Saddle-focus singular cycles
and prevalence of hyperbolicity

by

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ABSTRACT. – The objective of this work is to study the 2-parameter unfolding of an intricate bifurcating dynamical structure in dimension 3, namely a saddle-focus singular cycle. It is showed that hyperbolicity is a prevalent phenomena: the initial system is a total density point of hyperbolic dynamics. This dynamic exhibit either simple critical elements or nontrivial basic set. It seems to be the first time that non-trivial hyperbolicity (as well as its prevalence) is proved for the maximal invariant set for a perturbation of a vector field with a cycle containing a saddle-focus singularity.

RÉSUMÉ. – L'objectif de ce travail est l'étude du développement à 2 paramètres d'une structure dynamique complexe avec bifurcation en dimension 3, à savoir un cycle singulier « selle-foyer ». Nous montrons que l'hyperbolicité est un pénomène prévalant : le système initial est un point de densité totale de dynamiques hyperboliques. Cette dynamique présente ou bien des éléments critiques simples ou un ensemble fondamental non trivial. Cela semble le premier ensemble où l'hyperbolicité non triviale (et sa prévalence) est prouvée pour l'ensemble invariant maximal pour une perturbation d'un champ vecteur à un cycle contenant une singularité « selle-foyer ».

1. INTRODUCTION

The main aim of this work is to show that in the unfolding of a three-dimensional vector field having a saddle-focus singular cycle, hyperbolicity is a prevalent phenomena, *i.e.*, the set of parameters corresponding to Axiom A flows has full Lebesgue density at this bifurcation value.

Mechanisms involving diffeomorphisms in two-dimensional manifolds which unfold simple dynamics into a complicate one were studied, for instance in [3] and [5]. Essentially, the main phenomenon involved in these works is the appearance and unfolding of homoclinic tangencies. This yields non hyperbolicity for a positive Lebesgue measure set of parameters and even infinitely many sinks for a residual set on intervals close to the first bifurcation value in the parameter line (see [2] and [4]). Nevertheless, it was proved that hyperbolicity has Lebesgue density one at its bifurcation value ([8]).

A similar result is also true when the simple dynamic diffeomorphism is changed by a non-trivial Axiom A diffeomorphism, provided that the limit capacity or Hausdorff dimension of the basic sets involved in the cycle are not too large (see [8]).

The vector field case containing a cycle which has a singularity is quite different. For instance, in [1] was treated a certain class of cycles, called singular cycles, formed by periodic orbits and a unique singularity which is hyperbolic and has two negative different eigenvalues. It was proved, in the so-called expansive case, that the set of bifurcation values is included in a Cantor set with small limit capacity and, therefore, the hyperbolicity has total Lebesgue measure in the parameter space. In the contractive case, a similar result has been obtained for the measure of the parameter set corresponding to hyperbolic dynamics [6], [11].

The cycle studied here was motivated by the example in $\mathbb{R}^3$ pointed out in [12]. This consists of two singularities, one of them is a singularity with real eingenvalues as in the singular cycle case mentioned above, and the other is a saddle-focus index–two singularity. In that work it was proved that three dimensional vector field exhibiting such a cycle has a similar dynamics as the so-called Šil'nikov cycle (see [12]), provided that certain twist condition related to the intersection between the central manifolds of the singularities holds.

Now, what we call a saddle-focus singular cycle is just the above cycle with the oppositive twist condition. So that the cycle is now isolated: the cycle is the maximal invariant set in a neighborhood of it. It is a co-dimension-two cycle and we study it through generic two parameter

unfoldings. The novelty of this cycle is that it possible to prove, for small perturbations of the original field, that the maximal invariant set in a fixed neighborhood of the cycle is a hyperbolic nontrivial set. Moreover, the dynamic is hyperblolic for a set with full density at this bifurcation value, *i.e.*, the set of hyperbolic parameters has total density at the $(0,0)$ parameter. This extends the results that showed the existence of hyperbolic set like finite symbol subshift.

Let us now give the precise definition and statements of our results. Let $\mathcal{X}^r$ be the space of C^r vector fields on $\mathbb{R}^3$. Given $X \in \mathcal{X}^r$, we denote by $\Gamma(X)$ its chain recurrent set.

A cycle of a vector field $X \in \mathcal{X}^r$ is a compact invariant chain recurrent set of X, consisting of a finite family of critical elements and orbits whose α and $\omega - limit$ set are critical element of the family. Critical elements are periodic orbits or singularities.

We study here the vector fields that presents a simple saddle-focus singular cycle defined as follows.

DÉFINITION. – A simple saddle-focus singular cycle Λ for a vector field $X \in \mathcal{X}^r$ is a cycle satisfying

- Λ contains only two singularities p and q;
- the eigenvalues of $D_pX : \mathbb{R}^3 \hookleftarrow$ are real and satisfy $-\lambda_3 < -\lambda_1 < 0 < \lambda_2$;
- the eigenvalues of $D_qX : \mathbb{R}^3 \hookleftarrow$ are $a \pm ib$ and c where $a < 0, b > 0$ and $c > 0$;
- Λ has a unique non singular orbit γ_0 contained in $W^u(p)$ such that $\omega - \lim(\gamma_0)$ is q and a unique orbit γ_1 contained in $W^u(q)$ such that $\omega - \lim(\gamma_1)$ is p;
- for each $x \in \gamma_0$ and each invariant manifold W of X passing through p and tangent at p to the eigenspace spanned by the eigenvectors associated to $-\lambda_1$ and λ_2, we have

$$T_xW + T_xW^s(q) = \mathbb{R}^3;$$

- there exists a neighborhood $\mathcal{U}$ of X such that if $Y \in \mathcal{U}$, the continuations p_Y and q_Y of p and q are well defined and, the vector field Y is C^2-linearizable at p_Y and q_Y;
- Λ is isolated, *i.e.*, it has an isolating block. Recalling that an isolating block of an invariant set Λ of a vector field X is an open set $U \subset M$ such that $\Lambda = \cap_{t \in \mathbb{R}} X_t(U)$, where $X_t : M \hookleftarrow$ is the flow generated by X.
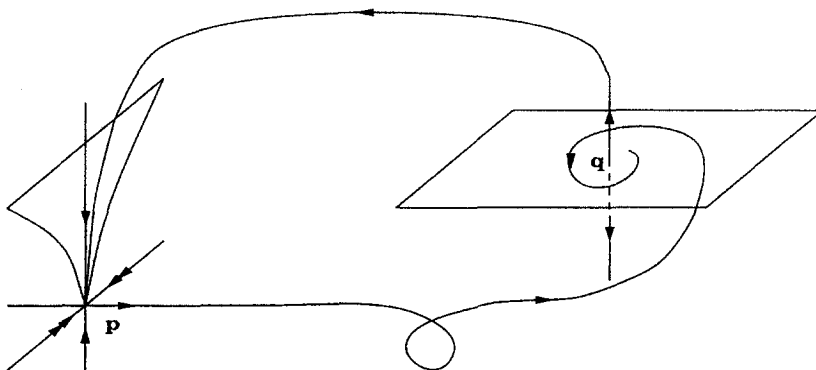
Fig. 1

Now we state the main theorems this work. We use the following notation: If $Y \in \mathcal{X}^r$ and $U \subset \mathbb{R}^3$ is an open set, we denote by $\Lambda(Y, U)$ the set $\cap_t Y_t(U)$, and we denote by $\Gamma(Y)$ the chain recurrent set of $Y|\Lambda(Y, U)$, where Y_t is the flow generated by the vector field Y.

THEOREM 1. – *Let Λ be a simple saddle-focus singular cycle of a vector field X and U be an isolating block of Λ. Then there exists a small neighborhood $\mathcal{U}$ of X and a co-dimension-two submanifold $\mathcal{N}$ through X contained in $\mathcal{U}$ such that if $Y \in \mathcal{N}$, then $\Lambda(Y, U)$ is a simple saddle-focus singular cycle which is topologically equivalent to Λ.*

Let S be a two-dimensional manifold which is transverse to $\mathcal{N}$ at the vector field X. The study of the hyperbolicity of the set $\Gamma(Y), Y \in S$ depends on the following eingenvalues conditions : $-\dfrac{a\lambda_1}{c\lambda_2} < 1$ and $-\dfrac{a\lambda_1}{c\lambda_2} > 1$.

DEFINITION. – Let Λ be a simple saddle-focus singular cycle for a vector field X. We will say that the cycle is **expansive** if it has the first inequality. In the other case, we will say that the cycle is **contractive**.

THEOREM 2. – *Let Λ be a contractive cycle and $\mathcal{H} \subset S$ be the set of vector fields such that $\Gamma(Y)$ is formed by the analytical continuations p_Y, q_Y and at most a unique attracting periodic orbit which is hyperbolic. Then*

$$\lim_{\rho \to 0} \frac{m(\mathcal{H} \cap V_\rho(X))}{\rho^2} = 1$$

*where $V_\rho(X)$ is a ρ-neighborhood of X in S.*

The dynamics in the expansive case is much richer than in the previous case because the maximal invariant set in the neighborhood of the cycle must contain a suspended horse-shoe equivalent to a finite-symbol subshifts. The following is the main result in this work.

THEOREM 3. – *Let Λ be a expansive cycle as above. Let $\mathcal{H}$ be subset of S formed by vector fields such that $\Gamma(Y)$ is formed by p_Y, q_Y and at most one hyperbolic saddle-type basic set. Then*

$$\lim_{\rho \to 0} \frac{m(\mathcal{H} \cap V_\rho(X))}{\rho^2} = 1.$$
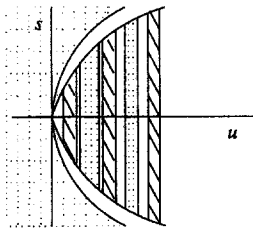
contractive case with c > a
Fig. 2a
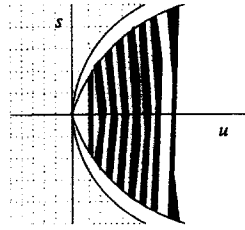
expansive case with $\lambda_1 > \lambda_2$
Fig. 2b

$\square$: $\Lambda(Y,U) = \{p_Y, q_Y\}$ $\boxed{\diagdown}$: $\Lambda(Y,U) = \{p_Y, q_Y\}+$ per. orb. $\blacksquare$: $\Lambda(Y,U) = \{p_Y, q_Y\}+$ H-S

This paper is divided in three sections. In section 2, we prove theorems 1 and 2. The first one is an immediate consequence of usual transversality arguments. The second is proved studying the Poincare map induced by the flow for every vector field belonging to S. It is showed that this map is actually a contracting map for most parameter values. Finally, in section 3 we give a proof of theorem 3. To do this we introduce an approach as the one presented in [8].

