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Minimal, rigid foliations by curves on $\mathbb{CP}^n$

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Abstract. We prove the existence of *minimal* and *rigid* singular holomorphic foliations by curves on the projective space $\mathbb{CP}^n$ for every dimension $n \geq 2$ and every degree $d \geq 2$. Precisely, we construct a foliation $\mathcal{F}$ which is induced by a homogeneous vector field of degree d, has a finite singular set and all the regular leaves are dense in the whole of $\mathbb{CP}^n$. Moreover, $\mathcal{F}$ satisfies many additional properties expected from chaotic dynamics and is rigid in the following sense: if $\mathcal{F}$ is conjugate to another holomorphic foliation by a homeomorphism sufficiently close to the identity, then these foliations are also conjugate by a projective transformation. Finally, all these properties are persistent for small perturbations of $\mathcal{F}$.

This is done by considering pseudo-groups generated on the unit ball $\mathbb{B}^n \subset \mathbb{C}^n$ by small perturbations of elements in Diff$(\mathbb{C}^n, 0)$. Under open conditions on the generators, we prove the existence of many pseudo-flows in their closure (for the C^0-topology) acting transitively on the ball. Dynamical features as minimality, ergodicity, positive entropy and rigidity may easily be derived from this approach. Finally, some of these pseudo-groups are realized in the transverse dynamics of polynomial vector fields in $\mathbb{CP}^n$.

Introduction

A vector field with homogeneous polynomial coefficients of degree d in $\mathbb{C}^{n+1}$

$$\mathcal{Z} = H_0(z)\partial_{z_0} + H_1(z)\partial_{z_1} + \cdots + H_n(z)\partial_{z_n}, \quad z = (z_0, z_1, \ldots, z_n)$$

defines a regular holomorphic foliation by complex curves in $\mathbb{C}^{n+1} \setminus \mathrm{Sing}(\mathcal{Z})$ where $\mathrm{Sing}(\mathcal{Z})$ stands for the common zero set of the coefficients H_i. The leaves

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are the complex trajectories (integral curves) of $\mathcal{Z}$. This foliation, as well as the singular set, are invariant under the radial action of $\mathbb{C}^*$ by homotheties. Therefore it induces a foliation on $\mathbb{CP}^n$ which is regular away from an analytic set $\mathrm{Sing}(\mathcal{F})$ corresponding to the projection of the set where the vector field $\mathcal{Z}$ is tangent to the radial vector field. Modulo dividing $\mathcal{Z}$ by the common factor of the polynomials H_i (which does not change the underlying foliation), we can assume without loss of generality that the singular set of $\mathcal{F}$, $\mathrm{Sing}(\mathcal{F})$, has codimension ≥ 2. Conversely, it turns out that any regular one-dimensional holomorphic foliation on $\mathbb{CP}^n \setminus \mathcal{S}$, where $\mathcal{S}$ is an analytic set of codimension ≥ 2, is obtained as above.

The set $\mathcal{F}^d(\mathbb{CP}^n)$ of such foliations inherits a natural structure of finite dimensional complex manifold (actually it is a Zariski-open subset of some projective space $\mathbb{CP}^N$, see Sect. 7). When the degree d is *zero* (resp. *one*), all these foliations are easily described. Indeed, they are conjugate under $PGL(n+1,\mathbb{C})$ to a foliation given in the main affine chart $(z_1,\dots,z_n)$ of $\mathbb{CP}^n$ by a constant ∂_{z_1} (resp. linear $\sum_{1\leq i,j\leq n} m_{i,j} z_i \partial_{z_j}$) vector field ($(m_{i,j}) \in GL(n,\mathbb{C})$). In contrast, very little is known about the dynamics of a generic foliation $\mathcal{F} \in \mathcal{F}^d(\mathbb{CP}^n)$ for $d \geq 2$. Roughly speaking, the generic foliation $\mathcal{F}$ of degree d has $\frac{d^{n+1}-1}{d-1}$ singular points which are all hyperbolic and no leaf is contained in an algebraic curve (see [LN,So]). In particular every leaf has an infinite limit set (consisting of a union of leaves and possibly singular points). Finally it is also known that every leaf, viewed as an abstract Riemann surface, is uniformized by the unit disc Δ (see [LN]). To be complete, one should also mention that the foliation is topologically linearizable on neighborhoods of its hyperbolic singularities (see [Ch]), so that the local dynamics is rather "poor" (see Sect. 7 for details). In fact, these properties are satisfied by any $\mathcal{F}$ belonging to some real Zariski-open subset of $\mathcal{F}^d(\mathbb{CP}^n)$ and, to the best of our knowledge, these are all the established facts about the dynamics of generic foliations in $\mathcal{F}^d(\mathbb{CP}^n)$. In particular it is not known whether or not every leaf must contain a singular point in its closure. This problem, namely the possible existence of *an exceptional minimal set* (see [Ca,LN,Sa]), has prevented further progress on the study of generic foliations for years and remains unsolved. However, in his report to the Helsinki Conference (cf. [Il2, p. 823]), Il'yashenko made some conjectures concerning the global dynamical behavior of "most of" these foliations. The purpose of the present work is to provide a partial (or local) affirmative answer to his conjectures by proving the following theorem:

Theorem A. *For any $d \geq 2$, there exists a non-empty open subset $\mathcal{U} \subset \mathcal{F}^d(\mathbb{CP}^n)$ such that any foliation $\mathcal{F}$ belonging to $\mathcal{U}$ has a finite number of singularities and satisfies:*

- **Minimality:** *every leaf is dense in $\mathbb{CP}^n$;*
- **Ergodicity:** *every measurable set of leaves has zero or total Lebesgue measure;*
- **Rigidity:** *if $\mathcal{F}' \in \mathcal{F}^d(\mathbb{CP}^n)$ is conjugate to $\mathcal{F}$ by an homeomorphism $\Phi : \mathbb{CP}^n \to \mathbb{CP}^n$ close to the identity, then $\mathcal{F}$ and $\mathcal{F}'$ are also conjugate under $PGL(n+1,\mathbb{C})$.*

Even for $n = 2$ this result is new, as far as ergodicity and topological rigidity are concerned. In higher dimensions no example of minimal foliations was previously known (cf. below). Besides, following a remarkable idea due to É. Ghys, we deduce the:

Corollary B. *For any $d \geq 2$, there exists a subset $E \subset \mathcal{F}^d(\mathbb{CP}^n)$ having total Lebesgue measure such that every element $\mathcal{F} \in E$ is tangent to no (strict) algebraic subset of $\mathbb{CP}^n$.*

In other words, all leaves of $\mathcal{F}$ are Zariski-dense in $\mathbb{CP}^n$. This result has been proved only for $n = 2$ (see [Jo]) and $n = 3$ (for a Zariski-open subset of $\mathcal{F}^d(\mathbb{CP}^3)$, see [So]). In higher dimensions it has been established only for analytic sets of dimension 1, as mentioned above.

Consider a germ X of degenerate singular analytic vector field defined around the origin of $\mathbb{C}^{n+1}$ and having Taylor expansion $X = X_d + X_{d+1} + \cdots$ (with X_i homogeneous of degree i). It follows from Corollary B that, if X_d is "generic", then X is tangent to no analytic set of dimension ≥ 2 on a neighborhood of the origin. Indeed, after blowing-up the origin, the tangent cone of a possible invariant analytic set is an algebraic subset which is invariant by the foliation $\mathcal{F}$ induced by X_d on the exceptional divisor. This fact should be compared to the well-known examples of analytic vector fields without invariant analytic curves due to [GM,Lu] and [Lu,Ol]. Nevertheless, these latter examples have no intersection with our construction since the foliations $\mathcal{F}$ in question have degenerate singular points.

The rest of the introduction is devoted to discussing the main ideas involved in the proof of our theorem as well as situating it with regard to previous work. In dimension $n = 2$, a great amount of work has been devoted to the dynamical behavior of foliations in $\mathcal{F}^d(\mathbb{CP}^2)$ *which are tangent to a projective line*, say the line L_∞ at infinity; let us denote by $\mathcal{F}^d(\mathbb{C}^2) \subset \mathcal{F}^d(\mathbb{CP}^2)$ the class of these foliations. A combination of remarkable results due to M.O. Hudai-Verenov and mainly to Yu. Il'yashenko in the 70's yields the following theorem:

Theorem (Il'yashenko). *For any $d \geq 2$, there is a set $\mathcal{A}^d \subset \mathcal{F}^d(\mathbb{C}^2)$ having total Lebesgue measure such that any foliation $\mathcal{F} \in \mathcal{A}^d$ has a finite number of singularities and satisfies:*

- **Minimality:** *each leaf (apart from the invariant line at infinity) is dense in $\mathbb{C}^2$;*
- **Ergodicity:** *any measurable set of leaves has zero or total Lebesgue measure;*
- **Rigidity:** *if $\mathcal{F}' \in \mathcal{F}^d(\mathbb{C}^2)$ is conjugate to $\mathcal{F}$ by an homeomorphism $\Phi : \mathbb{CP}^n \to \mathbb{CP}^n$ close to the identity, then $\mathcal{F}$ and $\mathcal{F}'$ are also conjugate by an affine transformation.*

Improvements due to A. Shcherbakov allow us to consider the class $\mathcal{A}^d$ as being open inside $\mathcal{F}^d(\mathbb{C}^2)$ (cf. [Sh1]). Further improvements have been made as the reader can check in [GM,OB], [GM] and [LN,Sa,Sc], always under the strong additional hypothesis that the foliation is tangent to an algebraic curve. Nonetheless, as already mentioned, the subclass of $\mathcal{F}^d(\mathbb{CP}^2)$ consisting of those foliations admitting an algebraic invariant curve is *very small* in the sense that it is contained in a Zariski-closed subset of high codimension. In particular, the results above fail to provide *an open set* of foliations (with fixed degree) exhibiting "chaotic" behavior. In fact, the open set $\mathcal{U}$ of Theorem A contains the class $\mathcal{F}^d(\mathbb{C}^2)$ in its boundary.

In [Mj], B. Mjuller constructed an open set of minimal foliations in $\mathcal{F}^d(\mathbb{CP}^2)$. He then derived examples of foliations in $\mathbb{CP}^3$ tangent to the projective plane at infinity and having all leaves dense in the affine part. Actually only recently B. Wirtz announced a new construction of stably minimal foliations *in dimension 2* having positive entropy. The analogous questions concerning ergodicity and topological rigidity were not addressed as far as we know.

The original approach of Il'yashenko to study elements of $\mathcal{F}^d(\mathbb{C}^2)$ is based on studying the holonomy group $\mathrm{Hol}(L_\infty)$ of the invariant line at infinity L_∞. Indeed $\mathrm{Hol}(L_\infty)$ is in general a "large" (e.g. non-solvable) subgroup of $\mathrm{Diff}(\mathbb{C}, 0)$ whose dynamics can be well-understood. Furthermore, every leaf of the foliation $\mathcal{F}$ in question must accumulate on L_∞ so that it is captured by the dynamics of $\mathrm{Hol}(L_\infty)$. In this way it is possible to deduce global properties of $\mathcal{F}$ from the local dynamics of $\mathrm{Hol}(L_\infty)$.

The generalization of this approach, however, involves mainly 3 difficulties. First, for studying *generic* foliations (even when $n = 2$), one should be able to deal with leaves having only "small" (e.g. cyclic) holonomy groups. We overcome this difficulty by considering perturbations of foliations having large linear holonomy group and showing the persistence of some "rich" transverse dynamics. On the other hand, for dimensions greater than 2, there is another additional difficulty, namely the holonomy groups involved are subgroups of $\mathrm{Diff}(\mathbb{C}^n, \underline{0})$ (or more generally, pseudo-groups acting on the unit ball of $\mathbb{C}^n$) whose study is much harder than the one-dimensional case. Finally, in dimensions greater than 2, a description of the dynamics of a foliation in a neighborhood of a curve does not automatically propagates to the entire projective space.

This paper is organized as follows. First we study the dynamics of certain pseudo-groups acting on the unit ball $\mathbb{B}^n$ of $\mathbb{C}^n$ without a common fixed point, which are obtained as "perturbations" of subgroups of $\mathrm{Diff}(\mathbb{C}^n, 0)$. These pseudo-groups will later embody pseudo-groups generated by the holonomy groups of several leaves taken together. Their dynamics are essentially investigated through their affine part. Indeed, under some assumptions, we prove that the pseudo-groups approximate many affine "pseudo-flows" as if they were a non-discrete subgroup of a Lie group (Sects. 2, 3, 4). This approach of the dynamics was already used in [Sh1], [Na], [Reb1] and [Be,Li,Lo1]. Using these "pseudo-flows" it is easy to conclude that the original dynamics are minimal, ergodic and rigid (as well as to show that they have positive entropy, Sects. 5, 6). After realizing the pseudo-groups considered above as holonomy of a foliation $\mathcal{F}$ in $\mathbb{CP}^{n+1}$, we shall obtain a good control of the dynamics of $\mathcal{F}$ in a certain region of $\mathbb{CP}^{n+1}$. We then use an "induction trick" to deduce the global behavior of $\mathcal{F}$ by means of these local data. An example of the nature of the results obtained in Sect. 5, is the following theorem.

Theorem C. *Let $f_1, \dots, f_d$ be in Diff$(\mathbb{C}^n, \underline{0})$. Suppose that their linear parts $A_1, \dots, A_d$ generate a dense subgroup of $GL(n, \mathbb{C})$. Then, there are constants $\varepsilon, r > 0$ such that any transformations $g_1, \dots, g_d : \mathbb{B}^n_r \to \mathbb{C}^n$ satisfying $\|g_i - f_i\|_{\mathbb{B}^n_r} \leq \varepsilon$ generate a minimal and ergodic pseudo-group either on $\mathbb{B}^n_r$, or on $\mathbb{B}^n_r \setminus \{p\}$ if the g_i's share a common fixed point p inside the ball.*

When $g_i \equiv f_i$ ($p = \underline{0}$), the fact that the pseudo-group in question acts minimally on the punctured ball was known only for the one-dimension case (see [Il1]) and was independently proved by S. Lamy in dimension two (see [La]). Finally we should say that our techniques of producing flows associated to pseudo-groups acting on $\mathbb{B}^n$ might be of interest for other purposes and, for this reason, are presented in the most general setting. After we started to work on this subject, by fall 1998, two works of our colleague M. Belliart improved our Theorem C (see [Be1] and [Be2]). He showed that under further generic conditions on the 3-jet of $g_1, \dots, g_d$, the corresponding minimal pseudo-group actually approximates any germ $g : (\mathbb{C}^n, q) \to (\mathbb{C}^n, q')$ of holomorphic diffeomorphism inside the domain of minimality $\mathbb{B}^n_r$ (resp. $\mathbb{B}^n_r \setminus \{p\}$).

1. Preliminary constructions within the linear group of $\mathbb{C}^n$

In this section, we shall recall some classical ideas which will be used and generalized to the non-linear context of pseudo-groups on the unit ball $\mathbb{B}^n \subset \mathbb{C}^n$ in Sects. 3, 4 and 5. Precisely, we are going to prove that, close to the identity $I \in GL(n, \mathbb{C})$, two generic matrices A and B generate a *large* subgroup G, accumulating on the whole of $SL(n, \mathbb{C})$. Throughout this section, the dimension is supposed to be $n \geq 2$.

Recall that two complex numbers $\lambda, \mu \in \mathbb{C}^*$ may generate a dense or a discrete subgroup Λ of $\mathbb{C}^*$ depending on their "multiplicative dependence over $\mathbb{Z}$". In particular, for subgroups of $\mathbb{C}^*$, properties as being discrete, non-discrete or dense are not persistent under perturbation of the generators. Nonetheless we point out that for "most" choices of the generators λ, μ (in the sense of Lebesgue measure on $(\mathbb{C}^*)^2$), the resulting group Λ is dense in $\mathbb{C}^*$.

Let us now consider the subgroup $G \subset GL(n, \mathbb{C})$, $n \geq 2$, generated by two matrices A and B. The determinant map, which associates to an element of G its determinant, defines a homomorphism from G to the subgroup $\det(G) = \{\delta = \det(C)\ ;\ C \in G\}$ of $\mathbb{C}^*$. Clearly $\det(G)$ is generated by the scalars $\lambda = \det(A)$ and $\mu = \det(B)$. Thus it follows again that A, B cannot persistently generate a dense subgroup of $GL(n, \mathbb{C})$. Moreover, consider two matrices A and B whose corresponding projective transformations $\widehat{A}, \widehat{B} \in PGL(n, \mathbb{C})$ generate a Schottky group on $\mathbb{CP}^n$. The group G is a discrete subgroup of $GL(n, \mathbb{C})$ and is, in fact, persistently discrete: the projective action of matrices A', B', respectively close to A, B will still generate a Schottky group on $\mathbb{CP}^n$. On the other hand the classical Zassenhaus Lemma (which holds for any finite-dimensional Lie group) ensures the existence of a neighborhood U of the identity such that any non-nilpotent subgroup G admitting a finite generating set contained in U is not discrete. Clearly, this statement enables us to find examples of groups $G \subseteq PGL(n, \mathbb{C})$ which are persistently non-discrete.

Let us equip $GL(n, \mathbb{C})$ with the distance $dist(M, N) = \|M - N\|$ induced by the norm $\|M\| = \sup_{|z|=1} |Mz|$. The key ingredient of Zassenhaus Lemma may be presented as follows:

Lemma 1.0. *For any $n \geq 2$, there exist constants $\varepsilon_0, C_0 > 0$ such that any matrices $A, B \in GL(n, \mathbb{C})$ which are ε_0-close to the identity I satisfy*

$$\|[A, B] - I\| \leq C_0 \cdot \|A - I\| \cdot \|B - I\| .$$

Proof. The differentiable map $(A, B) \mapsto [A, B]$ is equal to the the identity I on $GL(n, \mathbb{C}) \times \{I\}$ and on $\{I\} \times GL(n, \mathbb{C})$. Hence its differential at (I, I) is trivial. Then the Taylor's formula provides the required estimate. □

Equip $GL(n, \mathbb{C}) \times GL(n, \mathbb{C})$ with the product distance arising from the distance in $GL(n, \mathbb{C})$ introduced above.

Corollary 1.1. *For any $n \geq 2$, there is a Zariski-open subset $\mathcal{U}_1$ of the ε_1-neighborhood of (I, I) in $GL(n, \mathbb{C}) \times GL(n, \mathbb{C})$, for some $\varepsilon_1 > 0$, having the following property: if a pair of matrices (A, B) belongs to $\mathcal{U}_1$, then A, B generate a non-discrete subgroup G of $GL(n, \mathbb{C})$. In particular, the closure $\overline{G}$ of G (for the usual topology) contains a non-trivial real one-parameter group* $\exp(tM)$*, $t \in \mathbb{R}$.*

Proof. Choose $\varepsilon_1 \leq \varepsilon_0$ small enough such that $C_0\varepsilon_1 < 1/2$. According to Lemma 1.0 the sequence of iterated commutators, inductively given by

$$B_0 = B \quad \text{and} \quad B_{k+1} = [A, B_k] = AB_kA^{-1}B_k^{-1} \quad \text{for} \quad k \in \mathbb{N},$$

converges to the identity and $dist(B_k, I) < \varepsilon_1/2^k$.

Now, assume that A has only simple eigenvalues and choose linear coordinates where A is diagonal. Clearly, BAB^{-1} is also diagonal if and only if B permutes the eigendirections of A. We *claim* that B cannot non-trivially permute these directions provided that ε_1 was chosen small enough. Taking for grant this claim, it results that BAB^{-1} (or equivalently $[A, B]$) is diagonal if, and only if, B is so. By induction, the sequence $\{B_k\}$ is non-trivial, i.e. $B_k \neq I$ for every k, as long as B is not diagonal simultaneously with A.

Let us now prove our claim. Set $\varepsilon_1 < 1/n$. Then any matrix B which is ε_1-close to identity satisfies $|\mathrm{tr}(B) - n| < 1$ where $\mathrm{tr}(B)$ stands for the trace of B. On the other hand, given a basis of unit vectors v_i generating the permuted n directions, then clearly (these vectors are close to one another and) we have $B(v_i) = \lambda_i \cdot v_{\sigma(i)}$ for a permutation σ and scalars λ_i which are ε_1-close to 1. In the basis (v_i), the matrix B (may lie far from the identity but) has the following special form: on each row and each column, all coefficients but one are zero and, besides, the unique coefficient different from zero is equal to λ_i. The number of λ_i's appearing in the diagonal is equal to the number of unpermuted directions. If this number is not n, say $k \leq n - 2$, then $|\mathrm{tr}(B) - k| < k \cdot \varepsilon_1$ which gives a contradiction.

Finally, we have proved that the group G is non-discrete as long as A has only simple eigenvalues with at least one eigendirection which is not shared with B. The set $\mathcal{U}_1$ of (A, B) satisfying these conditions is clearly the complement of a finite number of algebraic relations among their coefficients, and thus it is Zariski-open. According to Cartan's theorem, the closure $\overline{G}$ of G is a real analytic Lie group which, being not discrete, must have a non-trivial Lie algebra $\mathfrak{G}$. The Corollary is proved. □

It is convenient to recall a direct construction of a non-trivial real one-parameter group (also called a *real flow*) contained in $\overline{G}$ (so that we can dispense with Cartan's theorem). This may be outlined as follows. For a suitable sequence $N_k \in \mathbb{N}$ (necessarily tending to $+\infty$), the renormalized matrices $C_k = B_k^{N_k}$ have distance to the identity upper and lower bounded by positive constants, say $r_1 < \|C_k - I\| < r_2$ for k large, $0 < r_1 < r_2$. Indeed, denote by $\mathbb{B}_t(M)$ the ball of radius t centered at M in $\mathfrak{gl}(n, \mathbb{C})$ and choose t_2, r_2, t_1, r_1, $0 < t_1 < \frac{t_2}{2}$, so that the exponential map takes the ball $\mathbb{B}_{t_2}(0)$ diffeomorphically onto a neighborhood of $\mathbb{B}_{r_2}(I)$ and takes the annulus $\mathbb{B}_{2t_1}(0) \setminus \mathbb{B}_{t_1}(0)$ inside the annulus $\mathbb{B}_{r_2}(I) \setminus \mathbb{B}_{r_1}(I)$. Since, every matrix $M \in \mathbb{B}_{t_1}(0)$ has an integer multiple nM belonging to the annulus $\mathbb{B}_{2t_1}(0) \setminus \mathbb{B}_{t_1}(0)$, the matrix $C = \exp(M) \in \mathbb{B}_{r_1}(I)$ escapes from this ball under iteration and its orbit intersects the annulus $\mathbb{B}_{r_2}(I) \setminus \mathbb{B}_{r_1}(I)$. Going back to the sequence C_k, modulo passing to a subsequence, it converges to some $C \in GL(n, \mathbb{C})$. Such C is the time-one map of the one-parameter family $C^t = \lim_{k\to\infty} B_k^{[t\cdot N_k]}$, $t \in \mathbb{R}$, where $[\ \cdot\ ]$ stands for the integral part.

Using the preceding statements, it is rather easy to ensure that, under further (open) generic assumptions on the matrices A, B, the closure $\overline{G}$ is as large as possible, namely it maps onto $PGL(n, \mathbb{C})$ under the natural projection. This is the contents of Corollary 1.3 which will follow from:

Lemma 1.2. *There is a (real) Zariski-open subset $\mathcal{U}_2$ of $GL(n, \mathbb{C}) \times GL(n, \mathbb{C})$ having the following property: if $(A, B) \in \mathcal{U}_2$, and $\mathfrak{G}$ is a real vector subspace of $\mathfrak{gl}(n, \mathbb{C})$ which is invariant under conjugation by A and B (i.e. $\mathfrak{G}$ satisfies $A\mathfrak{G}A^{-1} = B\mathfrak{G}B^{-1} = \mathfrak{G}$), then either $\mathfrak{G} \subset \mathbb{C} \cdot I$ or $\mathfrak{sl}(n, \mathbb{C}) \subset \mathfrak{G}$ (in the first case we say that $\mathfrak{G}$ is scalar and in the second that $\mathfrak{G}$ is large).*

Proof. We show, under generic assumptions on A and B, that any matrix $M_0 \in \mathfrak{gl}(n, \mathbb{C})$ which is not a scalar multiple of the identity (in particular M_0 does not have all entries equal to *zero*) together with its conjugates by A and B, generate a subspace $\mathfrak{G}$ over $\mathbb{R}$ which contains $\mathfrak{sl}(n, \mathbb{C})$. First we impose the condition of Corollary 1.1, namely that A has only simple eigenvalues $\lambda_1, \dots, \lambda_n$ and choose linear coordinates where A is diagonal. The action of A by conjugation on $\mathfrak{gl}(n, \mathbb{C})$

$$\mathfrak{gl}(n, \mathbb{C}) \to \mathfrak{gl}(n, \mathbb{C})\ ;\ M \mapsto AMA^{-1},$$

is linear diagonal with eigenvalues $\lambda_{i,j} = \lambda_i/\lambda_j$ in the Kronecker basis $(\delta_{i,j})$ of $\mathfrak{gl}(n, \mathbb{C})$. For $i = 1, \dots, n$, we clearly have $\lambda_{i,i} = 1$, and for $i \neq j$, we require that the $\lambda_{i,j}$'s are pairwise distinct in norm and non-real. Finally we suppose that B (resp. B^{-1}) takes the n eigendirections of A to the complement of the n invariant hyperplanes of A (in the coordinate where A is represented by a diagonal matrix, the preceding condition means that neither B nor B^{-1} has a vanishing entry). Clearly the set $\mathcal{U}_2$ of pairs (A, B) defined by those conditions is the complement of finitely many algebraic relations among the coefficients of the matrices. Indeed, the conditions on λ_i are equivalent to the non-vanishing of the algebraic function

$$F(A) = \prod_{i<j} \Im\mathrm{m}\left(\frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i}\right).$$

For example, when $n = 2$ and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, they become

$$\frac{\lambda_1}{\lambda_2} + \frac{\lambda_2}{\lambda_1} = \frac{(\lambda_1 + \lambda_2)^2}{\lambda_1\lambda_2} - 2 = \frac{(a+b)^2}{ad - bc} - 2 \notin \mathbb{R}.$$

Similarly the eigenvectors u_i of A may be obtained from the λ_i's and the coefficients of A by linear algebra. The second condition is then equivalent to the non-vanishing of the algebraic function

$$\prod_{i,j} \det(Bu_i, u_1, \ldots, \widehat{u_j}, \ldots, u_n).$$

Denote by Δ_A and $\Delta_A^{\perp}$ the complementary subspaces of $\mathfrak{gl}(n, \mathbb{C})$ respectively consisting of diagonal matrices and of matrices with zero entries in the diagonal. Denote by Π_A and $\Pi_A^{\perp}$ the respective linear projections.

Step 1: we prove that $\mathfrak{G}$ contains a non-zero matrix $M_1 \in \Delta_A^{\perp}$. If M_0 is not diagonal then it is enough to set $M_1 = AM_0A^{-1} - M_0$ which obviously satisfies our needs. Let us therefore suppose that M_0 is diagonal. In this case the new matrix $\tilde{M}_0 = BM_0B^{-1}$ cannot be diagonal. Indeed suppose for a contradiction that $\tilde{M}_0$ is also diagonal. Since M_0 is not a scalar multiple of the identity, it has at least 2 distinct eigenvalues and hence non-trivial eigenspaces which are direct sums of eigendirections of A. If $\tilde{M}_0$ is also diagonal, it follows that B takes an eigendirection of A to a direct sum of eigendirections of A whose dimension is strictly less than n. Therefore B takes an eigendirection of A to an invariant hyperplane. The resulting contradiction allows us to conclude that $\tilde{M}_0$ is not diagonal. Now Step 1 follows from letting $M_1 = A\tilde{M}_0A^{-1} - \tilde{M}_0$.

