Ann. Inst. Henri Poincaré,

Vol. 15, n° 5, 1998, p. 651-659 *Analyse non linéaire*

Contracting singular cycles

by

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ABSTRACT. – In the present paper we shall prove that the result in [2.2], concerning the measure of the bifurcating set for generic one-parameter families through a simple contracting singular cycle with only one periodic orbit, is still true for cycles with any number (finitely many) periodic orbits. The novelty is the choice of a proper sections to the flow and the corresponding Poincare maps.

RÉSUMÉ. – Dans cet article nous montrons que le résultat dans [2.2] sur la mesure de bifurcation pour des familles génériques à 1 paramètre à travers un simple cycle singulier contractant avec une orbite périodique est vrai aussi pour des cycles avec tout nombre (fini) d'orbites périodiques. La nouveauté réside dans le choix des sections appropriées du flot et des applications de Poincaré correspondantes.

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1. INTRODUCTION

In this work we continue the analysis of singular cycles started in [1] and continued in [2]. Recall that a singular cycle for a vector field is a finite set of singularities and periodic orbits, all of them hyperbolic, linked in a

* 1991 *Mathematics Suject Classifications.* 58 f 14, 58 f 12 58 f 13. Partially supported by CNPq (Brazil) and FONDECYT 1941080 (Chile).

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cyclic way by orbits in the intersections of the stable and unstable manifolds of the singularities and periodic orbits. Moreover, if the cycle have only one singularity and the expanding eigenvalue is smaller than the weakest contracting one, we call it contracting. Otherwise, we call it expanding. In [2], it was proved that in the unfolding of a contracting singular cycle with only one periodic orbit satisfying certain additional conditions, which holds for a large class of these fields, leads to the creation of at most one attracting periodic orbit. Moreover, the set of parameters corresponding to hyperbolic dynamics in the parameter space has full Lebesgue measure.

Now we will give the precise description of the problem and the results in [1] and [2]. Let M be a compact and boundaryless 3-manifold and let $\mathcal{X}^r$ be the Banach space of C^r vector fields on M. If $X \in \mathcal{X}^r$, denote by $\Gamma(X)$ its chain recurrent set. We say that X is simple when $\Gamma(X)$ is a union of finitely many hyperbolic critical orbits. By critical orbit we mean an orbit that is either periodic or singular. It is easy to see that the set S^r of simple C^r vector fields is an open subset on $\mathcal{X}^r$.

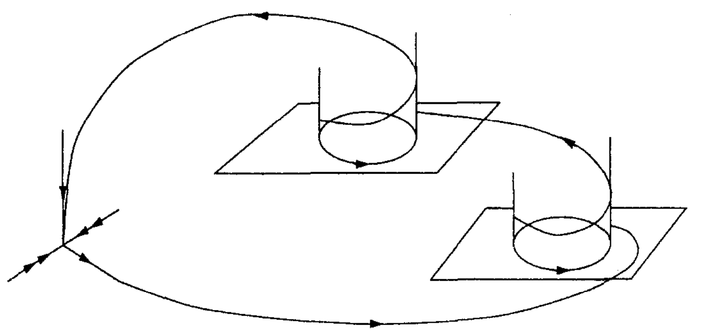

Fig. 1.

Our objet of study will be simple singular cycles. That is, a cycle Λ of $X \in \mathcal{X}^r$ satisfying:

- Λ contains a unique singularity σ_0;
- the eigenvalues of $D_{\sigma_0}X : T_{\sigma_0}M \hookleftarrow$ are real and satisfy $-\lambda_3 < -\lambda_1 < 0 < \lambda_2$;
- Λ contains a unique non singular orbit γ_0 contained in $W^u(\sigma_0)$ and such that $\omega - lim(\gamma_0)$ is a periodic orbit σ_1 and the other regular orbit are transversal intersection of stable and unstable manifolds;
- for each $p \in \gamma_0$ and each invariant manifold $W(\sigma_0)$ of X passing through σ_0 and tangent at σ_0 to the eigenspace spanned by the

eigenvectors associated to $-\lambda_1$ and λ_2, we have

$$T_pW(\sigma_0) + T_pW^s(\sigma_1) = T_pM;$$

– there exist neighborhood $\mathcal{U}$ of X such that if $Y \in \mathcal{U}$ the continuations $\sigma_i(Y)$, $i : 0, .., k$ of the critical orbits σ_i of the cycle are well defined and the vector field Y is C^2-linearizable in σ_0 as well as the Poincare maps of $\sigma_i(Y)$, $i : 1, .., k$;

– Λ is isolated. Here, this mean that there exist a neighborhood U of Λ, called isolating block, such that $\cap_t X_t(U)$ no contains orbit close γ_0, where $X_t : M \hookleftarrow$ is the flow generated by X.