2. PROOF OF THEOREMS 1 AND 2

Let Λ be a simple saddle-focus singular cycle for $X \in \mathcal{X}^r$, and let $(\tilde{x}, \tilde{y}, \tilde{z})$ and (x, y, z) be C^2-linearing coordinates in neighborhood of p and q respectively. X has the following form in these coordinates:

$$\begin{aligned}
\dot{\tilde{x}} &= -\lambda_3 \tilde{x} \\
\dot{\tilde{y}} &= \lambda_2 \tilde{y} \quad \forall |(\tilde{x}, \tilde{y}, \tilde{z})| \leq 2 \\
\dot{\tilde{z}} &= -\lambda_1 \tilde{z}
\end{aligned}$$

and

$$\dot{x} = ax - by$$
$$\dot{y} = bx + ay \quad \forall |(x, y, z)| \leq 2$$
$$\dot{z} = cz.$$

Let consider the following Poincare maps defined by the X-flow

$$\pi_L : \mathcal{D}(\pi_L) \subset \Sigma_0 \longrightarrow \Sigma_0$$
$$\pi_i : \mathcal{D}(\pi_i) \subset \Sigma_i \longrightarrow \Sigma_{i+1}$$

where

$$\Sigma_0 = \{(0, y, z) : y \geq 0, |z| \leq 1\}$$
$$\Sigma_1 = \{(x, y, 1) : |x|, |y| \leq 1\}$$
$$\Sigma_2 = \{(\tilde{x}, \tilde{y}, 1) : |\tilde{x}|, |\tilde{y}| \leq 1\}$$
$$\Sigma_3 = \{(\tilde{x}, 1, \tilde{z}) : |\tilde{x}|, |\tilde{z}| \leq 1\}$$

and $\Sigma_4 = \Sigma_0$.

For these maps we have the following properties.

A. $\pi_L(y, z) = (\lambda y, \sigma z)$, $(y, z) \in D(\pi_L) = \{(y, z) \in \Sigma_0; 0 \leq y \leq 1,$ $0 \leq z \leq \sigma^{-1}\}$, where $\lambda = e^{\frac{a2\pi}{b}}$ and $\sigma = e^{\frac{c2\pi}{b}}$.

B. $\pi_0(y, z) = (-z^{-\frac{a}{c}} y \sin(\text{Log}(z^{-\frac{b}{c}})), z^{-\frac{a}{c}} y \cos(\text{Log}(z^{-\frac{b}{c}})))$, for $(y, z) \in D(\pi_0) = \{(y, z) \in \Sigma_0; 0 \leq y \leq 1 \text{ and } \sigma^{-1} \leq z \leq 1\}$.

C. $\pi_1(x, y) = (A(x, y), B(x, y))$ is a C^2-diffeomorphism. We can take C^2 coordinates, iterating Σ_1 by the X-flow if necessary, such that $\partial_y B(0, 0) = 0$ and $\partial_x B(0, 0) < 0$. Therefore $\partial_y A(0, 0) \neq 0$.

D. $\pi_2(\tilde{x}, \tilde{y}) = (\tilde{y}^\beta \tilde{x}, \tilde{y}^\alpha)$, for all $-1 \leq \tilde{x} \leq 1$, and $0 \leq \tilde{y} \leq 1$, where $\beta = \frac{\lambda_3}{\lambda_2}$ and $\alpha = \frac{\lambda_1}{\lambda_2}$.

E. $\pi_3(\tilde{x}, \tilde{z}) = (\tilde{A}(\tilde{x}, \tilde{z}), \tilde{B}(\tilde{x}, \tilde{z}))$ is a C^2 diffeomorphism that satisfies $\partial_{\tilde{z}} \tilde{B}(0, 0) < 0$. This conditions follows from the definition.

Now, let $\mathcal{U}$ be a neighborhood of X as in the definition of simple saddle-focus singular cycle. Taking $\mathcal{U}$ small enough, we can define the maps $\pi_i(Y)$ and $\pi_L(Y)$ associated to $Y \in \mathcal{U}$ in a similar way as we did for the respective maps relative to X. In the case the equations for $\pi_L(Y), \pi_0(Y)$ and $\pi_2(Y)$ are like π_L, π_0 and π_2 changing the corresponding eigenvalues of X by the Y ones. On the other hand, $\pi_1(Y) = (A_Y, B_Y)$ and $\pi_3(Y) = (\tilde{A}_Y, \tilde{B}_Y)$ are diffeomorphisms C^2-close to π_1 and π_3 respectively, and therefore $\partial_x B_Y(0, 0)$ and $\partial_{\tilde{z}} \tilde{B}_Y(0, 0)$ are negatives.

In a similar way as we did for the vector field X, we can take linearizing coordinates for the vector field Y, such that $\partial_y B_Y(0, 0) = 0$.

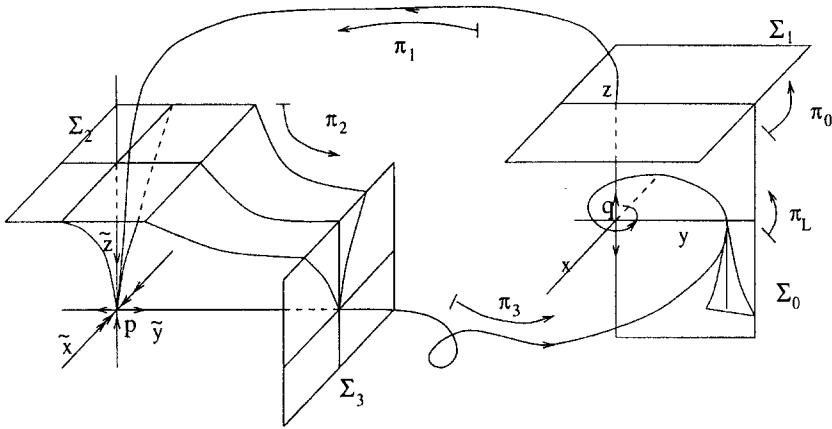*Obs.* – The Y-flow define a Poincare map π_Y from $\mathcal{D}(\pi_Y) \subset \Sigma_0$ into Σ_0.

Fig. 3.

Proof of Theorem 1. – Let $P(Y) = (P_1(Y), P_2(Y)) = \pi_1(Y)(0,0)$ and $Q(Y) = (Q_1(Y), Q_2(Y)) = \pi_3(Y)(0,0)$. Let define the map $F : \mathcal{U} \longrightarrow \mathbb{R}^2$ given by

$$F(Y) = (P_2(Y), Q_2(Y)).$$

Clearly F is a C^2 map and has X as a regular point, therefore $\mathcal{N} = F^{-1}(0,0) \cap \mathcal{U}$ is a co-dimension two manifold containing X and satisfying the wanted properties. ∎

Let $S \subset \mathcal{U}$ be a two-dimension manifold transversal to $\mathcal{N}$ at X. Since $F|_S : S \longrightarrow \mathbb{R}^2$ is C^2 and $D(F|_S)_X$ is a isomorphism, we obtain $Y(u, s) = (F|S)^{-1}(u, s)$ is a C^2-parametrization in a small neighborhood of X in S such that

(i) $Y(0,0) = X$

(ii) $P_2(Y(u, s)) = s$ and $Q_2(Y(u, s)) = u$.

In addition, we can take C^2-linearizing coordinates (depending smoothly on Y) such that $Q_1(Y) = c_0$, c_0 a constant with $\lambda < c_0 < 1$.

Obs. – From now on we will use the notation $p(u, s), q(u, s), A_{u,s}, \cdots$ etc. to mean $p(Y(u, s)), q(Y(u, s)), A_{Y(u,s)}, \cdots$ etc. To simplify the notation we will omit the dependence on u and s for the Poincare Maps.

The following lemma gives us information about the preimages by the map $\pi_3 \circ \pi_2$, of the horizontal lines in a small neighborhood of $(c_0, 0)$ in Σ_0. In particular of $(\pi_3 \circ \pi_2)^{-1}(W^s(q(u, s)))$.

LEMMA 1. – *For small u, s, t with $0 \leq t < u$ we have that the preimage set $\pi_3^{-1}\{(y, t) : y \in [c_0 - \varepsilon, c_0 + \varepsilon]\}$ is given by the graph of a C^2-map $\tilde{z} = \tilde{\Psi}_{u,s,t}(\tilde{x})$ such that*

(i) $|\tilde{\Psi}_{u,s,t}(\tilde{x}) - \tilde{\Psi}_{u,s,t}(0)| \leq d|\tilde{x}| \; \forall \; small \; \tilde{x}, \; where \; d \; is \; a \; positive \\ constant.$

(ii) $\dfrac{t-u}{\partial_{\tilde{z}}\tilde{B}_{0,0}(0,0)}\tilde{C}_2(u,s) \leq \tilde{\Psi}_{u,s,t}(0) \leq \dfrac{t-u}{\partial_{\tilde{z}}\tilde{B}_{0,0}(0,0)}\tilde{C}_1(u,s) \; with$

$$1 - \tilde{C}(|u| + |s|) \leq \tilde{C}_2(u,s) < \tilde{C}_1(u,s) \leq 1 + \tilde{C}(|u| + |s|).$$

(iii) $\pi_2^{-1}(graph\tilde{\Psi}_{u,s,t}) \; is \; given \; by \; the \; graph \; of \; a \; C^2\text{-}map, \\ \tilde{y} = \Psi_{u,s,t}(\tilde{x}) \; for \; all \; |\tilde{x}| \leq 1, \; which \; satisfies$

$$K(u-t)^{\frac{1}{\alpha}}C_2(u,s) \leq \Psi_{u,s,t}(\tilde{x}) \leq K(u-t)^{\frac{1}{\alpha}}C_1(u,s),$$

where $K = |\partial_{\tilde{z}}\tilde{B}_{0,0}(0,0)|^{-\frac{1}{\alpha}}$ and

$$[1 - \tilde{C}(|u| + |s| + |u|^p)]^{\frac{1}{\alpha}} \leq C_2(u,s) < C_1(u,s)$$
$$\leq [1 + \tilde{C}(|u| + |s| + |u|^p)]^{\frac{1}{\alpha}}$$

with $p = \frac{\beta}{\alpha} - 1$. Here, $\tilde{C}$ is a positive constant.

Proof. – It is clear that $(\tilde{x}, \tilde{z}) \in \pi_3^{-1}\{(y,t) : y \in [c_0 - \varepsilon, c_0 + \varepsilon]\}$ if and only if $\tilde{B}_{u,s}(\tilde{x}, \tilde{z}) = t$. Since $\partial_{\tilde{z}}\tilde{B}_{u,s}(0,0) < 0$ the Implicit Function Theorem give us a C^2 positive map $\tilde{z} = \tilde{\Psi}_{u,s,t}(\tilde{x})$ that satisfies $\tilde{B}_{u,s}(\tilde{x}, \tilde{z}) = t$. Moreover,

$$\left|\partial_{\tilde{x}}\tilde{\Psi}_{u,s,t}(\tilde{x})\right| = \left| -\frac{\partial_{\tilde{x}}\tilde{B}_{u,s}(\tilde{x}, \tilde{\Psi}_{u,s,t}(\tilde{x}))}{\partial_{\tilde{z}}\tilde{B}_{u,s}(\tilde{x}, \tilde{\Psi}_{u,s,t}(\tilde{x}))} \right| \leq d$$

for all $u, s, \tilde{x}$ small. Therefore we obtain (i).