Step 2: we prove that $\mathfrak{G}$ contains the complex line $\mathbb{C} \cdot M_2$ through some Kronecker matrix $M_2 = \delta_{i_0,j_0}$, $i_0 \neq j_0$. If S denotes the set of indices (i, j) corresponding to the non-vanishing entries $m_{i,j}$ of M_1, then the norm of $\lambda_{i,j}$ is maximized by a unique pair $(i_0, j_0) \in S$. It follows that the sequence of conjugates $A^kM_1A^{-k}$, $k \in \mathbb{N}$, tends *in direction* towards the complex line $\mathbb{C} \cdot \delta_{i_0,j_0}$. Recalling that no $\lambda_{i,j}$ is real (for $i \neq j$), it results that the complex line $\mathbb{C} \cdot \delta_{i_0,j_0}$ is completely accumulated by the sequence of "conjugate real lines" $\mathbb{R} \cdot A^kM_1A^{-k}$. These lines are clearly contained in $\mathfrak{G}$ and, since $\mathfrak{G}$ is closed, Step 2 follows.

Step 3: we prove that $\mathfrak{G}$ contains the whole of the complex subspace $\Delta_A^{\perp}$. Since $B = (b_{i,j})$ and $B^{-1} = (\widetilde{b}_{i,j})$ have no vanishing entry, the same holds for the conjugate $\tilde{M}_2 = BM_2B^{-1} = (b_{i,i_0}.\widetilde{b}_{j_0,j})$. Therefore $M_3 = A\tilde{M}_2A^{-1} - \tilde{M}_2$ has vanishing entries precisely on the diagonal and $\mathfrak{G}$ contains the complex line $\mathbb{C} \cdot M_3$. Since the $\lambda_{i,j}$'s are pairwise distinct (in norm) for $i \neq j$, it follows that the conjugates $A^kM_3A^{-k}$ are linearly independent over $\mathbb{C}$ for $k = 1, \ldots, n^2 - n$ and generate $\Delta_A^{\perp}$.

Step 4: we prove that $\mathfrak{G}$ contains the diagonal matrices Δ_A which have trace equal to zero. Consider the mapping from $\Delta_A^{\perp} \subset \mathfrak{G}$ to $\mathfrak{gl}(n, \mathbb{C})$ given by $M \mapsto \Pi_A(BMB^{-1})$. Because M has trace zero, it results that $\Pi_A(BMB^{-1})$ has trace zero as well. In other words, the image of the mapping in question is contained in the set of diagonal matrices with trace equal to zero. It is therefore sufficient to show that the rank of this map is equal to $n - 1$. To show this, it suffices to consider the restriction of this mapping to the space generated by $\delta_{1,2}, \ldots, \delta_{1,n}$ (which is clearly contained in $\Delta_A^{\perp}$). Indeed note that the corresponding $(n-1) \times n$−matrix $(b_{i,1}.\widetilde{b}_{j,i})_{i=1,\ldots,n;\,j=2,\ldots,n}$, whose columns are the coefficients of $\Pi_A(B\delta_{1,j}B^{-1})$

in the basis $(\delta_{i,i})$ of Δ_A, is such that the subdeterminant $\det(b_{i,1} \cdot \widetilde{b}_{j,i})_{i,j=2,\dots,n}$ equals $\frac{b_{1,1}\cdot\dots\cdot b_{n,1}}{\det(B)}$ which is not *zero* in view of the preceding assumptions on B. This completes the proof of the lemma. □

Corollary 1.3. *There is a (real) Zariski-open subset $\mathcal{U}_3$ of the ε_1-neighborhood of (I, I) in $GL(n, \mathbb{C}) \times GL(n, \mathbb{C})$ having the following property: the subgroup G generated by any $(A, B) \in \mathcal{U}_3$ contains a dense subgroup of $SL(n, \mathbb{C})$. In fact the closure $\overline{G}$ of G has the form*

$$\overline{G} = \Lambda \times SL(n, \mathbb{C}),$$

where $\Lambda \subset \mathbb{C}^* \cdot I$ is a closed subgroup of scalar matrices.

Proof. Set $\mathcal{U}_3 = \mathcal{U}_1 \cap \mathcal{U}_2$ so that any $(A, B) \in \mathcal{U}_3$ generates a non-discrete subgroup $G \subseteq GL(n, \mathbb{C})$ (Corollary 1.1) whose closure $\overline{G}$, and hence whose associated Lie algebra $\mathfrak{G}$, are both invariant by A and B. Since the non-trivial elements of $\mathfrak{G}$ were constructed by means of commutators, they clearly belong to $\mathfrak{sl}(n, \mathbb{C})$ (and hence are non-scalar). Therefore Lemma 1.2 implies that $\mathfrak{sl}(n, \mathbb{C}) \subseteq \mathfrak{G}$ and thus $SL(n, \mathbb{C}) \subset \overline{G}$. Since $SL(n, \mathbb{C})$ is *perfect* i.e. coincides with its derived group $D^1 G$, we conclude that $D^1 G$, is (contained and) dense in $SL(n, \mathbb{C})$. In fact, every element of $SL(n, \mathbb{C})$ can be written as a composition of commutators and, in turn, each commutator can be approximated by a commutator of elements in G since $SL(n, \mathbb{C}) \subset \overline{G}$. Now a simple argument involving the obvious short exact sequence shows that $\overline{G}/SL(n, \mathbb{C})$ can be identified with $\Lambda = \sqrt[n]{\det(\overline{G})}$. □

In the sequel, we will say that G is *large* if it fulfils the conclusions of Corollary 1.3, namely $\overline{G} = \Lambda \times SL(n, \mathbb{C})$. This property is persistent (stable) and generic for matrices close to I as follows from:

Corollary 1.4. *The subset $\mathcal{U}_4 \subset GL(n, \mathbb{C}) \times GL(n, \mathbb{C})$ of those (A, B) which generate a large subgroup $G \subseteq GL(n, \mathbb{C})$ is an open set. Furthermore, the intersection of $\mathcal{U}_4$ with a suitable neighborhood of (I, I) has total Lebesgue measure in this neighborhood.*

Proof. Only the part involving stability (i.e. the fact that these sets are open) needs further comments. If the group G generated by (A, B) is large, then it contains a dense subset of $SL(n, \mathbb{C})$ and one can find two words $(w_1(A, B), w_2(A, B)) \in \mathcal{U}_3$ which "persistently generate" a large subgroup of $GL(n, \mathbb{C})$. Since these words depend smoothly on the generators A and B, we immediately conclude the desired stability. □

In this sense, large subgroups are the largest "stable" subgroups of $GL(n, \mathbb{C})$. Since conditions of Lemma 1.2 make sense and are Zariski-open in $PGL(n, \mathbb{C}) \times PGL(n, \mathbb{C})$ or $SL(n, \mathbb{C}) \times SL(n, \mathbb{C})$, the same arguments show that the subsets of those pairs (A, B) which generate a dense subgroup of $PGL(n, \mathbb{C})$ (resp. $SL(n, \mathbb{C})$) are open sets and have total measure in a neighborhood of (I, I). We also have:

Corollary 1.5. *The subset $\mathcal{U}_5 \subset GL(n, \mathbb{C}) \times GL(n, \mathbb{C})$ of those (A, B) which generate a dense subgroup $G \subset GL(n, \mathbb{C})$ has total Lebesgue measure in a neighborhood of (I, I).*

Proof. If we denote by $\mathcal{U}$ the subset formed by those $(\lambda, \mu) \in \mathbb{C}^* \times \mathbb{C}^*$ which generate a dense subgroup Λ of $\mathbb{C}^*$, then $\mathcal{U}_5$ contains the set

$$\{(A, B) \in \mathcal{U}_3\ ;\ (\det(A), \det(B)) \in \mathcal{U}\}.$$

The corollary is therefore established since $\mathcal{U}$ has total Lebesgue measure in $\mathbb{C}^* \times \mathbb{C}^*$. □

We close this section with a simple lemma about large subgroups which will be used in the proof of the rigidity of pseudo-groups at the end of Sect. 6. Note that a large subgroup means a subgroup as in the statement of Corollary 1.3. Recall first that, identifying a complex scalar $\lambda = a + ib$ with the real matrix $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, we define a natural inclusion $GL(n, \mathbb{C}) \hookrightarrow GL(2n, \mathbb{R})$.

Lemma 1.6. *Let G be a large subgroup of $GL(n, \mathbb{C})$ and assume that G is conjugated to another subgroup $G' \in GL(n, \mathbb{C})$ by a real matrix $C \in GL(2n, \mathbb{R})$. If $n \geq 2$, then either C or its conjugate $\overline{C}$ is a complex matrix (i.e. it belongs to $GL(n, \mathbb{C})$).*

Proof. Denote by $J \in GL(2n, \mathbb{R})$ the conjugation by C of the complex scalar matrix $i.I$. By construction, $J^2 = -I$ and J does commute with all elements of G and, hence, with all elements of its closure $\overline{G}$. Since G is a large subgroup of $GL(n, \mathbb{C})$, it contains a copy of $SL(n, \mathbb{C})$ whose elements necessarily commute with J. In particular, J has to commute with the complex scalar matrix $\sqrt[n]{1} \cdot I \in SL(n, \mathbb{C})$. When $n \geq 3$, this matrix is not real and it follows that J is actually a complex matrix belonging to the center of $SL(n, \mathbb{C})$. Thus, $J = \lambda \cdot I$ and, since $J^2 = -I$, we obtain $\lambda = \pm i$. Finally, C conjugates $i.I$ to $\pm i.I$ and therefore is complex or anti-complex ending the proof. In the case $n = 2$, consider the commutator of J with a complex diagonal matrix $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \in SL(2, \mathbb{C})$. For a generic real coefficient $\lambda = a + i0 = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ it follows that J is block-diagonal with 2×2 (real) blocks. Then, setting $\lambda = i = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, we show that these blocks are actually complex, i.e. of the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$. Thus, $J \in GL(2, \mathbb{C})$ and the proof is completed as in the case $n \geq 3$. □

2. Pseudo-groups G on the unit ball $\mathbb{B}^n$ of $\mathbb{C}^n$, their closure $\overline{G}$ and their Lie pseudo-algebra $\mathfrak{G}$

In this section, we recall the definition of a pseudo-group G, define its closure $\overline{G}$ with respect to the uniform convergence on compact subsets and introduce the Lie pseudo-algebra $\mathfrak{G}$ consisting of those vector fields whose pseudo-flow is contained in $\overline{G}$. This way of associating vector fields to pseudo-groups was already considered in [Sh1], [Na], [Be,Li,Lo1] and [Reb1] in order to study the dynamics of certain pseudo-groups in dimension 1. These notions will be used throughout Sects. 3, 4, 5 and 6. In what follows $z = (z_1, \ldots, z_n)$ stands for the variable of $\mathbb{C}^n$ and $|z|$ for the usual norm. Denote by $\mathbb{B}^n_r$ the ball of radius $r > 0$ centered at the origin $\underline{0} \in \mathbb{C}^n$ and set $\mathbb{B}^n = \mathbb{B}^n_1$. Given a mapping $f : \mathbb{B}^n_r \to \mathbb{C}^n$, we denote by $\|f\|_r$ the supremum of $|f(z)|$ where $z \in \mathbb{B}^n_r$. A *pseudo-group* G on $\mathbb{B}^n$ is any collection of biholomorphic transformations $f : U \to V$ within the ball, $U, V \subset \mathbb{B}^n$, which is closed under:

– restrictions: if $(f : U \to V) \in G$ and $W \subset U$ then $f|_W \in G$,
– inversions: if $(f : U \to V) \in G$, then $(f^{-1} : V \to U) \in G$,
– compositions: if $(f : U \to V), (g : V \to W) \in G$, then $(g \circ f : U \to W) \in G$).

The pseudo-group G *generated on* $\mathbb{B}^n$ *by* a collection of injective holomorphic mappings $(f_i : U_i \hookrightarrow \mathbb{C}^n)_i$ is the smallest pseudo-group on $\mathbb{B}^n$ containing the dynamics induced by the f_i within $\mathbb{B}^n$, i.e. containing the restriction of each f_i to $f_i^{-1}(f_i(U_i \cap \mathbb{B}^n) \cap \mathbb{B}^n)$.

We shall say that an injective holomorphic transformation $f : U \to V$ within $\mathbb{B}^n$ is *approximated* by a sequence of elements $f_k : U_k \to V_k$ of G, $k \in \mathbb{N}$, (or by G for short) if, for any compact subset $K \subset U$, K is contained in U_k for sufficiently large k, and the sequence $f_k|_K$ restricted to K converges uniformly to $f|_K$. The collection $\overline{G}$ of all transformations approximated by G is a pseudo-group (trivially containing G) which will be called *the closure of* G. The pseudo-group Diff($\mathbb{B}^n$) of all holomorphic transformations within $\mathbb{B}^n$ can, indeed, be endowed with a topology for which $\overline{G}$ becomes the closure of G in Diff($\mathbb{B}^n$), but this will not be necessary to our purposes.

In order to follow the ideas developed in the linear case, we would like to consider the closure $\overline{G}$ as a real analytic Lie pseudo-group (as in Cartan's theorem), i.e. to define its (real) Lie pseudo-algebra $\mathfrak{G}$ in a reasonable sense.

Let X denote a real vector field defined on an open set $U \subset \mathbb{B}^n$ and consider the pseudo-flow $\phi^t_X : U_t \to V_t$, $t \in \mathbb{R}$ and $U_t, V_t \subset U$, obtained by integration of X (note that U_t might become empty when $|t|$ increases). Clearly, this pseudo-flow is a pseudo-group of holomorphic transformations if and only if X is the real part of a complex holomorphic vector field Z. Explicitly, setting $z_i = x_i + \sqrt{-1} \cdot y_i$, if the vector field Z is given as $Z = \sum_{i=1}^n f_i \partial/\partial z_i$ where the f_i's are holomorphic, then X has the form

$$X = \sum_{i=1}^{n} \Re e(f_i)\frac{\partial}{\partial x_i} + \Im m(f_i)\frac{\partial}{\partial y_i} \, .$$

Here $f_i = \Re e(f_i) + \sqrt{-1} \cdot \Im m(f_i)$. More generally, it will be proved later (Proposition 4.8) that any germ at $0 \in \mathbb{R}$ of continuous homomorphism from the additive group $(\mathbb{R}, +)$ into a pseudo-group of holomorphic transformations must be as above (i.e. it can be represented by the pseudo-flow of a vector field as X).

Now, define the Lie pseudo-algebra (or simply Lie algebra) $\mathfrak{G}$ associated to the closed pseudo-group $\overline{G}$ as the collection $\mathfrak{G}(U)$, for every open set $U \subset \mathbb{B}^n$, of the set of holomorphic vector fields X defined on U whose corresponding real pseudo-flow $\varphi^t_X : U_t \to V_t$ ($t \in \mathbb{R}$ small) is entirely contained in $\overline{G}$. It is easy to check that $\mathfrak{G}$ inherits a structure of a presheaf of real Lie algebras, i.e. the set of vector fields in $\mathfrak{G}(U)$ is stable under restrictions and any $\mathfrak{G}(U)$ is a real Lie algebra with respect to the Lie brackets of vector fields.

Finally, each $\mathfrak{G}(U)$ is closed under uniform convergence on compact subsets and the entire collection $\mathfrak{G}$ is invariant under $\overline{G}$: if $f : U \to V$ belongs to $\overline{G}$ then $\mathfrak{G}(V) = f_*\mathfrak{G}(U)$.

For instance, if we still denote by G the pseudo-group obtained by restricting all transformations of a given linear group $G \subset GL(n, \mathbb{C})$, then the closure $\overline{G}$ in the previous sense coincides with the pseudo-group induced by the closure of G with respect to the usual topology of $GL(n, \mathbb{C})$. Then, the Lie pseudo-algebra $\mathfrak{G}$ defined as above coincides with the Lie algebra of $\overline{G}$, viewed as a real analytic Lie subgroup of $GL(n, \mathbb{C})$, on the whole of $\mathbb{B}^n$.

In practice we just need to keep in mind that, given a pseudo-group G on $\mathbb{B}^n$, the Lie pseudo-algebra $\mathfrak{G}$ associated to its closure $\overline{G}$ consists of the complex holomorphic vector fields X defined on some $U \subset \mathbb{B}^n$ (actually in most applications U will be a sub-ball $\mathbb{B}^n_r$) and possessing the following property: every local diffeomorphism induced by the corresponding real pseudo-flow φ^t_X, $t \in \mathbb{R}$ fixed, can uniformly be approximated by a sequence h_k of elements of G on any compact subset of U_t (provided that φ^t_X is defined on U_t).

Now, let us explain why it is convenient to introduce these notions. First note that the transitivity of Lie algebra $\mathfrak{G}$ on the ball $\mathbb{B}^n$ automatically implies the minimality of the original pseudo-group G (i.e. all orbits are dense in $\mathbb{B}^n$). Indeed, the transitivity of $\mathfrak{G}$ means that, given any 2 points p and p' in $\mathbb{B}^n$, there exists a finite combination of pseudo-flows $\varphi = \varphi^{t_1}_{X_1} \circ \cdots \circ \varphi^{t_N}_{X_N}$ satisfying $\varphi(p) = p'$, $X_1, \ldots, X_N \in \mathfrak{G}$. Here, we omit the domain of definitions but we implicitly require that φ is defined at least on a neighborhood of p. By definition of $\overline{G}$ and $\mathfrak{G}$, φ (belongs to $\overline{G}$ and) is approximated on a neighborhood of p by some sequence φ_n of elements of G. Thus, p' is approximated by the sequence of points $\varphi_n(p)$ which obviously belong to the orbit of p by G.

In fact, the transitivity of the Lie algebra $\mathfrak{G}$ has many further consequences on the dynamics of G as it will be shown in Sect. 6. For instance, we may derive the ergodicity of the (pseudo-) action of G as well. Another motivation to work with vector fields rather than with elements of G is that we obtain an easier control of the domain of definition of the objects during their manipulation and several computations become linear.

The purpose of the next two sections is to provide *sufficient* conditions for a pseudo-group G, consisting of holomorphic transformations within the ball $\mathbb{B}^n$, to have non-trivial (real) Lie algebra. Our main result is:

Proposition 2.0. *There exists a constant $\varepsilon_2 > 0$ such that, for any scalar $0 < |\lambda| < 1$ satisfying $|\lambda - 1| < \varepsilon_2$, one can find a smaller constant $\varepsilon_\lambda > 0$ having the following property: every ε_λ-perturbations $f, g : \mathbb{B}^n \hookrightarrow \mathbb{C}^n$ respectively of the contracting homothety $f_0(z) = \lambda \cdot z$ and of the identity map $g_0(z) = z$ on $\mathbb{B}^n$ which do not have a common fixed point (i.e. they satisfy $\|f - f_0\|_1$, $\|g - g_0\|_1 < \varepsilon_\lambda$ and $f(z) = g(z) = z$ for no $z \in \mathbb{B}^n$), generate a pseudo-group G whose Lie algebra contains a non-trivial vector field $X \in \mathfrak{G}(\mathbb{B}^n)$.*

3. Catching non discrete pseudo-groups

The goal of this section is to provide sufficient conditions on two injective holomorphic mappings $f, g : \mathbb{B}^n \hookrightarrow \mathbb{C}^n$ so that they generate a pseudo-group G containing a sequence of non-trivial elements converging uniformly to the identity map Id on some ball $\mathbb{B}^n_r$. Precisely, in order to obtain convergence to the identity, we will require that f and g are close to the identity on $\mathbb{B}^n$. Besides f will be a contraction very close to a homothety. Additional generic conditions will be needed to ensure the non-triviality of the sequence. We begin with a non-linear analogue of Lemma 1.0 which is borrowed from [Gh].

Lemma 3.0 (Ghys). *Fix constants $r, \varepsilon, \tau > 0$ satisfying $4\varepsilon + \tau < r$. Let $f, g : \mathbb{B}^n_r \hookrightarrow \mathbb{C}^n$ be holomorphic mappings and denote by G the pseudo-group generated by them on $\mathbb{B}^n_r$. If f and g are ε-close to the identity on $\mathbb{B}^n_r$, then the commutator $[f, g] = f \circ g \circ f^{-1} \circ g^{-1}$ induces a mapping $\mathbb{B}^n_{r-4\varepsilon} \hookrightarrow \mathbb{B}^n$ belonging to G. Furthermore we have the estimate below:*

$$\|[f, g] - Id\|_{r-4\varepsilon-\tau} \quad \leq \quad \frac{2}{\tau} \cdot \|f - Id\|_r \cdot \|g - Id\|_r \,.$$

In Ghys's paper, [Gh], the right hand side of the corresponding estimate is given by $\sup (|| f - Id ||_r, || g - Id ||_r)^2$. However our sharper inequality follows from similar arguments as can easily be checked.

Proof. By assumption, the variations $\Delta_f = f - Id$ and $\Delta_g = g - Id$ of f and g are bounded by ε on $\mathbb{B}^n$. Clearly $f \circ g$ is well-defined as mapping from $\mathbb{B}^n_{r-2\varepsilon}$ to $\mathbb{B}^n$ and can be written as

$$f \circ g = Id + \Delta_f + \Delta_g + (\Delta_f \circ g - \Delta_f).$$

Cauchy Formula applied on small disks of radius τ, $0 < \tau < r$, inside $\mathbb{B}^n$ provides the following bound for the partial derivatives

$$\left\| \frac{\partial \Delta_f}{\partial z_i} \right\|_{r-\tau} \quad \leq \quad \frac{1}{\tau} \|\Delta_f\|_r \,.$$

There is also an analogous estimate for Δ_g. Now the Mean Value Theorem yields:

$$\|\Delta_f \circ g - \Delta_f\|_{r-2\varepsilon-\tau} \quad \leq \quad \sup_i \left\| \frac{\partial \Delta_f}{\partial z_i} \right\|_{r-\varepsilon-\tau} \|\Delta_g\|_{r-2\varepsilon-\tau} \quad \leq \quad \frac{1}{\tau} \|\Delta_f\|_r \|\Delta_g\|_r.$$

Therefore $f \circ g - g \circ f$, which is also well-defined on $\mathbb{B}^n_{r-2\varepsilon}$, satisfies

$$\|f \circ g - g \circ f\|_{r-2\varepsilon-\tau} \quad \leq \quad \|\Delta_f \circ g - \Delta_f\| + \|\Delta_g \circ f - \Delta_g\| \quad \leq \quad \frac{2}{\tau} \|\Delta_f\|_r \|\Delta_g\|_r.$$

On the other hand, $(g \circ f)^{-1}$ takes the ball of radius $r - 4\varepsilon - \tau$ to the interior of the ball of radius $r - 2\varepsilon - \tau$. Thus $[f, g] - Id = (f \circ g - g \circ f) \circ (g \circ f)^{-1}$ satisfies

$$\|[f, g] - Id\|_{r-4\varepsilon-\tau} \quad \leq \quad \frac{2}{\tau} \|\Delta_f\|_r \|\Delta_g\|_r \,.$$

The lemma is proved. □

In [Gh], this lemma is used to prove the convergence to the identity of some sequences h_k contained in the derived sequence of G.

Corollary 3.1 (Ghys). *Given holomorphic mappings $f, g : \mathbb{B}^n \hookrightarrow \mathbb{C}^n$, denote by G the corresponding pseudo-group on $\mathbb{B}^n$, and consider the sequence of subsets $\{Z_k\} \subset G$ defined by*

$$Z_0 = \{f, f^{-1}, g, g^{-1}\} \quad \textit{and} \quad Z_{k+1} = \{[h_k, h'_k]; h_k, h'_k \in Z_k\} \quad \textit{for} \quad k \in \mathbb{N}.$$

If f and g are $\frac{1}{32}$-close to the identity, then all elements of Z_k are defined at least on the ball $\mathbb{B}^n_{1/2}$ and are $\frac{1}{2^{k+5}}$-close to the identity on this ball, for all $k \in \mathbb{N}$.

Proof. We are going to prove that any $h_k \in Z_k$ is defined on the ball of radius $r_k = \frac{1}{2} + \frac{1}{2.2^k}$ and that the difference $h_k - Id$ is uniformly bounded on $\mathbb{B}^n_{r_k}$ by $\epsilon_k = \epsilon_0/2^k$, $\epsilon_0 = 1/32$. In order to inductively apply Lemma 3.0, notice that the sharp estimate will be obtained by choosing the constant $\tau_k > 0$ satisfying $r_{k+1} = r_k - 4\epsilon_k - \tau_k$, so that $\tau_k = \tau_0/2^k$ with $4\epsilon_0 + \tau_0 = 1/4$. Now Lemma 3.0 inductively produces

$$\|[h_k, h'_k] - Id\|_{r_{k+1}} \leq \frac{2}{\tau_k} \cdot \|h_k - Id\|_{r_k} \cdot \|h'_k - Id\|_{r_k} \leq \frac{2}{\tau_k} \cdot \epsilon_k^2 = \frac{2\epsilon_0}{\tau_0} \cdot \epsilon_k$$

which shows that $\frac{\epsilon_{k+1}}{\epsilon_k} = \frac{1}{2}$ for $\tau_0 = 4\epsilon_0$. Hence $\tau_0 = 1/8$ and $\epsilon_0 = 1/32$. □

In particular, any sequence $(h_k)_k$ contained in the sets above, $h_k \in Z_k$, converges uniformly to the identity on $\mathbb{B}^n_{1/2}$. Nevertheless, it may be difficulty to verify the non-triviality of such sequences. To require that G is non-solvable (when it makes sense) is in general not sufficient. Moreover the existence of sequences of elements in G which converge uniformly to the identity still does not guarantee the existence of pseudo-flows, as shown by the next example.

Example 3.2. Due to arithmetic reasons (see [Be,Ce,LN, Corollary 4.2, p. 262]), the subgroup $G_0 \subset \text{Diff}(\mathbb{C}, 0)$ generated by $z/(1-2z)$ and $z/\sqrt{1-4z^2}$ (where the determination $\sqrt{1} = 1$ was chosen) is a free group of rank 2. The mappings f and g are defined on $\mathbb{B}^n_{1/2}$ and we denote by G the pseudo-group generated by them within $\mathbb{B}^n_{1/2}$. Because G_0 is free, the series $\{Z_k\}$ considered in Corollary 3.1 does not degenerate into $\{Id\}$. Hence G contains sequences of non-trivial elements converging to the identity on a fixed neighborhood of $0 \in \mathbb{C}$. Nonetheless we claim that *G cannot approximate a non-trivial pseudo-flow on a neighborhood of* $0 \in \mathbb{C}$. In fact, suppose for a contradiction that φ_X is a non-trivial pseudo-flow defined around $0 \in \mathbb{C}$ and belonging to the Lie algebra of $\overline{G}$ (on a neighborhood of $0 \in \mathbb{C}$). Clearly $\varphi_X^t(\underline{0}) = \underline{0}$ for every $t \in \mathbb{R}$. Furthermore we can choose $t_0 \in \mathbb{R}$ so that the Taylor series based at $0 \in \mathbb{C}$ of the induced mapping $\varphi_X^{t_0}$ does not have integer coefficients. By assumption, there is a sequence $\{h_k\} \subset G$ converging uniformly to $\varphi_X^{t_0}$ on a neighborhood of $0 \in \mathbb{C}$. This implies that, for k large enough, the Taylor series of h_k based at $0 \in \mathbb{C}$ does not have integer coefficients. This is however impossible: since the Taylor series based at $0 \in \mathbb{C}$ of f, f^{-1}, g, g^{-1} all have integer coefficients, the same holds for any element of G_0.

The resulting contradiction proves our claim. Finally, recall that Nakai's theorem (see [Na]) asserts the existence of (many!) such pseudo-flows in the closure of G at the neighborhood of any point $z_0 \neq 0$ sufficiently close to 0. Thus, the associated Lie pseudo-algebra $\mathfrak{G}$ has "trivial germ" only at 0.