For simple singular cycles the following result holds [1]:

THEOREM 1. – *Let Λ_0 be a simple singular cycle for a vector field X_0, and let U be an isolating block of Λ_0. Then there are a neighborhood $\mathcal{U}$ of X_0 and a co-dimension one submanifold, $\mathcal{N} \subset \mathcal{X}^r$ containing X_0 such that*

– If $Y \in \mathcal{U} \cap \mathcal{N}$, then $\Lambda(Y, U) = \bigcap_t Y_t(U)$ contains a singular cycle topologically equivalent to Λ_0

– $\mathcal{U}\backslash\mathcal{N}$ has two connected components and one of them denoted by $\mathcal{U}^-$ is such that $Y \in \mathcal{U}^-$ implies that the chain recurrent set of $Y/\Lambda(Y, U)$ consists of the continuations $\sigma_i(Y)$, $0 \leq i \leq k$ of the critical orbits σ_i of Λ_0.

This means that the cycle persists in $\mathcal{N} \cap \mathcal{U}$ and breaks down in $\mathcal{U}^-$. Denote by $\mathcal{U}^+$ the other component of $\mathcal{U}\backslash\mathcal{N}$. Define $\mathcal{U}_H^+$ as the set of $Y \in \mathcal{U}^+$ such that the chain recurrent set for Y in $\Lambda(Y, U)$, consists of $\sigma_0(Y)$ plus a transitive hyperbolic set. Also let $\mathcal{U}_{H'}^+$ be the set of $Y \in \mathcal{U}^+$ such that the recurrent set of Y in U consists of the union of $\sigma_0(Y)$, a transitive hyperbolic set, and a unique attracting periodic orbit.

The study of $\mathcal{U}^+\backslash(\mathcal{U}_H^+ \cap \mathcal{U}_{H'}^+)$ depends on eigenvalues at the singularity σ_0 of the cycle. If $\lambda_1 - \lambda_2 < 0$, i.e. the expanding case, it was proved in [1] that $\mathcal{U}_{H'}^+$ is empty and $\mathcal{U}_H^+$ is dense in $\mathcal{U}^+$. In the contracting case, namely $\lambda_1 - \lambda_2 > 0$, we have the following result for cycles with only one periodic orbit and eigenvalues such that $\beta > \alpha + 2$, where $\beta = \frac{\lambda_3}{\lambda_2}$ and $\alpha = \frac{\lambda_1}{\lambda_2}$. This case is called strongly contracting.

THEOREM 2. – *([2]) If X_μ is a family transversal to $\mathcal{N}$ at X_0 and X_0 has a strongly contracting simple singular cycle with only one periodic orbit, then for some $t > 0$*

$$m\{0 \leq \mu \leq t : X_\mu \in \mathcal{U}^+\backslash(\mathcal{U}_H^+ \cap \mathcal{U}_{H'}^+)\} = 0$$

Here, the condition $\beta > \alpha + 2$ was used in [2] to ensure the existence of a C^3-stable foliation associated to the cycle Λ_0. So, the dynamics in the space of leaves of this foliation is given by a one-dimensional mapping which has negative Shwarzian derivative. However, it is enough to have a C^2-foliation, which exists if $\beta > \alpha + 1$, and to use that such one-dimensional mapping has monotonic derivative.

THEOREM 3. – *If* X_μ *crosses transversally at* $\mathcal{N}$ *in* X_0 *and* X_0 *has simple singular cycle* Λ_0 *such that* $\beta > \alpha + 1$, *then for some* $t > 0$

$$m\{0 \le \mu \le t : X_\mu \in \mathcal{U}\backslash(\mathcal{U}_H^+ \cap \mathcal{U}_{H'}^+)\} = 0$$

2. PROOF THEOREM 3

The method of proof is essentially the same used in [2]. So, we only need to show that lemma 1 there is still valid for the maps f_μ that we define below (see lemma 2).