On the other hand,

$$t = \tilde{B}_{u,s}(0,0) + \partial_{\tilde{z}}\tilde{B}_{u,s}(0,0)\tilde{\Psi}_{u,s,t}(0) + R_{u,s}(0, \tilde{\Psi}_{u,s,t}(0))$$

where $|R_{u,s}(0, \tilde{\Psi}_{u,s,t}(0))| \leq C[\tilde{\Psi}_{u,s,t}(0)]^2$. Thus

$$[\partial_{\tilde{z}}\tilde{B}_{u,s}(0,0) - C\tilde{\Psi}_{u,s,t}(0)]\tilde{\Psi}_{u,s,t}(0)$$
$$\leq t - u$$
$$\leq [\partial_{\tilde{z}}\tilde{B}_{u,s}(0,0) + C\tilde{\Psi}_{u,s,t}(0)]\tilde{\Psi}_{u,s,t}(0). \quad (1)$$

Furthermore, since $\tilde{\Psi}_{u,s,t}(0)$ is small, then $\tilde{\Psi}_{u,s,t}(0) \geq 0$. Hence

$$\frac{t-u}{\partial_{\tilde{z}}\tilde{B}_{u,s}(0,0) + \varepsilon} \leq \tilde{\Psi}_{u,s,t}(0) \leq \frac{t-u}{\partial_{\tilde{z}}\tilde{B}_{u,s}(0,0) - \varepsilon}. \quad (2)$$

Replacing (2) in (1) and arranging terms, we obtain that

$$(t - u)\frac{1 - \tilde{C}(u - t)}{\partial_{\tilde{z}}\tilde{B}_{u,s}(0,0)} \leq \tilde{\Psi}_{u,s,t}(0) \leq (t - u)\frac{1 + \tilde{C}(u - t)}{\partial_{\tilde{z}}\tilde{B}_{u,s}(0,0)}.$$

Now we obtain (ii) in the lemma taking $\tilde{C}_1(u, s) = (1 + \tilde{C}u)\frac{\partial_{\tilde{z}}\tilde{B}_{0,0}(0,0)}{\partial_{\tilde{z}}\tilde{B}_{u,s}(0,0)}$ and $\tilde{C}_2(u, s) = (1 - \tilde{C}u)\frac{\partial_{\tilde{z}}\tilde{B}_{0,0}(0,0)}{\partial_{\tilde{z}}\tilde{B}_{u,s}(0,0)}$.

To prove the third item of the lemma we observe that $\pi_2(\Sigma_2)$ is contained in the up-side of graph $\tilde{z} = |\tilde{x}|^{\frac{\alpha}{\beta}}$. Let define $\tilde{x}_1$ and $\tilde{x}_2$ by the equations

$$\tilde{x}_1^{\frac{\alpha}{\beta}} = \tilde{\Psi}_{u,s,t}(0) - d\tilde{x}_1,$$

$$\tilde{x}_2^{\frac{\alpha}{\beta}} = \tilde{\Psi}_{u,s,t}(0) + d\tilde{x}_2. \tag{3}$$

It follows from above and (i) that if $(\tilde{x}, \tilde{\Psi}_{u,s,t}(\tilde{x})) \in \pi_2(\Sigma_2)$ then $\tilde{x}_1^{\frac{\alpha}{\beta}} \leq \tilde{\Psi}_{u,s,t}(\tilde{x}) \leq \tilde{x}_2^{\frac{\alpha}{\beta}}$.

From (3) we have that

$$\tilde{x}_2^{\frac{\alpha}{\beta}} \leq \frac{t - u}{\partial_{\tilde{z}}\tilde{B}_{0,0}(0,0)}\tilde{C}_1(u, s)\left[1 + \tilde{C}\left(\frac{t - u}{\partial_{\tilde{z}}\tilde{B}_{0,0}(0,0)}\tilde{C}_1(u, s)\right)^p\right]$$

and

$$\tilde{x}_1^{\frac{\alpha}{\beta}} \geq \frac{t - u}{\partial_{\tilde{z}}\tilde{B}_{0,0}(0,0)}\tilde{C}_2(u, s)\left[1 - \tilde{C}\left(\frac{t - u}{\partial_{\tilde{z}}\tilde{B}_{0,0}(0,0)}\tilde{C}_2(u, s)\right)^p\right]$$

where $p = \frac{\alpha}{\beta} - 1$.

The Implicit Function Theorem implies that $\pi_2^{-1}(graph\tilde{\Psi}_{u,s,t})$ is the graph of a C^2-map $\Psi_{u,s,t}$ defined for all $|\tilde{x}| \leq 1$. By the inequalities above it follows that

$$K(u - t)^{\frac{1}{\alpha}}C_2(u, s) \leq \Psi_{u,s,t}(\tilde{x}) \leq K(u - t)^{\frac{1}{\alpha}}C_1(u, s)$$

where

$$C_2(u, s) = \left[\tilde{C}_2(u, s)\left(1 + \tilde{C}\left(\frac{-u}{\partial_{\tilde{z}}\tilde{B}_{0,0}(0,0)}\tilde{C}_2(u, s)\right)^p\right)\right]^{\frac{1}{\alpha}}$$

and

$$C_1(u, s) = \left[\tilde{C}_1(u, s)\left(1 + \tilde{C}\left(\frac{-u}{\partial_{\tilde{z}}\tilde{B}_{0,0}(0,0)}\tilde{C}_1(u, s)\right)^p\right)\right]^{\frac{1}{\alpha}}$$

which satisfy the required properties. ∎

Obs. – a) Applying the previous lemma in the case $t = 0$ we obtain that $(\pi_3 \circ \pi_2)^{-1}(W^s(q(u,s)))$ is given by the graph of a C^2-map $\tilde{y} = \Psi_{u,s}(\tilde{x})$ defined for all $|\tilde{x}| \leq 1$ which satisfies that

$$K u^{\frac{1}{\alpha}} C_2(u,s) \leq \Psi_{u,s}(\tilde{x}) \leq K u^{\frac{1}{\alpha}} C_1(u,s).$$

b) In particular Eq. (2) above implies that $\{(y,z) \in \pi_3 \circ \pi_2(\Sigma_2) : z \geq 0\}$ is contained in $[c_0 - \tilde{K}u, c_0 + \tilde{K}u] \times [0, u]$, where $\tilde{K}$ is some positive constant.

Proof of Theorem 2. – Clearly, if $u < 0$ then $\Gamma(Y_{u,s}) = \{p(u,s), q(u,s)\}$ where U is the isolating neighborhood of the cycle, given by the definition.

Let consider in the section Σ_0, the box B given by

$$B = [c_0 - \tau, c_0 + \tau] \times [0, 1]$$

with small τ.

Given s and $u > 0$, $n = n(u,s)$ will define from now on, the unique integer that satisfies the inequalities

$$\sigma_{(u,s)}^{-(n+1)} < u \leq \sigma_{(u,s)}^{-n}. \tag{4}$$

In addition let define the boxes

$$R_k = \pi_L^{-k}(R_0) \cap B$$

$$H_k = \pi_L^{-k}(H_0) \cap B$$

where $R_0 = [0,1] \times [\sqrt{\sigma_{(u,s)}^{-1}}, 1]$ and $H_0 = [0,1] \times [\sigma_{(u,s)}^{-1}, \sqrt{\sigma_{(u,s)}^{-1}}]$.

Let define the box $R_{u,s}$ by

$$R_{u,s} = \bigcup_{k \geq n}(R_k \cup H_k) \cup [c_0 - \tau, c_0 + \tau] \times \{0\} = [c_0 - \tau, c_0 + \tau] \times [0, \sigma_{(u,s)}^{-n}].$$

It is clear that every orbit in the new non-wandering set must intersects Σ_0 in the set $R_{u,s}$. **From this the $Y_{u,s}$-flow lead these orbits to intersect the section Σ_1 in the interior of the ball of radius $\lambda_{(u,s)}^n$ centered in** $(0,0)$. In the same way, we obtain that these orbits meet Σ_2 in the ball of radius $C\lambda_{(u,s)}^n$ (C positive constant) centered at the point $W^u(q(u,s)) \cap \Sigma_2$, that is, the point with coordinates $(P_1(u,s), s)$.

From the equation (4) we get:

$$-\frac{\text{Log}(u)}{\text{Log}(\sigma)} - 1 < n \leq -\frac{\text{Log}(u)}{\text{Log}(\sigma)}$$

$$u^{-\frac{\text{Log}(\lambda)}{\text{Log}(\sigma)}} \leq \lambda_{(u,s)}^{n} < \lambda_{(u,s)}^{-1} u^{-\frac{\text{Log}(\lambda)}{\text{Log}(\sigma)}}.$$

We will study four cases separately

1. $\mathcal{R}_1 = \{(u,s) : s \leq -a_1 u^{-\frac{\text{Log}(\lambda)}{\text{Log}(\sigma)}}\}$
2. $\mathcal{R}_2 = \{(u,s) : s \geq a_2 u^{\frac{1}{\alpha}}\}$
3. $\mathcal{R}_3 = \{(u,s) : a_3 u^{-\frac{\text{Log}(\lambda)}{\text{Log}(\sigma)}} \leq s \leq \tilde{a}_3 u^{\frac{1}{\alpha}}\}$
4. $\mathcal{R}_4 = \{(u,s) : |s| \leq a_4 u^{\delta - \frac{\text{Log}(\lambda)}{\text{Log}(\sigma)}}\}$

where the constants are independent of (u,s) and satisfy uniformly on (u,s) the inequalities $a_1, a_3 > \lambda_{(u,s)}^{-1} C$, $a_2 > 2K$, $0 < \tilde{a}_3 < \frac{K}{4}$, $0 < a_4 < 1$ and δ is a small positive real number.

Let observe that the set of parameters that belong to the complement of $\cup_{i=1}^{4} \mathcal{R}_i$ in $\{(u,s) : u \geq 0\}$ has zero Lebesgue density at the origin $(0,0)$.

CASE 1. – For $(u,s) \in \mathcal{R}_1$ we have that

$$s + C\lambda_{(u,s)}^{n} \leq s + C\lambda_{(u,s)}^{-1} u^{-\frac{\text{Log}(\lambda)}{\text{Log}(\sigma)}}$$
$$< s + a_1 u^{-\frac{\text{Log}(\lambda)}{\text{Log}(\sigma)}}$$
$$\leq 0$$

from this $\Gamma(Y_{u,s}) = \{p(u,s), q(u,s)\}$. Therefore $\mathcal{R}_1$ is a region filled by Morse-Smale systems.

CASE 2. – For $(u,s) \in \mathcal{R}_2$ we have that

$$s - C\lambda_{(u,s)}^{n} - Ku^{\frac{1}{\alpha}}C_1(u,s) \geq s - C\lambda_{(u,s)}^{-1} u^{-\frac{\text{Log}(\lambda)}{\text{Log}(\sigma)}} - Ku^{\frac{1}{\alpha}}C_1(u,s)$$
$$= s - u^{\frac{1}{\alpha}}(C\lambda_{(u,s)}^{-1} u^{-\frac{\text{Log}(\lambda)}{\text{Log}(\sigma)} - \frac{1}{\alpha}} + KC_1(u,s))$$
$$\geq s - 2Ku^{\frac{1}{\alpha}}$$
$$> s - a_2 u^{\frac{1}{\alpha}}$$
$$\geq 0$$

since $\dfrac{\text{Log}(\lambda)}{\text{Log}(\sigma)} + \dfrac{1}{\alpha} < 0$, because $-\dfrac{a\lambda_1}{c\lambda_2} > 1$.