Notice also that, in contrast with the finite dimensional case, Lemma 3.0 is not sufficient (without further assumptions) to imply that the sequences $h_k \in S_k$, $k \in \mathbb{N}$, contained in the "central" sequence of sets

$$S_0 = \{f, f^{-1}, g, g^{-1}\} \quad \text{and} \quad S_{k+1} = \{[h_0, h_k];\, h_0 \in S_0, h_k \in S_k\},$$

are well-defined as elements of the pseudo-group G and, furthermore, converge uniformly towards the identity. Indeed, if we choose f, g as translations arbitrarily close to the identity, then we can always find an integer k_0 for which any word $h_{k_0} \in S_{k_0}$ has empty domain of definition. Although this counterexample is somehow trivial (S_k consists of the identity transformation for $k > 0$) a small "non-nilpotent" perturbation of it, within the affine group, will have the same property and will provide serious obstructions to the existence of a "Zassenhaus Lemma" for pseudo-groups.

In order to ensure that the pseudo-group G is not discrete, our idea is to require also that f is a contraction. So, instead of dealing with the sequence $g_0 = g$, $g_{k+1} = [f, g_k]$ whose common domain of definition of the elements is shrinking, we are able to "restore" (i.e. "re-enlarge") domains by working with an alternate sequence of type $h_0 = g$, $h_{k+1} = f^{-N}[f, h_k]f^N$. This approach is successful only if the distortion of f can be bounded. More precisely, if we denote by $0 < \lambda_- \leq \lambda_+$ the lower and upper bounds for directional derivatives of f given by

$$\lambda_- = \inf_{|v|=1, |z|<1} \left| \frac{\partial}{\partial t} f(z+tv)\big|_{t=0} \right| \quad \text{and} \quad \lambda_+ = \sup_{|v|=1, |z|<1} \left| \frac{\partial}{\partial t} f(z+tv)\big|_{t=0} \right| ,$$

we then require that

$$(*) \quad 0 < (\lambda_+)^2 < \lambda_- \leq \lambda_+ < 1\,.$$

The assumption above is strong but sufficient for our purpose. The reader may notice that the perturbations f of $f_0(z) = \lambda \cdot z$ considered in Proposition 2.0 satisfy this requirement modulo shrinking the ball where they are defined. The condition $(*)$ will also imply that f can be linearized by a holomorphic change of coordinates at a neighborhood of its fixed point $\underline{0}$ (Poincaré Theorem, cf. Lemma 3.5).

Lemma 3.3. *There exists $\varepsilon_3 > 0$ such that, for any ε_3-close to the identity holomorphic mapping $f : \mathbb{B}^n \hookrightarrow \mathbb{C}^n$ fixing $\underline{0}$ and satisfying $(*)$, there is $N \in \mathbb{N}$ having the following property: consider the pseudo-group G generated on $\mathbb{B}^n$ by f and another ε_3-close to the identity holomorphic mapping $g : \mathbb{B}^n \hookrightarrow \mathbb{C}^n$. Then the sequence inductively given by*

$$g_0 = g \quad \textit{and} \quad g_{k+1} = f^{-N} \circ [f, g_k] \circ f^N$$

induces a sequence of mappings $\mathbb{B}^n_{1/3} \hookrightarrow \mathbb{B}^n$ belonging to G and converging uniformly towards the identity on this ball.

Further conditions will guarantee (cf. Lemma 3.4) in addition that $g_k \neq Id$ for every k.

Proof. The N^{th} iterate of f is well-defined on the ball $\mathbb{B}^n$ and satisfies

$$r\lambda_-^N \quad \leq \quad \|f^N\|_r \quad \leq \quad r\lambda_+^N.$$

for every $0 < r < 1$. Indeed, our assumptions ensure that $(\lambda_-(f))^N \leq \lambda_-(f^N) \leq \lambda_+(f^N) \leq (\lambda_+(f))^N$. Then, for $N = 1$, the estimates $r\lambda_- \leq r\|D_{\underline{0}}f\| \leq \|f\|_r \leq r\lambda_+$ follow respectively from Cauchy's inequality and Mean Value Theorem (f fixes $\underline{0}$).

Now, beginning with constants $r, \varepsilon_3, \tau > 0$ which satisfy $4\varepsilon_3 + \tau < r < 1$, we can apply Lemma 3.0 to f and g on the ball $\mathbb{B}^n_r \subset \mathbb{B}^n$ (these constants will be fixed later on)

$$\|[f, g] - Id\|_{r-4\varepsilon_3-\tau} \quad \leq \quad \frac{2\varepsilon_3}{\tau} \cdot \|g - Id\|_r \, .$$

Consider also an integer $N \in \mathbb{N}$ so large that $f^N(\mathbb{B}^n_r) \subset \mathbb{B}^n_{r-4\varepsilon_3-\tau}$. Then

$$\|[f, g] \circ f^N - f^N\|_r \quad \leq \quad \frac{2\varepsilon_3}{\tau} \cdot \|g - Id\|_r \, .$$

We now fix $\tau = \tau(r, \varepsilon_3, N) = r - 4\varepsilon_3 - \|f^N\|_r$ so that the preceding estimate is sharp.

The inverse mapping f^{-1} is defined on $\mathbb{B}^n_{1-\varepsilon_3}$ and satisfies $\frac{1}{\lambda_+} \leq \|f^{-1}\|_{1-\varepsilon_3} \leq \frac{1}{\lambda_-}$. Suppose for the time being that we are allowed to iterate N times f^{-1} from $[f, g] \circ f^N(\mathbb{B}^n_r)$ in G. Then, the estimate above gives

$$\|f^{-N} \circ [f, g] \circ f^N - Id\|_r \quad \leq \quad \frac{2\varepsilon_3}{\tau\lambda_-^N} \cdot \|g - Id\|_r \, .$$

Since $r - 4\varepsilon_3 - \tau = \|f^N\|_r \leq r\lambda_+^N$ and $\lambda_+^2 < \lambda_-$ (condition $(*)$), we finally obtain

$$\|f^{-N} \circ [f, g] \circ f^N - Id\|_r \quad \leq \quad \frac{2r^2\varepsilon_3}{\tau(r - 4\varepsilon_3 - \tau)^2} \cdot \|g - Id\|_r \, .$$

The coefficient $c(\tau) = 2r^2\varepsilon_3/\tau(r - 4\varepsilon_3 - \tau)^2$ attains its minimum for $\tau_0 = \frac{1}{3}(r - 4\varepsilon_3)$. Fix $\varepsilon_3 = \varepsilon_3(r)$ small enough so that the coefficient $c(\tau_0) = \frac{27}{2}\frac{\varepsilon_3}{r}(1 - 4\frac{\varepsilon_3}{r})^{-3}$ is less than 1. The proof of Lemma 3.3 will clearly follow by induction from these estimates applied to the sequence g_k provided that we show the existence of $N \in \mathbb{N}$ so that ($\tau = r - 4\varepsilon_3 - \|f^N\|_r \sim \tau_0$ and) $c(\tau) < 1$ and therefore that the above iteration of f^{-1} makes sense.

First verification. For $\tau_0 < \tau < \tau_1 = \frac{2}{3}(r - 4\varepsilon_3)$, we find $c(\tau) < c(\tau_1) = 27\frac{\varepsilon_3}{r}(1 - 4\frac{\varepsilon_3}{r})^{-3}$. Refine $\varepsilon_3 = \varepsilon_3(r)$ so that $c(\tau_1) < 1$. Then, we want to find $N \in \mathbb{N}$ satisfying

$$\frac{1}{3}(r - 4\varepsilon_3) \leq \|f^N\|_r \leq \frac{2}{3}(r - 4\varepsilon_3) \, .$$

Using the estimates $r\lambda_+^{2N} \leq r\lambda_-^N \leq \|f^N\|_r \leq r\lambda_+^N$, it is enough to find $N \in \mathbb{N}$ satisfying

$$\sqrt{\frac{1}{3}}\left(1 - 4\frac{\varepsilon_3}{r}\right)^{1/2} \leq \lambda_+^N \leq \frac{2}{3}\left(1 - 4\frac{\varepsilon_3}{r}\right).$$

There exists such integer N if, and only if, $\frac{\sqrt{3}}{2} < \lambda_+(1 - 4\frac{\varepsilon_3}{r})^{1/2}$. Since f is ε_3-close to the identity, we also have $1 - \varepsilon_3 \leq \lambda_+ < 1$, and N exists as long as $\varepsilon_3(r)$ is very small.

Second verification. From the estimates above, the domain $[f, g] \circ f^N(\mathbb{B}_r^n)$ is contained in the ball of radius $\|f^N\|_r + 4\varepsilon_3$. We just have to ensure that $f^{-k} \circ [f, g] \circ f^N(\mathbb{B}_r^n)$ remains in the domain of definition $\mathbb{B}_{1-\varepsilon_3}^n$ of f^{-1} for every $k = 0, \dots, N$. Since $\|f^{-1}\|_{1-\varepsilon_3} \leq \frac{1}{\lambda_-}$, it suffices to require that $\|f^N\|_r + 4\varepsilon_3 \leq \lambda_-^N(1 - \varepsilon_3)$. From the inequalities obtained for τ and N during the first verification, it is enough to impose $4\varepsilon_3 + \frac{2}{3}(r - 4\varepsilon_3) \leq \frac{1}{3}(1 - 4\frac{\varepsilon_3}{r})(1 - \varepsilon_3)$. This can be done by fixing $r = 1/3$ and ε_3 sufficiently small. □

Remark. The condition $(*)$ means that the distortion coefficient

$$\delta = \frac{\log(\lambda_-)}{\log(\lambda_+)}$$

which is always ≥ 1 for a uniform contraction, is actually bounded by 2. The preceding proof may be re-arranged so that it is possible to replace the condition $(*)$ by $\delta < \delta_0$ for a fixed $\delta_0 >> 0$. Nonetheless notice that ε_3 depends on this bound and asymptotically $\varepsilon_3(\delta_0) \sim 1/\delta_0$ so that we cannot dispense with some assumption concerning distortion.

The sequence g_k constructed in Lemma 3.3 is non-trivial under further generic assumptions on f and g. This is the contents of Lemma 3.4 below.

Lemma 3.4. *Let $f, g : \mathbb{B}^n \hookrightarrow \mathbb{C}^n$ be as in Lemma 3.3 and denote by $A = D_{\underline{0}}f$ and $B = D_{\underline{0}}g$ their differentials at $\underline{0} \in \mathbb{C}^n$. Suppose that one of the following conditions holds:*

(i) $g(\underline{0}) \neq \underline{0}$*;*
(ii) $g(\underline{0}) = \underline{0}$ *and* $[A, B] \neq I$*;*
(iii) $g(\underline{0}) = \underline{0}$, $[A, B] = I$ *and* $[f, g] \neq Id$*.*

Then, all the elements g_k of the sequence constructed in Lemma 3.3 also satisfy the same respective condition (i), (ii) or (iii). In particular none of them coincides with the identity.

Before proving it, let us complete the preceding statement with the following lemma.

Lemma 3.5. *Assume that $f : \mathbb{B}^n \hookrightarrow \mathbb{C}^n$ is a holomorphic map which fixes $\underline{0}$ and satisfies condition $(*)$. Then f is linearizable by a holomorphic change of coordinates. In other words, letting $A = D_{\underline{0}}f$, there exists a holomorphic embedding $\Phi : \mathbb{B}^n \hookrightarrow \mathbb{B}^n$ fixing $\underline{0}$ such that*

$$\Phi^{-1} \circ f \circ \Phi(z) = A \cdot z\ .$$

Moreover, Φ is unique up to composition on the right side with a matrix commuting with A.

In particular, when we are in the case (iii) with $g(\underline{0}) = \underline{0}$ and $[A, B] = I$, "most of" the mappings g still do not commute with f otherwise f and g would be simultaneously linearizable.

Proof of Lemma 3.5. Denote by $\lambda_1, \ldots, \lambda_n$ the spectrum of the linear part A of f. Modulo a permutation of indices, one has

$$0 < \lambda_- \leq |\lambda_1| \leq \cdots \leq |\lambda_n| \leq \lambda_+ < 1.$$

Furthermore condition $(*)$ implies that

$$0 < |\lambda_n|^2 < |\lambda_1| \leq \cdots \leq |\lambda_n| < 1 .$$

Since $|\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_k}| < |\lambda_{i_0}|$ as long as $k \geq 2$, there is no resonance among these eigenvalues. On the other hand, this spectrum obviously belongs to Poincaré domain since it consists only of eigenvalues whose norm is less than 1. Hence, thanks to Poincaré Theorem (see [Ar, Il; p. 72]), f is linearizable by a holomorphic germ of diffeomorphism Φ at $\underline{0}$. Clearly, Φ is uniquely defined up to composition on the right side with a germ of diffeomorphism commuting with A. In particular, after composition with a convenient homothety, we can assume that Φ is defined on $\mathbb{B}^n$. Finally, recall that the formal part of the proof of Poincaré Theorem relies on the fact that the absence of resonance implies absence of non-linear germ of diffeomorphism Ψ commuting with A. □

Proof of Lemma 3.4. Case (i): if g does not fix $\underline{0}$, then $g^{-1}(\underline{0}) \neq \underline{0}$ is not fixed by f, i.e. $g^{-1} \circ f^{-1}(\underline{0}) = g^{-1}(\underline{0}) \neq f^{-1} \circ g^{-1}(\underline{0})$. This implies that $[f, g](\underline{0}) \neq \underline{0}$ and hence $f^{-N} \circ [f, g] \circ f^N(\underline{0}) \neq \underline{0}$. Using induction, we conclude that $g_k(\underline{0}) \neq \underline{0}$ for every k.

Case (ii): since the linear part of $[f, g]$ is given by $[A, B]$, this case promtly follows from the proof of Corollary 1.1 when A has only simple eigenvalues. When A has Jordan blocks, the proof is similar (replacing invariant directions by invariant subspaces).

Case (iii): suppose that g does not commute with f. Since the linear part of $[f, g]$ is $[A, B] = I$, $[f, g]$ is a (non-trivial) map which is tangent to the identity. In the coordinate given by Lemma 3.5 where $f = A$ is linear, it is clear that $[f, g]$ is still a (non-trivial) map tangent to the identity and thus it is non-linear. Employing again Lemma 3.5, it follows that $[f, g]$ does not commute with $A = f$ and the proof follows by induction. □

In the next section we shall work through the coordinate Φ given by Lemma 3.5. Hence we shall deal only with the linear contraction A and the sequence $\{h_k = \Phi^{-1} \circ g_k \circ \Phi\}$ (which is converging to the identity as well). All these mappings can be supposed defined on $\mathbb{B}^n_{1/2}$ without loss of generality (just compose Φ on the right with a convenient homothety). Clearly, any pseudo-flow uniformly approximated on a neighborhood of $\underline{0}$ by words in A and h_k will give rise to a pseudo-flow in the closure of G which, after conjugation by a convenient iterate of the contraction f, will also be defined on the entire ball $\mathbb{B}^n$.

4. Catching pseudo-flows in the closure of non-discrete pseudo-groups

In this section, we consider a holomorphic pseudo-group G on the ball $\mathbb{B}^n$ which contains a non-trivial sequence $h_k : \mathbb{B}^n_{1/2} \hookrightarrow \mathbb{B}^n$ which tends uniformly to the identity. We are going to impose further conditions on G in order to construct a non-trivial pseudo-flow in the closure $\overline{G}$. As mentioned, the fact that the h_k's converge uniformly to the identity is not sufficient to imply the existence of pseudo-flows as shown by Example 3.2. The natural strategy (see the proof of Corollary 1.1) consisting of considering sequences $\varphi_k = (h_k)^{N_k}$ for suitable $N_k \in \mathbb{N}$ fails here. Indeed, in Example 3.2, the growth of the N_k needed to define and to bound the sequence φ_k on a small ball $\mathbb{B}^n_r$ implies the uniform convergence of the sequence $\{h_k\}$ to the identity on any compact subset of $\mathbb{B}^n_r$. Later on (cf. Proposition 4.6), we shall give useful additional sufficient conditions on h_k in order to avoid such phenomenon. As an application, we will prove in Proposition 4.8 that a continuous 1-parameter pseudo-group of holomorphic maps is the pseudo-flow of a holomorphic vector field. For the time being, thinking of G as the pseudo-group generated by f and g in Sect. 3, we propose an alternate strategy to construct pseudo-flows under the assumption that G contains a linear contraction $f = A$ close to a homothety (we do not need any longer the fact that $f = A$ is close to the identity). In this way we shall fastly obtain a proof of Proposition 2.0.

In the sequel, the map $f : \mathbb{B}^n \hookrightarrow \mathbb{B}^n$ is linear and will be denoted by A ($A = D_0 f$). For the sake of notations, we shall make no distinction between A thought as a map from $\mathbb{B}^n$ into itself or as a diagonal matrix having the form

$$A = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} .$$

The eigenvalues $\lambda_1, \dots, \lambda_n$ of A are supposed to satisfy the condition

$$(*) \quad 0 < |\lambda_n|^2 < |\lambda_1| \le \cdots \le |\lambda_n| < 1 .$$

Lemma 4.0. *Let A be a diagonal matrix which satisfies condition $(*)$ above and let $h_k : \mathbb{B}^n_{1/2} \hookrightarrow \mathbb{C}^n$ be a sequence of holomorphic mappings converging uniformly to the identity. Let G be the pseudo-group generated on $\mathbb{B}^n$ by these mappings and let $\overline{G}$ denote its closure. Suppose that one of the following additional conditions holds:*

(i) $h_k(\underline{0}) \neq \underline{0}$ *for every k;*
(ii) $h_k(\underline{0}) = \underline{0}$ *and the differential $D_{\underline{0}} h_k$ at $\underline{0}$ is not lower triangular for every k; moreover, the eigenvalues of A satisfy $|\lambda_1| < \cdots < |\lambda_n|$.*

Then, for any $\varepsilon > 0$, there exists an affine transformation $h_\infty : \mathbb{B}^n_{1/2} \hookrightarrow \mathbb{C}^n$ belonging to $\overline{G}$ which is ε-close to the identity but is not a lower triangular matrix (and thus, $h_\infty \not\equiv Id$). Precisely, h_∞ is uniformly approximated on $\mathbb{B}^n_{1/2}$ by the sequence of maps

$$A^{-N_k} \circ h_k \circ A^{N_k}$$

for a convenient sequence of positive integers $N_k \in \mathbb{N}$.

Proof. The proof relies on the fact that the action of A by conjugation increases the affine part of h_k because of (i) or (ii) while it decreases its non-linear part due to $(*)$. Let us make precise and prove these facts. Consider the decomposition of a map h as

$$h = h^+ + h^- \quad \text{with} \quad h^+(z) = T + C^+ \cdot z \quad \text{and} \quad h^-(z) = C^- \cdot z + h^{\geq 2}(z)\,,$$

where $T = (t_i)_i = h(\underline{0})$ stands for the translation part, $C^+ = (c_{i,j})_{i<j}$ and $C^- = (c_{i,j})_{i\geq j}$ denote the respective strictly upper triangular and lower triangular part of the differential $C^+ + C^- = (c_{i,j})_{i,j} = D_{\underline{0}}h$, and $h^{\geq 2}$ consists on the remainder non-linear terms. The sequence of affine mappings $T_k + C_k^+ + C_k^-$ (for the corresponding decomposition of h_k) converges to the identity as one can see from estimating their coefficients through Cauchy's formula. The sequence $h_k^{\geq 2}$ formed by the higher order terms also goes to *zero* and satisfies

$$\left|h_k^{\geq 2}(z)\right| \quad \leq \quad 4 \cdot |z|^2 \cdot \left\|h_k^{\geq 2}\right\|_{1/2}$$

for every $z \in \mathbb{B}^n_{1/2}$ and every k. Clearly the condition $(*)$ implies that

$$\left|A^{-1} \circ h_k^{\geq 2} \circ A(z)\right| \quad \leq \quad 4 \cdot \frac{|\lambda_n|^2}{|\lambda_1|} \cdot |z|^2 \cdot \left\|h_k^{\geq 2}\right\|_{1/2} \quad < \quad \left\|h_k^{\geq 2}\right\|_{1/2}\,.$$

On the other hand, for any affine transformation $T + C^+ + C^-$, we have

$$\begin{aligned} A^{-1} \circ (T + C^+ + C^-) \circ A &= A^{-1}T + \; A^{-1}C^+A \; + \; A^{-1}C^-A \\ &= \left(\tfrac{t_i}{\lambda_i}\right)_i \; + \left(\tfrac{\lambda_j}{\lambda_i}c_{i,j}\right)_{i<j} + \left(\tfrac{\lambda_j}{\lambda_i}c_{i,j}\right)_{i\geq j}\,. \end{aligned}$$

Clearly, the lower triangular part $A^{-N} \circ C_k^- \circ A^N$ of $D_{\underline{0}}h_k$ remains close to the identity while either condition (i), or (ii), guarantees that at least one of the Taylor coefficients of $A^{-N} \circ h_k^+ \circ A^N$ increases exponentially when $N \to +\infty$. Of course this implies that the norm $\|A^{-N} \circ h_k^+ \circ A^N\|_{1/2}$ increases too.

Equip the space V of those mappings $h^+ = T + C^+$ with the metric induced by $\|\cdot\|_{1/2}$ and, for $\varepsilon > 0$ small enough, denote by U_ε the ε-neighborhood of $0 \in V$. The action by conjugation of A on V fixes 0 so that there exists an open neighborhood $W_\varepsilon \subset U_\varepsilon$ of the 0 in V such that $A^{-1}W_\varepsilon A$ remains in U_ε. For k very large, h_k^+ belongs to W_ε. For N sufficiently large, we have seen that $A^{-N} \circ h_k^+ \circ A^N$ lies in the complement of U_ε, and hence away from W_ε. Thus, if one defines N_k as the smallest positive integer for which $A^{-N_k} \circ h_k^+ \circ A^{N_k}$ does not belong to W_ε, the sequence of affine mappings $A^{-N_k} \circ h_k^+ \circ A^{N_k}$ will remain in the relatively compact annulus $U_\varepsilon \setminus W_\varepsilon$. Passing to a subsequence, the sequence $A^{-N_k} \circ h_k^+ \circ A^{N_k}$ converges uniformly towards some affine transformation h_∞^+ on $\mathbb{B}^n_{1/2}$. Clearly h_∞^+ is ε-close to *zero* but lies in the complement of W_ε so that it is not trivial. Therefore, the affine transformation $h_\infty = I + h_\infty^+$ is not a lower triangular matrix and setting $\Delta_k = \|A^{-N_k} \circ h_k \circ A^{N_k} - h_\infty\|_{1/2}$, one has

$$\begin{aligned} \Delta_k &\leq \left\|A^{-N_k} \circ h_k^+ \circ A^{N_k} - h_\infty^+\right\|_{1/2} + \qquad \left\|A^{-N_k} \circ h_k^- \circ A^{N_k} - Id\right\|_{1/2} \\ &\leq \left\|A^{-N_k} \circ h_k^+ \circ A^{N_k} - h_\infty^+\right\|_{1/2} + \left\|A^{-N_k} \circ C_k^- \circ A^{N_k} - I\right\|_{1/2} + \left\|h_k^{\geq 2}\right\|_{1/2}. \end{aligned}$$

The proposition immediately follows from the estimates above. □

Remark 4.1. Here let us replace the condition $(*)$ by the weaker condition

$$0 < |\lambda_n|^\delta < |\lambda_1| \leq \cdots \leq |\lambda_n| < 1 ,$$

for some $\delta > 2$. We truncate $h_k = h_k^{<\delta} + h_k^{\geq\delta}$ where $h_k^{<\delta}$ denotes the Taylor jet of order $\delta - 1$ of h_k and $h_k^{\geq\delta}$ the remainder higher order terms which satisfy

$$\left\| h_k^{\geq\delta}(z) \right\| \leq |z|^\delta \cdot \left\| h_k^{\geq\delta} \right\|_1 .$$

Then the same proof shows that, under conditions (i) or (ii), for any $\varepsilon > 0$, some subsequence of the type $A^{-N_k} \circ h_k \circ A^{N_k}$ tends uniformly to a polynomial transformation $h_\infty^{<\delta}$ of degree $\delta - 1$. But $h_\infty^{<\delta}$ does not necessarily belong to a Lie group, unlike the affine case above, and the subsequent arguments cannot immediately be adapted.

Corollary 4.2. *Let A, h_k and G be as in Lemma 4.0. Then, there exists a non-trivial affine vector field $X \in \mathfrak{G}(\mathbb{B}^n)$ belonging to the Lie pseudo-algebra $\mathfrak{G}$ of the closure $\overline{G}$. Moreover, X has strictly upper triangular linear part.*

Proof. Notice first that the limit h_∞ constructed in Lemma 4.0 belongs to the Lie group $\mathrm{Aff}^+(\mathbb{C}^n)$ of those transformations whose linear part have coefficients 1 on the diagonal and 0 below the diagonal. Applying Lemma 4.0 to the sequence h_k, we conclude the existence of a non-trivial sequence $\varphi_k : \mathbb{B}^n_{1/2} \hookrightarrow \mathbb{B}^n$ in $\overline{G}$ consisting of elements of $\mathrm{Aff}^+(\mathbb{C}^n)$ and tending uniformly to the identity. Then the usual strategy to construct one-parameter subgroups of non-discrete Lie groups works out here, modulo checking that the resulting affine flow induces maps which are approximated, on appropriate domains, by elements in the pseudogroup G. In order to do this, we equip again $\mathrm{Aff}^+(\mathbb{C}^n)$ with the metric induced by $\|\cdot\|_{1/2}$. For $\varepsilon > 0$ small enough, the ε-neighborhood U_ε of Id in this Lie group is diffeomorphic to a neighborhood of zero of the Lie algebra via the exponential map and hence is such that any element $\varphi \neq Id \in U_\varepsilon$ escapes after finite iteration, namely there is $N \in \mathbb{N}$ such that $(\varphi)^N \notin U_\varepsilon$. Also there exists an open neighborhood $W_\varepsilon \subset U_\varepsilon$ of the Id in $\mathrm{Aff}^+(\mathbb{C}^n)$ such that any $\varphi \in W_\varepsilon$ satisfies $\varphi \circ \varphi \in U_\varepsilon$.

For k large enough, φ_k belongs to W_ε. Define N_k as the smallest positive integer for which $(\varphi_k)^{N_k}$ does not belong to W_ε, so that this renormalized sequence of affine mappings remains in the relatively compact annulus $U_\varepsilon \setminus W_\varepsilon$. Modulo passing to a subsequence, the sequence $(\varphi_k)^{N_k}$ tends uniformly to some affine transformation φ_∞ on $\mathbb{B}^n$ which is ε-close to the identity and lies in the complement of W_ε so that it does not coincide with the identity. By construction, φ_∞ is the time-one-map of the real pseudo-flow φ_∞^t given, for each $t \in [0, 1]$, by uniform convergence on compact subsets of an appropriate subsequence of $(\varphi_k)^{[t \cdot N_k]}$ where $[\cdot]$ stands for the integral part.