Let X_μ be a family as in Theorem 3. We assume that the cycle contains two periodic orbits σ_1 and σ_2.

Let S_1 and S_2 be cross sections to the flow X_0 at $q_1 \in \sigma_1$ and at $q_2 \in \sigma_2$ parametrized by linear coordinates $\{(x, y) : |x|, |y| \le 1\}$ and satisfying $W^s(\sigma_i) \supseteq \{(x, 0) : |x| \le 1\}$ and $W^u(\sigma_i) \supseteq \{(0, y) : |y| \le 1\}$.

A closed subset $C \subset S_i$ is called horizontal strip if it is bounded in S_i by two disjoint continuous curves connecting the vertical sides of S_i.

Since $W^u(\sigma_2)$ intersects $W^s(\sigma_0)$ and γ_0 has as ω-limit set σ_1, there is a horizontal strip $R_4 \subset S_2$ such that the Poincare map $F_0 : R_4 \to S_1$ is well defined. Clearly F_0 is defined in two horizontal strips $R_1 \subset S_1$ and $R_3 \subset S_2$ and coincides with the Poincare maps associated to the periodic orbits σ_1 and σ_2 respectively. Moreover, there is also a region $R_2 \subset S_1$ such that the Poincare map is defined from R_2 into R_3 (see Fig. 2).

Finally for $\mu > 0$ sufficiently small, S_i is still a cross section for X_μ at $\sigma_i(\mu)$ and $W^u(\sigma_0(\mu))$ intersects S_1 at $p_\mu = (x_\mu, y_\mu)$, where $\sigma_i(\mu)$ are the continuations of σ_i. As before, the Poincare map F_μ is defined on $R_1(\mu) \cup R_2(\mu) \cup R_3(\mu) \cup R_4(\mu)$ and the restrictions to $R_1(\mu)$ and $R_3(\mu)$ coincide with the Poincare maps $P_{\sigma_1(\mu)}$ and $P_{\sigma_2(\mu)}$ associated to the periodic orbits $\sigma_1(\mu)$ and $\sigma_2(\mu)$ respectively.

Up to replacing $R_2(\mu)$ by some negative iterate of it and $R_3(\mu)$ by some positive iterate, we may assume that the horizontal lines are very contracted and the verticals ones are very extended. Now, applying this fact and the same techniques as in [1] and [3], one proves the following lemma.

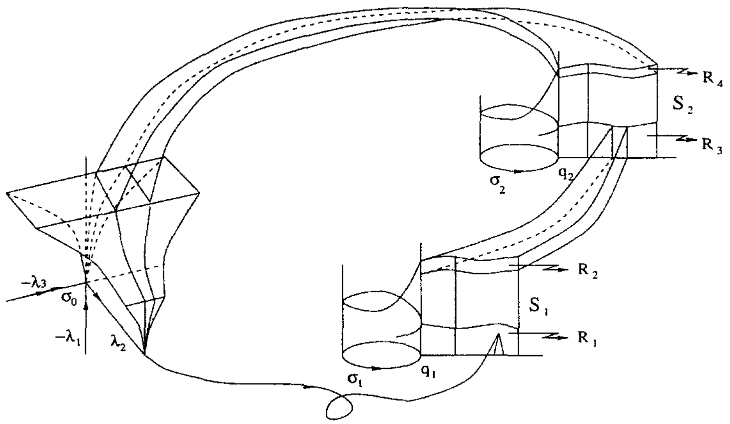

Fig. 2.

LEMMA 1. – *If* $\beta > \alpha + 1$, *then for every* $\mu \geq 0$ *sufficiently small there is an invariant* C^2*-stable foliation* $\mathcal{F}^s_\mu$ *by* F_μ *depending* C^1 *on* μ.