Therefore $\Gamma(Y_{u,s}) = \{p(u,s), q(u,s)\}$. Hence $\mathcal{R}_2$ is a region filled by Morse-Smale systems.

In the third case we will show the existence of one attracting periodic orbit, for which we will need to estimate the derivative of the Poincare map π_Y.

LEMMA 2. – *Let (y, z) be a point in $R_m \cup H_m$ such that $\tilde{y}_{(m)} > 0$, where $(\tilde{x}_{(m)}, \tilde{y}_{(m)}) = \pi_1 \circ \pi_0 \circ \pi_L^m(y, z)$ and m is a large positive integer. Then*

$$
D\pi(y, z) = D(\pi_3 \circ \pi_2 \circ \pi_1 \circ \pi_0 \circ \pi_L^m)(y, z)
$$
$$
= \begin{pmatrix} \lambda_{(u,s)}^m \tilde{y}_{(m)}^{\alpha-1} H_1 & \lambda_{(u,s)}^m \sigma_{(u,s)}^m \tilde{y}_{(m)}^{\alpha-1} H_2 \\ \lambda_{(u,s)}^m \tilde{y}_{(m)}^{\alpha-1} H_3 & \lambda_{(u,s)}^m \sigma_{(u,s)}^m \tilde{y}_{(m)}^{\alpha-1} H_4 \end{pmatrix}
$$

where H_i are uniformly bounded functions on both m and (u, s).

Proof. – It follows straightforward from the definitions of the maps involved in the computation. ∎

CASE 3. – If $(u, s) \in \mathcal{R}_3$ then

$$
\begin{aligned}
s + C\lambda_{(u,s)}^n - K u^{\frac{1}{\alpha}} C_2(u, s) &\leq s + C\lambda_{(u,s)}^{-1} u^{-\frac{\mathrm{Log}(\lambda)}{\mathrm{Log}(\sigma)}} - K u^{\frac{1}{\alpha}} C_2(u, s) \\
&= s + u^{\frac{1}{\alpha}} \left(C\lambda_{(u,s)}^{-1} u^{-\frac{\mathrm{Log}(\lambda)}{\mathrm{Log}(\sigma)} - \frac{1}{\alpha}} - K C_2(u, s) \right) \\
&\leq s - \frac{K}{2} u^{\frac{1}{\alpha}} \\
&< s - 2\tilde{a}_3 u^{\frac{1}{\alpha}} \\
&\leq -\tilde{a}_3 u^{\frac{1}{\alpha}} \\
&< 0
\end{aligned} \tag{5}
$$

and

$$
\begin{aligned}
s - C\lambda_{(u,s)}^n &\geq s - C\lambda_{(u,s)}^{-1} u^{-\frac{\mathrm{Log}(\lambda)}{\mathrm{Log}(\sigma)}} \\
&\geq a_3 u^{-\frac{\mathrm{Log}(\lambda)}{\mathrm{Log}(\sigma)}} - C\lambda_{(u,s)}^{-1} u^{-\frac{\mathrm{Log}(\lambda)}{\mathrm{Log}(\sigma)}} \\
&= \left(a_3 - C\lambda_{(u,s)}^{-1} \right) u^{-\frac{\mathrm{Log}(\lambda)}{\mathrm{Log}(\sigma)}} \\
&> 0.
\end{aligned} \tag{6}
$$

With this we obtain that $\pi(R_{u,s}) \subset \mathrm{int}\, R_{u,s}$. Moreover, the inequality (5) together with the definition of $K = |\partial_{\tilde{z}} \tilde{B}_{0,0}(0, 0)|^{-\frac{1}{\alpha}}$ give us that there exists a $\tilde{C} < 1$ such that, if $(y, z) \in \Sigma_0 \cap \pi(R_{u,s})$ then $y \geq \tilde{C} u$.

From now on $\tilde{C}$ will came denote differents constants. Next, lemma 2 implies for $(y, z) \in \pi(R_{u,s}) \cap (R_m \cup H_m)$, that:

$$
\|D\pi(y, z)\| \leq \tilde{C} \lambda_{(u,s)}^m \sigma_{(u,s)}^m \tilde{y}_{(m)}^{\alpha-1} \leq \begin{cases} \tilde{C} u^{-\frac{\mathrm{Log}(\lambda)}{\mathrm{Log}(\sigma)} - \frac{1}{\alpha}} & \alpha > 1 \\ \tilde{C} u^{-\alpha \frac{\mathrm{Log}(\lambda)}{\mathrm{Log}(\sigma)} - 1} & \alpha < 1 \end{cases}
$$

since

$$(a_3 - C\lambda_{(u,s)}^{-1})u^{-\frac{\mathrm{Log}(\lambda)}{\mathrm{Log}(\sigma)}} \le \tilde{y}_{(m)} \le (KC_2(u,s) - \tilde{a}_3)u^{\frac{1}{\alpha}}$$

and

$$\tilde{C}u \le y \le \sigma_{(u,s)}^{-m}.$$

Therefore $\Gamma(Y_{u,s})$ is formed by $\{p(u,s), q(u,s)\}$ plus a unique periodic orbit, which is an attracting hyperbolic one since $\|D\pi(y,z)\|$ is uniformly small in all region.

CASE 4. – Here we use the following lemmas.

LEMMA 3. – *Let define* $f_{m,u,s}(y_0, \cdot) : [\sigma_{(u,s)}^{-1}, 1] \mapsto \mathbb{R}$ *by*

$$f_{m,u,s}(y_0, z) = \frac{1}{\lambda_{(u,s)}^m}[B_{u,s} \circ \pi_0(\lambda_{(u,s)}^m y_0, z) - s]$$

where y_0 *is fixed. Then*

$$f_{m,u,s}(y_0, \cdot) \xrightarrow{C^2} f_{u,s}(y_0, z) = -\partial_x B_{u,s}(0,0)y_0 z^{-\frac{a}{c}} \sin(\mathrm{Log}(z^{-\frac{b}{c}})).$$

Moreover,

$$\|f_{m,u,s}(y_0, \cdot) - f_{u,s}(y_0, \cdot)\|_{C^2} \le \tilde{C}\lambda_{(u,s)}^m.$$

Proof. – It follows easily from the definitions. ∎

LEMMA 4. – *Let* $(u,s) \mapsto \lambda_{(u,s)} \in (0,1)$ *be a Lipschitz map defined in a neighborhood of* $(0,0) \in \mathbb{R}^2$. *Then, given* $\rho \in (0,1)$ *there are* $\epsilon > 0$ *and* m_0 *such that if* $|(u,s)| < \varepsilon$ *and* $m > m_0$ *then*

$$|\lambda_{(u,s)}^m - \lambda_{(0,0)}^m| < [\lambda_{(0,0)}^m]^\rho(|u| + |s|).$$

Proof.

$$\begin{aligned}
|\lambda_{(u,s)}^m - \lambda_{(0,0)}^m| &= \lambda_{(0,0)}^m \left| \left(\frac{\lambda_{(u,s)}}{\lambda_{(0,0)}}\right)^m - 1 \right| \\
&\le \lambda_{(0,0)}^m |(1 + K(|u| + |s|))^m - 1| \\
&\le \lambda_{(0,0)}^m Km[1 + K(|u| + |s|)]^{m-1}(|u| + |s|)
\end{aligned}$$

where K is some positive constant. For $|(u,s)|$ small and m_0 a large integer then, for any $m \ge m_0$

$$Km[1 + K(|u| + |s|)]^{m-1} \le (\lambda_{(0,0)}^m)^{\rho-1}.$$

Therefore

$$|\lambda^m_{(u,s)} - \lambda^m_{(0,0)}| < (\lambda^m_{(0,0)})^\rho (|u| + |s|).$$ ■

Now we are ready to study the case 4. If $(u, s) \in \mathcal{R}_4$ using lemmas 2 and 3 we obtain boxes $\tilde{R}_n \subset R_n$ and $\tilde{H}_n \subset H_n$ given by

$$\tilde{R}_n = [c_0 - \tau, c_0 + \tau] \times [\sigma^{-n}_{(u,s)} \sqrt{\sigma^{-1}_{(u,s)}} + \sigma^{-n}_{(u,s)} u^\gamma, \sigma^{-n}_{(u,s)} - \sigma^{-n}_{(u,s)} u^\gamma]$$

$$\tilde{H}_n = [c_0 - \tau, c_0 + \tau] \times [\sigma^{-(n+1)}_{(u,s)} + \sigma^{-n}_{(u,s)} u^\gamma, \sigma^{-n}_{(u,s)} \sqrt{\sigma^{-1}_{(u,s)}} - \sigma^{-n}_{(u,s)} u^\gamma]$$

such that

- π is defined in $\tilde{R}_n$ and $|D\pi(y,z)| < 1 \ \forall (y,z) \in \tilde{R}_n$,
- $\tilde{H}_n$ is wandering,

where $0 < \gamma < \delta$.

Moreover, $\pi(\tilde{R}_{u,s})$ is contained in the ball of radius $\tilde{C} u^{-\alpha \frac{\log(\lambda)}{\log(\sigma)}}$ centered at (c_0, u).

With this we obtain that

(i) If

$$\sigma^{-n}_{(u,s)} \sqrt{\sigma^{-1}_{(u,s)}} + \sigma^{-n}_{(u,s)} u^\gamma < u - \tilde{C} u^{-\alpha \frac{\mathrm{Log}(\lambda)}{\mathrm{Log}(\sigma)}}$$

and

$$u + \tilde{C} u^{-\alpha \frac{\mathrm{Log}(\lambda)}{\mathrm{Log}(\sigma)}} < \sigma^{-n}_{(u,s)} + \sigma^{-n}_{(u,s)} u^\gamma$$

then $\Gamma(Y_{(u,s)})$ is formed by $p(u,s), q(u,s)$, plus a unique attracting periodic orbit, whose orbit pass through $\tilde{R}_n$ and it is hyperbolic.

(ii) If

$$\sigma^{-(n+1)}_{(u,s)} + \sigma^{-n}_{(u,s)} u^\gamma < u - \tilde{C} u^{-\alpha \frac{\mathrm{Log}(\lambda)}{\mathrm{Log}(\sigma)}}$$

and

$$u + \tilde{C} u^{-\alpha \frac{\mathrm{Log}(\lambda)}{\mathrm{Log}(\sigma)}} < \sigma^{-n}_{(u,s)} \sqrt{\sigma^{-1}_{(u,s)}} - \sigma^{-n}_{(u,s)} u^\gamma$$

then $\Gamma(Y_{(u,s)}) = \{p(u,s), q(u,s)\}$.

Finally, applying lemma 4 to the function $\sigma^{-1}_{(u,s)}$ for ρ such that $1 - \rho < \delta - \dfrac{\mathrm{Log}(\lambda)}{\mathrm{Log}(\sigma)}$ we obtain that

(i) If

$$\sigma^{-n}_{(0,0)} \sqrt{\sigma^{-1}_{(0,0)}} + c(n) \leq u \leq \sigma^{-n}_{(0,0)} - c(n)$$

then $\Gamma(Y_{(u,s)}) = \{p(u,s), q(u,s), \gamma\}$, γ a periodic orbit.