Since the pseudo-flow φ_∞^t consists only of elements belonging to $\mathrm{Aff}^+(\mathbb{C}^n)$, it follows that the generating vector field X on $\mathbb{B}^n_{1/2}$, obtained as $X = \frac{\partial}{\partial t} \varphi_\infty^t|_{t=0}$, has strictly upper triangular linear part. After conjugation by a convenient iterate of A, the vector field actually becomes defined on the entire ball $\mathbb{B}^n$. □

Proof of Proposition 2.0. Let us begin with $\varepsilon_2 \leq \frac{\varepsilon_3}{4}$ given by Lemma 3.3. We also consider a scalar $0 < |\lambda| < 1$ satisfying $|\lambda - 1| \leq \varepsilon_2$. Then, fix $0 < \varepsilon_\lambda \leq \varepsilon_2$ so that the perturbations f and g remain $\frac{\varepsilon_3}{2}$-close to the identity. On the ball $\mathbb{B}^n_{1/2}$, the absolute value of any directional derivative of f remains $2\varepsilon_\lambda$-close to $|\lambda|$, as it can be seen by applying Cauchy Formula on the disk of radius $1/2$. It follows that, for $\varepsilon_\lambda < \frac{1-|\lambda|}{8}$, the perturbation satisfies the condition $(*)$ on the ball $\mathbb{B}^n_{1/2}$. After conjugating f and g by a homothety of ratio 2, these transformations are ε_3-close to the identity on $\mathbb{B}^n$ and f still satisfies condition $(*)$ on this ball. We then apply Lemmas 3.3 and 3.4 to f and g to construct some non-trivial sequence $g_{k+1} = f^{-N} \circ [f, g_k] \circ f^N$ converging uniformly to the identity on a sub-ball and satisfying $g_k(\underline{0}) \neq \underline{0}$. On the other hand, f is linear through the coordinate Φ given by Lemma 3.5. If the linear part A of f is diagonalizable, we can suppose, without loss of generality, that $\Phi^{-1} \circ f \circ \Phi = A$ is diagonal and satisfies $(*)$. Moreover $h_k = \Phi^{-1} \circ g_k \circ \Phi$ converges to the identity uniformly on $\mathbb{B}^n$ and satisfies condition (i) of Lemma 4.0. Thus Corollary 4.2 completes the proof in the diagonalizable case. Note that X has strictly upper triangular linear part in this case. To prove Proposition 2.0 in full generality, it remains to show that the proofs of Lemma 4.0 and of Corollary 4.2 in the case (i) also hold when A is no longer diagonal but has Jordan blocks. This is easy and left to the reader. □

Remark 4.3. It is possible to improve the estimates of Lemma 3.3 and 4.0 so as to ensure that the vector field constructed is actually non-singular at $\underline{0}$ and, in fact, is a translation. This can be carried out by writing $g = T \circ \tilde{g}$, where T stands for the translation part of g and $\tilde{g}$ is the remainder part fixing $\underline{0}$. We then consider the same decomposition for $g' = [A, g] = T' \circ \tilde{g}'$ with respect to the following formula

$$[A, T \circ \tilde{g}] = [A, T] \circ [T, [A, \tilde{g}]] \circ [A, \tilde{g}] \,.$$

It is possible to manage these terms so that the central double bracket $[T, [A, \tilde{g}]]$ becomes "very small" compared to the other ones and hence $T' \sim [A, T]$ and $\tilde{g}' \sim [A, \tilde{g}]$. So, considering the action by conjugation of A on T' and $\tilde{g}'$, as in Lemma 4.0, we are able to find some sequence $g_{k+1} = A^{-N_K} \circ [A, g_k] \circ A^{N_K}$ uniformly tending to a translation (maybe passing to a subsequence). In any case this would had led us to many more estimates, at least in order to control the domains of definition. In the sequel we shall construct non-singular vector fields just by considering conjugations under A and g with additional generic assumptions (needed later) in Proposition 5.1.

We also point out that the general discussion of Sect. 3 could have been slightly simplified by introducing the linearizing coordinate of f (given by Lemma 3.5) before Lemma 3.3. Also, we could have assumed from the beginning that $f = A$ is a diagonal matrix. This will be anyway required in Sect. 5. Nevertheless, notice that the diagonalizing coordinate Φ does not depend continuously on the map f near an homothety f_0. Thus, in the coordinate Φ, the map g may become very far from the identity preventing us from applying the strategy above.

The remaining part of the this section is devoted to complementary results that can easily be derived from our work but that are not strictly needed for the proof of our main theorem. The first one will be interpreted in Corollary 5.2 as an analogous of Proposition 2.0 for the common fixed point case.

Proposition 4.4. *Consider* $\varepsilon_4 > 0$ *and matrices* $A, B \in GL(n, \mathbb{C})$ ε_4*-close to the identity which satisfy*

– *A is diagonal with eigenvalues satisfying* $0 < |\lambda_n|^2 < |\lambda_1| < \cdots < |\lambda_n| < 1$,
– *B is not lower triangular.*

If ε_4 *is sufficiently small, then the pseudo-group G generated on a neighborhood of* $\underline{0}$ *by mappings* $f, g \in Diff(\mathbb{C}^n, \underline{0})$ *whose respective linear parts are A and B has non-trivial Lie algebra* $\mathfrak{G}$. *Precisely,* $\mathfrak{G}(\mathbb{B}^n_r)$ *contains a vector field X whose linear part is not lower triangular (in particular X is not trivial) for some ball* $r > 0$.

Proof. Fix $\varepsilon_4 > 0$ and A, B, f and g as in the statement. Up to a homothety, we can suppose that f and g are well-defined on the ball $\mathbb{B}^n$ and also arbitrarily close to A and B respectively. In particular, if ε_4 was chosen small enough, f and g are ε_3-close to identity, one-to-one and f is a contraction satisfying the condition $(*)$. Therefore Lemmas 3.3 and 3.4 apply to provide a non-trivial sequence g_k uniformly converging to the identity on some smaller ball. Note that, if the linear part of g_k is not lower triangular for every k, then the proof follows from Corollary 4.2 (similarly to the proof of Proposition 2.0 above).

In order to check that the linear part of g_k is not lower triangular notice that if B is not lower triangular then the same holds for $[A, B]$. Indeed if $[A, B] = T$ were lower triangular, then one would have $B^{-1}(T^{-1}A)B = A$. Employing an argument similar to the one used in the proof of Corollary 1.1 (replacing invariant directions by invariants flags), the last claim implies that B is the product of a lower triangular matrix and a permutation matrix. In fact, B will be lower triangular provided that ε_4 is sufficiently small. This gives us the desired contradiction. □

The lemma below is a variant of [Gh, Lemma 2.5].

Lemma 4.5. *Fix* $\varepsilon, \tau > 0$ *and a positive integer* $k \in \mathbb{N}^*$ *satisfying* $0 < k\varepsilon + \tau < 1$. *Then, given transformations* $f, g : \mathbb{B}^n \hookrightarrow \mathbb{C}^n$ *which are* ε*-close to the identity, the* k^{th} *iterates* f^k *and* g^k *are well-defined on* $\mathbb{B}^n_{1-k\varepsilon}$ *and the estimate below does hold*

$$\|f^k - g^k\|_{1-k\varepsilon-\tau} \quad \leq \quad k \cdot \|f - g\|_{1-\tau} \cdot \left(1 + \frac{(k-1)}{2\tau}\|g - Id\|_1\right).$$

Proof. It is similar to the proof of Lemma 3.0. Consider the decomposition

$$\begin{aligned} f^k - g^k &= (f^k - g \circ f^{k-1}) + (g \circ f^{k-1} - g^2 \circ f^{k-2}) + \cdots + (g^{k-1} \circ f - g^k) \\ &= (f - g) \circ f^{k-1} + (g \circ f - g^2) \circ f^{k-2} + \cdots + (g^{k-1} \circ f - g^k). \end{aligned}$$

Clearly

$$\begin{aligned} &\|f^k - g^k\|_{1-k\varepsilon-\tau} \\ &\leq \|f - g\|_{1-\varepsilon-\tau} + \|g \circ f - g^2\|_{1-2\varepsilon-\tau} + \cdots + \|g^{k-1} \circ f - g^k\|_{1-k\varepsilon-\tau}. \end{aligned}$$

On the other hand, for $l = 1, \ldots, k-1$, one has

$$\|g^l \circ f - g^{l+1}\|_{1-(l+1)\varepsilon-\tau} \leq \|f - g\|_{1-(l+1)\varepsilon-\tau} + \|\Delta_{g^l} \circ f - \Delta_{g^l} \circ g\|_{1-(l+1)\varepsilon-\tau}$$

where $\Delta_{g^l} = g^l - Id$. Applying Cauchy Formula to Δ_{g^l} on disks of radius τ, we conclude that

$$\begin{aligned}\|\Delta_{g^l} \circ f - \Delta_{g^l} \circ g\|_{1-(l+1)\varepsilon-\tau} &\leq \frac{1}{\tau} \cdot \|\Delta_{g^l}\|_{1-l\varepsilon} \cdot \|f - g\|_{1-(l+1)\varepsilon-\tau} \\ &\leq \frac{l}{\tau} \cdot \|\Delta_g\|_1 \cdot \|f - g\|_{1-\tau}\,.\end{aligned}$$

The lemma results at once. □

Proposition 4.6. *Let $h_k : \mathbb{B}^n \hookrightarrow \mathbb{C}^n$ be a sequence of injective holomorphic mappings converging uniformly to the identity and denote by G the pseudo-group generated by them on $\mathbb{B}^n$. Consider the sequence δ_k of positive real numbers defined by $\delta_k = \|\Delta_k\|_1$, where $\Delta_k = h_k - Id$ denote the variation of h_k. Let X be any limit obtained from the bounded family $\frac{\Delta_k}{\delta_k}$. Then X, viewed as a vector field, belongs to the Lie algebra $\mathfrak{G}(\mathbb{B}^n)$. Besides X is non-trivial unless the sequence $\frac{\Delta_k}{\delta_k}$ converges uniformly to the identity on compact sets $\mathbb{B}^n$ containing $\underline{0} \in \mathbb{C}^n$.*

Proof. Fix $\varepsilon, \tau > 0$ satisfying $0 < \varepsilon + \tau < 1$. By assumption, the sequence $\|\Delta_k\|_1 = \delta_k$ converges to *zero* and, up to passing to a subsequence, the sequence of maps $\frac{\Delta_k}{\delta_k}$ converges uniformly to a vector field X on compact subsets. Letting $N_k = [\frac{\varepsilon}{\delta_k}]$, where $[\cdot]$ stands for the integral part, we prove that the sequence of iterates $h_k^{N_k}$ converges uniformly on the ball $\mathbb{B}^n_{1-\varepsilon-\tau}$ to the ε-time map φ_X^ε of X. Note that both φ_X^ε and $h_k^{N_k}$ are defined on the ball $\mathbb{B}^n_{1-\varepsilon}$. Now Lemma 4.5 applied to $h_k^{N_k} - (\varphi_X^{\varepsilon/N_k})^{N_k}$ provides

$$\begin{aligned}\left\|h_k^{N_k} - \varphi_X^\varepsilon\right\|_{1-\varepsilon-\tau} &\leq \left\|h_k - \varphi_X^{\varepsilon/N_k}\right\|_{1-\tau} \cdot \left(N_k + \frac{N_k(N_k-1)}{2\tau}\|h_k - Id\|_1\right) \\ &\leq \left(\|N_k\Delta_k - \varepsilon X\|_{1-\tau} + \varepsilon\frac{\left\|\varphi_X^{\varepsilon/N_k} - Id - (\varepsilon/N_k)X\right\|_{1-\tau}}{\varepsilon/N_k}\right) \\ &\quad \cdot \left(1 + \frac{(N_k-1)\|h_k - Id\|_1}{2\tau}\right)\,.\end{aligned}$$

The term on the right hand side is bounded by $(1+\frac{\varepsilon}{2\tau})$. The term $\frac{\|\varphi_X^{\varepsilon/N_k} - Id - (\varepsilon/N_k)X\|_{1-\tau}}{\varepsilon/N_k}$ converges to zero by definition of the flow and, finally, $\|N_k\Delta_k - \varepsilon X\|_{1-\tau}$ converges to zero by assumption. □

Remark 4.7. The preceding proof shows in particular that the t-time map of a holomorphic vector field X on $\mathbb{B}^n$ is always obtained as uniform limit $\phi_X^t = \lim_{k\to\infty}(Id + \frac{t}{k}X)^k$ on any compact subset.

Proposition 4.8. *Let $\varphi_t : \mathbb{B}^n \hookrightarrow \mathbb{C}^n$, $t \in [-t_0, t_0]$, $t_0 > 0$, be a (real) 1-parameter family of holomorphic transformations satisfying*
– *the mapping $(z, t) \mapsto \varphi_t(z)$ is uniformly continuous on $\mathbb{B}^n \times [-t_0, t_0]$,*
– *for any $z \in \mathbb{B}^n$, we have $\varphi_t \circ \varphi_s(\underline{z}) = \varphi_{t+s}(z)$ whenever $s, t, s+t \in [-t_0, t_0]$,*
– *$\varphi^0 = Id$.*

Then, there exists a unique holomorphic vector field X on $\mathbb{B}^n$ such that $\varphi_t(z) = \phi^t_X(z)$ for any $z \in \mathbb{B}^n$ and $t \in [-t_0, t_0]$ whenever $\phi^t_X(z)$ is defined.

Proof. The uniqueness of X is clear since *a posteriori* it is obtained as $X = \frac{\partial}{\partial t}(\varphi_t - Id)|_{[t=0]}$ or equivalently as uniform limit on compact subsets of $\frac{\varphi_t - Id}{t}$. By virtue of the composition rules of φ_t and ϕ^t_X, if, on any subball $\mathbb{B}^n_r$, those families coincide for t small enough, then they will coincide whenever they are defined in the ball $\mathbb{B}^n$. So let us fix a ball $\mathbb{B}^n_r$, $0 < r < 1$.

It is indeed sufficient to show that the family $\frac{\varphi_{t_k} - Id}{t_k}$ is uniformly bounded for a convenient sequence $t_0 > t_1 > t_2 > \cdots > 0$ decreasing to 0. Let us first prove this claim. If this family is bounded, then it is relatively compact by Montel Theorem. Thus, maybe passing to a subsequence and maybe reducing r, the sequence $\frac{\varphi_{t_k} - Id}{t_k}$ converges uniformly on the ball $\mathbb{B}^n_r$ to a vector field X. In particular, as in Proposition 4.6, letting $\Delta_k = \varphi_{t_k} - Id$ and $\delta_k = \|\Delta_k\|_r$, it results that $\lim_{k\to\infty} \frac{\delta_k}{t_k} = \delta$ where $\delta = \|X\|_r$. Furthermore the sequence of renormalized maps $\frac{\Delta_k}{\delta_k}$ tends uniformly to the renormalized vector field X/δ. Set $N_k = [\frac{t\delta}{\delta_k}]$ where [.] stands for the integral part. For t sufficiently small, Proposition 4.6 states that the sequence of iterates $(\varphi_{t_k})^{N_k} = \varphi_{t_k N_k}$ tends uniformly to the element of flow $\phi^{t\delta}_{X/\delta} = \phi^t_X$. Finally, we have

$$\left\|\varphi_t - \phi^t_X\right\|_r \quad \leq \quad \|\varphi_t - \varphi_{t+\varepsilon_k}\|_r \quad + \quad \left\|\varphi^{N_k}_{t_k} - \phi^t_X\right\|_r$$

where $\varepsilon_k = t_k N_k - t \to 0$. The right hand side converges uniformly to zero on $\mathbb{B}^n_r$ when $k \to \infty$ so that φ_t coincides with ϕ^t_X. Clearly X is not trivial as long as $\varphi_t \not\equiv Id$ and this proves our claim.

So we just need to find a sequence t_k as before such that the corresponding maps $\{\frac{\varphi_{t_k} - Id}{t_k}\}$ are uniformly convergent on $\mathbb{B}^n_r$. We consider a sequence of the form $t_k = t/2^k$ with $0 < t < t_0$ very small. To simplify the notations, let us write $\varphi_t = Id + \Delta_t$. First let us show that the Δ_{t_k} are (at least) exponentially decreasing in norm on the intermediate ball $\mathbb{B}^n_R$, $R = \frac{r+1}{2}$. In order to do this, fix $\varepsilon = \frac{1-r}{8}$ and $\tau = \frac{1-r}{4}$ so that $1 - R = R - r = 2\varepsilon + \tau$ with $\tau = 2\varepsilon$. Let $\delta^t_r = \|\Delta_t\|_r$. Since $\delta^t_1 \to 0$ when $t \to 0$, maybe reducing t_0, we can assume that $\delta^t_1 < \varepsilon$ whenever $0 < t < t_0$. Reasoning as in the proof of Lemma 3.0 (with $f = g = \Delta_{t/2}$ on the ball $\mathbb{B}^n$), one gets

$$\|\Delta_t - 2\Delta_{t/2}\|_R \quad \leq \quad \frac{1}{\tau}\|\Delta_{t/2}\|_{1-\varepsilon}\|\Delta_{t/2}\|_R \quad \leq \quad \frac{\varepsilon}{\tau}\|\Delta_{t/2}\|_R \, .$$

By the triangle inequality (and $\frac{\varepsilon}{\tau} = \frac{1}{2}$), we obtain $2\|\Delta_{t/2}\|_R \leq \|\Delta_t\|_R + \frac{1}{2}\|\Delta_{t/2}\|_R$ and thus

$$\|\Delta_{t/2^k}\|_R \quad \leq \quad \left(\frac{2}{3}\right)^k \|\Delta_t\|_R \, ,$$

for all $k \in \mathbb{N}$ and $0 < t < t_0$. Using these estimates, we shall imply others on $\mathbb{B}^n_R$. Indeed, by repeating the discussion above, we now have

$$\begin{aligned} 2\|\Delta_{t_{k+1}}\|_r &\leq \|\Delta_{t_k}\|_r + \tfrac{1}{\tau}\|\Delta_{t_{k+1}}\|_{R-\varepsilon} \ \|\Delta_{t_{k+1}}\|_r \\ &\leq \|\Delta_{t_k}\|_r + \ \tfrac{1}{2}\left(\tfrac{2}{3}\right)^{k+1} \ \|\Delta_{t_{k+1}}\|_r \,. \end{aligned}$$

It results that

$$\left\|\frac{\Delta_{t_{k+1}}}{t_{k+1}}\right\|_r \left(1-\frac{1}{4}\left(\frac{2}{3}\right)^{k+1}\right) \ \leq \ \left\|\frac{\Delta_{t_k}}{t_k}\right\|_r ,$$

and thus

$$\limsup_{k\to\infty}\left\|\frac{\varphi_{t_k}-Id}{t_k}\right\|_r \ \leq \ \left(\lim_{k\to\infty}\prod_{l=0}^{k}\left(1-\frac{1}{4}\left(\frac{2}{3}\right)^{l+1}\right)^{-1}\right)\frac{\varepsilon}{t} \,.$$

Since the right hand side of the inequality above is convergent, we conclude that $\{\frac{\varphi_{t_k}-Id}{t_k}\}$ forms a relatively compact family as required. □

5. Deriving many pseudo-flows from a given one in the closure of a pseudo-group

In this section, G is the holomorphic pseudo-group generated on $\mathbb{B}^n$ by a contraction f close to a homothety (which satisfies $(*)$) and another transformation g as in Sect. 3 and in the beginning of Sect. 4. We can now assume that the Lie pseudo-algebra $\mathfrak{G}$ contains a non-trivial vector field $X \in \mathfrak{G}(\mathbb{B}^n)$ (cf. Propositions 2.0 and 4.4). The purpose of this section is to show that, in fact, $\mathfrak{G}$ is large under weak additional assumptions on f and g. Precisely, using Lemma 1.2, we are going to show that, through some coordinate $\Phi : \mathbb{B}^n \hookrightarrow \mathbb{C}$ (namely the linearizing coordinate for f), the image $\Phi_*\mathfrak{G}(\mathbb{B}^n)$ contains a copy of the affine Lie algebra $\mathfrak{sl}(n,\mathbb{C}) \ltimes \mathbb{C}^n$ generated by $\mathfrak{sl}(n,\mathbb{C})$ together with all the translations. The pseudo-algebra $\mathfrak{G}$ is said to have *large affine part* on the ball $\mathbb{B}^n$ if it contains a copy of $\mathfrak{sl}(n,\mathbb{C}) \ltimes \mathbb{C}^n$ (in appropriate coordinates). In the special case where G fixes $\underline{0}$, it will be shown that $\mathfrak{G}$ has *large linear part*, i.e. that $\Phi_*\mathfrak{G}(\mathbb{B}^n)$ contains $\mathfrak{sl}(n,\mathbb{C})$. As before, Φ stands for the linearizing coordinate of f. In the sequel f is thought of as the map $\mathbb{B}^n \hookrightarrow \mathbb{C}^n$ induced by a diagonal matrix

$$A = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

with eigenvalues satisfying condition

$$(*) \quad 0 < |\lambda_n|^2 < |\lambda_1| \leq \cdots \leq |\lambda_n| < 1 \,.$$

We also assume that g is well-defined at $\underline{0}$ viewed as an element of the pseudo-group G, i.e. $g(\underline{0}) \in \mathbb{B}^n$. Then, denoting by $B = D_{\underline{0}}g$ the linear part of g, the pair

(A, B) is supposed to satisfy the following additional conditions (which contains those of Lemma 1.2):

$$(**)\quad \begin{cases} \text{the } \lambda_i \text{ are pairwise distinct in norm, none of them being real;} \\ \text{in particular, we have } 0 < |\lambda_n|^2 < |\lambda_1| < \cdots < |\lambda_n| < 1\ ; \\ \text{the } \frac{\lambda_i}{\lambda_j},\ i \neq j, \text{ are pairwise distinct in norm, none of them being real;} \\ \text{finally, neither } B \text{ nor } B^{-1} \text{ admits } zero \text{ as entry.} \end{cases}$$

Let us begin with a continuous analogous of Lemma 4.0. This lemma also indicates how simpler the arguments become when diffeomorphisms (mappings) are replaced by vector fields.

Lemma 5.0. *Let A be a diagonal matrix satisfying $(*)$ and let G be a holomorphic pseudo-group on $\mathbb{B}^n$ containing A. Assume that the Lie pseudo-algebra $\mathfrak{G}$ of $\overline{G}$ contains a non-trivial vector field $X \in \mathfrak{G}(\mathbb{B}^n)$. If X does not vanish at $\underline{0}$, then $\overline{G}$ also contains some non-trivial translation pseudo-flow on $\mathbb{B}^n$ (i.e. the pseudo-flow induced by a constant vector field). On the other hand, if X vanishes at $\underline{0}$ but if its linear part is not strictly lower triangular, then $\overline{G}$ also contains a linear pseudo-flow on $\mathbb{B}^n$ which is not strictly lower triangular.*

Proof. Suppose first $X(\underline{0}) \neq \underline{0}$. Let us decompose X into a translation, a linear and a higher degree parts, $X = X^0 + X^1 + X^{\geq 2}$ in the obvious manner. Then the action of A by conjugation increases the translation part of X i.e.

$$\|A^{-1}X^0\|_1 \geq \frac{1}{|\lambda_n|}\|X^0\|_1\ ,$$

faster than the linear part since

$$\|A^{-1}X^1A\|_1 \leq \frac{|\lambda_n|}{|\lambda_1|}\|X^1\|_1\ ,$$

since we have by assumption that $1/|\lambda_n| > |\lambda_n|/|\lambda - 1|$. Assumption $(*)$ also implies that this action decreases the higher order terms

$$\|A^* X^{\geq 2}\|_1 \leq \frac{|\lambda_n|^2}{|\lambda_1|}\|X^{\geq 2}\|_1\ .$$

Therefore there exists a sequence of positive scalars $t_k \in \mathbb{R}^+$, $t_k \to 0$, such that the sequence of holomorphic vector fields defined on $\mathbb{B}^n$ by $(t_k \cdot (A^k)^* X)$ has translation part of constant (non-vanishing) norm. Clearly the higher order components of these vector fields converge to *zero*. The linear part of them, which has t_k as factor, converges to *zero* as well since the translation part of the original (i.e. non-renormalized) vectors fields grows faster than the linear one. Hence some subsequence uniformly tends to a constant vector field which, by construction, is non-trivial and contained in the closure of G.

Now, if $X(\underline{0}) = \underline{0}$ but if its linear part is not strictly lower triangular, then the action of A is linear diagonal on the entries of X^1, i.e. setting $X^1 = (v_{i,j})_{i,j}$ one has

$$A^{-1}X^1A = \left(\frac{\lambda_j}{\lambda_i}v_{i,j}\right)_{i,j}\ .$$

Since the eigenvalues have norm $\frac{|\lambda_j|}{|\lambda_i|} > 1$ for $i < j$, the norm of the non-zero upper entries increase under conjugacy by A. Similarly, the lower entries become closer to zero while the diagonal terms remain unchanged. Now the proof follows as above. □

Proposition 5.1. *Let G be a holomorphic pseudo-group on $\mathbb{B}^n$ whose Lie pseudo-algebra $\mathfrak{G}$ contains a vector field $X \in \mathfrak{G}(\mathbb{B}^n)$ with non-lower triangular linear part X_1 at $\underline{0}$. Assume that G contains also the linear contraction $f : \mathbb{B}^n \hookrightarrow \mathbb{B}^n$ induced by a matrix A as well as another transformation $g : \mathbb{B}^n_r \hookrightarrow \mathbb{B}^n$ (defined on some ball $\mathbb{B}^n_r$). Finally, assume that A together with the linear part $B = D_{\underline{0}} g$ fulfils the conditions $(**)$. Then $\mathfrak{G}(\mathbb{B}^n)$ actually contains the whole of $\mathfrak{sl}(n, \mathbb{C})$. Furthermore if $g(\underline{0}) \neq \underline{0}$, then $\mathfrak{G}(\mathbb{B}^n)$ contains also all translations and hence a copy of the affine Lie algebra $\mathfrak{sl}(n, \mathbb{C}) \ltimes \mathbb{C}^n$.*

Proof. Suppose first that $g(\underline{0}) = \underline{0}$ and $X(\underline{0}) = \underline{0}$. Assumption $(**)$ combined with Lemma 1.2 implies that the linear part X_1 of X along with a finite number of its conjugates under A and B generate $\mathfrak{sl}(n, \mathbb{C})$ over $\mathbb{R}$ on some ball $\mathbb{B}^n_r$ (on which the iterations of A and B needed for this construction are well-defined). Then, by a finite number of additional linear operations over $\mathbb{R}$, we can also find on $\mathbb{B}^n_r$ a collection $X_{i,j}$ of elements of $\overline{G}$ such that the corresponding linear part is the Kronecker matrix $X^1_{i,j} = \delta_{i,j}$. Now, we proceed as in the proof of the Lemma 5.0 in order to linearize the elements $X_{i,j}$ for which $i \geq j$. Namely one has

$$A^{-k} X^1_{i,j} A^k = \left(\frac{\lambda_j}{\lambda_i}\right)^k \delta_{i,j} \ .$$

On the other hand, $\|(A^k)^* X^{\geq 2}_{i,j}\|_r$ tends to $\underline{0}$ when $k \to +\infty$. Letting $t = \frac{|\lambda_i|}{|\lambda_j|} \leq 1$, the sequence of elements of $\overline{G}$ defined on $\mathbb{B}^n$ by $(t^k \cdot (A^k)^* X_{i,j})$ converges uniformly (maybe passing to an appropriate subsequence) to the linear Kronecker matrix $\delta_{i,j}$. The same construction can be carried out with purely imaginary Kronecker matrices $\sqrt{-1} \cdot \delta_{i,j}$. Thus $\overline{G}$ already contains on $\mathbb{B}^n_r$ the upper triangular complex Lie sub-algebra of $\mathfrak{sl}(n, \mathbb{C})$. A conjugation by a suitable power of A enables us to suppose that all these vector fields are defined on $\mathbb{B}^n$. In particular, $\overline{G}$ contains any diagonal element of $SL(n, \mathbb{C})$ sufficiently close to identity. Next we replace A by some of these elements, say $\tilde{A}$, with eigenvalues $\tilde{\lambda}_i$ now satisfying

$$0 < |\tilde{\lambda}_1|^2 < |\tilde{\lambda}_n| \leq \cdots \leq |\tilde{\lambda}_1| < 1 \ .$$

Therefore the same arguments show that $\mathfrak{G}$ contains lower triangular elements of $\mathfrak{sl}(n, \mathbb{C})$ as well.

Suppose now $X(\underline{0}) \neq \underline{0}$ (whether or not g fixes $\underline{0}$). The translation part of X together with its conjugates under A generate a real subspace $E \subset \mathbb{C}^n$ invariant by A. Employing a procedure of renormalization similar to the one explained above, we can suppose that the translations by elements in E also belong to $\overline{G}$. If $E \neq \mathbb{C}^n$, then E contains at least a translation parallel to some coordinate axis, say Y_0 (A has only simple eigenvalues). After conjugation by g, this translation becomes a vector field $Y = g^* Y_0$ whose (new) translation part is given by $B^{-1} Y_0$ (the possible translation part of g gives no contribution). By virtue of the condition $(**)$, $B^{-1}Y_0$ lies away from any A-invariant hyperplane. Repeating the same arguments, but using Y instead of X, we see that any translation pseudo-flow actually belongs to $\overline{G}$. We can then delete the possible translation part of g and X and conclude, as in the first case, that $\mathfrak{G}$ contains also $\mathfrak{sl}(n, \mathbb{C})$.