Let $\{f_\mu : \mathcal{D}(f_\mu) \subset [0,1] \cup [2,3] : \mu \geq 0\}$ be the family of one-dimensional map induced by $\{\mathcal{F}^s_\mu : \mu \geq 0\}$. Since $\mathcal{F}^s_\mu$ is C^2 each f_μ is also C^2, and it depends C^1 on μ. We can parametrize $\{f_\mu\}$ in such a way that $f_\mu(3) = \mu$. It is easy to see that the general form of f_μ is:

$$f_\mu(x) = \begin{cases} \rho_\mu \cdot x & x \in [0, \rho_\mu^{-1}] \\ g_\mu(x) & x \in [a, 1] \\ \sigma_\mu \cdot (x - 2) + 2 & x \in [2, 2 + \sigma_\mu^{-1}] \\ \mu - K_\mu(x) \cdot (3 - x)^{\alpha_\mu} & x \in [a_\mu, 3] \end{cases}$$

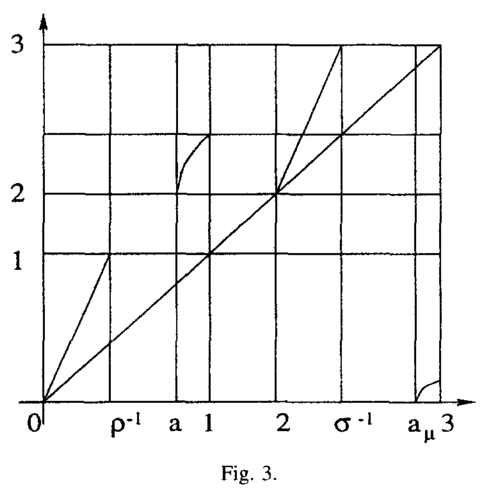

Fig. 3.

Remarks

a. – Actually the function g_μ is defined on one interval $[a, b_\mu]$ with b_μ close to 1, but without loss of generality we can assume that b_μ is equal to 1.

b. – ρ_μ is the expanding eigenvalue of the Poincare map associated to $\sigma_1(\mu)$, and σ_μ is the corresponding one for $\sigma_2(\mu)$.

c. – K_μ is C^2 and $K_\mu(3) > 0$.

d. – We may choose a close to 1 such that $\frac{4}{\rho_\mu} \cdot \frac{1-a}{a-\rho_\mu^{-1}} < \frac{1}{2}$ and $|g'_\mu(x)| > 1$ for all $x \in [a, 1]$.

e. – Since $f_\mu(a_\mu) = 0$ we have $a_\mu \to 3$ when $\mu \to 0$.

f. – We will suppose that f_μ is increasing.

The following properties of f_μ can be easily verified.

There exists $\overline{\mu} > 0$ such that for every $\mu \in [0, \overline{\mu}]$ we have:

i. – $|\frac{\partial}{\partial\mu} f_\mu(x) - 1| < \epsilon$ for all $x \in [a_\mu, 3]$ and small ϵ.

ii. – $0 < f'_\mu(x) < \epsilon$ for all $x \in I_4$.

iii. – For $f_\mu^m : I_2 \to I_2$, where this map has sense, $(f_\mu^m)''$ is negative, which implies that $(f_\mu^m)'$ is decreasing.

Without loss of generality we will assume $\rho_\mu = \rho$ and $\sigma_\mu = \sigma$. Let $I_1 = [0, \rho^{-1}]$, $I_2 = [a, 1]$, $I_3 = [2, 2 + \sigma^{-1}]$ and $I_4(\mu) = [a_\mu, 3]$. Define $\Delta_\mu = \{x : f_\mu^n(x) \in \cup_{i=1}^4 I_i(\mu) \forall n \geq 0\}$, $\tilde{\Delta} = \{\mu : 3 \notin \Delta_\mu\}$, and $H = \{\mu : f_\mu \text{ is hyperbolic }\}$

We now claim that

1) if $\mu \in \tilde{\Delta}$, then $\mu \in H$

2) if f_μ has a hyperbolic attracting periodic orbit in Δ_μ, then $\mu \in H$.

In fact, from (iii) we have that there exists at most one attracting periodic orbit in I_2 (Singer's theorem [4]) which attracts the critical point. Then, in both cases all periodic points are hyperbolic. Therefore, by Mañe's theorem [5], we have that f_μ is hyperbolic.

Denote by m the Lebesgue measure. Theorem 3 is a consequence of the following theorem.

THEOREM 4. – *There exists $\overline{\mu} > 0$ such that $m(H \cap [0, \overline{\mu}]) = \overline{\mu}$.*

To obtain this result we will prove that there exists $\overline{\mu} > 0$ such that for every $\mu_0 \in [0, \overline{\mu}]$,

$$\lim_{\epsilon \to 0} \frac{(H^c \cap [\mu_0 - \epsilon, \mu_0 + \epsilon])}{2\epsilon} < 1$$

where H^c is the complement of H in $\mathbb{R}$.