(ii) If

$$\sigma_{(0,0)}^{-(n+1)} + c(n) \le u \le \sigma_{(0,0)}^{-n} \sqrt{\sigma_{(0,0)}^{-1}} - c(n)$$

then $\Gamma(Y_{(u,s)}) = \{p(u,s), q(u,s)\}$

where $\lim_{n \to \infty} \dfrac{c(n)}{\sigma_{(0,0)}^{-n}} = 0.$

Fig. 4

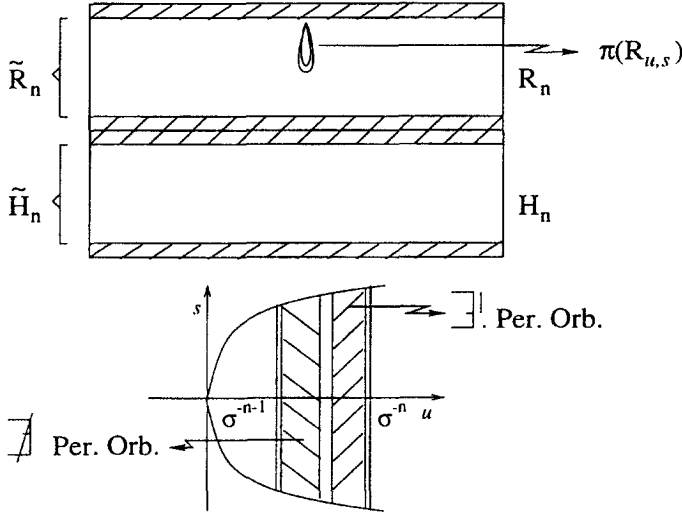The conclusion of Theorem 2 follows from the cases studied above.

3. PROOF THEOREM 3

Clearly, if $u < 0$ then $\Gamma(Y_{u,s}) = \{p_{u,s}, q_{u,s}\}$. The analysis for $u > 0$ will be divided in four cases.

1. $\mathcal{R}_1 = \{(u,s) : s \le -a_1 u^{-\frac{\text{Log}(\lambda)}{\text{Log}(\sigma)}}\}$

2. $\mathcal{R}_2 = \{(u,s) : s \ge a_2 u^{-\frac{\text{Log}(\lambda)}{\text{Log}(\sigma)}}\}$

3. $\mathcal{R}_3 = \{(u,s) : |s| \le u^{\frac{1}{\alpha}}\}$

4. $\mathcal{R}_4 = \{(u,s) : u^{\frac{1}{\alpha} - \delta} \le |s| \le a_4 u^{\delta - \frac{\text{Log}(\lambda)}{\text{Log}(\sigma)}(1+a)}\}$

where $a_1 > \tilde{K}\lambda_{(u,s)}^{-1}$, $a_2 > 2\tilde{K}\lambda_{(u,s)}^{-1}$ and $0 < a_3 < 1$.

Moreover, a and δ are chosen small so that the complement of $\cup_{i=1}^4 \mathcal{R}_i$ in $\{(u,s) : u \ge 0\}$ has zero Lebesgue density at the origin $(0,0)$.

We will use the notation of the previous section, and the fact that the maximal invariant set meets Σ_2 inside the ball centered on $(P_1(u, s), s)$ with radius $C\lambda_{(u,s)}^n$, where n in defined by the equation (4).

CASE 1. – Let $(u, s) \in \mathcal{R}_1$ be. Then we have that

$$
\begin{aligned}
s + C\lambda_{(u,s)}^n &\leq s + C\lambda_{(u,s)}^{-1} u^{-\frac{\operatorname{Log}(\lambda)}{\operatorname{Log}(\sigma)}} \\
&< s + a_1 u^{-\frac{\operatorname{Log}(\lambda)}{\operatorname{Log}(\sigma)}} \\
&\leq 0.
\end{aligned}
$$

Thus $\Gamma(Y_{u,s}) = \{p_{u,s}, q_{u,s}\}$, and from this $\mathcal{R}_1$ is a region of hyperbolicity.

CASE 2. – Let $(u, s) \in \mathcal{R}_2$. Then

$$
\begin{aligned}
s - C\lambda_{(u,s)}^n &- KC_1(u, s)u^{\frac{1}{\alpha}} \\
&> s - C\lambda_{(u,s)}^{-1} u^{-\frac{\operatorname{Log}(\lambda)}{\operatorname{Log}(\sigma)}} - KC_1(u, s)u^{\frac{1}{\alpha}} \\
&= s - u^{-\frac{\operatorname{Log}(\lambda)}{\operatorname{Log}(\sigma)}} (C\lambda^{-1} + KC_1(u, s)u^{\frac{1}{\alpha} + \frac{\operatorname{Log}(\lambda)}{\operatorname{Log}(\sigma)}}) \\
&> s - a_2 u^{-\frac{\operatorname{Log}(\lambda)}{\operatorname{Log}(\sigma)}} \\
&\geq 0
\end{aligned}
$$

since $\dfrac{1}{\alpha} + \dfrac{\operatorname{Log}(\lambda)}{\operatorname{Log}(\sigma)} > 0$.

Therefore $\Gamma(Y_{u,s}) = \{p_{u,s}, q_{u,s}\}$, thus $\mathcal{R}_2$ is a region of hyperbolicity.

In order to continue the analysis of our cases, we will give a more precise location of the non-wandering set.

LEMMA 5. – *Let u, s be such that $|s| < u^{\delta - \frac{\operatorname{Log}(\lambda)}{\operatorname{Log}(\sigma)}(1+a)}$. Then the non-wandering set is contained in the saturate of the set $C(u, s)$ defined by*

$$
\begin{aligned}
C(u, s) = \bigcup_{i=0}^{k} &[c_0 - \tau, c_0 + \tau] \times \{[\sigma_{(u,s)}^{-(n+i+1)}, \sigma_{(u,s)}^{-(n+i+1)} + \sigma_{(u,s)}^{-(n+i)} u^\gamma] \\
&\cup [\sigma_{(u,s)}^{-(n+i)} \sqrt{\sigma_{(u,s)}^{-1}} - \sigma_{(u,s)}^{-(n+i)} u^\gamma, \sigma_{(u,s)}^{-(n+i)} \sqrt{\sigma_{(u,s)}^{-1}} + \sigma_{(u,s)}^{-(n+i)} u^\gamma] \\
&\cup [\sigma_{(u,s)}^{-(n+i)} - \sigma_{(u,s)}^{-(n+i)} u^\gamma, \sigma_{(u,s)}^{-(n+i)}]\} \\
\bigcup &[c_0 - \tau, c_0 + \tau] \times [0, \sigma_{(u,s)}^{-(n+k+1)}]
\end{aligned}
$$

where k is the greater integer smaller than $n\,a$ and $0 < \gamma < \delta$. Moreover δ and a are taken so that $\delta < \dfrac{1}{\alpha} + \dfrac{\text{Log}(\lambda)}{\text{Log}(\sigma)}(1 + a)$.

Remark. – By observation b after lemma 1, we can change τ by $\tilde{K}u$.

Proof. – If $(y, w) \in (R_{n+i} \cup H_{n+i}) \cap \Gamma(Y_{(u,s)})$, then

$$0 \le B_{u,s} \circ \pi_0 \circ \pi_L^{n+i}(y, w) \le K u^{\frac{1}{\alpha}} C_1(u, s)$$

and from here

$$-s \le B_{u,s} \circ \pi_0(\lambda_{(u,s)}^{n+i} y, z) - s \le K u^{\frac{1}{\alpha}} C_1(u, s) - s$$

where $z = \sigma_{(u,s)}^{n+i} w$. Therefore

$$\frac{1}{\lambda_{(u,s)}^{n+i}} |B_{u,s} \circ \pi_0(\lambda_{(u,s)}^{n+i} y, z) - s| \le \tilde{C} u^{\gamma_1}$$

where $0 < \gamma_1 < \delta$.

From lemma 3

$$\begin{aligned}
|f_{u,s}(y,z)| &\le |f_{u,s}(y,z) - f_{n+i,u,s}(y,z)| + |f_{n+i,u,s}(y,z)| \\
&\le \tilde{C} \lambda_{(u,s)}^{n+i} + \tilde{C} u^{\gamma_1} \\
&\le \tilde{C} \lambda_{(u,s)}^{-1} u^{-\frac{\text{Log}(\lambda)}{\text{Log}(\sigma)}} + \tilde{C} u^{\gamma_1}.
\end{aligned}$$

Then the only possibilities for z are: $|z - 1| \le u^\gamma$ or $|z - \sqrt{\sigma_{(u,s)}^{-1}}| \le u^\gamma$ or $|z - \sigma_{(u,s)}^{-1}| \le u^\gamma$ with $0 < \gamma < \min\left\{\gamma_1, -\dfrac{\text{Log}(\lambda)}{\text{Log}(\sigma)}\right\}$. Iterating $(n+i)$-times by π_L^{-1} we obtain:

$$\begin{aligned}
w \in &[\sigma_{(u,s)}^{-(n+i+1)}, \sigma_{(u,s)}^{-(n+i+1)} + \sigma_{(u,s)}^{-(n+i)} u^\gamma] \\
&\cup [\sigma_{(u,s)}^{-(n+i)} \sqrt{\sigma_{(u,s)}^{-1}} - \sigma_{(u,s)}^{-(n+i)} u^\gamma, \sigma_{(u,s)}^{-(n+i)} \sqrt{\sigma_{(u,s)}^{-1}} + \sigma_{(u,s)}^{-(n+i)} u^\gamma] \\
&\cup [\sigma_{(u,s)}^{-(n+i)} - \sigma_{(u,s)}^{-(n+i)} u^\gamma, \sigma_{(u,s)}^{-(n+i)}]. \qquad\qquad \blacksquare
\end{aligned}$$

LEMMA 6. – *For all $\xi > 0$ there exists N_0 which satisfies the following property: for any $N > N_0$ and u, s with $0 < u \le u_0 = \sigma_{(0,0)}^{-N}, |s| < u^{\delta - \frac{\text{Log}(\lambda)}{\text{Log}(\sigma)}(1+a)}$ the nonwandering set is contained in the set*

$$[c_0 - \tilde{K} u_0, c_0 + \tilde{K} u_0] \times V_\xi(A_N)$$

where

$$V_\xi(A_N) = \{z \geq 0 \; s.t. \; d(z, A_N) \leq \xi u_0\}$$

with

$$A_N = \{\sigma_{(0,0)}^{-m}, \sigma_{(0,0)}^{-m}\sqrt{\sigma_{(0,0)}^{-1}}\}_{m \geq N}.$$

Proof. – From lemma 4 with $\rho > 1 - \delta + \dfrac{\text{Log}(\lambda)}{\text{Log}(\sigma)}(1 + a)$ and large N_0 we obtain that

$$
\begin{aligned}
|\sigma_{(0,0)}^{-N} - \sigma_{(u,s)}^{-N}| &\leq [\sigma_{(0,0)}^{-N}]^\rho (u + |s|) \\
&\leq u_0^{\rho+1} + u^{\rho+\delta-\frac{\text{Log}(\lambda)}{\text{Log}(\sigma)}(1+a)} \\
&\leq u_0 \tilde{\xi}
\end{aligned}
$$

where $\tilde{\xi}$ is a small real number.