Finally, suppose that $g(\underline{0}) \neq \underline{0}$ and $X(\underline{0}) = \underline{0}$. It is sufficient to show that we can replace X by an appropriate conjugate under A and g which does not vanish at $\underline{0}$. First, Lemma 5.0 allows us to suppose that X is linear and non-lower triangular. Assume that the conjugate $Y = g^*X$ still vanishes at $\underline{0}$. The linear part of Y is given by $Y_1 = B^*X$. Again assumption $(**)$ combined with Lemma 1.2 implies that a finite number of conjugates of X and Y under A have linear part generating $\mathfrak{sl}(n, \mathbb{C})$. Employing once again Lemma 5.0, we conclude that $\mathfrak{sl}(n, \mathbb{C})$ is contained in $\mathfrak{G}$. Therefore there exists an element $Z \in \mathfrak{sl}(n, \mathbb{C})$ which does not vanish at $g(\underline{0})$ so that g^*Z in turn does not vanish at $\underline{0}$. Now we proceed as in the preceding case. □

The following corollary is exactly what is needed for the proof of Theorem A (together with all the consequences establisehd in Sect. 6).

Corollary 5.2. *There exist a real Zariski open subset* $\mathcal{U}_5 \in (GL(n, \mathbb{C}))$ *and a constant* $\varepsilon_5 > 0$ *such that, for any scalar* $0 < |\lambda| < 1$ *satisfying* $|\lambda - 1| < \varepsilon_5$*, one can find a smaller constant* $0 < \varepsilon_\lambda << |\lambda - 1|$ *having the following property: all* ε_λ*-perturbations* $f, g : \mathbb{B}^n \hookrightarrow \mathbb{C}^n$ *respectively of the contracting homothety* $f_0(z) = \lambda \cdot z$ *and of the identity map* $g_0(z) = z$ *on* $\mathbb{B}^n$ *with* $f(\underline{0}) = \underline{0}$ *and derivatives* $(D_{\underline{0}} f, D_{\underline{0}} g)$ *at* $\underline{0}$ *lying in* $\mathcal{U}_5$ *satisfy:*
– *either* $g(\underline{0}) \neq \underline{0}$ *and* $\mathfrak{G}$ *has large affine part on the whole of* $\mathbb{B}^n$*,*
– *or* $g(\underline{0}) = \underline{0}$ *and* $\mathfrak{G}$ *has large linear part on the whole of* $\mathbb{B}^n$*.*

Proof. Let $\mathcal{U}$ be the set of pairs (A, B) satisfying $(**)$. In the first case, Proposition 2.0 ensures the existence of a non-trivial pseudo-flow X. In the second case, the existence of such X follows from Proposition 4.4. In both cases, X has strictly upper triangular part at $\underline{0}$ (which is trivial when X is constant). In the coordinate Φ given by Lemma 3.5, the map f is linear so that we can apply Proposition 5.1 to complete the proof of the lemma. □

As a direct application, we provide a generalization to arbitrary dimension $n \geq 1$ of a result due to Il'yashenko in the case $n = 1$ and Lamy in the case $n = 2$ (see [Ill] and [La]):

Corollary 5.3. *Suppose we are given holomorphic transformations* $f_1, \ldots, f_d$: $\mathbb{B}^n \hookrightarrow \mathbb{C}^n$ *fixing* $\underline{0} \in \mathbb{C}^n$ *whose derivatives at* $\underline{0}$ *generate a dense subgroup* $G_0 \subset GL(n, \mathbb{C})$*. Then* G *has large linear part on some ball* $\mathbb{B}^n_r$*. In particular the action of* G *on the punctured ball* $\mathbb{B}^n_r \setminus \{\underline{0}\}$ *is minimal (all orbits are dense) and ergodic (w.r.t. Lebesgue).*

Recall that two elements $f, g \in \mathrm{Diff}(\mathbb{C}^n, \underline{0})$ will generate such a pseudo-group G with dense linear part provided that their respective linear part $A, B \in GL(n, \mathbb{C})$ are close to I and "sufficiently generic" (cf. Corollary 1.5).

Proof. First we find elements $A, B \in G_0$ whose corresponding mappings $A, B : \mathbb{B}^n \hookrightarrow \mathbb{C}^n$ fulfil the assumptions of Corollary 5.2 for a given λ. Consider the corresponding elements $f, g \in G$ ($A = D_{\underline{0}} f$ and $B = D_{\underline{0}} g$). After conjugating f, g under an appropriate homothety (which will commute with the linear parts), these (non-linear) mappings are defined on the ball $\mathbb{B}^n$ and sufficiently close to A, B so that they also fulfil the assumptions of Corollary 5.2. Therefore the pseudo-group G has large linear (or affine) part. This imply in particular the statement about ergodicity (cf. Property 6.1). □

Corollary 5.4. *Let $f_1, \dots, f_d : \mathbb{B}^n \hookrightarrow \mathbb{C}^n$ be as in Corollary 5.3. Then there is $\varepsilon > 0$ such that the pseudo-group G generated on a neighborhood of $\underline{0}$ by any ε-small perturbation $g_1, \dots, g_d : \mathbb{B}^n \hookrightarrow \mathbb{C}^n$ of the original generators satisfies the following alternative:*

either: the g_i's do not have a common fixed point and $\mathfrak{G}$ has large affine part on $\mathbb{B}^n_r$,

or: the g_i's have a common fixed point p and $\mathfrak{G}$ has large linear part on $\mathbb{B}^n_r$.

The action of G is minimal and ergodic on $\mathbb{B}^n_r$ (or $\mathbb{B}^n_r \setminus \{p\}$).

Proof. The elements f and g constructed in the preceding proofs are expressed as words in terms of the generators $f_1, \dots, f_d$. Since assumptions of Corollary 5.2 are open, the same words in the new generators $g_1, \dots, g_d$ still verify the same assumptions, provided that the perturbation is sufficiently small. □

6. Pseudo-groups with large affine part

In this section we are going to complement the results of Sect. 5 by showing that a pseudo-group having large affine part necessarily possesses many dynamical properties. Precisely, the final results of Sect. 5 gave sufficient conditions on a pseudo-group G defined on the ball $\mathbb{B}^n$ to have large affine part on some ball $\mathbb{B}^n_r$. In other words, there is a holomorphic coordinate $\Phi : \mathbb{B}^n \hookrightarrow \mathbb{C}^n$ (namely the linearizing map of f mentioned in Sect. 5) where the Lie pseudo-algebra $\mathfrak{G}$ contains the affine Lie algebra $\mathfrak{sl}(n, \mathbb{C}) \ltimes \mathbb{C}^n$ on the open set $\Phi(\mathbb{B}^n_r)$. Equivalently the pseudo-group image $\Phi \circ G \circ \Phi^{-1}$ approximates (uniformly on compact subsets) the restriction to $\Phi(\mathbb{B}^n_r)$ of every translation as well as every element of $SL(n, \mathbb{C})$. To abbreviate notations, we shall simply say that $\mathfrak{G}$ contains a copy of $\mathfrak{sl}(n, \mathbb{C}) \ltimes \mathbb{C}^n$ on $\mathbb{B}^n_r$ (in the original coordinate). The "chaotic" properties of G will hold on the maximal domain where $\mathfrak{G}$ locally contains copies of $\mathfrak{sl}(n, \mathbb{C}) \ltimes \mathbb{C}^n$. Throughout the section, we consider a pseudo-group G on $\mathbb{B}^n$ whose Lie algebra $\mathfrak{G}(\mathbb{B}^n)$ contains the affine algebra $\mathfrak{sl}(n, \mathbb{C}) \ltimes \mathbb{C}^n$ on the whole $\mathbb{B}^n$ (this will be referred to by saying that *G has large affine part on the ball $\mathbb{B}^n$*).

First, since the action of $\mathfrak{sl}(n, \mathbb{C}) \ltimes \mathbb{C}^n$ is transitive in the ball $\mathbb{B}^n$, we automatically have (cf. Sect. 2):

Property 6.0. *Let G be a holomorphic pseudo-group having large affine part on $\mathbb{B}^n$. Then, G is minimal on $\mathbb{B}^n$, i.e. all orbits are dense.*

From the measure-theoretic point of view, the transitivity of $\mathfrak{G}$ also implies:

Property 6.1. *Let G be a holomorphic pseudo-group having large affine part on $\mathbb{B}^n$. Then, G is ergodic on the ball: every Lebesgue measurable subset which is invariant by G has null or total Lebesgue measure.*

We refer to [Reb1] for a simple proof.

Remark 6.2. Following ideas of [Be, Li, Lo1], we can prove that a holomorphic pseudo-group G having large affine part on $\mathbb{B}^n$ either preserves a volume form or the set of contracting points

$$\{p \in \mathbb{B}^n \ ; \ \text{there is } f \in G \text{ such that } f(p) = p \text{ and } D_p f \text{ is a contraction}\}$$

is dense in $\mathbb{B}^n$. If G preserves a volume form, then this volume form becomes (a constant multiple of) the usual Euclidean volume in the coordinate where G contains a copy of $\mathfrak{sl}(n, \mathbb{C}) \ltimes \mathbb{C}^n$.

More generally, when G does not preserve a volume form, there does not exist a σ-finite measure μ on the ball which is preserved by elements of the pseudo-group. A notion of geometric entropy for (regular) foliations and pseudo-groups is defined in [Gh,La,Wa]. The first example of a pseudo-group with strictly positive entropy is a Schottky configuration (see [Gh,La,Wa], p. 107). These dynamics can be recovered from translations and contractions so that they are contained in the dynamics of G. Since the entropy increases when we increase the set of generators, it follows immediately that the pseudo-group G has strictly positive entropy provided that it contains a contraction. The pseudo-group is also chaotic in the sense of Devaney in this case. Indeed, it is minimal, has dense periodic orbits (contracting points) and sensitivity may be derived from the affine motions in $\overline{G}$.

We may expect from generic foliations or pseudo-groups more complicated dynamics than affine dynamics. When $n = 1$ the main result of [Be,Li,Lo2] yields:

Property 6.3. *Assume that $n = 1$ and consider a holomorphic pseudo-group G having large affine part on the disc $\mathbb{B}^1$. This means that $\overline{G}$ contains the restriction of all translations within $\mathbb{B}^1$. Assume moreover that G is not conjugate to a subgroup of Mœbius transformations. Then any conformal transformation within the unit disc $\mathbb{B}^1$ is uniformly approximated by elements of G. In particular, no differential-geometric structure other than the conformal one is preserved by G.*

M. Belliart recently generalized this property for arbitrary dimensions $n \geq 2$ (see [Be1]). If the previous features are those expected by chaotic dynamics, less expected is the structural instability which immediately results from the topological rigidity.

Proposition 6.4. *Let G be a holomorphic pseudo-group having large affine part on $\mathbb{B}^n$. Assume moreover, when $n = 1$, that G contains some element whose derivative is not real at some point. Consider a homeomorphism $\Phi : \mathbb{B}^n \hookrightarrow \mathbb{C}^n$ onto its image conjugating G with an holomorphic pseudo-group $\tilde{G}$ on $\Phi(\mathbb{B}^n)$. Then Φ is either a holomorphic or an anti-holomorphic diffeomorphism.*

In dimension $n = 1$, results of measurable rigidity were obtained in [Reb2].

Proof. First let us show that Φ actually conjugates the respective Lie algebras $\mathfrak{G}$ and $\tilde{\mathfrak{G}}$. To make this notion of conjugation of Lie algebras precise, recall first that a holomorphic vector field X belongs to $\mathfrak{G}$ if, and only if, the corresponding pseudo-flow φ_X^t (induced in the domain of definition of X) is contained in $\overline{G}$. Then, we want to show that the image $\tilde{\varphi}_t = \Phi \circ \varphi_X^t \circ \Phi^{-1}$ of such pseudo-flow by Φ coincides with the pseudo-flow $\varphi_{\tilde{X}}^t$ of a vector field $\tilde{X}$ belonging to $\tilde{\mathfrak{G}}$. It is clear from the definition of the closure of a holomorphic pseudo-group (cf. Sect. 2) that Φ conjugates $\overline{G}$ with $\overline{\tilde{G}}$. This immediately implies that the 1-parameter family of homeomorphisms defined by $\tilde{\varphi}_t = \Phi \circ \varphi_X^t \circ \Phi^{-1}$ actually consists of holomorphic transformations belonging to $\overline{\tilde{G}}$. Then, it suffices to show that such 1-parameter family of holomorphic transformations always coincides with the pseudo-flow of a vector field $\tilde{X}$. Now, let us be more careful with the domains of definition. If X is defined on $U \subset \mathbb{B}^n$ and $V = \Phi(U)$ denotes its image, then for any relatively compact ball $B \subset V$ in V, there is a $t_0 > 0$ such that $\tilde{\varphi}_t$ is defined as a map $B \to V$ for all $-t_0 \le t \le t_0$. Moreover, the arrow $(z, t) \to \tilde{\varphi}_t(z)$ is uniformly continuous on $B \times [-t_0, t_0]$. Indeed, these properties are satisfied on $\Phi^{-1}(B)$ for the pseudo-flow φ_X^t and thus they result from the continuity of Φ. On the other hand, the family $\tilde{\varphi}_t$ automatically satisfies the composition rule $\tilde{\varphi}_t \circ \tilde{\varphi}_s(z) = \tilde{\varphi}_{t+s}(z)$ for any $z \in V$ and any $s, t \in \mathbb{R}$ provided that both expressions are defined. Thus, it is enough to prove that the restriction $\tilde{\varphi}_t|_B : B \to V$ coincides with the pseudo-flow of a vector field $\tilde{X}$ inside B for $|t|$ small and a fixed ball B as above. Nonetheless this is a consequence of Proposition 4.8.

Clearly it is sufficient to prove that Φ is holomorphic or anti-holomorphic in a local holomorphic coordinate. Therefore, our assumptions allow us to assume that G is defined on the ball $\mathbb{B}^n$ and that $\mathfrak{G}$ contains the affine Lie algebra $\mathfrak{sl}(n, \mathbb{C}) \ltimes \mathbb{C}^n$. In particular, there are $2n$ constant vector fields $X_1, \dots, X_{2n}$ in $\mathfrak{G}(\mathbb{B}^n)$ generating the translation part of $\mathfrak{G}$ over $\mathbb{R}$. By Lemma 6.5, Φ conjugates their pseudo-flow $\phi_{X_i}^t$ with the pseudo-flow $\phi_{\tilde{X}_i}^t$ of respective vector fields $\tilde{X}_1, \dots, \tilde{X}_{2n}$ belonging to $\tilde{\mathfrak{G}}$. The neighborhood of any point $p \in \mathbb{B}^n$ possesses a parametrization given by

$$(\mathbb{R}^{2n}, \underline{0}) \to (\mathbb{B}^n, p)\ ;\ (t_1, \dots, t_{2n}) \mapsto \phi_{X_1}^{t_1} \circ \cdots \circ \phi_{X_{2n}}^{t_{2n}}(p)\,.$$

Similarly the image under Φ of the neighborhood in question admits the parametrization

$$(\mathbb{R}^{2n}, \underline{0}) \to (\mathbb{C}^n, \Phi(p))\ ;\ (t_1, \dots, t_{2n}) \mapsto \phi_{\tilde{X}_1}^{t_1} \circ \cdots \circ \phi_{\tilde{X}_{2n}}^{t_{2n}}(p)\,.$$

Since Φ preserves the dimension as well as the regularity and the commutativity of the pseudo-flows, the corresponding vector fields $\tilde{X}_1, \dots, \tilde{X}_{2n}$ in $\tilde{\mathfrak{G}}$ are also $\mathbb{R}$-linearly independent at $\Phi(p)$. Through these (real) analytic parametrizations, Φ is the identity mapping at $(\mathbb{R}^{2n}, \underline{0})$ by construction. Hence, Φ is real analytic at p. In particular, Φ is (real) smooth at p. The differential of Φ, $D_p\Phi \in GL(2n, \mathbb{R})$, conjugates the differential $D_p\overline{G}^{\{p\}} \subset GL(n, \mathbb{C})$ at p of the isotropy subgroup

$\overline{G}^{\{p\}} = \{g \in \overline{G}\ ;\ g(p) = p\}$ of G to the corresponding subgroup $D_{\Phi(p)}\overline{\tilde{G}}^{\{\Phi(p)\}} \subset GL(n, \mathbb{C})$ associated to $\tilde{G}$. Since G is large, $D_p\overline{G}^{\{p\}}$ contains $SL(n, \mathbb{C})$. When the dimension is $n \geq 2$, Lemma 1.6 immediately implies that $C = D_p\Phi$ is complex or anti-complex which finishes the proof. When the dimension is $n = 1$, the additional assumption ensures that $D_p\overline{G}^{\{p\}}$ contains at least a non-real complex scalar $\lambda \in \mathbb{C}^* \setminus \mathbb{R}^*$. Then arguments similar to (but simpler than) those contained in the proof of Lemma 1.6 show again that $D_p\Phi$ is, in fact, complex or anti-complex. □

7. Singular holomorphic foliations by curves on $\mathbb{CP}^n$

We recall the basic definitions and properties of foliations by curves on $\mathbb{CP}^n$. Proofs and details can be found in [GM,OB], [LN] or [LN,So].

A 1-dimensional (regular) holomorphic foliation $\mathcal{F}$ on a complex manifold M is given by a collection of open sets U_i covering M and equipped with regular holomorphic vector fields X_i which satisfy the compatibility condition $X_i = f_{i,j} \cdot X_j$ on $U_i \cap U_j$, where $f_{i,j} : U_i \cap U_j \to \mathbb{C}$ are holomorphic functions. The leaf L_p passing through a point $p \in M$ is the orbit of p under the pseudo-group generated on M by all the local pseudo-flows $\phi^t_{X_i}$. The leaf L_p is a connected and injectively immersed Riemann surface $i : \Gamma \hookrightarrow M$ which is tangent to X.

We denote by $\mathbf{z} = (\mathbf{z}_0, \mathbf{z}_1, \ldots, \mathbf{z}_n)$ the usual coordinate of $\mathbb{C}^{n+1}$ and by $z = (z_1, \ldots, z_n)$ the coordinate of the main affine chart of $\mathbb{CP}^n$, where $z_i = \mathbf{z}_i/\mathbf{z}_0$. Given homogeneous polynomials $\mathbf{H}_0, \mathbf{H}_1, \ldots, \mathbf{H}_n$ of degree d in $\mathbb{C}^{n+1}$, the homogeneous vector field

$$\widetilde{\mathcal{Z}} = \mathbf{H}_0(\mathbf{z})\partial_{\mathbf{z}_0} + \mathbf{H}_1(\mathbf{z})\partial_{\mathbf{z}_1} + \cdots + \mathbf{H}_n(\mathbf{z})\partial_{\mathbf{z}_n}$$

defines a regular holomorphic foliation $\widetilde{\mathcal{F}}$ by complex curves in $\mathbb{C}^{n+1} \setminus \mathrm{Sing}(\widetilde{\mathcal{Z}})$ where $\mathrm{Sing}(\widetilde{\mathcal{Z}})$ denotes the common zero set of the coefficients $\mathbf{H}_i$. The leaf $\widetilde{L}_p$ passing through a point p is the complex trajectory (integral curve) of p under the pseudo-flow of $\widetilde{\mathcal{Z}}$ (of course, $\widetilde{L}_p = \{p\}$ if, and only if, $p \in \mathrm{Sing}(\widetilde{\mathcal{Z}})$). This regular foliation as well as the singular set are invariant under the radial action of $\mathbb{C}^*$ by homotheties. Indeed, if $\Phi(\tilde{z}) = \lambda \cdot \tilde{z}$ for some $\lambda \in \mathbb{C}^*$, then $\Phi^*\widetilde{\mathcal{Z}} = \lambda^{d-1} \cdot \widetilde{\mathcal{Z}}$. Hence it induces a singular foliation by curves $\mathcal{F}$ on $\mathbb{CP}^n$. In the main affine chart, $\mathcal{F}$ is also defined by the polynomial vector field $\mathcal{Z} = \sum_{i=1}^n (\mathbf{H}_i(1, z) - z_i\mathbf{H}_0(1, z))\partial_{z_i}$. Notice that $\mathcal{Z}$ has degree $d + 1$ with radial homogeneous component of degree $d + 1$

$$\mathcal{Z} = \mathcal{Z}_0 + \mathcal{Z}_1 + \cdots + \mathcal{Z}_d + H_d \cdot \mathcal{R}$$

$$\text{where } \begin{cases} \mathcal{Z}_k \text{ is an homogeneous vector field of degree } k; \\ H_d \text{ is an homogeneous polynomial of degree } d; \\ \mathcal{R} = z_1\partial_{z_1} + \cdots + z_n\partial_{z_n} \text{ is the radial vector field.} \end{cases}$$

The singular set $\mathrm{Sing}(\mathcal{F})$ of $\mathcal{F}$ is the projective algebraic subset of tangencies between $\widetilde{\mathcal{Z}}$ and the radial vector field $\widetilde{\mathcal{R}} = \tilde{z}_0\partial_{\tilde{z}_0} + \tilde{z}_1\partial_{\tilde{z}_1} + \cdots + \tilde{z}_n\partial_{\tilde{z}_n}$ (obviously containing $\mathrm{Sing}(\tilde{\mathcal{Z}})$). The intersection of $\mathrm{Sing}(\mathcal{F})$ with the main affine chart of $\mathbb{CP}^n$ mentioned above is given by

$$\mathrm{Sing}(\mathcal{F}) = \{\mathbf{H}_1 - z_1\mathbf{H}_0 = \mathbf{H}_2 - z_2\mathbf{H}_0 = \cdots = \mathbf{H}_n - z_n\mathbf{H}_0 = 0\}\,.$$

Similarly we can express $\mathrm{Sing}(\mathcal{F})$ in the other affine charts of $\mathbb{CP}^n$.

Another homogeneous vector field $\tilde{\mathcal{Z}}'$ of degree d defines the same foliation $\mathcal{F}$ (with the same singular set $\mathrm{Sing}(\mathcal{F})$) if, and only if, $\mathcal{Z}' = \lambda \cdot \mathcal{Z}$ for a scalar $\lambda \in \mathbb{C}^*$, i.e. $\tilde{\mathcal{Z}}' = \lambda \cdot (\tilde{\mathcal{Z}} + \mathbf{H}_{d-1}\tilde{\mathcal{R}})$ for a homogeneous polynomial $\mathbf{H}_{d-1}$ of degree d. It follows that $\mathrm{Sing}(\mathcal{F}) = \mathbb{CP}^n$ if, and only if, $\mathcal{Z}$ is vanishes identically, i.e. $\tilde{\mathcal{Z}} = \mathbf{H}_{d-1} \cdot \tilde{\mathcal{R}}$ is radial. Also, $\mathrm{Sing}(\mathcal{F})$ has a codimension 1 component $\{\mathbf{H}(\tilde{z}) = 0\}$ in $\mathbb{C}^{n+1}$ if, and only if, we have $\mathbf{H} \cdot \tilde{\mathcal{Z}}' = \lambda \cdot (\tilde{\mathcal{Z}} + \mathbf{H}_{d-1}\tilde{\mathcal{R}})$. In this case, the lower degree homogeneous vector field $\tilde{\mathcal{Z}}'$ defines a foliation $\mathcal{F}'$ that coincides with $\mathcal{F}$ away from $\mathrm{Sing}(\mathcal{F})$ but has a singular set $\mathrm{Sing}(\mathcal{F}') \subset \mathrm{Sing}(\mathcal{F})$ of codimension ≥ 2. The following lemma is simple and well-known.

Lemma 7.0. *Let $\mathcal{F}$ be a regular 1-dimensional holomorphic foliation defined on the complement $\mathbb{CP}^n \setminus \mathcal{S}$ of an analytic subset of codimension ≥ 2. Then, $\mathcal{F}$ coincides with the foliation induced by a homogeneous vector field of some degree d in $\mathbb{C}^{n+1}$.* □

In the sequel, such foliation $\mathcal{F}$ will simply be called a *foliation by curves on* $\mathbb{CP}^n$. The well-defined degree (d) of the homogeneous vector field $\tilde{\mathcal{Z}}$ inducing $\mathcal{F}$ coincides with the number of tangencies between $\mathcal{F}$ and a generic projective line $L \subset \mathbb{CP}^n$. The set $\mathcal{F}^d(\mathbb{CP}^n)$ of degree-d foliations on $\mathbb{CP}^n$ is naturally identified with the projectivization of the set of degree-$(d+1)$ vector fields of the form $\mathcal{Z} = \mathcal{Z}_0 + \mathcal{Z}_1 + \cdots + \mathcal{Z}_d + H_d \cdot \mathcal{R}$ which has a singular set in $\mathbb{C}^n$ of codimension ≥ 2 (i.e. whose polynomial coefficients have no common factor). Since the dimension of the set of homogeneous polynomials of degree d in $n+1$ variables, or, equivalently, of arbitrary polynomials of degree d in n variables, has dimension $\frac{(d+n)!}{d!n!}$, it follows that $\mathcal{F}^d(\mathbb{CP}^n)$ is a Zariski-open subset of the complex projective space of dimension

$$\dim_{\mathbb{C}}(\mathcal{F}^d(\mathbb{CP}^n)) = (d+n+1)\frac{(d+n-1)!}{d!(n-1)!} - 1.$$

Example 0. When $d = 0$, all (constant) vector fields $\tilde{\mathcal{Z}}$ are conjugate under $GL(n+1, \mathbb{C})$. Thus, any foliation $\mathcal{F}$ is conjugate by a projective automorphism to the "trivial" foliation $\mathcal{F}_0$ defined in the main affine chart by the radial vector field $\mathcal{R} = z_1\partial_{z_1} + \cdots + z_n\partial_{z_n}$. The origin $\underline{0} \in \mathbb{C}^n \subset \mathbb{CP}^n$ is the unique singular point.

Example 1. For $d = 1$ (and only in this case), the (linear) vector field $\tilde{\mathcal{Z}}$ is invariant by homotheties and defines a holomorphic vector field $\mathcal{Z}$ on $\mathbb{CP}^n$. The dynamic of the underlying foliation $\mathcal{F}$ is well-known and completely understood by means of the Jordan normal form for $\tilde{\mathcal{Z}}$ (see [Ca,Ku,Pa]). Notice that the singular points of $\mathcal{F}$ correspond to the eigendirections of $\tilde{\mathcal{Z}}$. For instance, apart from the radial vector field $\lambda \cdot \tilde{\mathcal{R}}$, the linear vector fields which do not belong to $\mathcal{F}^1(\mathbb{CP}^n)$ (i.e. those that have underlying foliation of degree zero) have an irreducible eigenspace of codimension 1. The "generic foliation of degree 1" has exactly $n+1$ singular points.