Suppose that $f_\mu^j(3) \in \cup_{i=1}^4 I_i(\mu)$, $0 \le j \le N$. Define the sequence $\xi_N(\mu)$ by

$$\xi_N(\mu)(j) = i \quad if \quad f_\mu^j(3) \in I_i(\mu); \quad 0 \le j \le N$$

LEMMA 2. – *Suppose that $\xi_N(\mu)$ is defined and constant on the interval $[\mu_0, \mu_1]$. Then, there exists $M > 0$ such that $\mathcal{X}_n(\mu) \le M\mathcal{X}_n(\mu_0)$ for every $\mu \in [\mu_0, \mu_1]$ where $\mathcal{X}_n(\mu) = \partial_\mu f_\mu^n(3)$.*

This lemma follows immediately from

LEMMA 3. – *Under the above conditions. If $f_\mu^n(3) \in I_1 \cup I_4(\mu)$, then*

$$\frac{\mathcal{X}_{n+1}(\mu)}{\mathcal{X}_{n+1}(\mu_0)} \le 2.$$

Proof. – Firstly, observe that $\mathcal{X}_n(\mu) > \frac{1}{2}$ for every $n \le N$. In fact

$$\begin{array}{ll}
\mathcal{X}_{n+1}(\mu) = \rho\mathcal{X}_n(\mu) & f_\mu^n(3) \in I_1 \\
\mathcal{X}_{n+1}(\mu) = g'_\mu(f_\mu^n(3))\mathcal{X}_n(\mu) + \partial_\mu g_\mu(x)|_{x=f_\mu^n(3)} & f_\mu^n(3) \in I_2 \\
\mathcal{X}_{n+1}(\mu) = \sigma\mathcal{X}_n(\mu) & f_\mu^n(3) \in I_3 \\
\mathcal{X}_{n+1}(\mu) = f'_\mu(f_\mu^n(3))\mathcal{X}(\mu) + \partial_\mu f_\mu(x)|_{x=f_\mu^n(3)} & f_\mu^n(3) \in I_4
\end{array}$$

Hence, using the properties of f_μ the assertion follows by inductions. Since, if $f_\mu^n(3) \in I_2$ then $\frac{\partial_\mu f_\mu(x)|_{x=f_\mu^n(3)}}{\mathcal{X}_n(\mu)}$ is small for μ small enough, lemma 2 it follows.

The hypotheses of lemma implies that

$$f_\mu^n(3) - f_{\mu_0}^n(3) \ge \frac{1}{2}(\mu - \mu_0)$$

Now we prove the lemma 2 by induction.

1. – If $f_\mu^n(3) \in I_1$ then $f_\mu^{n+1}(3) = \rho f_\mu^n(3)$. Hence we have

$$\frac{\mathcal{X}_{n+1}(\mu)}{\mathcal{X}_{n+1}(\mu_0)} \le 2.$$

2.- If $f_\mu^n(3) \in I_4$ then $f_\mu^n(3) = f_\mu^{n_0+1+k+1}(3)$ where $f_\mu^{n_0}(3) \in I_1$ and $f_\mu^{n_0+1}(3) \in I_2$. At once $f_\mu^{n+1}(3) = f_\mu(h_\mu(f_\mu^{n_0+1}(3)))$ where $h_\mu = \sigma^k(g_\mu - 2) + 2$. Thus we obtain

$$\begin{aligned}\mathcal{X}_{n+1}(\mu) =&\partial_\mu f_\mu(h_\mu(f_\mu^{n_0+1}(3))) + \partial_x f_\mu(h_\mu(f_\mu^{n_0+1}(3)))\partial_\mu h_\mu(f_\mu^{n_0+1}(3)) + \\ &\partial_x f_\mu(h_\mu(f_\mu^{n_0+1}(3)))\partial_x h_\mu(f_\mu^{n_0+1}(3))\mathcal{X}_{n_0+1}(\mu).\end{aligned}$$