In particular, it follows that

$$\sigma_{(u,s)}^{-N} \geq u_0(1 - \tilde{\xi}) \geq u(1 - \tilde{\xi}) > \sigma_{(u,s)}^{-(n+1)}(1 - \tilde{\xi})$$

for n given by the usual inequality $\sigma_{(u,s)}^{-(n+1)} < u \leq \sigma_{(u,s)}^{-n}$.

Since

$$n > N + \frac{\text{Log}(1 - \tilde{\xi})}{\text{Log}(\sigma_{(u,s)})} - 1.$$

Hence $n \geq N$, because $\tilde{\xi}$ is small. In this way applying the previous lemma we obtain the result. ∎

From the previous lemmas we obtain that $\Gamma(Y_{(u,s)})$ contains a suspended subshift of the finite type. Our objective here is to determine the parameters (u, s), and how abundant they are, for which the new non-wandering set is a suspended subshift of the finite type. For this we need to study the images, in Σ_2, through the Poincare map, of the vertical lines in a small neighborhood of $(c_0, 0) \in \Sigma_0$. In particular, we will need to compute the localizations of the criticalities of these images with respect to the horizontal lines in the section Σ_2.
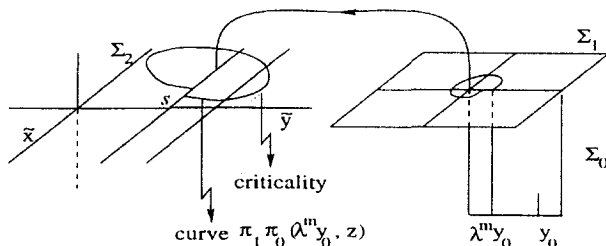

Fig. 5

LEMMA 7. – *Given u, s such that $|s| < u^{\delta - \frac{\text{Log}(\lambda)}{\text{Log}(\sigma)}(1+a)}$, $y_0 \in [c_0 - \tilde{K}u, c_0 + \tilde{K}u]$ and m a large integer, we have that $f_{m,u,s}(y_0, \cdot)$ has two critical points $z_{1,m,y_o}(u,s)$ and $z_{2,m,y_0}(u,s)$. Moreover, if we denote $\tilde{y}_{i,m,y_0}(u,s)$ for $B_{u,s} \circ \pi_0(\lambda^m y_0, z_{i,m,y_o}(u,s))$ $i = 1, 2$, then*

$$|\tilde{y}_{i,m,y_0}(u,s) - (s + \lambda_{(0,0)}^m c_0 t_i)| \le \tilde{C}\{\lambda_{(0,0)}^{2m} + [\lambda_{(0,0)}^m]^p (u + |s|)\}$$

where $c_0 t_i = f_{0,0}(c_0, z_i(0,0))$ $(t_1 > 0$ and $t_2 < 0)$ and $z_i(u,s)$, $i = 1, 2$, are the critical points of $f_{u,s}(y_0, \cdot)$, that independs on y_0.

Obs. – This lemma guarantees us that the criticalities are contained in a small neighborhood of the sequences $\{s + \lambda_{(0,0)}^m c_0 t_i\}_{m \ge 0}$.

Proof. – From definition we have that

$$\partial_z f_{u,s}(y_0, z)$$
$$= -\partial_x B_{u,s}(0,0) y_0 \frac{z^{-\frac{a}{c}-1}}{c} [-a \sin(\text{Log}(z^{-\frac{b}{c}})) - b \cos(\text{Log}(z^{-\frac{b}{c}}))].$$

Therefore, the solutions of equation

$$\partial_z f_{u,s}(y_0, z) = 0 \tag{7}$$

are the solutions of

$$tg(\text{Log}(z^{-\frac{b}{c}})) = -\frac{b}{a}. \tag{8}$$

Since $\sigma_{(u,s)}^{-1} \le z \le 1$, then $0 \le \text{Log}(z^{-\frac{b}{c}}) \le 2\pi$, therefore it has two solutions $z_1(u,s)$ and $z_2(u,s)$ with

$$\sqrt{\sigma_{(u,s)}^{-1}} < z_1(u,s) < 1 \text{ and } z_2(u,s) = e^{-\frac{c}{b}\pi} z_1(u,s).$$

Moreover, $\partial_{z^2} f_{u,s}(y_0, z_i(u,s)) \ne 0$.

From lemma 3, and for every large integer m, there are $z_{i,m,y_0}(u,s)$ close to $z_i(u,s)$ such that

$$\partial_z f_{m,u,s}(y_0, z_{i,m,y_0}(u,s)) = 0.$$

On the other hand, we have

$$|z_{i,m,y_0}(u,s) - z_i(u,s)| \le \tilde{C}\lambda_{(u,s)}^m. \tag{9}$$

In fact, from lemma 3 it follows that

$$|\partial_z f_{u,s}(y_0, z_{i,m,y_0}(u,s)) - \partial_z f_{m,u,s}(y_0, z_{i,m,y_0}(u,s))| \le \tilde{C}\lambda_{(u,s)}^m.$$

This implies

$$|\partial_z f_{u,s}(y_0, z_{i,m,y_0}(u,s)) - \partial_z f_{u,s}(y_0, z_i(u,s))| \le \tilde{C}\lambda_{(u,s)}^m$$

thus

$$|\partial_{z^2} f_{u,s}(y_0, \zeta)| \cdot |z_{i,m,y_0}(u,s) - z_i(u,s)| \le \tilde{C}\lambda_{(u,s)}^m.$$

From this inequality the claim (9) follows, because ζ is close to $z_i(u,s)$.

Now, using (9) we obtain:

$$|f_{m,u,s}(y_0, z_{i,m,y_0}(u,s)) - f_{u,s}(y_0, z_i(u,s))| \le \tilde{C}\lambda_{(u,s)}^m$$

and therefore

$$|\tilde{y}_{i,m,y_0}(u,s) - (s + \lambda_{(u,s)}^m) f_{u,s}(y_0, z_i(u,s))| \le \tilde{C}\lambda_{(u,s)}^{2m}.$$

Let $f_{u,s}(y_0, z_i(u,s)) = y_0 t_i(u,s)$. Now, arranging terms and applying lemma 4 to $\lambda_{(u,s)}$ we get

$$|\tilde{y}_{i,m,y_0}(u,s) - (s + \lambda_{(0,0)}^m c_0 t_i(0,0))| \le \tilde{C}\{\lambda_{(0,0)}^{2m} + [\lambda_{(0,0)}^m]^\rho(u + |s|)\}$$

This finishes the proof of the lemma.   ∎

CASE 3. – Let consider $(\tilde{x}_0, \tilde{y}_0) \in \Sigma_2 \cap \Gamma(Y_{(u,s)})$, and let m, y_0, z_0 be such that $(\tilde{x}_0, \tilde{y}_0) = \pi_1 \circ \pi_0(\lambda_{(u,s)}^m y_0, z_0)$
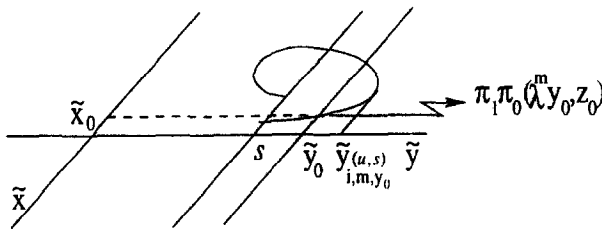

Fig. 6

Let $\varepsilon > 0$ fixed and $u_0 = \sigma_{(0,0)}^{-N}$ where N is a large integer and $|s| \le u_0^{\frac{1}{\alpha}}$. Let denote $\mathcal{B}_s(N,\varepsilon), \mathcal{B}(N,\varepsilon)$ and $\mathbf{T}_N$ the sets

$$\mathcal{B}_s(N,\varepsilon) = \{ u \in [\sigma_{(0,0)}^{-(N+1)}, \sigma_{(0,0)}^{-N}] : \text{there is } (\tilde{x}_0, \tilde{y}_0) \text{ like above,}$$

$$\text{such that } |\tilde{y}_0 - \tilde{y}_{i,m,y_0}(u,s)| \le \varepsilon u_0^{\frac{1}{\alpha}} \}$$

$$\mathcal{B}(N,\varepsilon) = \{(u,s) \text{ s.t. } |s| \le u^{\frac{1}{\alpha}}, u \in \mathcal{B}_s(N,\varepsilon)\}$$

and

$$\mathbf{T}_N = \{(u,s) : \sigma_{(0,0)}^{-(N+1)} \le u \le \sigma_{(0,0)}^{-N}, |s| \le u^{\frac{1}{\alpha}}\}.$$

Now we want to determine those parameter values (u,s) for which the chain recurrent set is a subshift of finite type. Afterwards we will show that these parameter values are prevalent. Our objective is to show the following assertions:

A. $\forall \varepsilon > 0$ $\exists N_0$ such that $\forall N \ge N_0$, $\dfrac{m(\mathcal{B}_s(N,\varepsilon))}{\sigma_{(0,0)}^{-N}} \le g(\varepsilon)$, where

$g(\varepsilon) \longrightarrow 0$ as $\varepsilon \longrightarrow o$.

B. If $(u,s) \in \mathbf{T}_N \setminus \mathcal{B}(N,\varepsilon)$ then $\Gamma(Y_{(u,s)})$ is a hyperbolic set.
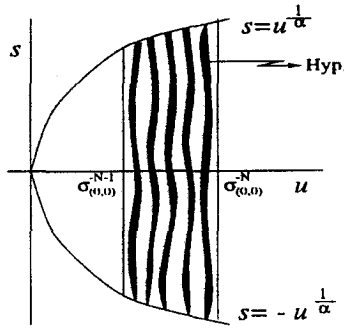

Fig. 7

Obs. – Using assertion **A** and **B** it is easy to prove that

$$\frac{m(\{(u,s) \in \mathbf{T}_N \text{ s.t. } \Gamma(Y_{(u,s)}) \text{ is hyperbolic}\})}{m(\mathbf{T}_N)} \ge 1 - \tilde{C}g(\varepsilon).$$

LEMMA 8. – *Given* $\xi > 0$ *there exists* N_0 *such that: for every* $N \ge N_0$ *and* (u,s) *such that* $\sigma_{(0,0)}^{-(N+1)} \le u \le \sigma_{(0,0)}^{-N} = u_0$, $|s| \le u_0^{\frac{1}{\alpha}}$; *and* m *satisfying* $|s + \lambda_{(0,0)}^m c_0 t_i(0,0)| \le C u_0^{\frac{1}{\alpha}}$, *we have: for every* $\tilde{x}', |\tilde{x}'| \le 1$

$$|\tilde{B}_{u,s} \circ \pi_2(\tilde{x}', \tilde{y}_{i,m,y_0}(u,s))$$
$$- (u + \partial_{\tilde{z}}\tilde{B}_{0,0}(0,0)[s + \lambda_{(0,0)}^m c_0 t_i(0,0)]^{\alpha(0,0)})| \le \xi u_0.$$

Proof. – From lemma 7 we know that

$$|\tilde{y}_{i,m,y_0}(u,s) - (s + \lambda_{(0,0)}^m c_0 t_i)| \le \tilde{C}\{\lambda_{(0,0)}^{2m} + [\lambda_{(0,0)}^m]^p(u + |s|)\}$$

On the other hand, for m like in the lemma we have that $\lambda_{(0,0)}^m \le k u_0^{\frac{1}{\alpha}}$ with k constant greater or equal than $\dfrac{C+1}{|c_0 t_i(0,0)|}$.