Let $V \subset \mathbb{CP}^n$ be an algebraic submanifold. We say that V is an invariant set for a foliation $\mathcal{F}$ if V consists of a union of leaves and singular points for $\mathcal{F}$. For instance, the hyperplane $\mathcal{H}_\infty$ at infinity is invariant by $\mathcal{F}$ if and only if the radial component $H_d \cdot \mathcal{R}$ of the vector field $\mathcal{Z} = \mathcal{Z}_0 + \mathcal{Z}_1 + \cdots + \mathcal{Z}_d + H_d \cdot \mathcal{R}$ vanishes identically ($H_d \equiv 0$). In this case, the foliation by curves induced by $\mathcal{F}$ on $\mathcal{H}_\infty \simeq \mathbb{CP}^{n-1}$ is the degree d foliation defined by $\mathcal{Z}_d$. If we denote by $\mathcal{F}^d(\mathbb{C}^n)$ the set of those foliations of degree d tangent to the hyperplane at infinity, then $\mathcal{F}^d(\mathbb{C}^n)$ is clearly is a submanifold of $\mathcal{F}^d(\mathbb{CP}^n)$ of dimension

$$\dim_{\mathbb{C}}(\mathcal{F}^d(\mathbb{C}^n)) = (d+n)\frac{(d+n-1)!}{d!(n-1)!} - 1$$

and codimension $\frac{(d+n-1)!}{d!(n-1)!}$. It is important to notice that the degree of $\mathcal{Z}$ is d or $d+1$ depending on whether or not the $\mathcal{H}_\infty$ is invariant. In [Il1], [Il2] the set of all foliations on $\mathbb{CP}^2$ is stratified by the degree of the corresponding vector field $\mathcal{Z}$ in the affine chart. This leads to consider the fact of leaving $\mathcal{H}_\infty$ invariant as a generic property among foliations having *affine degree d*. Obviously, what is generic with respect to the degree of $\mathcal{Z}$ is not generic with respect to the degree of $\widetilde{\mathcal{Z}}$ and vice-versa. Here is another well-known fact:

Lemma 7.1. *For any degree $d \geq 2$, there is a Zariski open subset $\mathcal{U} \subset \mathcal{F}^d(\mathbb{CP}^n)$ such that any $\mathcal{F} \in \mathcal{U}$ has exactly $\frac{d^{n+1}-1}{d-1} = d^n + d^{n-1} + \cdots + d + 1$ singular points in $\mathbb{CP}^n$. Similarly there is a Zariski open subset $\mathcal{U} \subset \mathcal{F}^d(\mathbb{C}^n)$ such that any $\mathcal{F} \in \mathcal{U}$ has exactly d^n singular points in the affine chart $\mathbb{C}^n$ and $\frac{d^n-1}{d-1} = d^{n-1} + d^{n-2} + \cdots + d + 1$ singular points in the invariant hyperplane $\mathcal{H}_\infty$ at infinity.*

The first assertion is an application of the Baum-Bott formula (see [GM]). The second one is even easier. The $\frac{d^n-1}{d-1} = d^{n-1} + d^{n-2} + \cdots + d + 1$ singular points in $\mathcal{H}_\infty$ result from the first assertion applied in lower dimension. The d^n singular points in $\mathbb{C}^n$ are a consequence of the Bézout Theorem.

An invariant irreducible curve $\Gamma \subset \mathbb{CP}^n$ for a foliation $\mathcal{F}$ (which is not totally contained in the singular set) always consists of a regular leaf and finitely many singular points. Conversely, consider the closure $\overline{L_p}$ of a (regular) leaf L_p with respect to the usual topology. Clearly $\overline{L_p}$ always consists of a union of leaves and singular points. Then, we have:

Lemma 7.2. *Let L_p be a regular leaf of some foliation $\mathcal{F} \in \mathcal{F}^d(\mathbb{CP}^n)$ and assume that its closure has the form $\overline{L_p} = L_p \cup \{s_1, \ldots, s_k\}$ for a finite set of (necessarily singular) points $s_1, \ldots, s_k \in Sing(\mathcal{F})$. Assume moreover that the leaf L_p becomes an embedded Riemann surface in $\mathbb{CP}^n \setminus \{s_1, \ldots, s_k\}$ after deleting these points. Then, the closure $\overline{L_p} = L_p \cup \{s_1, \ldots, s_k\}$ is an algebraic curve.*

The second assumption means that L_p is not contained in its limit set, i.e. it does not accumulate on itself or on another regular leaf $L_{p'}$ (in the sense of foliations).

Proof. We know that $\overline{\mathcal{F}_p}$ is a smooth analytic submanifold of $\mathbb{CP}^n \setminus \{s_1, \dots, s_k\}$. Now, the Remmert-Stein Theorem implies that the extension $\Gamma = \overline{L}_p$ is also analytic in $\mathbb{CP}^n$. Finally Chow Lemma ensures it is, in fact, algebraic. □

The next lemma is rather useful and will be needed later on.

Lemma 7.3. *Let $\mathcal{F}$ be a foliation by curves on $\mathbb{CP}^n$. Then, the closure $\overline{L}_p$ of any regular leaf L_p has non-empty intersection with the hyperplane at infinity (i.e. $\mathcal{H}_\infty \cap \overline{L}_p \neq \emptyset$).*

Naturally $\overline{L}_p$ also has non-empty intersection with every hyperplane $\mathcal{H} \subset \mathbb{CP}^n$.

Proof. Let L_p be a bounded leaf in $\mathbb{C}^n$. Consider a holomorphic vector field $\mathcal{Z}$ defining the foliation $\mathcal{F}$. For any ball $\mathbb{B}^n_R$ and any $\varepsilon > 0$, the pseudo-flow ϕ^t_X is well-defined as a holomorphic family of mappings $\{|z| < \varepsilon/M\} \times \mathbb{B}^n_{R-\varepsilon} \to \mathbb{B}^n_R$ where $M = \|X\|_R$ is the supremum of $|X(z)|$ on $\mathbb{B}^n_R$. For R large enough and ε small enough, the leaf L_p is totally contained in $\mathbb{B}^n_{R-\varepsilon}$. Then, using the composition rule $\phi^t_X \circ \phi^s_X = \phi^{t+s}_X$, the previous estimate shows that the pseudo-flow restricted to L_p is indeed defined for all $t \in \mathbb{C}$. Thus, we obtain a parametrization $\mathbb{C} \to \mathcal{F}_p$; $t \mapsto \phi^t_X(p)$. However, thanks to Liouville Theorem, this parametrization must be constant and therefore L_p is a singular point. □

Let X be a germ of vector field at $\underline{0} \in \mathbb{C}^n$ with an isolated singularity at $\underline{0}$ and denote by $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ the spectrum of its linear part. We say that X is *hyperbolic* at $\underline{0}$ if none of the quotients λ_i/λ_j is real. The following result is a more accurate version of a proposition of [LN,So].

Lemma 7.4. *Let X be a germ of vector field with a hyperbolic singularity at $\underline{0} \in \mathbb{C}^n$ and denote by $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ its spectrum. Then, there are exactly n germs of irreducible invariant analytic curves $\Gamma_1, \Gamma_2, \dots, \Gamma_n$ at $\underline{0}$ where each Γ_i is smooth and tangent to the eigendirection corresponding to λ_i. If another leaf L has $\underline{0}$ in its closure $\underline{0} \in \overline{L}$, then it accumulates exactly on two invariant curves $\overline{L} = L \cup \Gamma_i \cup \Gamma_j$.*

Proof. It is proved in [Ch] that a hyperbolic singular point is always topologically linearizable. In other words, there exists a local homeomorphism $\Phi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ sending the foliation $\mathcal{F}$ defined by X to the one defined by $X_0 = \lambda_1 z_1 \partial_{z_1} + \lambda_2 z_2 \partial_{z_2} + \cdots + \lambda_n z_n \partial_{z_n}$. In fact, it is actually shown that the pseudo-flows are conjugate, but this will not be needed. The pull-back through Φ of the coordinate axis correspond to n special leaves $L_1, \dots, L_n$ of the local foliation $\mathcal{F}$ whose closure $\Gamma_i = L_i \cup \{\underline{0}\}$ is a germ of analytic curve (here, we use Remmert-Stein Theorem). In the linearizing coordinates, we consider the leaf passing through a point $p = (z_1, z_2, \dots, z_n)$ which does not belong to the hyperplane $\{z_1 = 0\}$ and denote by $I(p) \subset \{1, 2, \dots, n\}$ the set of indices where the corresponding coordinate of p is not zero.

Then L is parametrized by $t \mapsto (e^{t\lambda_1} z_1, e^{t\lambda_2} z_2, \dots, e^{t\lambda_n} z_n)$ and $\underline{0}$ is in the closure of L if, and only if, the corresponding set $\Lambda = \{\lambda_i \ ; \ i \in I\}$ lies in some half-plane $\Lambda(p) \subset \{\theta < \arg(\lambda) < \theta + \pi\}$. Then, considering the holonomy map of Γ, namely $(z_2, \ldots, z_n) \mapsto (e^{2i\pi\lambda_2/\lambda_1} z_2, \ldots, e^{2i\pi\lambda_n/\lambda_1} z_n)$, it also follows that Γ_1 is in the closure of L if, and only if, λ_1 has extremal argument in $\Lambda(p)$, i.e. either $\Lambda(p) \subset \{\arg(\lambda_1) \leq \arg(\lambda) < \arg(\lambda_1) + \pi\}$, or $\Lambda(p) \subset \{\arg(\lambda_1) - \pi < \arg(\lambda) \leq \arg(\lambda_1)\}$. Repeating this discussion for all the eigenvalues λ_i, we arrive to the following conlusion. Given a point $p \neq \underline{0}$, define $I(p)$ and $\Lambda(p)$ as above. Then, $\underline{0}$ is in the closure $\overline{L}$ of L if, and only if, $\Lambda(p)$ lies in some half-plane of $\mathbb{C}$ (i.e. in Poincaré domain) and $\overline{L} = L \cup \Gamma_{i_1} \cup \Gamma_{i_2}$ where $\lambda_{i_1}, \lambda_{i_2} \in \Lambda(p)$ are the eigenvalues having extremal argument: $\Lambda(p) \subset \{\arg(\lambda_{i_1}) \leq \arg(\lambda) \leq \arg(\lambda_{i_2})\}$ with $\lambda_{i_2} - \lambda_{i_1} < \pi$. In particular, X lies in Poincaré domain if, and only if, every leaf accumulates on $\underline{0}$. □

It should be noted that the domain of definition of Chaperon's homeomorphism Φ used in the preceding proof cannot be uniformly chosen with respect to a local deformation of the vector field X. Nevertheless, an alternate construction of the invariant curves Γ_i viewed as intersections of the stable manifolds of some elements of the pseudo-flow ϕ_X^t provides the following stability result (see [LN,So] for details).

Lemma 7.5. *Consider a foliation $\mathcal{F}_0 \in \mathcal{F}^d(\mathbb{CP}^n)$ having a hyperbolic singular point $p_0 \in \mathbb{CP}^n$ and consider one of its local invariant curves Γ_0. Then there exists a neighborhood $\mathcal{U}$ of $\mathcal{F}_0$ in $\mathcal{F}^d(\mathbb{CP}^n)$ and a holomorphic map*

$$\phi : \Delta \times \mathcal{U} \to \mathbb{CP}^n$$

such that the image $\phi(\Delta, \mathcal{F}_0)$ of the unit disc $\Delta \in \mathbb{C}$ coincides with Γ_0 and $\phi(\Delta, \mathcal{F})$ coincides, for every $\mathcal{F} \in \mathcal{U}$, with a local invariant curve Γ of $\mathcal{F}$ through the persistent hyperbolic singularity $p = \widetilde{\phi}(0, \mathcal{F})$.

An application of Lemma 7.4 is the:

Proposition 7.6. *Consider a foliation $\mathcal{F} \in \mathcal{F}^d(\mathbb{C}^n)$ with a finite singular set $Sing(\mathcal{F})$. Assume moreover that all the singularities contained in the invariant hyperplane $\mathcal{H}_\infty$ at infinity are hyperbolic. If the closure $\overline{L}_p$ of a leaf L_p has finite intersection with $\mathcal{H}_\infty$, then $\overline{L}_p$ is an algebraic curve.*

Proof. If L_p is analytic in $\mathbb{CP}^n \setminus \mathrm{Sing}(\mathcal{F})$, then we just need to apply Remmert-Stein's theorem. So we assume for a contradiction that L_p is not analytic at a point q which is regular for $\mathcal{F}$. Therefore L_p accumulates on every point of the leaf L_q (in the sense of foliations, note that the possibility of having $L_p = L_q$ is not excluded). By virtue of Lemma 7.3, the closure $\overline{L}_q$ intersects $\mathcal{H}_\infty$ at some point q' and this point is therefore an accumulation point of L_p. When q' is smooth for $\mathcal{F}$, the closure $\overline{L}_p$ has an infinite intersection with $\mathcal{H}_\infty$ on a neighborhood of q' as it can be seen by considering a local trivialization of $\mathcal{F}$ at q' (the case $L_q \subset \mathcal{H}_\infty$ is trivial).

Now, assume that q' is a (hyperbolic) singular point of $\mathcal{F}$. Then, q' is also hyperbolic for the restricted foliation $\mathcal{F}|_{\mathcal{H}_\infty}$ and $\mathcal{H}_\infty$ contains $n-1$ local invariant curves at q'. The n^{th} local invariant curve is transverse to $\mathcal{H}_\infty$ at q'. If L_q is this transverse invariant curve, then L_p also accumulates on another invariant curve (cf. Lemma 7.4) which should be contained in $\mathcal{H}_\infty$. It follows a contradiction. If L_q is not the local transverse invariant curve, it still accumulates on q' and (cf. Lemma 7.4) on two local invariant curves. One of them is contained in $\mathcal{H}_\infty$ and is accumulated by L_p as well. In any case, we obtain a contradiction establishing the lemma. □

The following lemma will also be needed and can be proved by similar arguments. We omit the proof.

Lemma 7.7. *Consider a foliation $\mathcal{F} \in \mathcal{F}^d(\mathbb{CP}^n)$ with a finite singular set Sing($\mathcal{F}$). Assume that the closure $\overline{L}_p$ of a leaf L_p has 1-dimensional analytic intersection with a neighborhood U of the hyperplane at infinity $\mathcal{H}_\infty$. Then $\overline{L}_p$ is an algebraic curve in $\mathbb{CP}^n$.* □

Finally, the main result of [LN,So] is the:

Theorem 7.8 (Lins Neto, Soares). *Given $n, d \geq 2$, there exists a Zariski open subset $\mathcal{U}_7 \subset \mathcal{F}^d(\mathbb{CP}^n)$ such that any $\mathcal{F} \in \mathcal{U}_7$ satisfies:*

(i) *$\mathcal{F}$ has exactly $\frac{(d+n)!}{d!n!}$ hyperbolic singularities and is regular on the complement;*
(ii) *$\mathcal{F}$ has no invariant algebraic curve.*

8. The holonomy pseudo-group near a special family of degree d foliations

Let us describe a family of foliations coming from the world of linear differential equations for which the holonomy may be computed. The foliations of Theorem A will be obtained as perturbations of foliations in this family. The main result of this section will enable us to "prescribe" the linear part of the holonomy group of the foliation obtained by perturbation. For the sake of notations, we shall work with $\mathbb{CP}^{n+1}$ ($n \geq 1$) instead of $\mathbb{CP}^n$, denoting by (w, z) the coordinates in the main affine chart $\mathbb{C}^{n+1}$ where $z = (z_1, \ldots, z_n)$. In this way, the first coordinate is distinguished. Given $d \geq 2$, we fix pairwise distinct scalars $w_1, \ldots, w_d \in \mathbb{C}$ and consider the family of rational vector fields

$$X(M_1, \ldots, M_d) = \partial_w + \sum_{k=1}^{d} \frac{M_k}{w - w_k} z\partial_z \ , \quad M_1, \ldots, M_d \in M(n, \mathbb{C}) \ .$$

Here, the notation $Mz\partial_z$ has to be understood as $\sum_{i,j} m_{i,j} z_j \partial_{z_i}$ where $M = (m_{i,j})$. Denote by $\mathcal{F}(M_1, \ldots, M_d)$ the foliation induced on $\mathbb{CP}^{n+1}$. We want to describe some dynamical features of foliations close to $\mathcal{F}_I = \mathcal{F}(I, \ldots, I)$ where $I \in GL(n, \mathbb{C})$ is the identity matrix.

Lemma 8.1. *Assume that $M_1, \ldots, M_d \neq (0)$ and $M_1 + \cdots + M_d \neq I$. Then the foliation $\mathcal{F} = \mathcal{F}(M_1, \ldots, M_d)$ has projective degree d, is tangent to the projective line $L_0 : \{z = 0\}$ and has $d + 1$ isolated singularities $p_k = (w_k, \underline{0})$, $k = 1, \ldots, d$, and $p_{d+1} = (\infty, \underline{0})$ belonging to L_0. Furthermore on a neighborhood* *of a singularity p_k, the foliation is defined by a holomorphic vector field whose linear part at p_k is respectively given (in matricial notation) by*

$$\begin{pmatrix}1 & 0\\ 0 & M_1\end{pmatrix}, \quad \dots, \quad \begin{pmatrix}1 & 0\\ 0 & M_d\end{pmatrix}, \quad \begin{pmatrix}1 & 0\\ 0 & I - M_1 - \cdots - M_d\end{pmatrix}.$$

Finally the hyperplanes $\{w = w_1\}, \dots, \{w = w_d\}$ and the hyperplane at infinity $\{w = \infty\}$ are all tangent to the foliation and intersect over a degenerate codimension 2 singularity at infinity. There are no other singularities.

Denote by $\mathcal{V}$ the (smooth) complex submanifold of $\mathcal{F}^d(\mathbb{CP}^{n+1})$, which is parametrized by $(M_1, \dots, M_d) \mapsto \mathcal{F}(M_1, \dots, M_d)$, satisfying the assumptions of Lemma 8.1 above. Note that the dimension of this submanifold is dn^2.

Proof. The conditions $M_1, \dots, M_d \neq 0$ and $M_1 + \cdots + M_d \neq I$ imply that the polynomial vector field of degree $d \prod_{k=1}^{d}(w - w_k) \cdot X$, where $X = X(M_1, \dots, M_d)$, is irreducible. Also the homogeneous part of degree d of the polynomial vector field in question is not a multiple of the the radial vector field $w\partial_w + z\partial_z$. Using new projective coordinates $(t = 1/w, \tilde{z} = z/w)$, the foliation is defined by the rational vector field

$$t\partial_t + \left(I - \sum_{k=1}^{d} \frac{M_k}{1 - w_k t}\right)\tilde{z}\partial_{\tilde{z}}.$$

This proves the lemma. □

A foliation $\mathcal{F} \in \mathcal{V}$ possesses a *holonomy* (or *monodromy*) *group* associated to the leaf L_0. Note that we have already mentioned “local” holonomies relative to hyperbolic singularities. However the holonomy group in question has a more global nature and we shall brief recall its construction. This construction will be extended after the next proposition so as to include small perturbation of $\mathcal{F}$. Fix a point $p_0 = (w_0, \underline{0})$ in the leaf $L_0^* = L_0 \setminus \{p_1, \dots, p_{d+1}\}$ and consider the vertical affine hyperplane $\Sigma = \{w = w_0\}$. For a very “small” complex time T, the complex flow ϕ_X^T is well-defined on some affine tubular neighborhood of Σ and induces a linear map $\Sigma \to \{w = w_0 + T\}$. This map is also given by integrating the non-autonomous vector field $Y(t) = \sum_{k=1}^{d} \frac{M_k}{w+t-w_k} z\partial_z$ over the segment $t \in [w, w+T]$. Analogously, given any smooth loop $\gamma : [0,1] \to L_0^*$ with extremities at $\gamma(0) = \gamma(1) = p_0$, we define a linear map $f_\gamma : \Sigma \to \Sigma$ by integrating the non-autonomous vector field $Y(t) = \sum_{k=1}^{d} \frac{M_k}{\gamma(t)-w_k} z\partial_z$ over $[0, 1]$. The resulting map f_γ depends only on the homotopy class of γ in the fundamental group $\pi_1(L_0^*, p_0)$. Now, choose a collection $\gamma_1, \dots, \gamma_d : [0,1] \to L_0^*$ of generators for $\pi_1(L_0^*, p_0)$ so that each γ_k has index 1 around p_k, is homotopic to 0 in $L_0^* \cup \{p_k\}$, and, moreover, $\gamma_1 \cdot \gamma_2 \cdot \cdots \cdot \gamma_d$ is homotopic to 0 in $L_0^* \cup \{p_\infty\}$. Finally, denote by $A_k \in GL(n, \mathbb{C})$ the matrix defining $f_{\gamma_k} : \Sigma \to \Sigma$.

Proposition 8.2. *The holomorphic mapping from $\mathcal{V}$ to $(GL(n,\mathbb{C}))^d$, $d \geq 2$, given by*

$$\begin{cases} \mathcal{F} = \mathcal{F}(M_1, \dots, M_d) \mapsto (A_1, \dots, A_d); \\ \mathcal{F}_I = \mathcal{F}(I, \dots, I) \quad \mapsto \quad (I, \dots, I), \end{cases}$$

is a local diffeomorphism at $\mathcal{F}_0$.

We have discovered later that the proposition above is a variant of the famous 1920's Lappo-Danilevskii's affirmative answer to the Riemann-Hilbert problem in the case where the monodromy representation is close to the identity.

Proof. Because the M_k's are close to I, each of the $d+1$ singularities of the initial vector field $X = X(M_1, \dots, M_d)$ lying in the projective line L_0 are non-resonant in Poincaré domain (see [Ar,Il], p. 72). Hence they can be linearized by a local holomorphic change of coordinates. Using these linearizations, it becomes clear that each f_{γ_k} is locally conjugate to the (local) holonomy of the corresponding linear vector field. In other words, there is $g_k \in \mathrm{Diff}(\mathbb{C}^n, 0)$ such that $f_{\gamma_k} = g_k^{-1} e^{2i\pi M_k} g_k$. However, since f_{γ_k} is linear, the previous equation still holds when g_k is replaced by its linear part $D_{\underline{0}} g_k = B_k \in GL(n, \mathbb{C})$. Thus

$$A_k = B_k^{-1} e^{2i\pi M_k} B_k \quad \text{for a suitable} \quad B_k \in GL(n, \mathbb{C}), \quad k = 1, \dots, d.$$

It should be noted that the transition matrices B_k are generally distinct, each of them depending non-linearly on all the coefficients (entries) of M_k so that they cannot be explicited.

Nevertheless, when only one of the M_k differs from the identity by exactly one of its entries, the A_k are computable as follows. Let $k_0 \in \{1, \dots, d\}$ and $i_0, j_0 \in \{1, \dots, n\}$. Let also $M_k = I$ for $k \neq k_0$ and $M_{k_0} = I + t\delta_{i_0, j_0}$ where δ_{i_0, j_0} stands for the Kronecker matrix. In the main affine chart and away from the poles $\{w = w_k\}$, $k = 1, \dots, d$, the integral curves of the vector field $X(M_1, \dots, M_d)$ are locally parametrized by $w \mapsto (w, z(w))$ where the functions $z_i(w)$ satisfy the following system of differential equations

$$\begin{cases} \frac{dz_i}{dw} = \sum_{k=1}^{d} \frac{z_i}{w - w_k}, \quad i \neq i_0 \\ \frac{dz_{i_0}}{dw} = \sum_{k=1}^{d} \frac{z_{i_0}}{w - w_k} + t \frac{z_{j_0}}{w - w_{k_0}} . \end{cases}$$

Beginning with initial data $p = (w_0, z(w_0))$, a direct integration gives for $i \neq i_0$

$$z_i(w) = z_i(w_0) \cdot \prod_{k=1}^{d} \left(\frac{w - w_k}{w_0 - w_k} \right),$$

and for $i_0 = j_0$

$$z_{i_0}(w) = z_{i_0}(w_0) \cdot \left(\prod_{k \neq k_0} \left(\frac{w - w_k}{w_0 - w_k} \right) \right) \cdot \left(\frac{w - w_{k_0}}{w_0 - w_{k_0}} \right)^{1+t} .$$

In this last case (i.e. $i_0 = j_0$), we obtain by continuation over γ_k

$$\begin{cases} A_k = I, \quad k \neq k_0 \\ A_{k_0} = \begin{pmatrix} I^{i_0 - 1} & & 0 \\ & e^{2i\pi t} & \\ 0 & & I^{n - i_0} \end{pmatrix}. \end{cases}$$

However, if $j_0 \neq i_0$, then the differential equation satisfied by $z_{i_0}(w)$ becomes

$$\frac{dz_{i_0}}{dw} = \sum_{k=1}^{d} \frac{z_{i_0}}{w - w_k} + t\frac{z_{j_0}(w_0)}{w - w_{k_0}} \prod_{k=1}^{d} \left(\frac{w - w_k}{w_0 - w_k} \right).$$

Replacing $z_{i_0}(w) = c(w) \cdot \prod_{k=1}^{d} (\frac{w-w_k}{w_0-w_k})$ in the last equation with initial value $c(w_0) = z_{i_0}(w_0)$, the function $c(w)$ may be computed by a direct integration providing

$$z_{i_0}(w) = \left[z_{i_0}(w_0) + tz_{j_0}(w_0) \log \left(\frac{w - w_{k_0}}{w_0 - w_{k_0}} \right) \right] \prod_{k=1}^{d} \left(\frac{w - w_k}{w_0 - w_k} \right).$$

Therefore, by analytic continuation over γ_k, we obtain

$$\begin{cases} A_k = I, \quad k \neq k_0, \\ A_{k_0} = I + 2i\pi t \delta_{i_0, j_0}. \end{cases}$$

These computations mean that the two holomorphic maps

$$\begin{aligned} (M_1, \ldots, M_d) &\mapsto (A_1, \ldots, A_d) \\ (M_1, \ldots, M_d) &\mapsto (e^{2i\pi M_1}, \ldots, e^{2i\pi M_d}) \end{aligned}$$

do coincide near $\mathcal{F}_I$ along the coordinate axis (relative to the parametrization given by Kronecker matrices) and then that they are tangent at $\mathcal{F}_I$. □

Now, we show that the construction of the return maps f_k remains valid in the context of holonomy pseudo-groups of arbitrary foliations $\mathcal{F}$ sufficiently close to the family $\mathcal{V}$. Fix $\mathcal{F}_1 \in \mathcal{V}$ and $R > 0$. Consider a foliation $\mathcal{F} \in \mathcal{F}^d(\mathbb{CP}^{n+1})$ very close to $\mathcal{F}_1$. Then all the hypersurfaces $\{(w, z) \, ; \, w = \gamma_k(t), \, z \in \mathbb{B}^n_R\}$ are transverse to $\mathcal{F}$ for $t \in [0, 1]$ and $k = 1, \ldots, d$. Indeed, this is clearly true for $\mathcal{F}_1$. Since we can uniformly control the dependence of the slope of the leaves of $\mathcal{F}$ on compact sets where the foliation is regular, this transversality property is persistent for small perturbations of $\mathcal{F}_1$. Consider the immersion

$$\Sigma_R : \mathbb{B}^n_R \hookrightarrow \mathbb{CP}^{n+1} \; ; \; z \mapsto (w_0, z)$$

of the ball $\mathbb{B}^n_R$ of radius $r > 0$ in the vertical affine hyperplane $\Sigma \subset \mathbb{C}^{n+1}$. Since no misunderstanding is possible, we also denote by Σ_R the image $\Sigma_R(\mathbb{B}^n_R) \subset \mathbb{C}^{n+1}$. A simple compactness argument shows that, for sufficiently small r ($0 < r < R$) and for an arbitrary point $p \in \Sigma_r$, the path γ_k can be lifted in the leaf L_p through p (with respect to the transverse fibration above) as a path $\gamma_{k,p}$ verifying $\gamma_{k,p}(0) = p$ and $\gamma_{k,p}(1) \in \Sigma_R$. This allows us to define the *return map* f_k *around the singularity* p_k (relative to the choices of the homotopy classes $\gamma_1, \ldots, \gamma_d$ and r, R) without ambiguity by

$$f_{k,\mathcal{F}} : \Sigma_r \to \Sigma_R \; ; \; p \mapsto \gamma_{k,p}(1).$$

Moreover, these return maps depend holomorphically on $\mathcal{F}$. In particular, if $\mathcal{F}$ happens to belong to $\mathcal{V}$, then the maps $f_{k,\mathcal{F}}$ coincide with the restriction to Σ_r of the original linear return maps constructed at the beginning of the present section.