Now we claim that

a) $$\partial_\mu f_\mu(h_\mu(f_\mu^{n_0+1}(3))) + \partial_x f_\mu(h_\mu(f_\mu^{n_0+1}(3)))\partial_\mu h_\mu(f_\mu^{n_0+1}(3)) \leq 2[\partial_\mu f_\mu(h_\mu(f_\mu^{n_0+1}(3))) + \partial_x f_\mu(h_\mu(f_\mu^{n_0+1}(3)))\partial_\mu h_\mu(f_\mu^{n_0+1}(3))]|_{\mu=\mu_0}$$

b) $$\partial_x f_\mu(h_\mu(f_\mu^{n_0+1}(3)))\partial_x h_\mu(f_\mu^{n_0+1}(3)) \leq \partial_x f_{\mu_0}(h_{\mu_0}(f_{\mu_0}^{n_0+1}(3)))\partial_x h_{\mu_0}(f_{\mu_0}^{n_0+1}(3))$$

Then the lemma follows by induction.

Proof of (a):

Let $x_0 = f_{\mu_0}^{n_0+1}(3)$, $x_1 = f_\mu^{n_0+1}(3)$; x_0 , $x_1 \in I_2$, $y_0 = h_{\mu_0}(x_0)$, $y_1 = h_\mu(x_1)$; y_0 , $y_1 \in I_4$

So using **i** we have that

$$2\partial_\mu f_\mu(y_0)|_{\mu=\mu_0} - \partial_\mu f_\mu(y_1) \approx 1.$$

On the other hand, $\partial_\mu g_\mu(a) = 0$ we have that for every $x \in I_2$ such that $f_\mu^k(g_\mu(x)) = y \in I_4$

$$\partial_\mu g_\mu(x) = \partial^2_{\mu,x} g_\mu(z)[\partial_x g_\mu(\tilde{z})]^{-1}\sigma^{-k}(y-2)$$

where $z, \tilde{z} \in I_2$

Thus $|\partial_\mu h_\mu(x)| = |\sigma^k \partial_\mu g_\mu| \leq C$, C independing of k. From here and **ii** we obtain that

$$\partial_x f_\mu(y)\partial_\mu h_\mu(x) - 2[\partial_x f_{\mu_0}(y_0)\partial_\mu h_\mu(x_0)]|_{\mu=\mu_0} \approx 0$$

Thus **(a)** is proved.

Proof of (b):

Let $F(\mu, x) = \partial_x(f_\mu \circ h_\mu)(x)$. In order to have **(b)** is sufficient to prove that $DF_{(\mu,x)}(1, v) < 0$ for all $v > \frac{1}{2}$. This is clear since

$$DF_{(\mu,x)}(1, v) = \sigma^{2k}(3-\tilde{x})^{\alpha-2}[(3-\tilde{x})M_k(\mu, x) + L_k(\mu, x)v]$$

and $L_k(\mu, x) \leq c < 0$, c is independent of k, M_k are uniformly bounded in k and $\tilde{x} = f_\mu^k(g_\mu(x)) \in I_4$.

To finish the proof of the theorem we can follow [2].

ACKNOWLEDGEMENTS

The author would like to express his grateful to Prof. J Palis Jr. for all the comments during the preparation of the present paper. Also want to express his acknowledgments to C. Morales and I. Huerta by all the fruitful conversations related with this work.

REFERENCES

[1] R. Bamon, R. Labarca, R. Mañe and M. J. Pacifico , The explosion of singular cycles. I.H.E.S., *Publications Mathématiques*, Vol. **78**, 1993, pp. 207-232.

[2] M. J. Pacifico and A. Rovella, Unfolding contracting cycles. *Ann. Scient. Ec. Norm. Sup.*, 4^e serie t.26, 1993, pp. 691-700.

[3] A. Rovella, The Dynamics of Perturbations of the contracting Lorenz attractor, *Bol. Soc. Bras. Math.*, vol **24**, N. 2, 1993, pp. 233-259.

[4] S. Singer, Stable orbits and bifurcations of map of the interval, *SIAM J. Appl. Math.*, Vol. **35**, 1978, pp. 260-267.

[5] R. Mañe, Hyperbolicity, sinks and measure in one dimensional dynamics, *Comm. Math. Phys.*, Vol. **100**, 1985, pp. 495-524 and Erratum, *Comm. Math. Phys.*, Vol. **112**, 1987, pp. 721-724.

(Manuscript received July 10, 1996.)