Thus

$$|\tilde{y}_{i,m,y_0}(u,s) - (s + \lambda_{(0,0)}^m c_0 t_i)| \le \tilde{C}\{k^2 u_0^{\frac{1}{\alpha}} + k^\rho u_0^{\frac{\rho}{\alpha} - \frac{1}{\alpha}}(u_0 + u_0^{\frac{1}{\alpha}})\} \cdot u_0^{\frac{1}{\alpha}}$$

therefore if ρ is such that $\rho > 1 - \alpha$ we obtain, for every $\xi' > 0$, that there exists N_0 such that

$$|\tilde{y}_{i,m,y_0}(u,s) - (s + \lambda_{(0,0)}^m c_0 t_i)| \le \xi' u_0^{\frac{1}{\alpha}}.$$

Moreover, it is easy to show that

$$|(\tilde{y}_{i,m,y_0}(u,s))^\alpha - (s + \lambda_{(0,0)}^m c_0 t_i)^\alpha| \le \tilde{\xi}(\xi') u_0$$

and

$$|(s + \lambda_{(0,0)}^m c_0 t_i)^\alpha - (s + \lambda_{(0,0)}^m c_0 t_i)^{\alpha(0,0)}| \le \tilde{\xi}(\xi') u_0$$

where $\tilde{\xi}(\xi') \to 0$ when $\xi' \to 0$.

Let $(\tilde{x}, \tilde{z}) = \pi_2(\tilde{x}', \tilde{y}_{i,m,y_0}(u,s))$, then

$$
\begin{aligned}
&|\tilde{B}_{u,s}(\tilde{x},\tilde{z}) - (u + \partial_{\tilde{z}}\tilde{B}_{0,0}(0,0)[s + \lambda_{(0,0)}^m c_0 t_i]^\alpha)| \\
&= |\tilde{B}_{u,s}(\tilde{x},\tilde{z}) - \tilde{B}_{u,s}(0,0) - \partial_{\tilde{z}}\tilde{B}_{0,0}(0,0)[s + \lambda_{(0,0)}^m c_0 t_i]^\alpha| \\
&\le |\tilde{B}_{u,s}(\tilde{x},\tilde{z}) - \tilde{B}_{u,s}(0,\tilde{z})| \\
&\quad + |\tilde{B}_{u,s}(0,\tilde{z}) - \tilde{B}_{u,s}(0,0) - \partial_{\tilde{z}}\tilde{B}_{0,0}(0,0)[s + \lambda_{(0,0)}^m c_0 t_i]^\alpha| \\
&\le \tilde{C}|\tilde{x}| + |\partial_{\tilde{z}}\tilde{B}_{u,s}(0,\tilde{w})\tilde{z} - \partial_{\tilde{z}}\tilde{B}_{0,0}(0,0)[s + \lambda_{(0,0)}^m c_0 t_i]^\alpha| \\
&\le \tilde{C}|\tilde{x}| + |\partial_{\tilde{z}}\tilde{B}_{u,s}(0,\tilde{w})\tilde{z} - \partial_{\tilde{z}}\tilde{B}_{u,s}(0,\tilde{w})[s + \lambda_{(0,0)}^m c_0 t_i]^\alpha| \\
&\quad + |\partial_{\tilde{z}}\tilde{B}_{u,s}(0,\tilde{w}) - \partial_{\tilde{z}}\tilde{B}_{0,0}(0,0)|[s + \lambda_{(0,0)}^m c_0 t_i]^\alpha \\
&\le \tilde{C}|\tilde{x}| + \tilde{C}|\tilde{z} - (s + \lambda_{(0,0)}^m c_0 t_i)^\alpha| \\
&\quad + |\partial_{\tilde{z}}\tilde{B}_{u,s}(0,\tilde{w}) - \partial_{\tilde{z}}\tilde{B}_{0,0}(0,0)|[s + \lambda_{(0,0)}^m c_0 t_i]^\alpha.
\end{aligned}
$$

Since $|\tilde{x}| \le \tilde{C}u_0^{\frac{\beta}{\alpha}}$ we obtain that $|\tilde{B}_{u,s}(\tilde{x},\tilde{z}) - (u + \partial_{\tilde{z}}\tilde{B}_{0,0}(0,0)[s + \lambda_{(0,0)}^m c_0 t_i]^\alpha|$ is smaller than $\tilde{C}u_0^{\frac{\beta}{\alpha}} + \tilde{C}\tilde{\xi}(\xi')u_0 + |\partial_{\tilde{z}}\tilde{B}_{u,s}(0,\tilde{w}) - \partial_{\tilde{z}}\tilde{B}_{0,0}(0,0)|C^\alpha u_0$.

From this inequality we obtain for u_0 small that

$$|\tilde{B}_{u,s} \circ \pi_2(\tilde{x}', \tilde{y}_{i,m,y_o}(u,s)) - (u + \partial_{\tilde{z}}\tilde{B}_{0,0}(0,0)[s + \lambda_{(0,0)}^m c_0 t_i]^{\alpha(0,0)})| \le \xi u_0$$

as we claimed.   ∎

Proof of the claim A. – Given $\xi > 0$, $u_0 = \sigma_{(0,0)}^{-N}$, N a large integer and $|s| \leq u_0^{\frac{1}{\alpha}}$. Let denote by $\bar{B}_N(s)$ the set

$$\bar{B}_N(s) = \{\partial_{\tilde{z}}\tilde{B}_{0,0}(0,0)[s + \lambda_{(0,0)}^m c_0 t_i]^{\alpha(0,0)}\}_{m \geq 0} \cap [-2u_0, 0] \cup \{0\}$$

and let $V_\xi(\bar{B}_N(s))$ be the neighborhood given by

$$V_\xi(\bar{B}_N(s)) = \{z : d(z, \bar{B}_N(s)) \leq \xi \cdot u_0\}.$$

Let define

$$\bar{\mathcal{B}}_s(N, \xi) = \{u \in [\sigma_{(0,0)}^{-(N+1)}, \sigma_{(0,0)}^{-N}] : d(u + V_\xi(\bar{B}_N(s)), V_\xi(A_N)) \leq \xi u_0\}.$$

The next proposition give us the proof of claim A.

PROPOSITION 1. – *Let ξ and N_0 be as in the previous lemma. Then for all $N \geq N_0$ and $|s| \leq u_0 = \sigma_{(0,0)}^{-N}$ we have that*

$$m(\bar{\mathcal{B}}_s(N, \xi) \leq \tilde{g}(\xi)\sigma_{(0,0)}^{-N}$$

where $\tilde{g}(\xi) \to 0$ when $\xi \to 0$.

Proof. – We have that $u \in \bar{\mathcal{B}}_s(N, \xi)$ if and only if $[u + V_\xi(\bar{B}_N(s))] \cap V_{2\xi}(A_N) \neq \emptyset$. Thus implies $\bar{\mathcal{B}}_s(N, \xi) \subset V_{2\xi}(A_N) - V_\xi(\bar{B}_N(s))$. It is easy to prove that $V_{2\xi}(A_N)$ can be cover with $\tilde{C} - \dfrac{\text{Log}(\xi)}{\text{Log}(\sigma_{(0,0)})}$ intervals of length $2\xi u_0$ and that $V_\xi(\bar{B}_N(s))$ can be cover by $\tilde{C}(1 + \dfrac{\text{Log}(\xi)}{\text{Log}(\lambda_{(0,0)})})$ intervals of lenght ξu_0.

From here it is obtained that

$$m(V_\xi(A_N) - V_\xi(\bar{B}_N(s))) \leq \left(\tilde{C} - \frac{\text{Log}(\xi)}{\text{Log}(\sigma_{(0,0)})}\right)\tilde{C}\left(1 + \frac{\text{Log}(\xi)}{\text{Log}(\lambda_{(0,0)})}\right)3\xi u_0$$

therefore

$$m(\bar{\mathcal{B}}_s(N, \xi)) \leq \tilde{g}(\xi)\sigma_{(0,0)}^{-N}$$

with $\tilde{g}(\xi) = (\tilde{C} + \dfrac{\text{Log}(\xi)}{\text{Log}(\sigma_{(0,0)})})\tilde{C}(1 + \dfrac{\text{Log}(\xi)}{\text{Log}(\lambda_{(0,0)})})3\xi$. This complete the proof of the proposition. ∎

To finish the proof of claim A, we will show that $\forall \varepsilon > 0$ there is $\xi = \xi(\varepsilon) > 0$ which satisfies $\xi \longrightarrow 0$ when $\varepsilon \longrightarrow 0$ and N_0 such that $\forall N \geq N_0$

$$\mathcal{B}_s(N, \varepsilon) \subset \bar{\mathcal{B}}_s(N, \xi).$$

Let $u \in \mathcal{B}_s(N, \varepsilon)$. Then there are $(\tilde{x}_0, \tilde{y}_0) \in \Sigma_2 \cap \Gamma(Y_{(u,s)})$ and m, y_0, z_0 such that $(\tilde{x}_0, \tilde{y}_0) = \pi_1 \circ \pi_0(\lambda^m y_0, z_0)$ and

$$|\tilde{y}_0 - \tilde{y}_{i,m,y_0}(u, s)| \leq \varepsilon u_0^{\frac{1}{\alpha}}.$$

Following the argument used in the proof of lemma 8 we obtain

$$|\tilde{y}_0^\alpha - \tilde{y}_{i,m,y_0}^\alpha(u, s)| \leq \tilde{\xi}(\varepsilon) u_0.$$

From lemma 8 and using that $|\tilde{y}_0^\beta - \tilde{y}_{i,m,y_0}^\beta(u, s)| \leq |\tilde{y}_0^\alpha - \tilde{y}_{i,m,y_0}^\alpha(u, s)|$, we have that

$$|\tilde{B}_{u,s} \circ \pi_2(\tilde{x}_0, \tilde{y}_0) - (u + \partial_{\tilde{z}} \tilde{B}_{0,0}(0,0)[s + \lambda_{(0,0)}^m c_0 t_i(0,0)]^{\alpha(0,0)})| \leq \xi u_0$$

for large N_0 where $\xi \longrightarrow 0$ if $\varepsilon \longrightarrow 0$.

This implies that $\tilde{B}_{u,s} \circ \pi_2(\tilde{x}_0, \tilde{y}_0) \in u + V_\xi(\bar{\mathcal{B}}_N(s))$. Moreover, $\tilde{B}_{u,s} \circ \pi_2(\tilde{x}_0, \tilde{y}_0) \in V_\xi(A_N)$. From here it follows that

$$u \in \bar{\mathcal{B}}_s(N, \xi)$$

and in this way we have that $\dfrac{m(\mathcal{B}_s(\mathcal{N}, \xi))}{\sigma_{(0,0)}^{-\mathcal{N}}} \leq \tilde{g}(\xi) = g(\varepsilon)$.

