C^{0,\alpha}(B_R) \quad \text{for all } \alpha \in (0,1).$$

Proof. – Let $(C_\infty, f_\infty, S_\infty)$ be a minimizer of E_∞. We have seen that f_∞ is not in $H^1(B_R)$, but as in [1], we can find g_κ in $H^1(B_R)$ such that

$$\lim_{\kappa \to \infty} E_\kappa(C_\infty \kappa, g_\kappa, S_\infty) = E_\infty(C_\infty, f_\infty, S_\infty).$$

Let $(N_\kappa, f_\kappa, S_\kappa)$ be a minimizer of E_κ. Energy comparisons give:

$$E_\infty(C_\infty, f_\infty, S_\infty) \leq E_\infty\left(\frac{N_\kappa}{\kappa}, f_\kappa, S_\kappa\right) \leq E_\kappa(N_\kappa, f_\kappa, S_\kappa),$$

$$E_\kappa(N_\kappa, f_\kappa, S_\kappa) \leq E_\kappa(C_\infty \kappa, g_\kappa, S_\infty).$$

We let κ tend to infinity and obtain

$$\lim_{\kappa \to \infty} \int_{B_R} \frac{1}{\kappa^2} f'^2_\kappa = 0$$
$$\text{and} \lim_{\kappa \to \infty} E_\infty\left(\frac{N_\kappa}{\kappa}, f_\kappa, S_\kappa\right) = E_\infty(C_\infty, f_\infty, Q_\infty). \quad (44)$$

It implies that $(1/r)S_\kappa$ is bounded in D_S. As $\|f_\kappa\|_{L^\infty} \leq 1$, up to the extraction of a subsequence,

$$f_\kappa^2 \rightharpoonup f^2 \quad \text{weakly in } L^2(B_R),$$
$$\frac{1}{r}S'_\kappa \rightharpoonup \frac{1}{r}S' \quad \text{weakly in } L^2,$$
$$\frac{1}{r}S_\kappa \to \frac{1}{r}S \quad \text{strongly in } L^p \text{ for all finite } p,$$

where $(f, S) \in L^4(B_R) \times D_S$. Weak convergence and lower semi continuity give

$$\int_{B_R} \frac{1}{2}(f^2-1)^2 + \left(\frac{1}{r}S' - H_0\right)^2 \leq \liminf_{\kappa \to \infty} \int_{B_R} \frac{1}{2}(f_\kappa^2 - 1)^2 + \left(\frac{1}{r}S'_\kappa - H_0\right)^2. \tag{45}$$

(i) If $f \equiv 0$, it is easy to see that $f_\kappa \to 0$ a.e. and in L^p for all finite p. Then (44) and (45) imply

$$E_\infty(0, f, S) \leq \liminf_{\kappa \to \infty} E_\kappa(N_\kappa, f_\kappa, S_\kappa) = E_\infty(C_\infty, f_\infty, S_\infty).$$

So it means that the normal state is a minimizer of E_∞, which is not the case as shown in Theorem 3.9.

(ii) So $f \not\equiv 0$, and as in the proof of Theorem 3.6, since

$$\left(\frac{N_\kappa}{\kappa}\right)^2 \int_{B_R} \frac{1}{r^2} f_\kappa^2$$

is bounded independently of κ, then the sequence N_κ/κ is bounded. So, up to the extraction of a subsequence, $N_\kappa/\kappa \to C$. Let $\mathbf{A}_\kappa(r) = (1/r)S_\kappa(r)\mathbf{e}_\theta$. We have

$$\Delta \mathbf{A}_\kappa = \left(\mathbf{A}_\kappa - \frac{N_\kappa}{\kappa r}\mathbf{e}_\theta\right) f_\kappa^2.$$

Since $\mathbf{A}_\kappa$ is bounded in H^1 and $\Delta \mathbf{A}_\kappa$ is bounded in L^p for $p < 2$, it implies that $\mathbf{A}_\kappa$ is bounded in $W^{2,p}$ and we infer from elliptic estimates that $\mathbf{A}_\kappa$ converges to $\mathbf{A}$ in $W^{1,p}(B_R)$ for $p < 2$ and $C^{0,\alpha}(B_R)$ for $\alpha \in (0, 1)$. For all small ϵ, we have

$$\liminf_{\kappa \to \infty} \int_{B_R \setminus B_\epsilon} \frac{1}{r^2}\left(S_\kappa - \frac{N_\kappa}{\kappa}\right)^2 f_\kappa^2 \geq \int_{B_R \setminus B_\epsilon} \frac{1}{r^2}(S - C)^2 f^2. \tag{46}$$

We can easily derive from (44)-(45)-(46) that (C, f, S) is a minimizer of E_∞. So thanks to Theorem 3.9, there exists r_1 such that $f \equiv 0$ on $(0, r_1)$ and $f^2 = 1 - (S - C)^2/r^2$ on (r_1, R). Then we proceed as in [1] to show that f_κ tends to f strongly in L^p, now that we know the shape of f.