Set $r = 1$ and $R = 2\sup_k \|f_{k,\mathcal{F}_1}\|_1$. Since the return maps of foliation in $\mathcal{V}$ are globally defined, we can suppose that the return maps $f_{k,\mathcal{F}}$ are well-defined and one-to-one on the unit ball $\Sigma_1 \simeq \mathbb{B}^n$ provided that $\mathcal{F}$ is sufficiently close to $\mathcal{F}_1 \in \mathcal{V}$. Repeating this construction for every $\mathcal{F}_1 \in \mathcal{V}$, we obtain the following lemma:

Lemma 8.3. *There exists a neighborhood $\mathcal{U} \subset \mathcal{F}^d(\mathbb{CP}^{n+1})$ containing $\mathcal{V}$, such that the return maps constructed above are well-defined and injective on the unit ball $f_{k,\mathcal{F}} : \mathbb{B}^n \hookrightarrow \mathbb{C}^n$ for every $\mathcal{F} \in \mathcal{U}$. Furthermore they depend holomorphically on $\mathcal{F}$.* □

Now, denote by

$$A_{k,\mathcal{F}} = D_{\underline{0}} f_{k,\mathcal{F}} \in GL(n, \mathbb{C})$$

the differential at $\underline{0} \in \mathbb{B}^n_r \simeq \Sigma_r$ of the return map $f_{k,\mathcal{F}}$. The matrices $A_{k,\mathcal{F}}$ depend also holomorphically on $\mathcal{F}$ and an immediate consequence of Proposition 8.2 is the:

Corollary 8.4. *The holomorphic mapping*

$$\begin{cases} \mathcal{U} \to & (GL(n,\mathbb{C}))^d \\ \mathcal{F} \mapsto & (A_{1,\mathcal{F}}, \ldots, A_{d,\mathcal{F}}) \end{cases}$$

defined by the differential of the return maps as above is a submersion at $\mathcal{F}_I$. □

In other words, for $\mathcal{F}$ sufficiently close to $\mathcal{F}_I$ in $\mathcal{V}$, or more generally in $\mathcal{F}^d(\mathbb{CP}^{n+1})$, the linear parts of all the return maps $f_{k,\mathcal{F}}$ may be deformed arbitrarily and independently by a deformation of $\mathcal{F}$.

In the sequel, we will denote by $G_{\mathcal{F}}$ the pseudo-group generated on $\mathbb{B}^n$ by the $f_{k,\mathcal{F}}$. Given a point $p \in \Sigma_1$, the pseudo-orbit of p under $G_{\mathcal{F}}$ is totally contained in the intersection $L_p \cap \Sigma_1$ of the corresponding leaf with the transversal. Thus L_p is dense in a neighborhood of Σ_1 provided that the pseudo-orbit of p is dense in Σ_1. Next we are going to construct an open set of foliations close to $\mathcal{F}_I$ which have dense leaves on a certain open set. In Sect. 9, it will be shown how to ensure that the leaves are, in fact, dense in the entire $\mathbb{CP}^{n+1}$.

Consider $\mathcal{F}_{\alpha I} = \mathcal{F}(\alpha I, \ldots, \alpha I) \in \mathcal{V}$ where $I \in GL(n,\mathbb{C})$ denote the identity matrix and $\alpha \in \mathbb{C}$ is a scalar belonging to the upper half-plane $\Im m(\alpha) > 0$. The associated return maps can be computed by explicitly integration in this case and all of them coincide with the linear contraction $A_{k,\mathcal{F}_{\alpha I}} = \lambda I$, $\lambda = e^{2i\pi\alpha}$. Assume that $\mathcal{F}_{\alpha I}$ is sufficiently close to $\mathcal{F}_I$ (i.e. α is close to 1) so that the map $\mathcal{F} \mapsto (A_{1,\mathcal{F}}, \ldots, A_{d,\mathcal{F}})$ of Corollary 8.4 is still a submersion on the neighborhood of $\mathcal{F}_{\alpha I}$ and the contrations $A_{k,\mathcal{F}_{\alpha I}} = \lambda I$ are close to the identity.

Proposition 8.5. *If $\mathcal{F}_{\alpha I}$ as before is sufficiently close to $\mathcal{F}_I$, then there exists an open neighborhood $\mathcal{U}_8 \subset \mathcal{F}^d(\mathbb{CP}^{n+1})$ of $\mathcal{F}_{\alpha I}$ such that any $\mathcal{F} \in \mathcal{U}_8$ satisfy the following alternative:*

- *either the pseudo-group $G_{\mathcal{F}}$ has a fixed point in $\mathbb{B}^n$ and then, $\mathcal{F}$ has an invariant projective line $L_{0,\mathcal{F}}$ close to L_0,*
- *or the pseudo-group $G_{\mathcal{F}}$ accumulate a non-trivial (real) pseudo-flow on $\mathbb{B}^n$.*

Moreover, there is a (real analytic) Zariski-open subset $\mathcal{U}_8' \subset \mathcal{U}_8$ such that the pseudo-group $G_{\mathcal{F}}$ of any $\mathcal{F} \in \mathcal{U}_8'$ has respectively large linear or affine part on $\mathbb{B}^n$.

Proof. The $d+1$ singular points $p_1, \ldots, p_{d+1}$ of $\mathcal{F}_{\alpha I}$ belonging to L_0 are hyperbolic. At each of these points p_k, the foliation admits exactly $n+1$ local invariant curves. For a sufficiently small ball W_k centered at p_k, denote by $L_k = L_0 \cap W_k$ the local invariant curve contained in L_0.

Given a foliation $\mathcal{F}$ sufficiently close to $\mathcal{F}_{\alpha I}$, those $d+1$ hyperbolic singularities will persist as singularities $p_{1,\mathcal{F}}, \ldots, p_{d+1,\mathcal{F}}$ of $\mathcal{F}$ (cf. Lemma 7.5) and we denote by $L_{k,\mathcal{F}}$ the corresponding persistent invariant curves in W_k. The persistent fixed point of the k^{th} return map $f_{k,\mathcal{F}}$ within $\mathbb{B}^n$ necessarily corresponds to the intersection with Σ_1 of the leaf $L_{k,\mathcal{F}}$. Then, if the unique fixed point of $f_{1,\mathcal{F}}$, is also fixed by the other return maps, this means that the branches $L_{k,\mathcal{F}}$ are parts of a common leaf which turns out to be an embedded sphere close to L_0 and hence a projective line.

On the other hand, if one of the return maps $f_{k,\mathcal{F}}$ does not fix any longer the unique fixed point of $f_{1,\mathcal{F}}$, then we apply Proposition 2.0 to $f = f_{1,\mathcal{F}}$ and $g = (f_{k,\mathcal{F}})^{-1} \circ f_{1,\mathcal{F}}$ and the alternative is proved. The last assertion follows immediately from Corollary 5.2. □

9. Construction of minimal foliations on $\mathbb{CP}^{n+1}$

We keep the notations of Sect. 8 and start with the simpler 2-dimensional case. Given a foliation $\mathcal{F}$ and an open set $\mathcal{U} \subset \mathcal{F}^d(\mathbb{CP}^{n+1})$, we say that $\mathcal{U}$ *approximates* $\mathcal{F}$ if $\mathcal{F}$ lies in the boundary of $\mathcal{U}$.

Theorem 9.1. *Given $d \geq 2$, there exists an open subset $\mathcal{U} \subset \mathcal{F}^d(\mathbb{CP}^2)$ (i.e. $n = 1$) approximating $\mathcal{F}_I$ such that every foliation $\mathcal{F} \in \mathcal{U}$ satisfies:*

(i) *$\mathcal{F}$ has exactly $\frac{(d+2)(d+1)}{2}$ hyperbolic singularities and is regular on the complement,*
(ii) *every leaf of $\mathcal{F}$ is dense in the whole of $\mathbb{CP}^2$,*
(iii) *the closure $\overline{G}_{\mathcal{F}}$ of the holonomy pseudo-group $G_{\mathcal{F}}$ is transitive on $\Sigma_1 \simeq \Delta$.*

In fact, since the holonomy pseudo-group of a foliation $\mathcal{F} \in \mathcal{A}^d$ contains contractions arbitrarily close to 1 (cf. Introduction and [Il1]), the following construction can be carried out on a neighborhood of any $\mathcal{F} \in \mathcal{A}^d$. Since $\mathcal{A}^d$ is dense in $\mathcal{F}^d(\mathbb{C}^2)$, it follows that *the subset of those foliations $\mathcal{F} \in \mathcal{F}^d(\mathbb{CP}^2)$ satisfying properties (i), (ii) and (iii) above contains an open set $\mathcal{U} \subset \mathcal{F}^d(\mathbb{CP}^2)$ approximating the whole of $\mathcal{F}^d(\mathbb{C}^2)$*. In other words, the intersection $\mathcal{U} \cap \mathcal{F}^d(\mathbb{C}^2)$ is open and dense in $\mathcal{F}^d(\mathbb{C}^2)$.

Proof. We keep the notations of Proposition 8.5. There exists a neighborhood W_0 a compact part of $L_0 \setminus (\bigcup_{k=1}^{d+1} L_k)$ such that any leaf intersecting W_0 will also meet the transversal Σ. The existence of W_0 can easily be established by considering a finite covering by trivialization boxes. For each $k = 1, ldots, d+1$, recall that W_k is a small ball around p_k as in the proof of Proposition 8.5. Without loss of generality we can suppose that $W = W_0 \cup \bigcup_{k=1}^{d+1} W_k$ defines a neighborhood of L_0. Because of the hyperbolicity of $p_{k,\mathcal{F}}$, the "horizontal and vertical" invariant curves $L_{k,\mathcal{F}}$ and $\Gamma_{k,\mathcal{F}}$ in W_k depend holomorphically on $\mathcal{F}$ (where $L_{k,\mathcal{F}_{\alpha I}} = L_0 \cap W_k$ and $\Gamma_{k,\mathcal{F}_{\alpha I}} = \{w = w_k\} \cap W_k$). Hence we can assume that $L_{k,\mathcal{F}}$ intersects W_0. Since $p_{k,\mathcal{F}}$ is non-resonant in Poincaré domain, we can also suppose that this singularity is linearizable in the whole of W_k for every $\mathcal{F}$ close to $\mathcal{F}_{\alpha I}$. Thus any leaf in W_k, other than $\Gamma_{k,\mathcal{F}}$, will accumulate on $L_{k,\mathcal{F}}$, and hence intersect W_0 as well. As a consequence it will, indeed, meet Σ. Using Lemma 7.7, we then obtain the following alternative: *any leaf L of $\mathcal{F}$ either has algebraic closure or meets Σ.*

If $\mathcal{F}$ lies in the Zariski-open subset $\mathcal{U}_7 \subset \mathcal{F}^d(\mathbb{CP}^2)$ given by Theorem 7.8, then every leaf L of $\mathcal{F}$ meets Σ and is captured by the dynamics generated by the return maps $f_{1,\mathcal{F}}, \dots, f_{d,\mathcal{F}}$ on Σ. Furthermore the second alternative of Proposition 8.5 has to occur. In addition, if $\mathcal{F} \in \mathcal{U}_8'$, then the closure of $G_{\mathcal{F}}$ possesses many translation pseudo-flows. It results that $G_{\mathcal{F}}$ acts minimally on Σ. Thus, *if $\mathcal{F} \in \mathcal{U}_7 \cap \mathcal{U}_8'$, then any leaf L of $\mathcal{F}$ is dense in a neighborhhod of Σ.*

We prove that the leaves are, in fact, dense in $\mathbb{CP}^2$ in the following way. Given a leaf L and a regular point $p \in \mathbb{CP}^2$ of $\mathcal{F}$, denote by L' the leaf passing through p and by $\gamma(t)$ a path in L' joining $\gamma(0) = p$ to $\gamma(1) \in L' \cap \Sigma$. We see that L must accumulate on $\gamma(1)$. By using a simple argument involving flow-boxes along γ, one easily concludes that L accumulates on p as well. So we just need to set $\mathcal{U} \subset \mathcal{U}_7 \cap \mathcal{U}_8'$. The theorem is proved. □

The only reason for which the previous proof cannot immediately be adapted to the general case is that, for $n \geq 2$, there is no reason why an arbitrary leaf must accumulate on the line L_0 and hence meet Σ. Thus we are not able to deduce that the leaves are globally dense. Hidden behind the recursive proof below (Theorem 9.3) is the idea that, for a foliation in $\mathbb{CP}^{n+1}$ tangent to a projective flag

$$L_0 = \mathcal{H}^1 \subset \mathcal{H}^2 \subset \cdots \subset \mathcal{H}^n$$

(where $\mathcal{H}^i$ stands for some $i-$dimensional linear projective space), we can ensure that every leaf has to accumulate on $\mathcal{H}^n$ (Lemma 7.3). With a few restrictions on the foliation, we may conclude that L is either contained in an algebraic curve, or actually accumulate on a regular part of $\mathcal{H}^n$ and thus on a leaf L' belonging to $\mathcal{H}^n$. On the other hand, the leaf L' (and thus L) will accumulate on $\mathcal{H}^{n-1}$, again by the Lemma 7.3. Proceeding inductively we eventually conclude that any leaf either accumulates on L_0, or gives rise to an invariant algebraic curve. Then, by perturbing the original foliation, we can destroy the invariant flag as well as all the invariant curves. The difficulty in this construction is to ensure that the leaves still intersect a neighborhood of L_0 after the perturbation. Precisely, we have to pay particular attention to the possible invariant curves which could give rise, after perturbation, to a set of leaves staying far away from L_0. In order to do this, we shall need a last easy lemma:

Lemma 9.2. *Let $A \in GL(n, \mathbb{C})$ be a hyperbolic matrix (i.e. all eigenvalues λ_i satisfy $|\lambda_i| \neq 1$). Consider $R > r > 0$ such that both $A(\mathbb{B}_r^n)$, $A^{-1}(\mathbb{B}_r^n)$ are contained in $\mathbb{B}_R^n$. Let $g : \mathbb{B}_r^n \hookrightarrow \mathbb{C}^n$ be a holomorphic mapping sufficiently close to A on $\mathbb{B}_r^n$. Then the following holds:*

(1) *g has a unique fixed point in $\mathbb{B}^n_r$ which will be denoted by p_g.*
(2) *If $z \in \mathbb{B}^n_r \setminus \{p_g\}$, then there exists $n_z \in \mathbb{Z}$ such that $g^{n_z}(z)$ is well-defined and lies in $\mathbb{B}^n_{4R/3} \setminus \mathbb{B}^n_{3r/4}$.*

Proof. The persistence of a hyperbolic fixed point for g as well as its smooth dependence on parameters is rather well-known (cf. for example [Ar,Il]). Thus we just need to prove the assertion 2. A version with parameters of Grobman-Hartman linearization theorem would be sufficient for the conclusion, but we have not found such statement in the literature. In any case, we give a direct and elementary proof. Modulo conjugating g by a translation close to the identity, we may assume without loss of generality $p_g = \underline{0}$ and g close to A. Note also that $A' = D_{\underline{0}} g$ is also close to A.

By assumption, every point $z \in \mathbb{B}^n_r$ distinct from $\underline{0}$ escapes from $\mathbb{B}^n_r$ and intersects $\mathbb{B}^n_R \setminus \mathbb{B}^n_r$ through positive or negative iteration of A. By a compactness argument, there is $N \in \mathbb{N}$ such that for every $z \in \mathbb{B}^n_r \setminus \mathbb{B}^n_{r/2}$, the truncated orbit $\{A^k(z) \; ; \; -N \leq k \leq N\}$ intersects $\mathbb{B}^n_R \setminus \mathbb{B}^n_r$. If g was sufficiently close to A on $\mathbb{B}^n_r$, then the truncated orbit $\{g^k(z) \; ; \; -N \leq k \leq N\}$ (or the well-defined part of this pseudo-orbit) still intersects the larger domain $\mathbb{B}^n_{4R/3} \setminus \mathbb{B}^n_{3r/4}$.

Next, observe that the map $g_k(z) = (\frac{3}{2})^k g((\frac{2}{3})^k z)$, $k \in \mathbb{N}$, is closer than g to the linear map A', and thus remains close to A. In particular, the previous discussion shows that the pseudo-orbit of any point in $\mathbb{B}^n_r \setminus \mathbb{B}^n_{r/2}$ under g_k escapes from $\mathbb{B}^n_{3r/4}$. This also means that the pseudo-orbit of every point in $\mathbb{B}^n_{(\frac{2}{3})^k r} \setminus \mathbb{B}^n_{(\frac{2}{3})^k r/2}$ under g escapes from $\mathbb{B}^n_{(\frac{2}{3})^{k-1} r/2}$. The natural recursive argument then shows that it escapes from $\mathbb{B}^n_{(\frac{2}{3})^{k-2} r/2}, \ldots, \mathbb{B}^n_{(\frac{2}{3}) r/2} = \mathbb{B}^n_{r/3}, \mathbb{B}^n_{r/2}$ and finally $\mathbb{B}^n_{(\frac{2}{3})^{-1} r/2} = \mathbb{B}^n_{3r/4}$. □

Theorem 9.3. *Given $n \geq 1$ and $d \geq 2$, there exists an open subset $\mathcal{U} \subset \mathcal{F}^d(\mathbb{CP}^{n+1})$ approximating $\mathcal{F}_I$ such that every foliation $\mathcal{F} \in \mathcal{U}$ satisfies:*

(i) *$\mathcal{F}$ has exactly $\frac{(d+n)!}{d!n!}$ hyperbolic singularities and is regular on the complement,*
(ii) *every leaf of $\mathcal{F}$ is dense in the whole of $\mathbb{CP}^{n+1}$,*
(iii) *the closure $\overline{G}_{\mathcal{F}}$ of the holonomy pseudo-group $G_{\mathcal{F}}$ has large affine part on Σ_1.*

Proof. As mentioned before, the proof is recursive on n and Theorem 9.1 already provides the first step $n = 1$. Assume now that the degree $d \geq 2$ and the dimension $n \geq 2$ are fixed. Let us consider the $\frac{(d+n)!}{d!n!}$-codimensional subspace $\mathcal{F}^d(\mathbb{CP}^{n+1}, \mathcal{H})$ consisting of those foliations of degree d in $\mathbb{CP}^{n+1}$ which are tangent to the horizontal hyperplane $\mathcal{H} : \{z_n = 0\}$. Since $\mathcal{H} \simeq \mathbb{CP}^n$, we also have the natural restriction map

$$\begin{cases} \mathcal{F}^d(\mathbb{CP}^{n+1}, \mathcal{H}) \to \mathcal{F}^d(\mathbb{CP}^n) \\ \qquad \mathcal{F} \qquad\quad \mapsto \widehat{\mathcal{F}} = \mathcal{F}|_H . \end{cases}$$

In the sequel the restriction to $\mathcal{H}$ of any object relative to $\mathcal{F}$ will be assigned with a hat. The main steps of the proof are contained in the following two lemmas.

Lemma 9.4. *Arbitrarily close to* $\mathcal{F}_1 = \mathcal{F}_{\alpha I}$*, there exists a foliation* $\mathcal{F}_2 \in \mathcal{F}^d(\mathbb{CP}^{n+1}, \mathcal{H})$ *satisfying:*

- $\mathcal{F}_2$ *has only hyperbolic singular points, namely* $\frac{d^{n+1}-1}{d-1}$ *in* $\mathcal{H}$ *and* d^{n+1} *in the complement;*
- $\mathcal{F}_2$ *has finitely many irreducible invariant algebraic curves* $\Gamma_1, \dots, \Gamma_r$*, each of them intersecting* $\mathcal{H}$ *transversely at singular points;*
- *every leaf of* $\mathcal{F}_2$ *apart from the* Γ_l*'s intersects* Σ_1.

Proof of Lemma 9.4. Recall first that $\mathcal{F}_{\alpha I} = \mathcal{F}(\alpha I, \dots, \alpha I) \in \mathcal{F}^d(\mathbb{CP}^{n+1}, \mathcal{H})$ (cf. Sect. 8). Thus, its restriction $\widehat{\mathcal{F}}_{\alpha I} = \mathcal{F}(\alpha \widehat{I}, \dots, \alpha \widehat{I})$ is a foliation (in lower dimension) possessing properties similar to those of $\mathcal{F}_{\alpha I}$ itself. We now make our induction assumption namely, *we suppose that we have already constructed an open subset* $\widehat{\mathcal{U}} \subset \mathcal{F}^d(\mathbb{CP}^n)$ *approximating* $\widehat{\mathcal{F}}_{\alpha I}$ *whose elements* $\widehat{\mathcal{F}} \in \widehat{\mathcal{U}}$ *satisfy conclusions (i), (ii) and (iii) of the statement of Theorem 9.3.*

Recall that affine polynomial vector fields giving rise to degree d foliations in $\mathbb{CP}^{n+1}$ where characterized in Sect. 7. We have also characterized when the hyperplan at infinity is invariant. Modulo identifying, for this immediate purpose, $\mathcal{H}$ with the hyperplan at infinity, it becomes clear that we can find a foliation $\mathcal{F}_2$ in $\mathcal{F}^d(\mathbb{CP}^{n+1}, \mathcal{H})$, close to $\mathcal{F}_{\alpha I}$, whose restriction $\widehat{\mathcal{F}_2}$ to $\mathcal{H}$ lies in $\widehat{\mathcal{U}}$. In particular the singularities of $\widehat{\mathcal{F}_2}$ in $\mathcal{H}$ are hyperbolic. Without loss of generality we can suppose that, indeed, all singularities of $\mathcal{F}_2$ are hyperbolic (in particular isolated). In fact, Lemma 7.1 allows us to suppose that all the singularities of $\mathcal{F}_2$ are isolated (and even simple as it follows from the sketched proof, see also [GM]). It is now easy to see that a small perturbation $\mathcal{F}_2$ turns these singularities into hyperbolic ones without affecting the fact that $\mathcal{H}$ must be invariant by $\mathcal{F}_2$. Since the set $\widehat{\mathcal{U}}$ is open, this is sufficient for our purposes.

Notice that the importance of our induction assumption relies only on the fact that every leaf of $\mathcal{F}_2$ which is contained in $\mathcal{H}$ must intersect $\widehat{\Sigma}_1$. By Proposition 7.6, any leaf L, apart from those contained in $\mathcal{H}$, either is contained in an invariant algebraic curve, or accumulates on a regular point p of $\widehat{\mathcal{F}_2}$. In the second case, the leaf L automatically accumulates on the leaf L' passing through p. Since L' intersects $\widehat{\Sigma}_1$, it results that L intersects Σ_1. Finally, any invariant algebraic curve Γ intersects $\mathcal{H}$ at singular points as a local invariant curve. Hence, there are at most $\frac{d^{n+1}-1}{d-1}$ distinct invariant curves. This finishes the proof. □

Lemma 9.5. *Every foliation* $\mathcal{F} \in \mathcal{F}^d(\mathbb{CP}^{n+1})$ *sufficiently close to* $\mathcal{F}_2$ *satisfies:*

- $\mathcal{F}$ *has* $\frac{d^{n+2}-1}{d-1}$ *hyperbolic singular points and is regular on the complement;*
- $\mathcal{F}$ *has at most r irreducible invariant algebraic curves, each of them arising as a "perturbation" of one of the* Γ_l*'s;*
- *every leaf of* $\mathcal{F}$*, apart from the persistent invariant curves, intersects* Σ_1.

Proof. Denote by p_s the singularities of $\mathcal{F}_2$ while s runs over $s = 1, \dots, \frac{d^{n+2}-1}{d-1}$ and, for each s, let W_s be a small ball centered at p_s. Let V_l be a tubular neighborhood of the compact part of Γ_l given by $\Gamma_l \setminus \bigcup_{p_s \in \Gamma_l} (\Gamma_l \cap W_s)$. Finally, denote by U a neighborhood of the compact remaining part $K = \mathbb{CP}^{n+1} \setminus (\bigcup_s W_s \cup \bigcup_l V_l)$. In fact, we shall prove that this open covering of $\mathbb{CP}^{n+1}$ can be chosen so that every $\mathcal{F}$ sufficiently close to $\mathcal{F}_2$ satisfies the following:

- every leaf of $\mathcal{F}$ in U intersects Σ_1;
- every leaf of $\mathcal{F}$ in V_l intersects U with possible exception of one leaf which gives rise to a persistent invariant curve close to Γ_l;
- every leaf of $\mathcal{F}$ in W_s escapes from W_s.

First notice that any ball W_s satisfies the property above because of Lemma 7.3. In any case, to control the leaves after the perturbation in the neighborhood $V_l \cup \bigcup_{p_s \in \Gamma_l} W_s$ of each invariant curve Γ_l, we shall follow more or less the ideas developed in the proof of Theorem 9.1 (neighborhood of L_0). Besides we are going to impose one further condition on W_s. To state this condition, consider the local invariant curve defined by Γ_l at p_s. Here, we omit the possibility that Γ_l consists of several local invariant curves in W_s, but the same discussion can be carried out in this case with many additional subscripts in the notation. Denote by $\phi_{\mathcal{F}} : \Delta \to \mathbb{CP}^{n+1}$ the parametrization of the persistent local invariant curve given by Lemma 7.5. Then we require that the holomorphic disc $\phi_{\mathcal{F}_2}(\Delta)$ contains all of the intersection $\Gamma_l \cap W_s$. This means that the boundary of the disc $\phi_{\mathcal{F}}(\Delta)$ will lie in the complement of W_s, i.e. in V_l, for every $\mathcal{F}$ sufficiently close to $\mathcal{F}_2$. This can be done by choosing W_s very small.

Now fix a small transverse section Σ_l to Γ_l parametrized by some ball $i_l : \mathbb{B}^n_R \hookrightarrow \Sigma_l$. Let $f_{l,s} : \mathbb{B}_r \hookrightarrow \mathbb{B}_R$ denote the return maps around each of the singular (hyperbolic) points. Then, maybe reducing the radius $0 < r < R$, $f_{l,s}$ has a unique fixed point $\underline{0} = (i_l)^{-1}(\Sigma_l \cap \Gamma_l)$ and the corresponding differential $A_{l,s} = D_{\underline{0}} f_{l,s}$ is hyperbolic, i.e. it has no eigenvalue of norm one. Maybe rescaling the parametrization, we can assume that $f_{l,s}$ is arbitrarily close to the linear map $A_{l,s}$. Using Lemma 9.2, we may ensure that the foliation $\mathcal{F}_2$, as well as any small perturbation of $\mathcal{F}_2$, satisfies the following property: every point of $\mathbb{B}^n_r$, other than the unique (persistent) fixed point, reaches $\mathbb{B}^n_R \setminus \mathbb{B}^n_r$ either by positive or negative iteration of f. On the other hand, recall that an algebraic invariant curve Γ_l contains at least one singularity p_s. While this is a general fact, in our context such point simply arises at the intersection between Γ_l and the hyperplane $\mathcal{H}$. Now, choose a neighborhood V_l of $\Gamma_l \setminus \bigcup_{p_s \in \Gamma_l} \Gamma_l \cap W_s$ so that, $(\Gamma_l \cap \Sigma_l)$ is contained in $i_l(\mathbb{B}^n_r)$ for every sufficiently small perturbation $\mathcal{F}$ of $\mathcal{F}_2$. Note that $i_l(\mathbb{B}^n_r)$ is in turn contained in the "stably repelling" part of the dynamics of $f_{l,s}$ in view of the discussion above. Thus $i_l(\mathbb{B}^n_R \setminus \mathbb{B}^n_r)$ is contained in the compact K. Furthermore every leaf in V_l intersects the transversal Σ_l. Thus, every leaf necessarily escapes from V_l and meets K with possible exception of one leaf when the persistent fixed points of all the $f_{l,s}$ coincide. In this case, the special leaf in V_l corresponding to this common fixed point glue together with the persistent local invariant curves in the balls W_s (the discs $\phi_{\mathcal{F}}(\Delta)$ mentioned above) into a global invariant curve, that must be algebraic by Chow's lemma.