Thus, claim A is proved.   ∎

Proof claim B – To proof the claim B, we will determine lower bounds for the angles formed, in the section Σ_2, between the horizontal lines and the candidates to be unstables manifolds, *i.e.*, the images of vertical lines in a small neighborhood of $(c_0, 0)$ in the section Σ_0 through the respective Poincare Maps. Next, we will prove hyperbolicity constructing a cone field over $\Sigma_2 \cup \Gamma(Y_{(u,s)})$, *i.e.*, for each $(\tilde{x}_0, \tilde{y}_0) \in \Sigma_2 \cup \Gamma(Y_{(u,s)})$ we take a cone $C_{u,s}(\tilde{x}_0, \tilde{y}_0) \subset \Sigma_2$, such that

$$D(\pi_1 \circ \pi_0 \circ \pi_L^m \circ \pi_3 \circ \pi_2)_{(\tilde{x}_0, \tilde{y}_0)} C_{u,s}(\tilde{x}_0, \tilde{y}_0) \subset C_{u,s}(\pi_1 \circ \pi_0 \circ \pi_L^m \circ \pi_3 \circ \pi_2(\tilde{x}_0, \tilde{y}_0))$$

and, there exists constant $\Theta > 1$ such that both $\forall (\tilde{v}_1, \tilde{v}_2) \in C_{u,s}(\tilde{x}_0, \tilde{y}_0)$

$$\|D(\pi_1 \circ \pi_0 \circ \pi_L^m \circ \pi_3 \circ \pi_2)_{(\tilde{x}_0, \tilde{y}_0)}(\tilde{v}_1, \tilde{v}_2)\| \geq \Theta \|(\tilde{v}_1, \tilde{v}_2)\|$$

and

$\forall(\tilde{v}_1, \tilde{v}_2)$ in the complement of $D(\pi_1 \circ \pi_0 \circ \pi_L^m \circ \pi_3 \circ \pi_2)^{-1}_{(\tilde{\tilde{x}}_0, \tilde{\tilde{y}}_0)} C_{u,s}(\tilde{\tilde{x}}_0, \tilde{\tilde{y}}_0)$

$$\|D(\pi_1 \circ \pi_0 \circ \pi_L^m \circ \pi_3 \circ \pi_2)_{(\tilde{x}_0, \tilde{y}_0)}(\tilde{v}_1, \tilde{v}_2)\| \leq \Theta^{-1}\|(\tilde{v}_1, \tilde{v}_2)\|$$

where $(\tilde{\tilde{x}}_0, \tilde{\tilde{y}}_0) = \pi_1 \circ \pi_0 \circ \pi_L^m \circ \pi_3 \circ \pi_2(\tilde{x}_0, \tilde{y}_0)$ hold.

Step 1. – Determining the angles

Recall that $\sigma_{(0,0)}^{-(N+1)} \leq u \leq \sigma_{(0,0)}^{-N}, |s| \leq u^{\frac{1}{\alpha}}, \lambda_{(u,s)}^{\alpha}\sigma_{(u,s)} > 1$ and m are those integers satisfying $\lambda_{(u,s)}^m < \lambda_{(u,s)}^{-1} u^{-\frac{\text{Log}(\lambda)}{\text{Log}(\sigma)}}$.

Let $(\tilde{x}_0, \tilde{y}_0) \in \Sigma_2 \cap \Gamma(Y_{(u,s)})$ and let y_0, z_0 and m be such that $(\tilde{x}_0, \tilde{y}_0) = \pi_1 \circ \pi_0(\lambda_{(u,s)}^m y_0, z_0)$. Then, the angle is given by

$$|Tg \not{\star}| = \frac{|\partial_z(B_{u,s} \circ \pi_0(\lambda_{(u,s)}^m y_0, z))/_{z=z_0}|}{|\partial_z(A_{u,s} \circ \pi_0(\lambda_{(u,s)}^m y_0, z))/_{z=z_0}|}.$$

From here $|Tg \not{\star}| \geq \tilde{C}|\partial_z f_{m,u,s}(y_0, z_0)|$.

Now, it is easy to see that if $d = |\tilde{y}_0 - \tilde{y}_{i,m,y_0}(u,s)|$ we have two different bounds for $|Tg \not{\star}|$:

(i) $|Tg \not{\star}| \geq L$ when $\dfrac{d}{\lambda_{(u,s)}^m} \geq \zeta$.

L is a some constant and ζ is a small positive real number.

(ii) $|Tg \not{\star}| \geq \tilde{C}\sqrt{\dfrac{d}{\lambda_{(u,s)}^m}}$ when $\dfrac{d}{\lambda_{(u,s)}^m} \leq \zeta$.

From this, we have that for every $(u,s) \in \mathbf{T}_N \setminus \mathcal{B}(N,\varepsilon)$ either $|Tg \not{\star}| \geq L$ or $|Tg \not{\star}| \geq \tilde{C}\sqrt{\varepsilon}\sqrt{\dfrac{1}{2C}}$, because $\dfrac{d}{\lambda_{(u,s)}^m} \leq \zeta$ implies that $\lambda_{(u,s)}^m < 2Cu_0^{\frac{1}{\alpha}}$ (C some poitive constant).

Step 2. – Contruction of the cone field

Let define $C_{u,s}(\tilde{x}_0, \tilde{y}_0)$ by the vectors $(\tilde{v}_1, \tilde{v}_2) \in \Sigma_2$ such that

$$\frac{|\tilde{v}_2|}{|\tilde{v}_1|} \geq \frac{L}{2} \text{ or } \frac{|\tilde{v}_2|}{|\tilde{v}_1|} \geq \frac{\tilde{C}\sqrt{\varepsilon}}{2}\sqrt{\frac{1}{2C}}$$

depending on the previous cases (i) and (ii).

From this, it follows that if $(\tilde{v}_1, \tilde{v}_2) \in C_{u,s}(\tilde{x}_0, \tilde{y}_0)$ and $(\tilde{w}_1, \tilde{w}_2)$ defined by $(\tilde{w}_1, \tilde{w}_2) = D\pi_{2(\tilde{x}_0, \tilde{y}_0)}(\tilde{v}_1, \tilde{v}_2)$ then

$$\frac{|\tilde{w}_2|}{|\tilde{w}_1|} \geq \tilde{C}u_0^{\frac{\alpha-\beta}{\alpha}}.$$

This implies that $D(\pi_3 \circ \pi_2)_{(\tilde{x}_0, \tilde{y}_0)} C_{u,s}(\tilde{x}_0, \tilde{y}_0)$ is contained in a cone centered in a vertical line passing through $(y'_0, w'_0) = \pi_3 \circ \pi_2(\tilde{x}_0, \tilde{y}_0)$ with angle upper bounded by some positive constant. Now, we must apply π_L^m, where m is such that $\sigma_{(u,s)}^m w'_0 \in [\sigma_{(u,s)}^{-1}, 1]$. Therefore, $D(\pi_L^m \circ \pi_3 \circ \pi_2)_{(\tilde{x}_0, \tilde{y}_0)} C_{u,s}(\tilde{x}_0, \tilde{y}_0)$ is containing in a small cone centered in a vertical line passing through $(\lambda_{(u,s)}^m y'_0, z'_0)$, $z'_0 = \sigma_{(u,s)}^m w'_0$, with angle smaller than $\tilde{C} \lambda_{(u,s)}^m \sigma_{(u,s)}^{-m}$. It is easy to see that this cone is led away inside the cone centered in the tangent space to the curve $z \longmapsto \pi_1 \circ \pi_0(\lambda_{(u,s)}^m y'_0, z)$ at the point z'_0, with angle smaller than $\tilde{C} \sigma_{(u,s)}^{-m}$.

Now, if we take large N,

$$D(\pi_1 \circ \pi_0 \circ \pi_L^m \circ \pi_3 \circ \pi_2)_{(\tilde{x}_0, \tilde{y}_0)} C_{u,s}(\tilde{x}_0, \tilde{y}_0) \subset C_{u,s}(\pi_1 \circ \pi_0 \circ \pi_L^m \circ \pi_3 \circ \pi_2(\tilde{x}_0, \tilde{y}_0))$$

because in this way $\tilde{C}\sigma_{(u,s)}^{-m}$ is smaller than both $\frac{L}{2}$ in the first case and $\frac{\tilde{C}\sqrt{\varepsilon}}{2}\sqrt{\frac{1}{2K}}$ in the second one.

Moreover, the expansivity of the vectors in $C_{u,s}(\tilde{x}_0, \tilde{y}_0)$, follows straightforward from its definition. In a similar way we obtain the desired contractivity in the corresponding place.

CASE 4. – We need to study this case only when $\frac{1}{\alpha} > 1$ and $-\frac{\text{Log}(\lambda)}{\text{Log}(\sigma)} < 1$.

Let $\varepsilon > 0$ be and $u_0 = \sigma_{(0,0)}^{-N}$ with N a large integer. Given $u \leq u_0$ let define the sets

$$\tilde{\mathcal{B}}_u(N, \varepsilon) = \{|s| \leq u_0 \text{ such that there is } (\tilde{x}_0, \tilde{y}_0) = \pi_1 \circ \pi_0(\lambda_{(u,s)}^m y_0, z_0)$$
$$\in \Sigma_2 \cup \Gamma(Y_{(u,s)}) \text{ with } |\tilde{y}_{i,m,y_0}(u, s)| \leq \varepsilon u_0\}|$$
$$\tilde{\mathcal{B}}(N, \varepsilon) = \{(u, s) : u \in V_\varepsilon(A_N) \text{ or } s \in \tilde{\mathcal{B}}_u(N, \varepsilon)\}$$
$$\tilde{\mathbf{T}}_N = [0, u_0] \times [-u_0, u_0].$$

As in case 3 we can prove, in a more simple way, the following claims for a suitable choose of δ, a and ρ.

A. $\forall \varepsilon > 0 \; \exists \; N_0$ such that $\forall \; N \geq N_0$, $\frac{m(\tilde{\mathcal{B}}(N, \varepsilon))}{2 u_0^2} \leq g(\varepsilon)$ where $g(\varepsilon) \longrightarrow 0$ as $\varepsilon \longrightarrow o$.

B. If $(u, s) \in \tilde{\mathbf{T}}_N \setminus \tilde{\mathcal{B}}(N, \varepsilon)$ then $\Gamma(Y_{(u,s)})$ is a hyperbolic set.

In this case the proof is easier, because the width of the region that contains $\Sigma_2 \cup \Gamma(Y_{(u,s)})$, which has order $u_0^{\frac{1}{\alpha}}$ is smaller than εu_0. Moreover, the ratio between u_0 and the length of the interval around $s + \lambda_{(0,0)}^m c_0 t_i$ that contains the criticalities, for m such that $\lambda_{(0,0)}^m \leq C u_0$ (C some constant), tends to 0 as $u_0 \longrightarrow 0$.

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