3.4. Numerical Study

We use the same scheme as before. We compute the solutions with $R = 1$ and $H_0 = 3$. We are interested in what happens with large κ. Figure 3

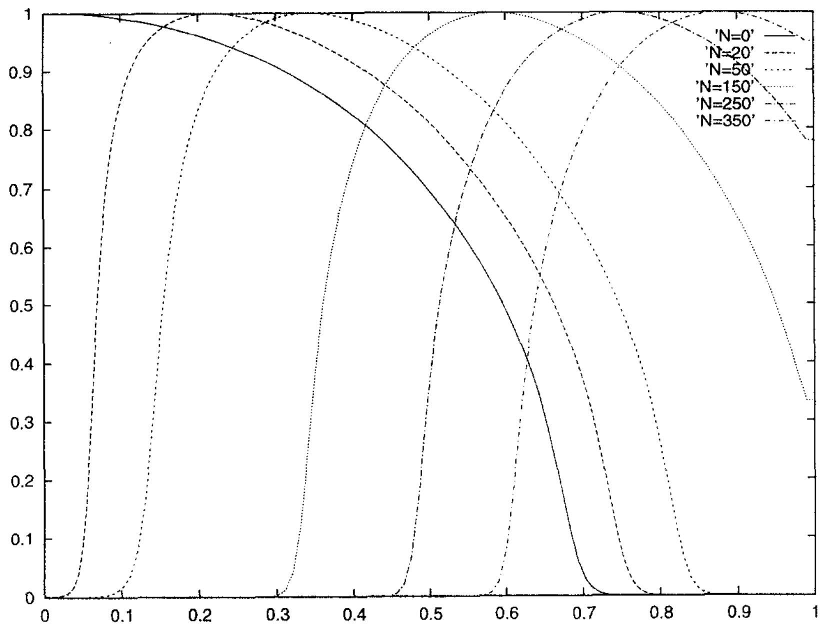

Figure 3. – Radial solutions with vortices $f(r)$ for $R = 1$, $\kappa = 300$, $H_0 = 3$.

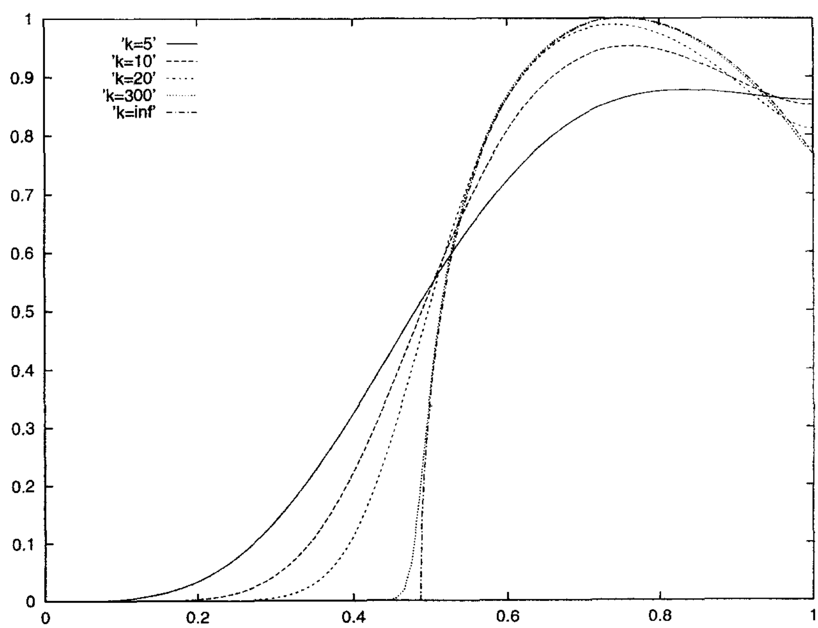

Figure 4. – Minimizers of E_κ with vortices $f(r)$ for $R = 1$, $H_0 = 3$.

illustrates the different types of radial solutions f to the Ginzburg-Landau equations according to the number of vortices:

– when N is too large, the only solution is $f \equiv 0$;

– for N of the same order as κ, f is equal to zero in a small ball around the origin;

– for small N, f is equal to zero in an exterior ring.

Figure 4 illustrates Theorem 3.12: it shows the convergence of minimizers f_κ as κ tends to infinity. We can see that the vortex core in an inside ball and there is an outside annulus where superconductivity remains.

4. CONCLUSION

We have proved in the case of a ball, that for large κ, the minimizer among radially symmetric solutions has N vortices concentrated at the origin, N being of order κ. It should be interesting to make a stability analysis of these minimizers. What we may expect for the global minimizer of E_κ is to have of order κ vortices, but not necessarily concentrated at one point.

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