Finally, the fact that Σ persistently intersects all the leaves which contain points of the compact remaining part $K = \mathbb{CP}^{n+1} \setminus (\bigcup_s W_s \cup \bigcup_l V_l)$ (on which $\mathcal{F}_2$ is regular) is standard. By the third property of Lemma 9.4, given point $p \in K$, there is a path γ_p contained in the leaf of $\mathcal{F}_2$ through p and joining $\gamma_p(0) = p$

to $\gamma_p(1) \in \Sigma_1$. Considering a finite covering of γ by trivialization boxes for the foliation $\mathcal{F}_2$, it follows that any point q sufficiently close to p is also linked to Σ_1 by a path γ_q close to γ_p. Indeed, this is just the classical construction of the holonomy map associated to γ_p. Since the trivialization boxes depend continuously on $\mathcal{F}$, we can ensure the existence of $\varepsilon_p, r_p > 0$ such that, for every foliation $\mathcal{F} \in \mathcal{F}^d(\mathbb{CP}^{n+1})$ which is ε_p-close to $\mathcal{F}_2$ and every point $q \in \mathbb{CP}^{n+1}$ which is r_p close to p, there is a path $\gamma_{q,\mathcal{F}}$ close to γ_p joining q to Σ_1 in the leaf of $\mathcal{F}$ passing through q. Finally, using the compactness of K, the desired neighborhood U can be obtained by selecting a finite covering of K from those r_p-neighborhoods. Denoting by ε the minimum of the corresponding ε_p, we have proved that for any foliation $\mathcal{F}$ which is ε_p-close to $\mathcal{F}_2$, every leaf intersecting U will also intersect Σ_1. This finishes the proof of the lemma. □

End of the proof of Theorem 9.3. We conclude as in the proof of Theorem 9.1. If $\mathcal{F}_{\alpha I}$ was chosen sufficiently close to $\mathcal{F}_I$ and $\mathcal{F}_2$ very close to $\mathcal{F}_{\alpha I}$, then $\mathcal{F}_2$ is approximated by the open subset $\mathcal{U}_7 \cap \mathcal{U}'_8$ given by Theorem 7.8 and Proposition 8.5. Thus, every foliation $\mathcal{F}$ sufficiently close to $\mathcal{F}_2$ belonging to this open set has no algebraic invariant curve and is minimal. Indeed, each leaf intersects Σ_1 by the preceding discussion and is dense on the neighborhood of Σ_1 by Proposition 8.5. The same arguments as in the proof of Theorem 9.1 show that the leaves are, in fact, dense on the entire $\mathbb{CP}^{n+1}$. Furthermore $G_{\mathcal{F}}$ has large affine part on Σ_1. □

Remark 9.6. Since $G_{\mathcal{F}}$ has large affine part, it follows that the foliations constructed above are also ergodic with respect to the Lebesgue measure. More generally they possess all the dynamical properties discussed in Sect. 6.

10. Proof of the topological rigidity and of Corollary B

This last section is mainly devoted to proving the rigidity part of the statement of Theorem A. In the sequel, a minimal foliation $\mathcal{F} \in \mathcal{F}^d(\mathbb{CP}^{n+1})$ contained in the open set $\mathcal{U}$ of Theorem 9.3 is fixed. In particular, the pseudo-group $G_{\mathcal{F}}$ on Σ_1 has large affine part. Given $\varepsilon > 0$, let $\mathcal{U}_\varepsilon$ denote the ε-neighborhood of $\mathcal{F}$ in $\mathcal{F}^d(\mathbb{CP}^{n+1})$. Furthermore ε is supposed to be so small that $\mathcal{U}_\varepsilon$ is contained in the open set $\mathcal{U}$ of Theorem 9.3. In particular, given $\mathcal{F}' \in \mathcal{U}_\varepsilon$, the return maps $f_{k,\mathcal{F}'} : \Sigma_1 \simeq \mathbb{B}^n \to \mathbb{C}^n$ are well-defined and the pseudo-group generated by them on Σ_1 also has large affine part. However we are going to deal with another (larger) pseudo-group too. The definition of this pseudo-group is given below. Denote by p_s, $s = 1, \dots, \frac{d^{n+2}-1}{d-1}$, the singular points of $\mathcal{F}$ and by W_s a small ball around p_s. Consider the regular foliation $\widetilde{\mathcal{F}}$ induced by $\mathcal{F}$ on the open complement $V = \mathbb{CP}^n \setminus \overline{W}$ of the union $W = \cup_s W_s$ of these balls. Without loss of generality (modulo reducing ε), we can assume that the singularities of all the foliations $\mathcal{F}' \in \mathcal{U}_\varepsilon$ remain contained in W. Denote also by $\widetilde{\mathcal{F}}'$ the foliation induced by $\mathcal{F}'$ on V. The pseudo-group induced by $\widetilde{\mathcal{F}}$ on Σ_1 is defined as follows. Given an open subset $U \subset \Sigma_1$, a continuous map

$$U \times [0, 1] \to V \; ; \; (z, t) \mapsto \gamma_z(t)$$

satisfying $\gamma_z(0) = z$, $\gamma_z(1) \in \Sigma_1$ and a path γ_z contained in the leaf of $\widetilde{\mathcal{F}}$ passing through z, we know that the return map (or holonomy map) from U to Σ_1 defined by $z \mapsto \gamma_z(1)$ is holomorphic. Besides this map depends only on the homotopy class γ of γ_z. In the sequel we denote by $f_\gamma : U \to \Sigma_1$ the return map induced by γ. The set $G_{\widetilde{\mathcal{F}}}(\Sigma_1)$ consisting of all such holonomy maps is called the pseudo-group induced by $\widetilde{\mathcal{F}}$ on Σ_1. Note that this pseudo-group contains (sometimes strictly) the pseudo-group $G_{\mathcal{F}}$ considered in the preceding section (cf. Remark 9.6).

Lemma 10.0. *If $\varepsilon > 0$ is sufficiently small, then there exist finitely many holonomy maps $f_{\gamma_{k,\mathcal{F}'}} : U_k \to \Sigma_1$ (as above), $k = 1, \ldots, \tilde{d}$, depending holomorphically on $\mathcal{F}' \in \mathcal{U}_\varepsilon$, which generate the pseudo-group $G_{\widetilde{\mathcal{F}}'}(\Sigma_1)$ induced by $\widetilde{\mathcal{F}}'$ on Σ_1.*

Indeed the path $\gamma_{k,\mathcal{F}'}$ depends continuously on $\mathcal{F}'$. The proof is standard and relies on the existence of a finite covering of U by trivialization boxes (with parameter) for the foliation. We omit it. In the sequel, we will simply denote by $f_{k,\mathcal{F}'}$ the return map $f_{\gamma_{k,\mathcal{F}'}}$ induced by $\gamma_{k,\mathcal{F}'}$. Also we assume without loss of generality that, for $k = 1, \ldots, d$, these holonomy maps are the return maps constructed in Sect. 8 (in general $d < \tilde{d}$). Using Chaperon's linearizing coordinates, it is easy to see that the entire pseudo-group induced by $\mathcal{F}'$ on Σ_1 is also finitely generated as above. We can denote this last pseudo-group by $G_{\mathcal{F}'}^{\mathrm{Total}}$ so that we have the inclusions

$$G_{\mathcal{F}} \subseteq G_{\widetilde{\mathcal{F}}'}(\Sigma_1) \subseteq G_{\mathcal{F}'}^{\mathrm{Total}} .$$

Nevertheless for $G_{\mathcal{F}'}^{\mathrm{Total}}$ we cannot find a family of generators depending continuously on the foliation.

On the other hand, denote by Homeo$(\mathbb{CP}^{n+1})$ the set of homeomorphisms $\Phi : \mathbb{CP}^{n+1} \to \mathbb{CP}^{n+1}$ equipped with the C^0-topology (uniform convergence on $\mathbb{CP}^{n+1}$).

Lemma 10.1. *Given $\varepsilon > 0$ sufficiently small, there exists an open neighborhood of the identity $\mathcal{W} \subset$ Homeo$(\mathbb{CP}^{n+1})$ having the following property: if $\Phi \in \mathcal{W}$ conjugates the foliation $\mathcal{F}$ (of degree d) to another holomorphic foliation by curves $\mathcal{F}'$ on $\mathbb{CP}^{n+1}$, then $\mathcal{F}'$ also has degree d and is close to $\mathcal{F}$ in $\mathcal{F}^d(\mathbb{CP}^n)$ (i.e. by reducing ε we can make $\mathcal{F}$, $\mathcal{F}'$ arbitrarily close in $\mathcal{F}^d(\mathbb{CP}^n)$). Furthermore Φ induces a conjugacy $\Phi : \Sigma_{1/2} \hookrightarrow \Sigma_1$ between the respective holonomy maps $\Phi_* f_{k,\mathcal{F}} = f_{k,\mathcal{F}'}$ for $k = 1, \ldots, \tilde{d}$.*

Remark 10.2. In the statement above, the induced conjugacy $\Phi : \Sigma_{1/2} \to \Sigma_1$ between the return maps is a priori only a homeomorphism onto its image. However it is, in fact, either holomorphic, or anti-holomorphic thanks to Proposition 6.4. Moreover, assuming that ε is very small, then we may conclude that Φ is too close to the identity to be anti-holomorphic.

Proof. Consider a homeomorphism Φ conjugating $\mathcal{F}$ to another singular holomorphic foliation by curves $\mathcal{F}'$. Clearly Φ preserves the number of singular points, namely $\frac{d^{n+2}-1}{d-1}$, as well as their Milnor number which are all equal to 1 for $\mathcal{F}$. This implies that the singularities of $\mathcal{F}'$ are simple as well. An application of

Baum-Bott Formula then shows that the number of singularities of $\mathcal{F}'$ is equal to $\text{Deg}\,(\mathcal{F}')^{n+1} + \text{Deg}\,(\mathcal{F}')^{n} + \cdots + 1$, where $\text{Deg}\,(\mathcal{F}')$ stands for the degree of $\mathcal{F}'$ (cf. [LN,So], Remark 4.1). It immediately results that $\text{Deg}\,(\mathcal{F}') = d$, so that $\mathcal{F}$, $\mathcal{F}'$ have both degree d.

Now, fix a ball B far away from the singular points of $\mathcal{F}$. If Φ is sufficiently close to the identity, then $\mathcal{F}'$ is regular and C^0-close to $\mathcal{F}$ on B. This means that the maximal distance between the leaves L_p and L'_p inside B is uniformly bounded by a constant that can be assumed arbitrarily small. Due to Cauchy's inequality, this implies that $\mathcal{F}'$ is C^1-close to $\mathcal{F}$ on some smaller ball $B' \subset B$. Thus the slope of a vector field $\mathcal{Z}'$ defining $\mathcal{F}'$ in the affine chart containing B' is close to the slope of a corresponding vector field $\mathcal{Z}$ for $\mathcal{F}$. However this C^1-topology coincides with the topology on $\mathcal{F}^d(\mathbb{CP}^n)$. Indeed, if the slope of a sequence of polynomial vector fields $\mathcal{Z}'_k$ (of constant degree) tends uniformly on B' to the slope of $\mathcal{Z}$, then $[\mathcal{Z}'_k]$ tends to $[\mathcal{Z}]$ where $[\cdot]$ denotes the point of the appropriate projective space associated with the coefficients of $\mathcal{Z}$, $\mathcal{Z}'_k$. In other words, the images of $\mathcal{Z}'_k$ in $\mathbb{CP}^N$ (which contains the set $\mathcal{F}^d(\mathbb{CP}^n)$) converge to the image of $\mathcal{Z}$ in $\mathbb{CP}^N$. In order to check this, we just need to observe that, otherwise, the compactness of $\mathbb{CP}^N$ would enable us to find a limit point $[\mathcal{Z}^\infty]$ for the sequence $[\mathcal{Z}'_k]$ which is different from $[\mathcal{Z}]$. Such point corresponds to a polynomial vector fields, of degree at most $d+1$, tangent to $\mathcal{Z}$ on B' which is impossible.

The homeorphism Φ sends Σ_1 to a section Σ' which is topologically transverse to the foliation $\mathcal{F}'$. Moreover Σ' is close to Σ_1. By a standard argument involving a trivialization box for the foliation $\mathcal{F}'$ (see [Il1]), maybe composing Φ with homeorphism close to the identity preserving the leaves of $\mathcal{F}'$ and coincinding with the identity outside the trivialization box, we can assume without loss of generality that Φ sends, say $\Sigma_{1/2}$, into Σ_1.

Given a path γ in a leaf of $\mathcal{F}$ with $\gamma(0), \gamma(1) \in \Sigma_{1/2}$, the homeomorphism Φ takes γ to a path γ' contained in a leaf of $\mathcal{F}'$ with $\gamma(0), \gamma(1) \in \Sigma$. It is also well-known that the holonomy map $f_\gamma : U \to \Sigma$, $U \in \Sigma_{1/2}$, is conjugated to the corresponding holonomy map of γ' by the restriction $\Phi|_{\Sigma_{1/2}}$ on the image $\Phi(U)$. For the sake of notations, in what follows, we shall make no distinction between Φ and the homeomorphism $\Phi|_{\Sigma_{1/2}} : \Sigma_{1/2} \to \Phi(\Sigma_{1/2})$ induced on the transversal.

Finally, if Φ is very close to the identity, then Φ conjugates each return map $f_{k,\mathcal{F}}$ to the holonomy map f_γ of a path γ very close to γ_k and thus coincides with the corresponding return map $f_{k,\mathcal{F}'}$. □

Lemma 10.3. *For $\varepsilon > 0$ sufficiently small, there exists a holomorphic family*

$$\mathcal{U}_\varepsilon \times \Sigma_1 \to \mathbb{C}^{n+1} \; ; \; (\mathcal{F}', z) \mapsto \Phi_{\mathcal{F}'}(z)$$

of injective mappings $\Phi_{\mathcal{F}'} : \Sigma_1 \to \mathbb{C}^n$ and, for some $m \in \mathbb{N}^*$, a holomorphic map

$$\mathcal{M} \; : \; \mathcal{U}_\varepsilon \to \mathbb{C}^m \; ; \; \mathcal{F} \mapsto \underline{0}$$

having the following property: given $\mathcal{F}', \mathcal{F}'' \in \mathcal{U}_\varepsilon$, there exists a holomorphic mapping $\Phi : \Sigma_{1/2} \to \Sigma$ conjugating their holonomy maps $\Phi_* f_{k,\mathcal{F}'} = f_{k,\mathcal{F}''}$ for $k = 1, \ldots, \tilde{d}$ if, and only if, $\mathcal{M}(\mathcal{F}') = \mathcal{M}(\mathcal{F}'')$. Moreover, in this case, Φ can be obtained as $\Phi = (\Phi_{\mathcal{F}''})^{-1} \circ \Phi_{\mathcal{F}'}$.

Remark 10.4. In particular, maybe reducing ε, the analytic set $\mathcal{T} = \mathcal{M}^{-1}(\underline{0})$ is connected (containing at least the germ of the orbit of $\mathcal{F}$ under $PGL(n+2,\mathbb{C})$). For $\mathcal{F}'' = \mathcal{F}$, the preceding statement can be interpreted as follows. If a foliation $\mathcal{F}'$ is conjugated to $\mathcal{F}$ by an element $\Phi \in \mathcal{W}$ (given by Lemma 10.1), then there exists a connected analytic deformation $T \mapsto \mathcal{F}_T$, $T \in \mathcal{T}$, of $\mathcal{F} = \mathcal{F}_{\underline{0}}$ containing $\mathcal{F}'$ together with an analytic deformation $t \mapsto \Phi_T : \Sigma_{1/2} \to \Sigma$ of the identity $\Phi_{\underline{0}} \equiv Id$ conjugating the return maps $(\Phi_T)_* f_{k,\mathcal{F}} = f_{k,\mathcal{F}_T}$.

Proof. The statement will result from the existence of a normal form for the return maps $f_{k,\mathcal{F}'}$. By the construction of $\mathcal{F}$, the first return map $f_{1,\mathcal{F}}$ is close to a homothety which has simple eigenvalues. Thus it is linearizable. Precisely, by Lemma 3.5, there is an injective holomorphic map $\Phi_{\mathcal{F}} : \Sigma_1 \to \mathbb{C}^{n+1}$ conjugating $f_{1,\mathcal{F}}$ to a diagonal linear map

$$A = \begin{pmatrix} \lambda_1(\mathcal{F}) & & 0 \\ & \ddots & \\ 0 & & \lambda_n(\mathcal{F}) \end{pmatrix}.$$

After we have indexed the eigenvalues, this diagonalizing map is unique up to composition with a diagonal map. We can use this last degree of freedom to normalize another holonomy map. For instance, since the return maps $f_{1,\mathcal{F}}, \dots, f_{d,\mathcal{F}}$ generate a pseudo-group with large affine part, there exists a word in these elements, say $f_{2,\mathcal{F}}$ to abridge notations, taking $\underline{0}$ to the complement of all the invariant hypersurfaces $\Phi^{-1}(\{z_i = 0\})$ of $f_{1,\mathcal{F}}$, $i = 1, \dots, n-1$. We then set (for instance) the normalization $\Phi \circ f_{2,\mathcal{F}}(\underline{0}) = (1, \dots, 1)$. The map $\Phi_{\mathcal{F}} : \Sigma_1 \to \mathbb{C}^n$ is therefore unique and depends holomorphically on $\mathcal{F}$ (as long as the spectrum remains in Poincaré domain and far from resonances).

Given $\mathcal{F}'$ and $\mathcal{F}''$ close to $\mathcal{F}$, any conjugacy between their respective return maps $\Phi_* f_{k,\mathcal{F}'} = f_{k,\mathcal{F}''}$ must induce the identity through the normalizing maps, i.e.

$$\Phi_{\mathcal{F}''} \circ \Phi \circ (\Phi_{\mathcal{F}'})^{-1} \equiv Id.$$

This strictly follows from the uniqueness of the normalizing map. Conversely, any transformation Φ inducing the identity as above will provide a conjugacy between the return maps. For each $k = 1, \dots, \tilde{d}$, choose a point $p_k \in U_k$ and consider the Taylor coefficients $a_{k,i,\underline{k}}(\mathcal{F})$ at p_k of the return map $f_{k,\mathcal{F}} = (f_{k,1,\mathcal{F}}, \dots, f_{k,n,\mathcal{F}})$ viewed through the normalizing coordinates

$$\Phi_{\mathcal{F}} \circ f_{k,\mathcal{F}} \circ (\Phi_{\mathcal{F}})^{-1} = \left(\sum_{|\underline{k}| \geq 0} a_{k,1,\underline{k}}(\mathcal{F})(z - p_k)^{\underline{k}}, \dots, \sum_{|\underline{k}| \geq 0} a_{k,n,\underline{k}}(\mathcal{F})(z - p_k)^{\underline{k}} \right).$$

Here $\underline{k} = (k_1, \dots, k_n) \in \mathbb{N}^n$ denotes a multi-index, $|\underline{k}| = k_1 + \cdots + k_n$ is the corresponding length and $z^{\underline{k}} = (z_1^{k_1}, \dots, z_n^{k_n})$. For instance, given any $i = 1, \dots, n$, we have $a_{1,i,\underline{k}}(\mathcal{F}) = \lambda_i(\mathcal{F})$ for $k_i = 1$ and $k_j = 0$, $j \neq i$, and $a_{1,i,\underline{k}}(\mathcal{F}) = 0$ for any other multi-index $\underline{k}$. Consider the ideal $\mathcal{I} \subset \mathcal{O}(\mathcal{U}_\varepsilon \times \mathcal{U}_\varepsilon)$ of holomorphic functions on $\mathcal{U}_\varepsilon \times \mathcal{U}_\varepsilon$ generated by the differences

$$\Delta_{k,i,\underline{k}}(\mathcal{F}', \mathcal{F}'') = a_{k,i,\underline{k}}(\mathcal{F}'') - a_{k,i,\underline{k}}(\mathcal{F}').$$

Thanks to Hilbert Basis Theorem, there is a finite set $\{\Delta_{k_1,i_1,\underline{k}_1}, \ldots, \Delta_{k_m,i_m,\underline{k}_m}\}$ generating the ideal on $\mathcal{U}_\varepsilon \times \mathcal{U}_\varepsilon$. Denote by $\mathcal{M} : \mathcal{U}_\varepsilon \to \mathbb{C}^m$ the holomorphic function defined by

$$\mathcal{M}(\mathcal{F}) = (a_{k_1,i_1,\underline{k}_1}(\mathcal{F}), \ldots, a_{k_m,i_m,\underline{k}_m}(\mathcal{F})).$$

By construction, $\mathcal{M}(\mathcal{F}') = \mathcal{M}(\mathcal{F}'')$ if, and only if, the respective normal forms for the return maps coincide

$$\Phi_{\mathcal{F}'} \circ f_{k,\mathcal{F}'} \circ (\Phi_{\mathcal{F}'})^{-1} = \Phi_{\mathcal{F}''} \circ f_{k,\mathcal{F}''} \circ (\Phi_{\mathcal{F}''})^{-1}$$

which proves the lemma. □

The orbit of $\mathcal{F}$ under $PGL(n+2,\mathbb{C})$ is smooth on $\mathcal{U}_\varepsilon$ and has (maximal) dimension n^2+4n. This is a consequence of the fact that no holomorphic vector field on $\mathbb{CP}^n$ fixes $\mathcal{F}$. Indeed, such vector field cannot be tangent to all the leaves, otherwise the foliation would have degree 1. Thus a vector field preserving $\mathcal{F}$ should induce a transversal symmetry, i.e. a non-trivial holomorphic vector field on Σ_1 commuting with the corresponding pseudo-group. However this is impossible because of Lemma 10.3. The analytic set $\mathcal{T}$ (defined in Remark 10.4) consisting of those foliations $\mathcal{F}' \in \mathcal{U}_\varepsilon$ whose pseudo-group $G_{\widetilde{\mathcal{F}}'}$ is conjugate to $G_{\widetilde{\mathcal{F}}}$ on Σ clearly contains the $PGL(n+2,\mathbb{C})$-orbit of $\mathcal{F}$. It can be shown that $\mathcal{T}$, in fact, coincides with this orbit. Here we shall prove this only in the case where $\mathcal{T}$ is smooth at $\mathcal{F}$. The general case would lead to several additional difficulties and is not necessary to the proof of Theorem A. Indeed, $\mathcal{T}$ is smooth at a generic point. Thus, maybe replacing $\mathcal{F}$ by a perturbation, we can assume without loss of generality that $\mathcal{T}$ smooth at $\mathcal{F}$. Now the rigidity part of our statement immediately follows from the combination of Lemma 10.1, Lemma 10.3 and the lemma below.

Lemma 10.5. *Assume that the fiber $\mathcal{T} = \{\mathcal{F}' ; \mathcal{M}(\mathcal{F}') = \mathcal{M}(\mathcal{F})\}$ of the "modular function" $\mathcal{M}$ given by Lemma 10.3 containing $\mathcal{F}$ is smooth at $\mathcal{F}$. Then, for $\varepsilon > 0$ sufficiently small, $\mathcal{T}$ coincides with the $PGL(n+2,\mathbb{C})$-orbit of $\mathcal{F}$ in $\mathcal{U}_\varepsilon$.*

Proof. Choosing ε very small, we can assume that $\mathcal{T}$ is a smooth connected submanifold of dimension ν in $\mathcal{U}_\varepsilon$. The main part of the proof consists of constructing a $\nu+1$-dimensional singular foliation $\mathcal{G}$ on $\mathcal{T} \times \mathbb{CP}^{n+1}$ having some special properties. Namely the singular set of $\mathcal{G}$ should coincide with the $\frac{d^{n+2}-1}{d-1}$ analytic submanifolds Γ_s parametrized as $T \mapsto p_{s,T}$ and given by the persistence $p_{s,T} \in \mathrm{Sing}(\mathcal{F}_T)$ of the singular points $p_s \in \mathrm{Sing}(\mathcal{F})$. Let $\Gamma = \cup_s \Gamma_s$. The foliation $\mathcal{G}$ is transverse to all the projective planes $\{T\} \times \mathbb{CP}^{n+1}$ away from Γ and induces the foliations $\mathcal{F}_T$. In this sense, the family $\mathcal{F}_T$ will be made into an integrable unfolding of $\mathcal{F}$. Then, the main result of [GM] states that this unfolding is holomorphically trivial, i.e. that there exists a holomorphic family $\mathcal{T} \to PGL(n+2,\mathbb{C})$; $T \mapsto \Phi_T$ satisfying $(\Phi_T)_*\mathcal{F}_T = \mathcal{F}$ which establishes the proof.

The foliation $\mathcal{G}$ is constructed as follows. Recall that we have (cf. Remark 10.4) a family of maps $\Phi_T : \Sigma_{1/2} \to \Sigma_1$ conjugating the pseudo-group $G_{\widetilde{\mathcal{F}}_T}$ to $G_{\widetilde{\mathcal{F}}}$. Recall also that the intersection of any leaf of $\widetilde{\mathcal{F}}_T$ with Σ_1 coincides with an orbit under $G_{\widetilde{\mathcal{F}}_T}$. Thus, Φ_T induces a one-to-one correspondence between the leaves of

$\widetilde{\mathcal{F}}_T$ and those of $\widetilde{\mathcal{F}}$. The existence of a contraction $f_{1,\mathcal{F}_T} : \Sigma_1 \to \Sigma_1$ in $G_{\widetilde{\mathcal{F}}_T}$ shows that the exact domains of definitions of the Φ_T are not relevant as long as they contain the contracting fixed point. The restriction of $\mathcal{G}$ to $\mathcal{T} \times (\mathbb{CP}^n \setminus \cup_s W_s)$, is the regular foliation defined by the previous correspondance. Precisely, if we denote by $\widetilde{\mathcal{F}}_{T,p}$ the leaf of $\widetilde{\mathcal{F}}_T$ passing through a point $(T, p) \in \{T\} \times \Sigma$, then the leaf $\mathcal{G}_{(T_0,p)}$of $\mathcal{G}$ passing through (T_0, p) is given by $\mathcal{G}_{(T_0,p)} = \cup_{T\in\mathcal{T}} \widetilde{\mathcal{F}}_{T,\Phi_T^{-1}(p)}$. Now, the $\nu+1$-dimensional regular foliation $\mathcal{G}$ extends to $\mathcal{T} \times \mathbb{CP}^{n+1}$ as a $\nu+1$-dimensional singular foliation (as follows, for example, from Levy Extension Theorem). This extension $\mathcal{G}$ still satisfies the required properties. This completes the proof of the lemma and, therefore, the proof of Theorem A. □

Let us finish the paper with the proof of Corollary B in the Introduction.

Proof of Corollary B. The fact that a foliation $\mathcal{F} \in \mathcal{F}^d(\mathbb{CP}^{n+1})$ possesses an invariant algebraic subset $V \subset \mathbb{CP}^{n+1}$ may be expressed as a system of algebraic equations satisfied by the Taylor coefficients of the (degree-d) homogeneous vector field $\mathcal{Z} = \sum_{i=0}^n H_i(z)\partial_{z_i}$ defining the foliation. Indeed, if $F_1, \dots, F_m$ is a reduced family of homogeneous polynomials defining V, then V is invariant by $\mathcal{F}$ if, and only if, for any $k = 1, \dots, m$, $\mathcal{Z} \cdot F_k$ belongs to the ideal $(F_1, \dots, F_m)$ defining V, i.e. if

$$\sum_{i=0}^{n} H_i \cdot \frac{\partial F_k}{\partial z_i} \quad = \quad G_1 \cdot F_1 + \cdots + G_m \cdot F_m$$

for polynomials $G_1, \dots, G_m$. Now, we can apply an idea which was used by E. Ghys to prove that any two matrices $A, B \in GL(n, \mathbb{C})$ with algebraically independent coefficients generate a free group of rank 2. Consider a homogeneous vector field $\mathcal{Z}$ of degree d in $\mathbb{C}^{n+2}$ whose Taylor coefficients are algebraically independent. The set E of such vector fields has obviously total Lebesgue measure. Given another vector field $\mathcal{Z}' \in E$, there exists an automorphism σ of the field $\mathbb{C}$ of the complex numbers over $\mathbb{Q}$ sending each Taylor coefficients of $\mathcal{Z}$ to the corresponding one for $\mathcal{Z}'$. If $\mathcal{Z}$ admits an invariant algebraic subset, its coefficients will satisfy an algebraic relation as above. The corresponding algebraic relations obtained after applying σ will show that $\mathcal{Z}'$ also has an invariant algebraic subset (of the same dimension). Finally, because of the denseness of E, we can choose $\mathcal{Z}'$ in the open subset $\mathcal{U}$ defined by our Theorem A. It follows that the algebraic relation considered above is impossible for $\mathcal{Z}'$ and, therefore also for any $\mathcal{Z} \in E$. The proof of the corollary is over. □ □

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